CHAPTER III

SOLUTION METHOD

In this chapter, we briefly summarize numerical procedures employed to approximate the flexural buckling load of structures. Two key steps involved in the implementation are associated with the calculation of the minimum eigenvalue of the eigenvalue problem (2.22) and the iterative algorithm to achieve the converged (exact) buckling shape and buckling load. Before we address those two steps, an explicit expression for all involved elements stiffness matrices is given.

3.1 Element stiffness matrices

Since the shape functions $\psi_i(x)$ and $\hat{\psi}_i(x)$ are given in terms of exponential functions, all elements stiffness matrices $\mathbf{K}_{bi}$, $\mathbf{K}_{si}$, $\mathbf{K}_{l1i}$, $\mathbf{K}_{l2i}$, and $\mathbf{K}_{gi}$ can readily be obtained in an explicit form via the direct integration. Entries of these matrices are given by

$$[\mathbf{K}_{bi}]_{mn} = E_i I_i \sum_{j=1}^{4}\sum_{k=1}^{4} a_j a_k r_j r_k \Gamma_{jm}\Gamma_{kn}\left(e^{(r_j+r_k)L}-1\right)/\left(r_j+r_k\right) \tag{3.1}$$

$$[\mathbf{K}_{si}]_{mn} = \lambda_i G_i A_i \sum_{j=1}^{4}\sum_{k=1}^{4} (r_j-a_j)(r_k-a_k)\Gamma_{jm}\Gamma_{kn}\left(e^{(r_j+r_k)L}-1\right)/\left(r_j+r_k\right) \tag{3.2}$$

$$[\mathbf{K}_{l1i}]_{mn} = k_{1i} \sum_{j=1}^{4}\sum_{k=1}^{4} \Gamma_{jm}\Gamma_{kn}\left(e^{(r_j+r_k)L}-1\right)/\left(r_j+r_k\right) \tag{3.3}$$

$$[\mathbf{K}_{l2i}]_{mn} = k_{2i} \sum_{j=1}^{4}\sum_{k=1}^{4} r_j r_k \Gamma_{jm}\Gamma_{kn}\left(e^{(r_j+r_k)L}-1\right)/\left(r_j+r_k\right) \tag{3.4}$$

$$[\mathbf{K}_{gi}]_{mn} = P_i \sum_{j=1}^{4}\sum_{k=1}^{4} r_j r_k \Gamma_{jm}\Gamma_{kn}\left(e^{(r_j+r_k)L}-1\right)/\left(r_j+r_k\right) \tag{3.5}$$

It is worth noting that while roots of the characteristic equation (i.e. r_1, r_2, r_3 and r_4) can be complex numbers, it can readily be verified that all entries of the matrices $\mathbf{K}_{bi}$, $\mathbf{K}_{si}$, $\mathbf{K}_{l1i}$, $\mathbf{K}_{l2i}$, and $\mathbf{K}_{gi}$ shown above are real numbers. The arithmetic involving complex numbers can readily be treated using any standard computer languages. Explicit results for other special cases are shown in Appendix B and Appendix C.

3.2 Determination of minimum eigenvalue and corresponding eigenvector

In this investigation, a numerical technique based on a power method supplemented by the Rayleigh quotient scheme (e.g. Hamming, 1987; Chapra and Canale, 1990; and Notay, 2001) is adopted to estimate the minimum eigenvalue of the eigenvalue problem (2.22). Key steps for this iterative technique can be summarized as follows:

(i) Start the iteration by choosing an initial guess vector $\mathbf{v}_k$

(ii) Construct a vector $\mathbf{b}_k$ from a simple matrix-vector multiplication

$$\mathbf{b}_k = \hat{\mathcal{K}}_g \mathbf{v}_k \tag{3.6}$$

(iii) Obtain the update vector $\mathbf{v}_{k+1}$ by solving a system of linear equations

$$\left(\hat{\mathcal{K}}_b + \hat{\mathcal{K}}_s + \hat{\mathcal{K}}_{11} + \hat{\mathcal{K}}_{12}\right)\mathbf{v}_k = \mathbf{b}_k \tag{3.7}$$

using LU decomposition

(iv) Estimate the minimum eigenvalue, P_{0k}, by forming the Rayleigh quotient

$$P_{0k} = \frac{\mathbf{v}_{k+1}^T\left(\hat{\mathcal{K}}_b + \hat{\mathcal{K}}_s + \hat{\mathcal{K}}_{11} + \hat{\mathcal{K}}_{12}\right)\mathbf{v}_{k+1}}{\mathbf{v}_{k+1}^T \hat{\mathcal{K}}_g \mathbf{v}_{k+1}} \tag{3.8}$$

(v) Check convergence of the estimated eigenvalue from following criteria

$$\left|\frac{P_{0k} - P_{0k-1}}{P_{0k-1}}\right| < \varepsilon \tag{3.9}$$

where ε is a specified tolerance (use 10^{-9} in the present study). If the above criterion is satisfied, the iterative is terminated and the minimum eigenvalue is obtained; otherwise, return to step (II).

A flowchart demonstrating the iterative procedure for the power method and the Rayleigh quotient is shown in Figure 3.1.

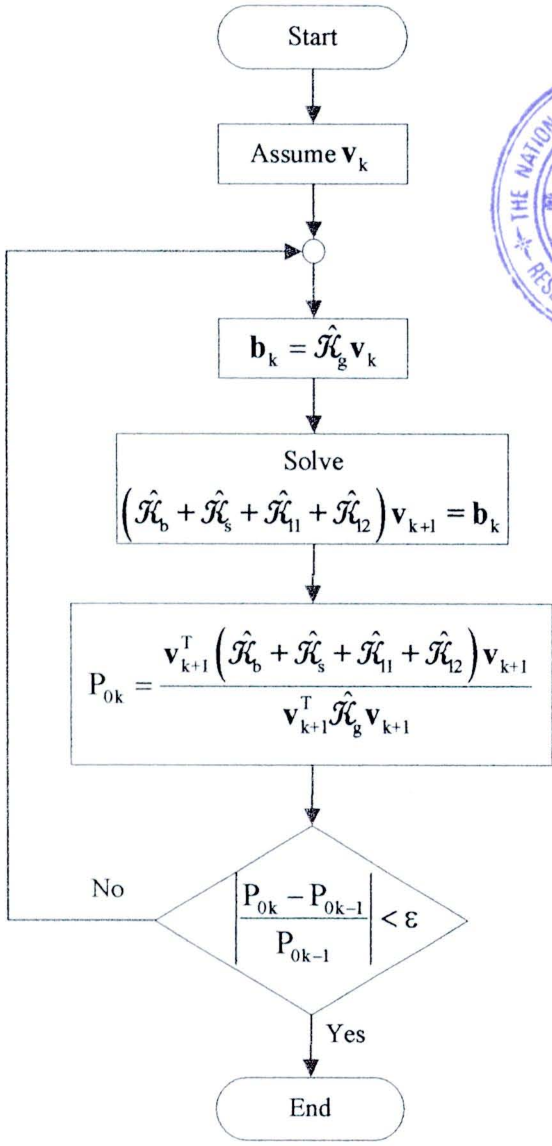

Figure 3.1 Flowchart demonstrating the power method and Rayleigh quotient for determining the minimum eigenvalue

3.3 Iterative procedure to improve buckling load

It is apparent from section 2.5 that special basis functions utilized in the approximation allow automatic adaptivity in the sense that the involved axial load parameter $\overline{P}_i$ can be varied in an arbitrary manner. In addition, based on a means that these functions were derived, they can form a space of trial functions that contain the exact buckling shape if the axial load P_i is chosen to be identical to the buckling load of the member. By using these two positive features, proper iterative schemes can be developed to improve the approximate buckling shape and, at the same time, enhance the accuracy of the buckling load estimation. In this study, we propose an iterative procedure based on a following conjecture: shape functions based on the axial load parameter $\overline{P}_i$ computed from the previous estimated buckling load provides a better approximation of the buckling shape. While the validity of this conjecture has not been confirmed mathematically, the iterative procedure implemented in this study by following this idea has been found compromising and robust (see results and discussion in Chapter 4).

The iterative procedure begins first with guessing the buckling load of all members and using this information to compute the axial load parameter $\overline{P}_i$ for each member. The shape functions based on this set of axial load parameters (i.e. those given by equations (2.37) and (2.38)) are then used in the Rayleigh-Ritz approximation for the first estimation of the buckling load. Based on above conjecture, the estimated buckling load can now be used to update the axial load parameter for each member and the shape functions based on this new axial load parameter should improve the estimation of the buckling load in the next iteration. Due to the anticipated improvement of the buckling load and buckling shape estimation in any iteration, the scheme should eventually yield a converged result comparable to the exact solution.

To update of the axial parameter $\overline{P}_i = P_i L_i^2 / E_i I_i$, it is also required to replace the modulus E_i by the tangent modulus E_{Ti}. To estimate the new tangent modulus for the next iteration, we assume that the effective length factor of each member is identical to that for the current iteration and the E_{Ti} can be obtained from (2.39) by setting $E_T = E_{Ti}$ and $\sigma = \pi^2 E_{Ti} I_i/(K_i L_i)^2 A_i$ where A_i is the cross-sectional area of the i^{th} member and K_i is the effective length factor of the i^{th} member obtained in the current iteration. The explicit expression for E_{Ti} is given by

$$E_{Ti} = \begin{cases} \sigma_0/\varepsilon_0 & ; \ \sigma/\sigma_0 \le 1 \\ \dfrac{\sigma_0/\varepsilon_0}{\sqrt[n]{n(1-B)\left(\dfrac{\pi^2 I_i}{\sigma_0 A_i (K_i L_i)^2}\right)^{n-1}}} & ; \ \sigma/\sigma_0 > 1 \end{cases} \tag{3.10}$$

Two different iterative procedures to obtain the converged buckling load are proposed in the current study as indicated in Figure 3.2 and Figure 3.3. The detailed descriptions for each procedure are given below.

First iterative procedure:

(i) Input essential data, e.g. structure geometry, member and material properties, axial load factor, etc.

(ii) Set j = 1 and N = 1, where j and N are the iteration number for updating the tangent modulus and the number of adaptive steps, and then guess the axial load parameter for all members

(iii) Compute element stiffness matrices $\mathbf{K}_{bi}$, $\mathbf{K}_{si}$, $\mathbf{K}_{11i}$, $\mathbf{K}_{12i}$, and $\hat{\mathbf{K}}_{gi}$ for all members

(iv) Assemble element stiffness matrices to obtain the unconstrained global stiffness matrices $\mathcal{K}_b$, $\mathcal{K}_s$, $\mathcal{K}_{11}$, $\mathcal{K}_{12}$ and $\mathcal{K}_g$

(v) Remove rows and columns of $\mathcal{K}_b$, $\mathcal{K}_s$, $\mathcal{K}_{11}$, $\mathcal{K}_{12}$ and $\mathcal{K}_g$ associated with degrees of freedom where the essential boundary conditions are prescribed to obtain $\hat{\mathcal{K}}_b$, $\hat{\mathcal{K}}_s$, $\hat{\mathcal{K}}_{11}$, $\hat{\mathcal{K}}_{12}$ and $\hat{\mathcal{K}}_g$

(vi) Solve the eigenvalue problem (2.22) to obtain the minimum eigenvalue P_0^N by using the iterative procedure shown in Figure 3.1

(vii) Determine the approximate buckling load for each member from $P_{N+1}^{(i)} = \alpha_i P_0^N$

(viii) Check convergence of the approximate buckling load using the criteria $\left|\dfrac{P_0^N - P_0^{N-1}}{P_0^N}\right| < \varepsilon$ where ε is a specified tolerance (use 10^{-6} in the present study). If the convergence is not achieved, then update the axial load parameter $P_{N+1}^{(i)} L^2 / E_{Ti}^j I_i$ and return to step (iii); otherwise, go to step (ix)

(ix) Obtain the converged buckling load for each member $P_{N+1}^{(i)}$ and use $P_{N+1}^{(i)}$ to calculate the tangent modulus for the next iteration E_{Ti}^{j+1} from equation (3.10)

(x) Check convergence of the tangent modulus using the criteria $\left|\frac{E_T^j - E_T^{j+1}}{E_T^j}\right| < \varepsilon$ where ε is specified tolerance (use 10^{-6} in the present study). If the convergence is achieved, obtain the final approximate buckling load for each member and terminate the procedure; otherwise, update the new axial load parameter for all members using the new tangent modulus obtained from step (ix) and return to step (iii).

Second iterative procedure:

(i) Input essential data, e.g. structure geometry, member and material properties, axial load factor, etc.

(ii) Set N = 1, where N is the number of adaptive steps, and then guess the axial load parameter for all members

(iii) Compute element stiffness matrices $\mathbf{K}_{bi}$, $\mathbf{K}_{si}$, $\mathbf{K}_{11i}$, $\mathbf{K}_{12i}$, and $\hat{\mathbf{K}}_{gi}$ for all members

(iv) Assemble element stiffness matrices to obtain the unconstrained global stiffness matrices $\mathcal{K}_b$, $\mathcal{K}_s$, $\mathcal{K}_{11}$, $\mathcal{K}_{12}$ and $\mathcal{K}_g$

(v) Remove rows and columns of $\mathcal{K}_b$, $\mathcal{K}_s$, $\mathcal{K}_{11}$, $\mathcal{K}_{12}$ and $\mathcal{K}_g$ associated with degrees of freedom where the essential boundary conditions are prescribed to obtain $\hat{\mathcal{K}}_b$, $\hat{\mathcal{K}}_s$, $\hat{\mathcal{K}}_{11}$, $\hat{\mathcal{K}}_{12}$ and $\hat{\mathcal{K}}_g$

(vi) Solve the eigenvalue problem (2.22) to obtain the minimum eigenvalue P_0^N by using the iterative procedure shown in Figure 3.1

(vii) Determine the approximate buckling load for each member from $P_{N+1}^{(i)} = \alpha_i P_0^N$ and use $P_{N+1}^{(i)}$ to calculate the tangent modulus for the next iteration E_{Ti}^{N+1} from equation (3.10)

(viii) Check convergence of the approximate buckling load using the criteria $\left|\frac{P_0^N - P_0^{N-1}}{P_0^N}\right| < \varepsilon$ and convergence of the tangent modulus using the criteria $\left|\frac{E_{Ti}^N - E_{Ti}^{N+1}}{E_{Ti}^N}\right| < \varepsilon$ where ε is a specified tolerance (use 10^{-6} in the present study). If the convergence is not achieved, then update the axial load parameter $P_{N+1}^{(i)}L^2 / E_{Ti}^{N+1}I_i$ and return to step (iii); otherwise, go to step (ix)

(ix) Obtain the converged buckling load for each member $P_{N+1}^{(i)}$ and show results

It should be noted that the key difference between the two iterative procedures is associated with the update of the tangent modulus for each member. For the first scheme, the tangent modulus for each member (E_{Ti}^{j}) is estimated before enter the loop for updating the adaptive shape functions using the information of the previous converged buckling load and this tangent modulus is held constant for all iterations in the inside loop. For the second scheme, the tangent modulus and the adaptive shape functions are updated simultaneously within a single loop for iterations. From extensive numerical experiments, both schemes yielded the same converged results but the second procedure requires less computational time.

We also remark that to accelerate the convergence rate of the minimum eigenvalue computation, an initial guess of the eigenvector in the N^{th} adaptive step is chosen from the converged eigenvector obtained in the $(N-1)^{th}$ adaptive step. The number of adaptive steps required for obtaining the converged buckling load for a specified tolerance and the number of iterations required in the computation of the minimum eigenvalue in each adaptive step are thoroughly investigated to demonstrate the computational efficiency of the developed technique.

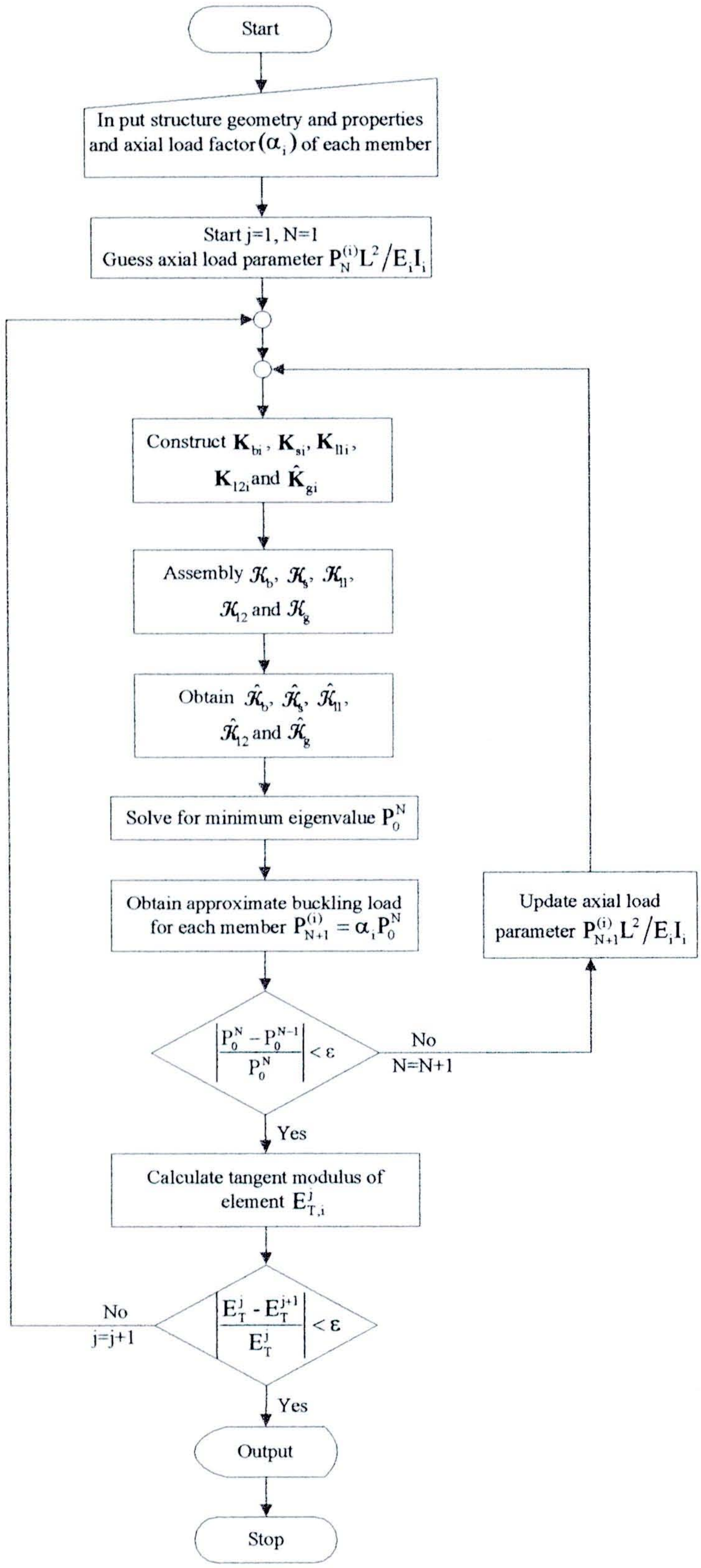

Figure 3.2 Flowchart demonstrating iterative procedure to obtain converged buckling load of first numerical scheme

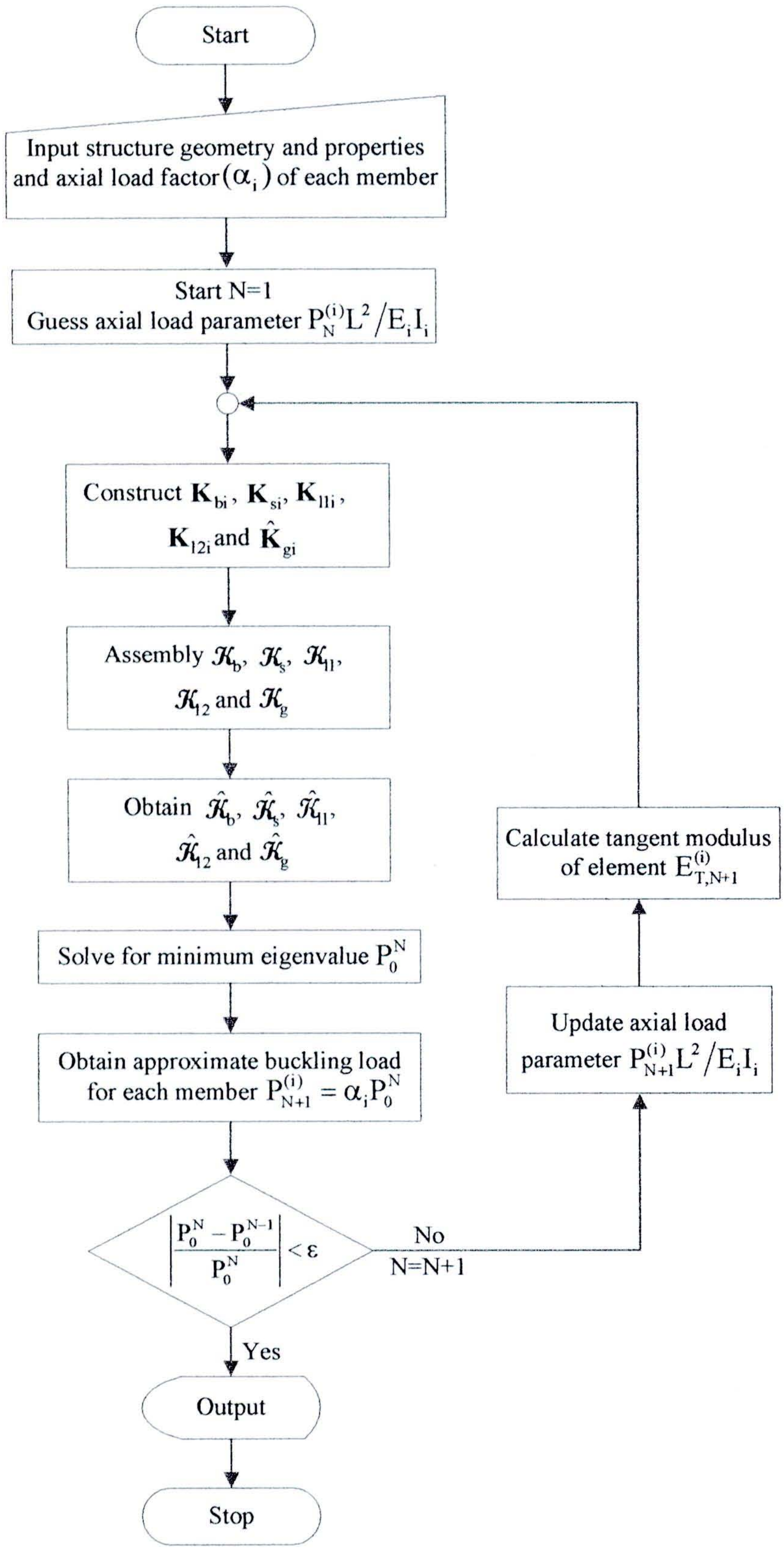

Figure 3.3 Flowchart demonstrating iterative procedure to obtain converged buckling load of second numerical scheme