CHAPTER II

THEORETICAL CONSIDERATIONS

In this chapter, several theoretical aspects relevant to the present study including the problem statement, the variational formulation for a buckling problem by the principle of stationary total potential energy, construction of an approximate characteristic equation by a Rayleigh-Ritz approximation scheme, a direct assembly procedure to form a discretized eigenvalue problem for the entire structure, and the construction of adaptive shape functions used in the approximation of the buckling shape, are summarized.

2.1 Problem statement

Consider a two-dimensional, axially loaded, initial-imperfection free, skeleton structure as shown schematically in Figure 2.1. The structure can consist of either a single prismatic member or multiple prismatic members with different cross sectional properties. In addition, for each individual member, restraints against the lateral movement or rotation, modeled either by a concentrated elastic spring or the uniformly distributed elastic spring, may be present. All members are assumed to be made of homogeneous and isotropic materials. The overall structure is properly constrained to prevent all possible in-plane rigid body motions whereas it is fully constrained against the out-of-plane displacement.

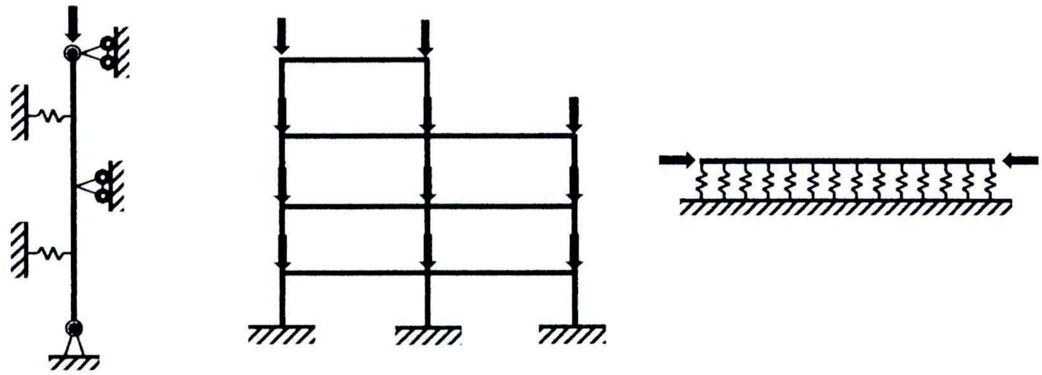


Figure 2.1 Two dimensional axially loaded structures focused in the current investigation

The problem statement, for this particular study, is to determine the flexural buckling load of structures described above by including the influence of the lateral restraint, shear deformation and nonlinear behavior of constituting materials in the mathematical model. The influence of axial deformation is assumed to be relatively small and is therefore discarded in the current investigation.

2.2 Variational formulation

A variational formulation governing the flexural buckling load of a structure defined in the problem statement is established using the principle of stationary total potential energy as briefly described below.

At the onset of buckling, the load potential functional associated with the axial load in the member undergoing the axial shortening resulting only from the curvature, denoted by W, is given by

$$W = -\sum_{i=1}^{m} \frac{1}{2} \int_{0}^{L_i} P_i \left(\frac{dv}{dx} \right)^2 dx \tag{2.1}$$

where $v = v(x)$ is the transverse displacement (or deflection) of a member, m is the number of axially loaded members in the structure, and L_i and P_i are the length and axial load of the i^{th} member, respectively. The strain energy functional of the structure, denoted by U, due to bending deformation, shear deformation and deformation of elastic lateral restraints is given by (see also the work of Seemapholkul, 2000)

$$U = U_b + U_s + U_l \tag{2.2}$$

where

$$U_b = \sum_{i=1}^{m} \frac{1}{2} \int_{0}^{L_i} E_i I_i \kappa_i^2 dx \tag{2.3a}$$

$$U_s = \sum_{i=1}^{m} \frac{1}{2} \int_{0}^{L_i} \lambda_i G_i A_i \gamma_i^2 dx \tag{2.3b}$$

$$U_l = \sum_{i=1}^{m} \frac{1}{2} \int_0^{L_i} k_{1i} v^2 dx + \sum_{i=1}^{m} \frac{1}{2} \int_0^{L_i} k_{2i} \left(\frac{dv}{dx} \right)^2 dx \tag{2.3c}$$

where U_b is the strain energy due to bending deformation, κ_i is the curvature, and $E_i I_i$ is the flexural rigidity of the cross section at the onset of buckling; U_s is the strain energy due to shear deformation, γ_i is the shear angle of the cross section, and λ_i and $G_i A_i$ are the shear correction factor and the shear rigidity of the cross section; and U_l is the strain energy due to deformation of elastic lateral restraints and k_{1l} and k_{2l} are constants associated with the two-parameter foundation model (1.3). From kinematics, the curvature and shear angle at any cross section can be related to the transverse displacement v and the rotation of the cross sectional β by

$$\gamma_i = \frac{dv}{dx} - \beta \tag{2.4}$$

$$\kappa_i = \frac{d\beta}{dx} \tag{2.5}$$

where dv/dx represents the slope of the member axis. The total potential energy of the structure at the onset of the buckling, denoted by Π, is given by

$$\Pi = U + W \tag{2.6}$$

From the principle of stationary total potential energy, the deformed state is an equilibrium state if and only if the total potential energy is stationary or, equivalently, the first variation of Π must identically vanish, i.e.

$$\delta\Pi = \delta U + \delta W = 0 \tag{2.7}$$

where δ denotes the first variation of a functional. It is worth noting that equation (2.7) is in fact the static equilibrium equation of the structure formulating based on the deformed configuration. By inserting equations (2.1) and (2.2) into equation (2.7), it leads to

$$\sum_{i=1}^{m} \int_0^{L_i} E_i I_i \kappa_i \delta\kappa_i dx + \sum_{i=1}^{m} \int_0^{L_i} \lambda_i G_i A_i \gamma_i \delta\gamma_i dx + \sum_{i=1}^{m} \int_0^{L_i} k_{1i} v \delta v dx + \sum_{i=1}^{m} \int_0^{L_i} k_{2i} v' \delta v' dx = \sum_{i=1}^{m} \int_0^{L_i} P_i v' \delta v' dx \tag{2.8}$$

The weak-form equation (2.8) forms a basis for the development of an approximate characteristic equation to estimate the flexural buckling load.

2.3 Characteristic equation for single element

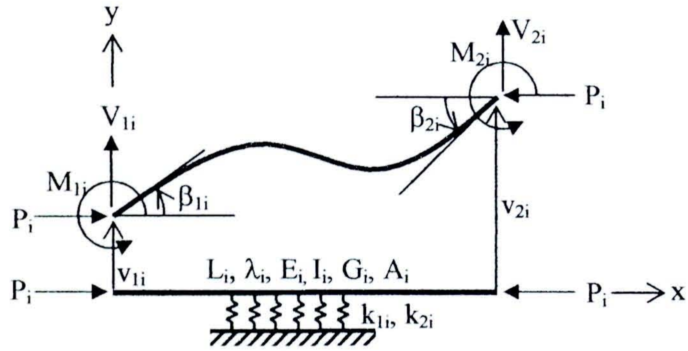


Figure 2.2 Undeformed and deformed configurations of generic i^{th} element of length L_i, axial load P_i, and properties $\{E_i, I_i, \lambda_i, G_i, A_i, k_{1i}, k_{2i}\}$

Consider the generic i^{th} element of length L_i, axial load P_i, and properties $\{E_i, I_i, \lambda_i, G_i, A_i, k_{1i}, k_{2i}\}$. Its undeformed configuration and deformed configuration at the onset of buckling are illustrated in Figure 2.2. The total potential energy of this generic member due to all effects at the buckling state is given by

$$\Pi_i = \frac{1}{2}\left\{\int_0^{L_i} E_i I_i \kappa_i^2 dx + \int_0^{L_i} \lambda_i G_i A_i \gamma_i^2 dx + \int_0^{L_i} k_{1i} v^2 dx + \int_0^{L_i} k_{2i} v'^2 dx - \int_0^{L_i} P_i v'^2 dx\right\}$$
$$-V_{1i} v_{1i} - M_{1i}\beta_{1i} - V_2 v_{2i} - M_2 \beta_{2i} \qquad (2.9)$$

where $\{V_{1i}, M_{1i}, V_{2i}, M_{2i}\}$ denote the shear forces and bending moments at both ends of the member induced in the deformed state and $\{v_{1i}, \beta_{1i}, v_{2i}, \beta_{2i}\}$ denote the transverse displacements and cross sectional rotations at both ends of the member. Note that the last four terms on the right hand side of (2.9) are associated with the load potential produced by $\{V_{1i}, M_{1i}, V_{2i}, M_{2i}\}$. For the deformed state $v = v(x)$ and $\beta = \beta(x)$ to be an equilibrium state, the total potential energy Π_i must be stationary, i.e.

$$\delta\Pi_i = \int_0^{L_i} E_i I_i \kappa_i \delta\kappa_i dx + \int_0^{L_i} \lambda_i G_i A_i \gamma_i \delta\gamma_i dx + \int_0^{L_i} k_{1i} v\delta v dx + \int_0^{L_i} k_{2i} v'\delta v' dx - \int_0^{L_i} P_i v'\delta v' dx$$

$$-V_{1i}\delta v_{1i} - M_{1i}\delta\beta_{1i} - V_2\delta v_{2i} - M_2\delta\beta_{2i} = 0 \tag{2.10}$$

By following Rayleigh-Ritz approximation, the buckling shape of the member is approximated by

$$v(x) = \psi_1(x)v_{1i} + \psi_2(x)\beta_{1i} + \psi_3(x)v_{2i} + \psi_4(x)\beta_{2i} \tag{2.11}$$

$$\beta(x) = \hat{\psi}_1(x)v_{1i} + \hat{\psi}_2(x)\beta_{1i} + \hat{\psi}_3(x)v_{2i} + \hat{\psi}_4(x)\beta_{2i} \tag{2.12}$$

where $\psi_i(x)$ and $\hat{\psi}_i(x)$ are prescribed shape functions. By substituting (2.11) and (2.12) into (2.10) and then employing arbitrariness of $\{\delta v_{1i}, \delta\beta_{1i}, \delta v_{2i}, \delta\beta_{2i}\}$, it leads to a set of characteristic equations for the $\mathbf{i}^{th}$ member:

$$\left(\mathbf{K}_{bi} + \mathbf{K}_{si} + \mathbf{K}_{l1i} + \mathbf{K}_{l2i} - \mathbf{K}_{gi}\right)\mathbf{u} = \mathbf{f} \tag{2.13}$$

where $\mathbf{u} = \{v_{1i}, \beta_{1i}, v_{2i}, \beta_{2i}\}^T$ is a vector containing nodal degrees of freedom of the member, $\mathbf{f} = \{V_{1i}, M_{1i}, V_{2i}, M_{2i}\}^T$ is a vector of member end forces, $\mathbf{K}_{bi}$, $\mathbf{K}_{si}$, $\mathbf{K}_{l1i}$, $\mathbf{K}_{l2i}$, and $\mathbf{K}_{gi}$ are element stiffness matrices with their entries defined by

$$[\mathbf{K}_{bi}]_{mn} = \int_0^{L_i} E_i I_i \hat{\psi}'_m \hat{\psi}'_n dx \tag{2.14}$$

$$[\mathbf{K}_{si}]_{mn} = \int_0^{L_i} \lambda_i G_i A_i \left(\psi'_m - \hat{\psi}_m\right)\left(\psi'_n - \hat{\psi}_n\right) dx \tag{2.15}$$

$$[\mathbf{K}_{l1i}]_{mn} = \int_0^{L_i} k_{1i} \psi_m \psi_n dx \tag{2.16}$$

$$[\mathbf{K}_{l2i}]_{mn} = \int_0^{L_i} k_{2i} \psi'_m \psi'_n dx \tag{2.17}$$

$$[\mathbf{K}_{gi}]_{mn} = \int_0^{L_i} P_i \psi'_m \psi'_n dx \tag{2.18}$$

where $[\mathbf{A}]_{mn}$ denotes an entry located at the $\mathbf{m}^{th}$ row and the $\mathbf{n}^{th}$ column of a matrix $\mathbf{A}$. By introducing a relation $P_i = \alpha_i P_0$ where P_0 is a reference axial load and α_i is a load scaling factor for the $\mathbf{i}^{th}$ member, the characteristic equations (2.13) now become

$$\left(\mathbf{K}_{bi} + \mathbf{K}_{si} + \mathbf{K}_{11i} + \mathbf{K}_{12i} - P_0 \hat{\mathbf{K}}_{gi}\right)\mathbf{u} = \mathbf{f} \tag{2.19}$$

where $\hat{\mathbf{K}}_{gi}$ is defined by

$$\left[\hat{\mathbf{K}}_{gi}\right]_{mn} = \int_0^{L_i} \alpha_i \psi'_m \psi'_n dx \tag{2.20}$$

2.4 Discretized eigenvalue problem for entire structure

By employing equilibrium and continuity at all nodes of the structure, the characteristic equations (2.19) for all members can be assembled into a set of characteristic equations for the entire structure using the same procedure as that for the direct stiffness method (e.g. Gallagher et al., 2000; Kassimali, 2005). The global characteristic equation can be expressed in a matrix form by

$$\left(\mathcal{K}_b + \mathcal{K}_s + \mathcal{K}_{11} + \mathcal{K}_{12} - P_0 \mathcal{K}_g\right)\mathbf{U} = \mathbf{F} \tag{2.21}$$

where $\mathcal{K}_b$, $\mathcal{K}_s$, $\mathcal{K}_{11}$, $\mathcal{K}_{12}$ and $\mathcal{K}_g$ are unconstrained stiffness matrices of the structure resulting from the direct assembly of $\mathbf{K}_{bi}$, $\mathbf{K}_{si}$, $\mathbf{K}_{11i}$, $\mathbf{K}_{12i}$, and $\hat{\mathbf{K}}_{gi}$, respectively, $\mathbf{U}$ is a vector of nodal degrees of freedom of the corresponding unconstrained structure, and $\mathbf{F}$ is a vector of nodal forces. It is worth noting that for buckling problems, all entries of the vector $\mathbf{F}$ vanishes except those associated with the constrained degrees of freedom where non-zero reactions induced during buckling may exist. By further enforcing the essential boundary conditions at all supports via proper removal of rows and columns of $\mathcal{K}_b$, $\mathcal{K}_s$, $\mathcal{K}_{11}$, $\mathcal{K}_{12}$ and $\mathcal{K}_g$, it leads to a discretized eigenvalue problem governing the approximate reference buckling load of the structure P_0:

$$\left(\hat{\mathcal{K}}_b + \hat{\mathcal{K}}_s + \hat{\mathcal{K}}_{11} + \hat{\mathcal{K}}_{12} - P_0 \hat{\mathcal{K}}_g\right)\hat{\mathbf{U}} = \mathbf{0} \tag{2.22}$$

where $\hat{\mathcal{K}}_b$, $\hat{\mathcal{K}}_s$, $\hat{\mathcal{K}}_{11}$, $\hat{\mathcal{K}}_{12}$, $\hat{\mathcal{K}}_g$ and $\hat{\mathbf{U}}$ are reduced stiffness matrices and a vector of free degrees of freedom after the treatment of essential boundary conditions.

2.5 Construction of special basis functions

Another crucial component of the present study is a set of adaptive shape functions used in the Rayleigh-Ritz approximation (2.11)-(2.12). These shape functions can be constructed directly from an exact solution of the ordinary differential equations governing the buckling shape of each member as described below.

First, the differential equations governing the buckling shape $v = v(x)$ and $\beta = \beta(x)$ of the i^{th} member can readily be obtained by applying the stationary principle (2.10) along with the relations (2.4)-(2.5) and the arbitrariness of $\delta v(x)$ and $\delta\beta(x)$ (see details of derivation in Seemapholkul, 2000). This finally leads to a pair of fully coupled differential equations

$$(P_i - k_{2i} - \lambda_i G_i A_i)\frac{d^2 v}{dx^2} + \lambda_i G_i A_i \frac{d\beta}{dx} + k_{1i} v = 0 \tag{2.23}$$

$$E_i I_i \frac{d^2\beta}{dx^2} + \lambda_i G_i A_i \left(\frac{dv}{dx} - \beta\right) = 0 \tag{2.24}$$

By taking derivative of (2.23) with respect to x and then solving for $d^2\beta/dx^2$, it yields

$$\frac{d^2\beta}{d\bar{x}^2} = \left(1 - \bar{\eta}_i \bar{P}_i + \bar{\eta}_i \bar{k}_{2i}\right)\frac{d^3\bar{v}}{d\bar{x}^3} - \bar{\eta}_i \bar{k}_{1i}\frac{d\bar{v}}{d\bar{x}} \tag{2.25}$$

where $\bar{x} = x/L$, $\bar{v} = v/L$, $\bar{\eta}_i = E_i I_i / \lambda_i G_i A_i L_i^2$, $\bar{P}_i = P_i L_i^2 / E_i I_i$, $\bar{k}_{1i} = k_{1i} L_i^4 / E_i I_i$ and $\bar{k}_{2i} = k_{2i} L_i^2 / E_i I_i$. By inserting (2.25) into (2.24), we then obtain the explicit expression for the rotation β in terms of the displacement v as

$$\beta = \bar{\eta}_i \left(1 - \bar{\eta}_i \bar{P} + \bar{\eta}_i \bar{k}_{2i}\right)\frac{d^3\bar{v}}{d\bar{x}^3} + \left(1 - \bar{\eta}_i^2 \bar{k}_{1i}\right)\frac{d\bar{v}}{d\bar{x}} \tag{2.26}$$

By substituting (2.26) into (2.23), it leads to a governing differential equation for the displacement $\bar{v}$:

$$\frac{d^4\bar{v}}{d\bar{x}^4} + 2\bar{\varpi}\frac{d^2\bar{v}}{d\bar{x}^2} + \bar{\xi}\bar{v} = 0 \tag{2.27}$$

where

$$\bar{\varpi} = \frac{\left(\bar{P}_i - \bar{k}_{2i} - \bar{\eta}_i \bar{k}_{1i}\right)}{2\left(1 - \bar{\eta}_i \bar{P}_i + \bar{\eta}_i \bar{k}_{2i}\right)} \tag{2.28}$$

$$\bar{\xi} = \frac{\bar{k}_{1i}}{\left(1 - \bar{\eta}_i \bar{P}_i + \bar{\eta}_i \bar{k}_{2i}\right)} \tag{2.29}$$

The general solution of equation (2.27) takes the form

$$\bar{v}(\bar{x}) = C_1 e^{r_1 \bar{x}} + C_2 e^{r_2 \bar{x}} + C_3 e^{r_3 \bar{x}} + C_4 e^{r_4 \bar{x}} \tag{2.30}$$

where C_1, C_2, C_3 and C_4 are arbitrary constant and r_1, r_2, r_3 and r_4 are distinct roots of the following characteristic equation:

$$r^4 + 2\bar{\varpi} r^2 + \bar{\xi} = 0 \tag{2.31}$$

By substituting (2.30) into (2.26), we then obtain

$$\beta(\bar{x}) = \bar{C}_1 e^{r_1 \bar{x}} + \bar{C}_2 e^{r_2 \bar{x}} + \bar{C}_3 e^{r_3 \bar{x}} + \bar{C}_4 e^{r_4 \bar{x}} \tag{2.32}$$

where

$$\bar{C}_m = \left\{\bar{\eta}_i \left(1 - \bar{\eta}_i \bar{P}_i + \bar{\eta}_i \bar{k}_{2i}\right) r_m^3 + \left(1 - \bar{\eta}_i^2 \bar{k}_{2i}\right) r_m \right\} C_m \tag{2.33}$$

By enforcing following essential boundary conditions at the both ends of a member

$$\bar{v}(0) = v_{1i}/L \tag{2.34a}$$

$$\beta(0) = \beta_{1i}/L \tag{2.34b}$$

$$\bar{v}(1) = v_{2i}/L \tag{2.34c}$$

$$\beta(1) = \beta_{2i}/L \tag{2.34d}$$

along with using the relation (2.34), it yields a system of linear algebraic equations for C_i. Once C_i are solved and $\bar{C}_i$ are determined from (2.33), they are inserted into (2.30) and (2.32) to obtain the buckling shape in terms of the nodal degrees of freedom $\{v_{1i}, \beta_{1i}, v_{2i}, \beta_{2i}\}$:

$$\bar{v}(\bar{x}) = \psi_1(\bar{x})\bar{v}_{1i} + \psi_2(\bar{x})\beta_{1i} + \psi_3(\bar{x})\bar{v}_{2i} + \psi_4(\bar{x})\beta_{2i} \quad (2.35)$$

$$\beta(\bar{x}) = \hat{\psi}_1(\bar{x})\bar{v}_{1i} + \hat{\psi}_2(\bar{x})\beta_{1i} + \hat{\psi}_3(\bar{x})\bar{v}_{2i} + \hat{\psi}_4(\bar{x})\beta_{2i} \quad (2.36)$$

where $\psi_i(x)$ and $\hat{\psi}_i(x)$ are given by

$$\psi_i(\bar{x}) = \sum_{m=1}^{4} \Gamma_{mi} e^{r_m \bar{x}} \quad (2.37)$$

$$\hat{\psi}_i(\bar{x}) = \sum_{m=1}^{4} a_m \Gamma_{mi} e^{r_m \bar{x}} \quad (2.38)$$

in which the constants a_m and Γ_{mi} are given explicitly in Appendix A. It should be noted that the shape functions (2.37) and (2.38) are applicable for the case that r_1, r_2, r_3 and r_4 are all distinct. The shape functions $\psi_i(x)$ and $\hat{\psi}_i(x)$ for some special cases that (2.31) admits repeated roots are shown in Appendix B and Appendix C.

It is apparent that the shape functions obtained above can be used to generate a trial function that assumes the same function form as the exact buckling shape of the structure. The only difference is that the axial load parameter $\bar{P}_i$ appearing in such shape functions takes arbitrary value and is generally not the same as the buckling load which is unknown a priori. This special trial function, when incorporated with a selected iterative procedure to improve the axial load parameter, can converge to the exact buckling shape. Once the trial function converges to the exact buckling shape, the approximate buckling load estimated by the principle of stationary total potential energy also converges to the exact buckling load.

2.6 Inelastic material model

To model the inelastic flexural buckling, the tangent modulus theory proposed by Engesser (1889) is employed. A model for the stress-strain relationship selected for the present investigation consists of both linear and inelastic regimes described by

$$\varepsilon/\varepsilon_0 = \begin{cases} \sigma/\sigma_0 & ; \ \sigma/\sigma_0 \le 1 \\ B + (1-B)(\sigma/\sigma_0)^n & ; \ \sigma/\sigma_0 > 1 \end{cases} \quad (2.39)$$

where σ_0 and ε_0 are reference stress and strain and B and n are material constants. It is evident that this constitutive model includes following special cases: (i) a linear stress-strain relation if choosing $B = 0$ and $n = 1$, (ii) a bilinear stress-strain relation if choosing for $B < 0$ and $n = 1$, and a nonlinear stress-strain relation with a continuous tangent modulus at $\sigma/\sigma_0 = 1$ if choosing $B = 1 - 1/n$ and $n > 1$. Plots of the stress-strain relation (2.39) for various exponent n are shown in Figure 2.3(a) for the general case and in Figure 2.3(b) for the case that the tangent modulus is entirely continuous (i.e. $B = 1 - 1/n$). It is evident that the extent of material nonlinearity is governed by the exponent n; more specifically, a material exhibits higher nonlinearity for larger n.

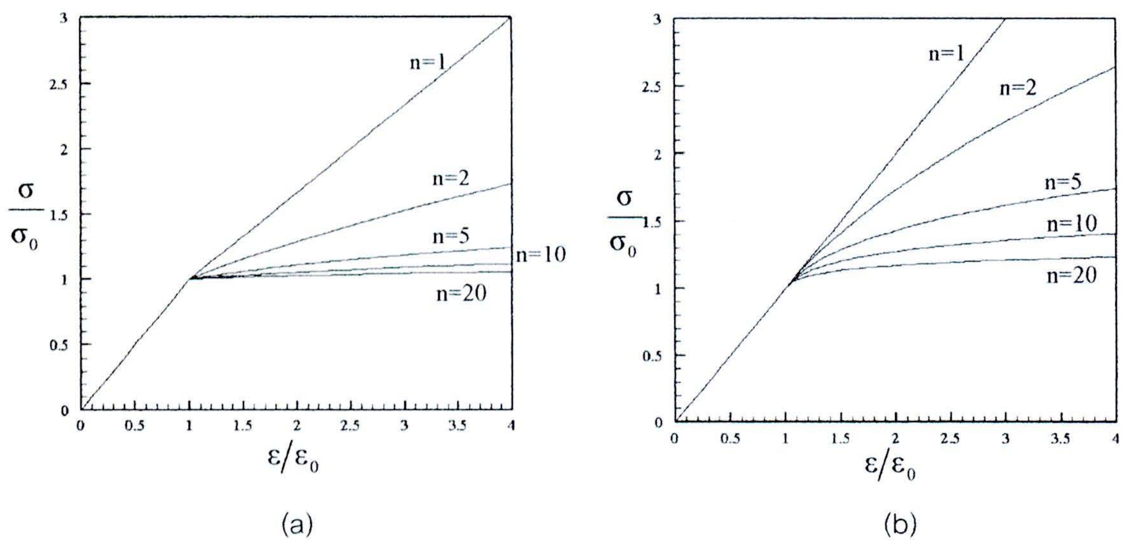


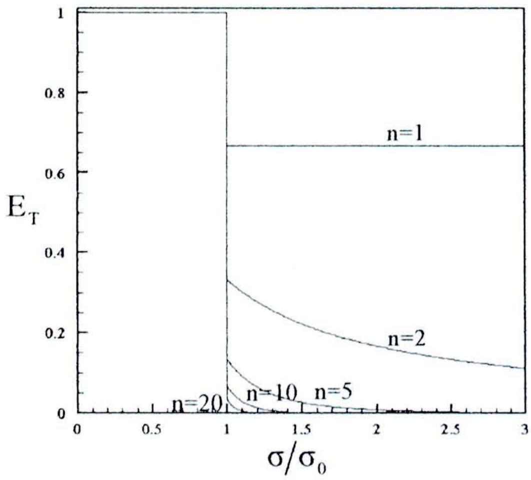
Figure 2.3 Stress-strain relation governed by (2.39): (a) general case with $B = -0.5$ and (b) $B = 1 - 1/n$

The tangent modulus, denoted by E_T, of the stress-stress model (2.39) can readily be obtained by a direct differentiation and the final result is given by

$$E_T = \begin{cases} \sigma_0/\varepsilon_0 & ; \ \sigma/\sigma_0 \le 1 \\ \dfrac{\sigma_0/\varepsilon_0}{n(1-B)(\sigma/\sigma_0)^{n-1}} & ; \ \sigma/\sigma_0 > 1 \end{cases} \qquad (2.39)$$

Plots of the tangent modulus versus the stress level (σ/σ_0) are shown in Figure 2.4 for both the general case and the case that $B = 1 - 1/n$. It is clear that in general, the model

(2.39) yields a finite jump of the tangent modulus at $\sigma/\sigma_0 = 1$ except for the case that B is chosen equal to $1 - 1/n$.



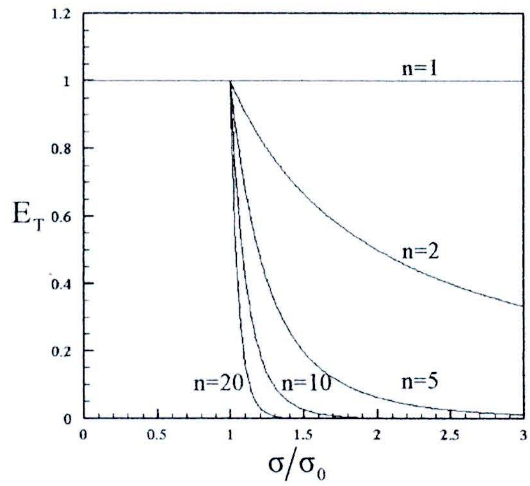


Figure 2.4 Tangent modulus versus stress level (σ/σ_0): (a) general case with $B = -0.5$ and (b) $B = 1 - 1/n$