CHAPTER I

INTRODUCTION

1.1 General

In design practice, buckling is commonly known as one of dominant modes of failure of structures consisting of slender, axially loaded members. Buckling is particularly dangerous because it can lead to catastrophic failure that generally provides no warning. A value of load associated with the buckling state is commonly defined as the load at which the structure switches from an equilibrium configuration in which all members remains straight to other equilibrium configurations where either certain or all members possess non-straight or twist configurations. In general, the load at buckling (as defined above) and the corresponding deformed configuration, known as the buckling mode shape, of structures containing perfectly straight, axially loaded members always exists but is not unique and depends primarily on various factors such as the geometry of the structure, loading patterns, boundary conditions, behavior of constituting materials, lateral constraints, etc. The lowest value among these loads is commonly termed the *buckling load* of the structure. Knowledge of the buckling load is not only useful in the design consideration of axially loaded slender structures but also essential in the analysis and design of structures subjected to combined axial and bending loads. For instance, in the recent design specification for steel buildings (e.g. ANSI/AISC 360-05), information of the elastic buckling load must be supplied to the design equations, in terms of the effective length factor, for both compression members and members in flexure and compression. Similarly, an interaction equation recommended by ACI 318-05 for designing long reinforced concrete columns under combined compression and bending moment also necessitates the effective length factor of those members. While the overall buckling behavior (e.g. the entire structure loosing their stability) depends primarily on the symmetry of the cross section and can be classified into several modes (e.g. ANSI/AISC 360-05; Salmon and Johnson, 1996; McCormac, 1994), e.g. flexural buckling, torsional buckling, flexural-torsional buckling, the flexural buckling has become one of the failure modes that is mostly encountered in the design practice of compression members and beam-columns and is the main focus of the current investigation.

A method used to calculate the buckling load must be properly selected in order to provide reasonably accurate results with acceptable computational cost and effort. In general, techniques used in the analysis for the buckling load can be classified into three main categories, i.e. analytical techniques, semi-analytical technique, and numerical techniques. Analytical techniques, introduced since the toddler age of this area and continuously used until now, are based primarily on solving the governing differential equation along with determining a solution of the exact eigenvalue problem. Besides its positive feature to yield exact value of the buckling load, the method itself poses several drawbacks. For instance, the associated eigenvalue problem generally yields nonlinear equations involving functions of a transcendental form and, more importantly, it experiences mathematical difficulty when geometry of the structure, member properties and boundary conditions become increasingly complicated. In particular, as the complexity and number of characteristic equations increase, determination of the minimum eigenvalue in an analytical fashion is impossible.

To avoid the direct solving of governing differential equations and corresponding eigenvalue problems, an attractive alternative is to seek an explicit expression to estimate the buckling load. The most recognized, ready-for-use, analytical-based expression is Euler's formula, i.e. $P_{cr} = \pi^2 EI/(KL)^2$ where E represents Young's modulus, I is the area moment of inertia of the cross section, L denotes the unsupported length of the member, and K is a parameter reflecting the end conditions commonly termed the effective length factor (e.g. Timoshenko and Gere, 1961). Besides the popularity gaining from its simple form and the explicit indication of factors affecting the buckling load, this formula still possesses a major drawback associated with the difficulty to estimate the effective length factor K especially for columns in multi-story frames. A rough approximation of the effective length factor for columns in both sway and non-sway frames can be achieved by using alignment charts (e.g. Gaylord et al., 1992; McCormac, 1994). It should be noted however that due to several simplified assumptions employed in the construction of such charts, the estimated effective length factor can be substantially deviated from the analytical solution. Work towards the improvement of the estimation of the effective length factor has been carried out continuously by various researchers who still fall in love with the beauty of Euler's formula (e.g. Aristizabal-Ochoa, 1994a; Aristizabal-Ochoa, 1994b; Hellesland et al., 1996a; Hellesland et al., 1996b; Gantes and Mageirou, 2005).

An improved version of the analytical technique, termed a semi-analytical technique, is to employ certain numerical procedures to aid the massive and complex computations associated with solving nonlinear equations while still maintain the analytical nature of the solution. The buckling load obtained from this technique is basically of comparable quality to the exact solution. However, similar to the analytical technique, its practical applications are still limited to structures of simple configurations.

Most powerful techniques applicable to the buckling analysis of various types of structures are based upon approximation theories (e.g. Galerkin approximation, Rayleigh-Ritz approximation, finite element approximation, etc.) along with appropriate numerical procedures. The formulation of the boundary value problem is normally established in a form well-suited for the approximation to be carried out in the general setting (e.g. weak formulation by either the weight residual technique or the principle of virtual work, variational formulation by the principle of stationary total potential energy, work and energy conservation equation, etc.). While techniques in this category possess less mathematical complexity in comparison with the analytical and semi-analytical methods and the rapid growth of their applications has been recognized nowadays, it still requires consideration of various computational aspects such as the approximation strategy, the solution method, and the implementation in terms of the computer software. The quality of approximate solutions depends primarily on the strategy and level of approximation and this requires special care to ensure the convergence and accuracy of the computed buckling loads.

It has no strong evidence to solidly identify the best among the three groups of techniques used for determining the buckling load of structures. It is generally problem dependent and, sometimes, the matter of preference. The key motivation of this proposed study is to seek a means to improve existing techniques for better estimation of the buckling load of a broad class of columns and frames. In the following section, results from extensive literature survey are presented in order to clearly define the objective and the scope of this study.

1.2 Literature Review

In this section, a brief overview of the background and existing work that are relevant to the current study is provided. The key objective is to demonstrate the sequence of historical development in this specific area and also provide sufficient evidence to identify available gap of knowledge. Results from literature survey are separated into three parts regarding to their main focus; the first part is associated with studies of elastic and inelastic flexural buckling loads of structures, the second part devoted to investigations of the influence of shear deformation on the flexural buckling behavior, and the last part summarizes work on buckling analysis of members with restraints against the lateral movement.

1.2.1 Elastic and inelastic flexural buckling analysis of structures

For several decades, mathematicians, researchers and engineers have proposed various approaches for determining flexural buckling load of column and frame structures. In 1744, Euler showed that there exists a critical load associated with the state where a perfectly straight, slender column under compression starts to admit another deformed equilibrium configuration; this critical load is later known as the buckling load. In his study, the column is only supported against the lateral movement at both ends and is compressed within the elastic range of a constituting material. For any value of axial load less than the buckling load, the column remains its straight and stable equilibrium configuration while, for any value of axial load larger than the buckling load, the straight equilibrium configuration becomes unstable and infinitesimally small perturbation can push the column to a new stable equilibrium configuration. Since the Euler's era, the elastic buckling load (sometimes called the Euler's load to honor his first study in this area) of single columns with various end conditions and more complex structures have been extensively investigated (see Timoshenko and Gere, 1961).

One important drawback of the Euler's formula is its limited practical applications resulting from the linear elasticity assumption. More precisely, the constituting material is assumed to remain in a linear regime both prior to and at the onset of buckling. Elastic buckling can occur only for very slender columns while most columns found in practices buckle within an inelastic range. To enable the Euler's formula to treat inelastic buckling, the concept of variable modulus of elasticity has been introduced (e.g. Engesser, 1889; Engesser, 1891; Considère, 1891). In 1889, Engesser proposed a well-known tangent modulus theory. In his investigation, the column was assumed to remain straight until the onset of buckling, and the tangent modulus was assumed to be constant throughout the cross section. Based on the tangent modulus theory, the Euler's buckling formula can be modified by simply replacing the Young modulus by the tangent modulus at a stress level at the onset of buckling. Later, Engesser (1995) pointed out that his original tangent modulus theory is invalid, and he then replaced it by the reduced modulus or the double modulus theory. Based on the latter theory, fibers on the convex side of a bent column undergo elastic unloading (or strain reversal) while those on the concave side experience inelastic loading. With this new assumption, the theory was anticipated to better predict the inelastic buckling load; however, experimental evidences tended to flavor the tangent modulus theory while the reduced modulus theory generally yields higher buckling loads than test results. Later, in 1946, Shanley drew significant attention to the erroneous assumption of the reduced modulus theory; i.e. a column is always assumed to remain perfectly straight up to the reduced modulus load. To support his argument, Shanley proposed a model of two columns connecting at its two rigid ends by a spring at the center. He pointed out that an initially straight column will buckle at the tangent modulus load and will continue to bend with increasing axial load. With the Shanley concept, the tangent modulus theory provides a lower bound of the column strength, i.e. the load at which an initially straight column will start to bend. On the contrary, the reduced modulus theory leads to the upper bound of the buckling load since the reduced modulus load can be achieved only when the column is temporarily supported until reaching that load.

To estimate the elastic and inelastic flexural buckling loads of both single columns and frames, various techniques have been proposed since the first study by Euler in 1744. A classical approach that has been utilized extensively and continuously since its early age is based upon an analytical technique. The key step is to solve the governing differential equation for a correct function form describing the buckling shape and then employ the boundary conditions to form an eigenvalue problem. This technique has proven successful for determining the buckling load of single columns with various end conditions and frames with simple configurations (e.g. Timoshenko and Gere, 1961; Chajes, 1974; Chen and Lui, 1987). To treat more complex structures, Mahfouz (1999) proposed a semi-analytical technique using stability functions of each member to form a set of exact characteristic equations of the entire structure and the minimum eigenvalue (elastic buckling load) was searched by increasing an axial loading parameter from zero until reaching the point where the determinant of the characteristic matrix changes sign. While Mahfouz's approach can yield results of comparable accuracy to the exact solution, the computational cost related to calculations of the matrix determinant and a large number of iteration can be significant.

To enhance the capability of the analytical and semi-analytical techniques to treat a broader class of structures, various approximate techniques have been proposed. Two of these techniques include the use of Rayleigh-Ritz strategy to approximate the buckling shape in the conservation of work and energy equation (e.g. Chajes, 1974) and in the principle of stationary total potential energy (e.g. Dawe, 1984; Hughes, 1987). In such techniques, the buckling shape of the structure was assumed a priori to establish a set of characteristic equations governing the approximate buckling load. While they are of less mathematical complexity in comparison with the analytical method, they generally yield the buckling load higher than the exact value. Another key disadvantage of the Rayleigh-Ritz approximation is that there is no systematic means to choose the space of trial functions to ensure the accuracy and convergence of the approximate solution. Another powerful numerical procedure for buckling load analysis is the finite element method (FEM) (e.g. Dawe, 1984; Yang, 1986; Hughes, 1987); this particular technique can be viewed as the improved version of the Rayleigh-Ritz approximation. A space of trial buckling shapes is systematically constructed based on simple functions defined in an element-wise fashion. Nevertheless, convergence and accuracy of the approximate buckling load must still be confirmed by numerical experiments on a series of meshes. It is also important to emphasize that use of simple functions to represent the buckling shape can pose a potential drawback to this technique; for instance, a large number of elements may be required to accurately capture the complex buckling shape and this can result in a substantial computational cost. Other numerical and approximate techniques used to investigate flexural buckling problems have also been adopted; some of them are summarized below.

Gantes and Mageirou (2005) proposed a scheme to improve the estimation of an effective length factor of columns in multi-story sway frames. In their technique, a frame is modeled as an individual column with a rotational spring at both ends. A slope-deflection method was utilized to derive the expression of the spring rotational stiffness for all possible boundary conditions at the far end, with and without the axial force. The simplified version of the derived rotational stiffness is also obtained via the use of Taylor series expansion. In 2007, Girgin and Ozmen presented a simplified procedure for determining the buckling loads of three-dimensional frames. In their work, the principle of virtual work and Betti's reciprocal theorem were employed and it finally led to a single equation governing the buckling load:

$$W_1 = W_2 \tag{1.1}$$

where W_1 is the virtual work of forces from a system I (under axial loading) due to the displacement from a system II (under lateral loading) and W_2 is the virtual work of forces from the system II due to the displacement from the system I. It is worth noting that while the displacement from both systems were taken to be identical in such calculations, the displacement from the system I represents the relative column displacement whereas that from the system II corresponds to the story displacement. This proposed technique has been found applicable to both regular and irregular structures; however, it still requires to compute the displacements of the system II and values of the approximate buckling load depends primarily on the choice of lateral loads applied to that system.

Later, Yoo and Choi (2008) proposed a new method for analysis of inelastic buckling of steel frames. Their method utilized standard eigenvalue analysis along with the tangent modulus theory and a column strength curve. The first iteration of this method requires performing linear stress analysis to determine the internal force and bending moments of all members. The eigenvalue at the first step was set equal to the eigenvalue obtained from elastic buckling analysis and the minimum eigenvalue was then employed to obtain the flexural and axial resistances from a column strength curve. Next, the tangent modulus of each member was obtained from the axial and flexural information and then used to construct the stiffness matrix for the next search of the minimum eigenvalue. When the convergence was achieved, the computed eigenvalue was utilized to find the critical load of the steel frame. Note in addition that the geometric imperfection present in each member can be treated via the use of a column strength curve.

Recently, Choi and Yoo (2009) developed a technique to improve the accuracy of the effective length factor for multi-story frames. The traditional iterative buckling approach can predict reasonably accurate effective length factors only for columns in the weakest story or the weakest member of the frame. The weakest story or the weakest member was defined as a story or a member that is critical and controls the overall buckling of the frame or, equivalently, a story or a member possessing the maximum stiffness parameter, $L\sqrt{P/EI}$. To enhance performance of the traditional approach, they introduced a fictitious axial force by considering both the most influential member (with the maximum stiffness parameter) and the least influential member (with the minimum stiffness parameter). A formula proposed for estimating such fictitious axial force was given by

$$\delta P = \frac{E_{li} I_{li}}{E_{mi} I_{mi}} \left(\frac{K_{mi} L_{mi}}{\overline{K}_{li} L_{li}} \right)^2 P_{mi} - P_{li} \qquad (1.2)$$

The first step of this approach involves solving a conventional eigenvalue problem to obtain an increment of the fictitious axial force by comparing the stiffness parameter given by (1.2). Next, the axial force for all members is modified and the new geometric stiffness matrix is recalculated for the next search of the minimum eigenvalue. Once the new minimum eigenvalue is obtained, the effective length factor for all members is computed following by their convergence check. For the next iteration, the increment of the fictitious force is not required for members whose effective length factor is already converged. The process is to be continued until the convergence of the effective length factor is achieved for all members.

1.2.2 Buckling analysis of structures with consideration of shear deformation

Shear deformation has been considered as one of important factors that play an important role in the behavior of flexural buckling of columns and frames. Engesser (1891) was recognized the first who investigated the influence of shear deformation on the buckling load of a straight bar and suggested the modification to the original Euler's differential equation that governs the buckling shape. All internal force measures in Engesser's approach were based on the undeformed state as shown in Figure 1.1(a); more specifically, the axial force N_1 acts in the direction of the member axis and the shear force Q_1 acts in the tangential direction of the cross section. The other different and well-recognized model was proposed by Haringx in 1948. In Haringx's approach, all internal forces were measured based on the deformed state as depicted in Figure 1.1(b). Unlike the former approach, the axial force N_2 was assumed to be normal to the rotating cross section and the shear force Q_2 was assumed to be in the tangential direction of the rotating cross section.

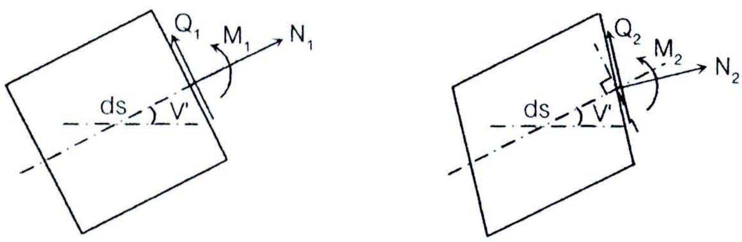

Figure 1.1 Two different measures for all internal forces within cross section: (a) Engesser's approach and (b) Haringx's approach

These two models have been extensively employed by various investigators to study the influence of shear deformation on the buckling behavior (e.g. Timoshenko and Gere, 1961; Ziegler, 1982; Djukic and Atanackovic, 1993; Wang et al., 2002; Blaauwendraad, 2008). Timoshenko and Gere (1961) utilized both Engesser and Haringx's approaches in the buckling analysis of columns. They pointed out that the Haringx's approach can lead to more accurate results when the effect of shear deformation is significantly large (e.g. the buckling of helical springs) while the Engesser's approach yields results on the safe side. Ziegler (1982) further examined those two approaches by comparing with a more fundamental method based on analytical mechanics. He concluded in his study that the Engesser's approach is superior to the Haringx's approach for analysis of buckling of bars. In particular, he also explained why the Haringx's method predicts accurate results for the buckling of springs. Later, Djukic and Atanackovic (1993) investigated the buckling behavior of a hinged-hinged column by taking shear deformation into account. In their approach, the axial force was assumed to direct along the rod axis (the same as the Engesser's approach) while the shear force assumed the direction normal to the deformed axis. It was found from this study that results were in close agreement with those obtained from the Engesser's approach.

Wang et al. (2002) employed an analytical technique (i.e. solving the differential equation for the buckling shape and exact eigenvalue problem for the buckling load) to establish the exact stability criteria and obtain the buckling load of

Timoshenko columns subjected to interior and end axial loads for various types of boundary conditions. The influence of shear deformation, boundary conditions, and magnitudes and positions of the interior load on values of the buckling load was fully investigated. Results revealed that the effect of transverse shear deformation becomes significant when a column is subjected to the interior load near its base. In this particular situation, the column behaves in the same way as a stocky column. It was also found that the effect of transverse shear deformation in lowering the buckling load is more apparent for columns with greater restraints at their ends. In 2008, Blaauwendraad showed that the Haringx's approach yields a wrong limit of the buckling load for very weak-in-shear beam-columns. He supported his argument by considering a simply-supported Timoshenko beam-column with a semi-rigid connection and a spring support at its mid-span. His results indicated that the buckling load obtained from the Engesser's and Haringx's approaches are comparable in magnitude when the shear rigidity of a column is large. However, when a column has the weak shear rigidity, the buckling load obtained from the Haringx's approach significantly deviates from that for the limiting case of a column with infinitely large flexural rigidity and finite shear rigidity.

1.2.3 Buckling analysis of members with restraints against lateral movement

Beams and columns supported laterally along their length are very common in structural configuration, e.g. beams resting on an elastic foundation and columns braced against the lateral movement. A well-known mathematical model used to describe an elastic support is proposed by Winkler (1867) and later named to honor him as *Winkler* foundation. In this model, the foundation acts as if it consists of an infinite number of closely spaced linear springs, and its constitutive behavior is completely characterized by a single parameter termed the foundation modulus k. To enhance the Winkler model, some investigators included, in addition, the interactions between the elastic spring and the foundation and this, therefore, leads to one additional parameter. Several equivalent two-parameter elastic foundation models have been found in the literature, e.g. Filonenko-Borodich foundation, Pasternak foundation, generalized foundation and Vlasov foundation. The Filonenko-Borodich foundation was first proposed by Filonenko-Borodich (1940). In this model, the top end of springs is attached to an elastic membrane that is pre-stretched by a constant tension T. The Pasternak foundation, proposed by Pasternak (1954), takes the shear interactions among the springs into account. Specifically, the top end of the springs is attached to an incompressible layer that can resist only the transverse shear deformation. In 1964, Kerr proposed the generalized foundation model by assuming that at each contact point, there are both the pressure and moment acting to the foundation. The Vlasov foundation, developed by Vlasor and Leontiev (1966), was mathematically complicated for its original version. A simplified model was later introduced and has been widely used. For all two-parameter models described above, their behavior is governed by the same equation as follows:

$$p(x) = k_1 w(x) - k_2 \frac{d^2 w(x)}{dx^2} \tag{1.3}$$

where $p(x)$ denotes the reaction normal to the foundation, $w(x)$ represents the lateral or transverse deflection, and k_1 and k_2 are model parameters. Note that for the Winkler foundation, the parameter k_2 is taken to be zero.

On the basis of extensive literature survey, above models have been used extensively in the analysis of beams and columns resting on the elastic foundation. Zhaohua and Cook (1983) employed the finite element method to analyze beams on both single-parameter and two-parameter foundations. Two types of elements, one is based on the exact displacement function and the other is based on a cubic displacement function, were developed in their study. It was found that use of elements based on the exact displacement function in the discretization yields exact solution for all deformation and internal forces without mesh refinement but use of elements based on the cubic displacement function gives only approximate solutions with their accuracy depending upon the level of refinement. Later, Yankelevsky and Eisenberger (1986) applied a direct analytical technique to derive an exact stiffness matrix for a beam-column resting on an elastic Winkler foundation. Eisenberger et al. (1986) also derived the elastic and geometric stiffness matrices for a beam-column resting on an elastic

Winkler foundation. By using these matrices, they were able to determine the buckling loads and the corresponding buckling mode shapes of a continuous column on an elastic foundation.

In 1988, Cheng and Pantelides presented the buckling load and buckling mode shape of a simple Timoshenko beam-columns supported laterally by an elastic foundation. In their study, two approaches were employed to derive the governing differential equations, stiffness coefficients, and fixed-end forces. The first approach was based upon the Haringx's model with the shear component being calculated from the total slope, while the second approach was based upon the Engesser's model with the shear component being computed differently from the bending slope. They observed from this study that values of the buckling load for columns with relatively small slenderness ratio are significantly reduced when the shear deformation is included, and the first approach always yields the buckling load less than the second approach. In particular, when the slenderness ratio is reduced, the buckling load predicted by the Haringx's and Engesser's models exhibits significant discrepancy.

In 1995, Naidu and Rao used the finite element method to study the stability behavior of prismatic columns resting on a two-parameter elastic foundation. The constant of a shearing layer for the two halves of a column was taken to have different values and various boundary conditions were considered. In 2000, Seemapholkul developed a technique based on the finite element method to determine the buckling load of a Timoshenko beam-column resting on a two-parameter Filonenko-Borodich foundation with consideration of shear deformation via the Engesser's model. In the finite element approximation, the exact element shape functions obtained by solving a Timoshenko beam analytically (in the absence of an axial load) were utilized. Their technique was found promising and yielded accurate results upon proper mesh refinement. Recently, Xia and Zhang (2009) derived a governing differential equation for a simply-supported beam-column resting on the Winkler foundation. By directly solving the differential equation and the corresponding eigenvalue problem, they could obtain the buckling load of such beam-column and then confirmed their results with those by a reliable finite element program.

Based on extensive review of relevant work in this area and the great contribution of knowledge to practical applications, it has no doubt that the development of accurate and powerful numerical techniques to compute the buckling load of both columns and frames by taking various factors such as the inelastic effect, shear deformation, and restraints against the lateral movement into account is essential and still requires further investigations. One potential improvement to existing methods, and is the main focus of this study, is to supply the automatic adaptivity of the approximation that allows the exact buckling load be achieved without any mesh refinement.

1.3 Research Objective

The key objective of this investigation is to develop an efficient and accurate numerical technique to estimate the buckling load of two-dimensional skeleton structures.

1.4 Scope of Research

The present research has been carried out within following context and assumptions:

1) Structures are two-dimensional and consist of straight and prismatic one-dimensional members.
2) Initial imperfections such as initial crookedness, eccentric loads, and residual stress are not included.
3) Only flexural buckling is considered.
4) The constituting material can be either linear elastic or inelastic. For elastic materials, the Young's modulus is prescribed and for inelastic materials, the tangent modulus is known.
5) Effect of shear deformation is included by using Engesser' model.
6) Influence of point restraints against the lateral movement and rotation is considered using translational and rotational spring models.

7) Influence of uniformly distributed lateral restraints is considered using a two-parameter foundation model.
8) Influence of axial deformation is neglected.

1.5 Research Methodology

The key task of this research is associated with the development of a numerical procedure to have capabilities for analysis of the buckling load of structures. Methodology employed to accomplish such task can be described as follows:

1) The principle of stationary total potential energy is employed to establish the variational formulation governing the buckling problem,
2) The Rayleigh-Ritz approximation is adopted to derive the approximate characteristic equations for an individual element,
3) Space of trial functions used in the approximation of the buckling shape is based on adaptive basis functions derived from the exact solution of the ordinary differential equation governing the buckling shape,
4) Standard assembly procedure is employed to form an approximate eigenvalue problem for the entire structure,
5) A power method supplemented by Rayleigh quotient is utilized to calculate the minimum eigenvalue, and
6) A selected iterative procedure is employed, along with adaptive buckling shape, to achieve the accurate buckling load.

1.6 Research Significance

An important output gained from this study is an accurate and efficient numerical procedure capable of determining the flexural buckling loads and other relevant buckling information such as the effective length factor of various structures typically encountered in practices, e.g. multi-story non-sway and sway frames, columns resting on elastic foundations, buckling of piles, etc. The most attractive feature of the proposed technique is the use of a special space of trial functions that allows the automatic adaptivity to enhance the accuracy of approximate solutions. Results obtained from this technique are of high quality and, generally, comparable to the analytical solutions. As a consequence, results generated from the current technique can be used as benchmark solutions for verification and comparison purposes. Another direct application is to employ this technique to enhance the estimation of the effective length factor, instead of using an old-style approach via the alignment charts, in the design of compression members and members in combined flexure and compression.