

Chapter 3

Main Results

In this chapter, we investigate the existence of free α -labelings, ρ -tripartite labelings, and graceful labelings of various classes of cubic graphs. We also investigate the existence of 2-fold ρ^+ -labelings of the class of cubic multigraphs consisting of even cycles with alternating double edges. In Section 3.2, we show that if G is a bipartite prism, a bipartite Möbius ladder, or a connected cubic bipartite graph of order at most 14, then G admits a free α -labeling. In Section 3.3, we show if G is an odd prism, an even Möbius ladder, or a connected cubic tripartite graph of order at most 10, then G admits a ρ -tripartite labeling. In Section 3.4, we show that every cubic graph of order at most 12, other than $2K_4$ and $3K_4$, is graceful. Finally, in Section 3.5, we show that \tilde{C}_{2n} , the $2n$ -cycle with alternating double edges, admits a 2-fold ρ^+ -labeling for every integer $n \geq 2$. In each of these cases, we give the resulting cyclic designs and offer some conjectures.

3.1 Additional Notation and Terminology

We denote the path P_{k+1} with vertices x_0, x_1, \dots, x_k , where x_i is adjacent to x_{i+1} , $0 \leq i \leq k-1$, by (x_0, x_1, \dots, x_k) . In using this notation, we are thinking of traversing the path from x_0 to x_k so that x_0 is the first vertex, x_1 is the second vertex, and so on. Let $G_1 = (x_0, x_1, \dots, x_j)$ and $G_2 = (y_0, y_1, \dots, y_k)$. If G_1 and G_2 are vertex-disjoint except for $x_j = y_0$, then by $G_1 + G_2$ we mean the path $(x_0, x_1, \dots, x_j, y_1, y_2, \dots, y_k)$. If the only vertices they have in common are $x_0 = y_k$ and $x_j = y_0$, then by $G_1 + G_2$ we mean the cycle $(x_0, x_1, \dots, x_j, y_1, y_2, \dots, y_{k-1}, x_0)$.

Let $P(2k)$ be the path with $2k$ edges and $2k+1$ vertices $0, 1, \dots, 2k$ given by $(0, 2k, 1, 2k-1, 2, 2k-2, \dots, k-1, k+1, k)$. Note that the set of vertices of this graph is $A \cup B$, where $A = [0, k]$, $B = [k+1, 2k]$, and every edge joins a vertex

from A to one from B . Furthermore the set of labels of the edges of $P(2k)$ is $[1, 2k]$.

Let a and b be nonnegative integers and k , d_1 , and d_2 be positive integers such that $a + kd_1 < b$. Let $\hat{P}(2k, d_1, d_2, a, b)$ be the path with $2k$ edges and $2k+1$ vertices given by $(a, b + (k-1)d_2, a + d_1, b + (k-2)d_2, a + 2d_1, \dots, a + (k-1)d_1, b, a + kd_1)$. Note that $\hat{P}(2k, 1, 1, 0, k+1)$ is the graph $P(2k)$. Note that this graph $\hat{P}(2k, d_1, d_2, a, b)$ has the following properties:

- P1: $\hat{P}(2k, d_1, d_2, a, b)$ is a path with first vertex a , second vertex $b + (k-1)d_2$, and last vertex $a + kd_1$.
- P2: Each edge of $\hat{P}(2k, d_1, d_2, a, b)$ joins a vertex from $A = \{a + id_1 : 0 \leq i \leq k\}$ to a vertex with a larger label from $B = \{b + id_2 : 0 \leq i \leq k-1\}$.
- P3: The set of edge labels of $\hat{P}(2k, d_1, d_2, a, b)$ is $\{b - a - kd_1 + i(d_1 + d_2) : 0 \leq i \leq k-1\} \cup \{b - a - (k-1)d_1 + i(d_1 + d_2) : 0 \leq i \leq k-1\}$.

The path $\hat{P}(10, 2, 4, 14, 40)$ is shown in Figure 3.10 below.

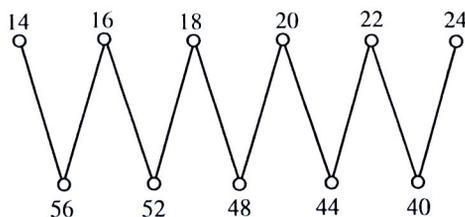


Figure 3.10: The path $\hat{P}(10, 2, 4, 14, 40)$.

3.2 Free α -labelings of Some Cubic Graphs

Let G be a graph and let $h: V(G) \rightarrow N$ be a labeling of G . If $v \in V(G)$, we call $h(v)$ the *label* of v . If $W \subseteq V(G)$, we let $h(W) = \{h(v) : v \in W\}$. If h is a labeling of G , we define a function $\bar{h}: E(G) \rightarrow \mathbb{Z}^+$ by $\bar{h}(e) = |h(u) - h(v)|$, where $e = \{u, v\} \in E(G)$. The number $|h(u) - h(v)|$ is called the *label* of the edge $\{u, v\}$. If $F \subseteq E(G)$, then $\bar{h}(F) = \{\bar{h}(e) : e \in F\}$. For convenience, we will often let $h(G)$ denote the labeled graph G ; that is, $h(G)$ is the graph with vertex set $h(V(G))$ and edge set $\{\{h(u), h(v)\}; \{u, v\} \in E(G)\}$. Recall that a labeling h of a graph G with q edges is a β -labeling of G if $h(V(G)) \subseteq [0, q]$ and $\bar{h}(E(G)) = [1, q]$. Also, recall that an α -labeling is a β -labeling having the additional property that there exists an integer k such that if $\{u, v\} \in E(G)$, then $\{u, v\} = \{a, b\}$, where $h(a) \leq k < h(b)$. The number k , which is unique, is called the *critical value* of the α -labeling. Note that necessarily $0, k, k + 1$, and $|E(G)|$ are in $h(V(G))$. Moreover, G must be bipartite.

If h is an α -labeling of G , then $h' = |E(G)| - h$ is also an α -labeling of G with critical value $k' = |E(G)| - (k + 1)$. We shall refer to h' as the *complementary α -labeling* of h .

Numerous large classes of bipartite graphs have α -labelings; examples include complete bipartite graphs, caterpillars, d -cubes, bipartite prisms, and cycles of length $4k$ (see Section 2.3). Labelings of graphs are particularly interesting because of their applications to graph decompositions. It is well known that if a graph G with q edges admits an α -labeling, then the edge-sets of K_{2qx+1} , $K_{2qx+2} - I$ (where I is a 1-factor), and $K_{qx, qy}$ can be partitioned into subgraphs isomorphic to G for all positive integers x and y (see [41] and [36]). One may not be able to obtain these same results with the less restrictive graceful labelings of G .

Let G be a bipartite graph on p vertices and with q edges. We list some known necessary conditions for G to admit an α -labeling:

- I) *The Order-Size Condition:* $q \geq p - 1$.
- II) *The Parity Condition:* If every vertex of G has even degree, then we must have $q \equiv 0 \pmod{4}$.

III) *Wu's Condition*: If d_1, d_2, \dots, d_p is the degree sequence of G , then we must have $\gcd(d_1, d_2, \dots, d_p, q)$ divides $q(q-1)/2$.

This last condition is unpublished and is due to Wu according to Gallian (see [28]). Numerous graphs satisfy these conditions, but do not admit α -labelings.

The conditions above can help decide which regular bipartite graphs G might admit α -labelings. For example, rK_2 admits an α -labeling if and only if $r = 1$ (since $|V(rK_2)| > |E(rK_2)| + 1$ for $r \geq 2$). Half of the 2-regular bipartite graphs cannot admit an α -labeling by the parity condition. The graph $3C_4$ is the only 2-regular bipartite graph that satisfies the parity condition and is known not to admit an α -labeling. All other 2-regular bipartite graphs that satisfy the parity condition and have at most 3 components admit α -labelings (see [2] and [25]). Moreover, Abraham and Kotzig [1] proved that rC_4 admits an α -labeling for all $r \neq 3$.

As for 3-regular graphs, it is known that $K_{3,3}$, the 3-cube, and all bipartite prisms [27, 26] and bipartite Möbius ladders [38] admit α -labelings. We note that none of the three forbidding conditions above apply to 3-regular graphs. The parity condition does not apply to 4-regular bipartite graphs; however, Wu's condition does. For example, the graph $K_{5,5} - I$, where I is a 1-factor does not admit an α -labeling.

In [22], El-Zanati and Vanden Eynden introduced the concept of a free α -labeling as follows. Let G be a graph with an α -labeling h and critical value k . We say that h is *free* if $k > 2$, and neither 1 nor $k-1$ is in $h(V(G))$. They showed that if both G_1 and G_2 admit free α -labelings, then the vertex-disjoint union of G_1 and G_2 also admits a free α -labeling.

Theorem 3.2.1 *Let G_i be a graph with a free α -labeling h_i and critical value k_i for $i \in \{1, 2\}$. Then the vertex-disjoint union $G_1 \cup G_2$ admits a free α -labeling h with critical value $k_1 + k_2 - 1$.*

We illustrate how Theorem 3.2.1 works by showing how the labeling of $G = G_1 \cup G_2$ is obtained. Let $V(G_i) = X_i \cup Y_i$, where if $v \in X_i$, then $h_i(v) \leq k_i$, and if $v \in Y_i$, then $h_i(v) > k_i$, $i = 1, 2$. Define h on $V(G)$ to be h_1 on X_1 , $h_2 + k_1 - 1$

on $X_2 \cup Y_2$, and $h_1 + |E(G_2)|$ on Y_1 . Then h is an α -labeling for G with critical value $k = k_1 + k_2 - 1 > 2$.

we will show that if G is a bipartite prism, a bipartite Möbius ladder or a connected cubic bipartite graph of order at most 14, then G admits a free α -labeling. We also conjecture that every bipartite cubic graph admits a free α -labeling.

3.2.1 Free α -labelings of Bipartite Prisms

Recall that by a *prism* D_n ($n \geq 4$) we mean the cartesian product $C_n \times P_2$ of a cycle with n vertices and a path with 2 vertices. For convenience, we let $D_n = C_n \cup C'_n \cup F$, where $C_n = (v_1, v_2, \dots, v_n, v_1)$, $C'_n = (v'_1, v'_2, \dots, v'_n, v'_1)$, and $F = \{\{v_i, v'_i\} : 1 \leq i \leq n\}$. We shall refer to C_n as the *outer cycle*, to C'_n as the *inner cycle*, and to F as the *spokes*. Figure 3.11 shows the prism D_8 . We note that D_{2n} , $n \geq 2$, is necessarily bipartite with bipartition $(O \cup W', W \cup O')$, where $O = \{v_{2i-1} : 1 \leq i \leq n\}$, $W' = \{v'_{2i} : 1 \leq i \leq n\}$, $W = \{v_{2i} : 1 \leq i \leq n\}$, and $O' = \{v'_{2i-1} : 1 \leq i \leq n\}$. We will show that D_n admits a free α -labeling for all even integers $n \geq 4$.

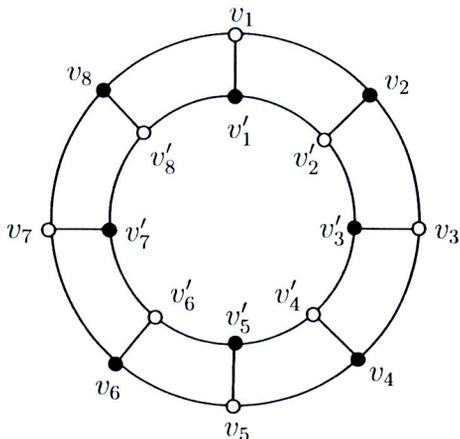


Figure 3.11: The prism D_8 .

Theorem 3.2.2 *The prism D_n admits a free α -labeling for all even $n \geq 4$.*

Proof. We separate the proof into 3 cases. In each case, we give an α -labeling f such that the complementary labeling of f is free.

Case 1: $n \equiv 0 \pmod{6}$.

Let $n = 6t$. Thus, $|V(D_n)| = 12t$ and $|E(D_n)| = 18t$. A free α -labeling of D_6 is given in Table 3.1 (graph Bc7). For $t \geq 2$, define a one-to-one function $f : V(D_{6t}) \rightarrow [0, 18t]$ as follows:

$$\begin{aligned}
 f(v_i) &= i - 1, & v_i \in O &= \{v_i : i \text{ odd}, 1 \leq i < 6t\}, \\
 f(v_2) &= 18t, \\
 f(v_i) &= 18t - 2i + 2, & v_i \in W_1 &= \{v_i : i \text{ even}, 2 < i \leq 2t\}, \\
 f(v_i) &= 18t - 2i, & v_i \in W_2 &= \{v_i : i \text{ even}, 2t < i \leq 4t\}, \\
 f(v_i) &= 18t - 2i - 2, & v_i \in W_3 &= \{v_i : i \text{ even}, 4t < i < 6t\}, \\
 f(v_{6t}) &= 12t - 3, \\
 f(v'_1) &= 18t - 3, \\
 f(v'_i) &= i - 1, & v_i \in W' &= \{v_i : i \text{ even}, 1 < i \leq 6t\}, \\
 f(v'_i) &= 18t - 2i + 2, & v_i \in O'_1 &= \{v_i : i \text{ odd}, 1 < i \leq 2t + 1\}, \\
 f(v'_i) &= 18t - 2i, & v_i \in O'_2 &= \{v_i : i \text{ odd}, 2t + 1 < i \leq 4t - 1\}, \\
 f(v'_i) &= 18t - 2i - 2, & v_i \in O'_3 &= \{v_i : i \text{ odd}, 4t - 1 < i < 6t\}.
 \end{aligned}$$

Note that $W = \{v_2\} \cup W_1 \cup W_2 \cup W_3 \cup \{v_{6t}\}$ and $O' = \{v'_1\} \cup O'_1 \cup O'_2 \cup O'_3$. Thus the domain of f is indeed $V(D_{6t})$.

Next, we confirm that f is one-to-one. We compute

$$\begin{aligned}
 f(O) &= \{0, 2, \dots, 6t - 2\}, \\
 f(W_1) &= \{18t - 6, 18t - 10, \dots, 14t + 2\}, \\
 f(W_2) &= \{14t - 4, 14t - 8, \dots, 10t\}, \\
 f(W_3) &= \{10t - 6, 10t - 10, \dots, 6t + 2\}, \\
 f(W') &= \{1, 3, \dots, 6t - 1\}, \\
 f(O'_1) &= \{18t - 4, 18t - 8, \dots, 14t\}, \\
 f(O'_2) &= \{14t - 6, 14t - 10, \dots, 10t + 2\}, \\
 f(O'_3) &= \{10t - 4, 10t - 8, \dots, 6t\}.
 \end{aligned}$$

Note that f is piecewise strictly increasing by 2 or strictly decreasing by 4 and that all labels are distinct. Thus f is one-to-one. Moreover, $f(O \cup W') = [0, 6t - 1]$ and for $v_i \in W \cup O'$, $6t \leq f(v_i) \leq 18t$.

To help compute the edge labels, we will describe $f(D_{6t})$ in terms of the paths $\hat{P}(2k, d_1, d_2, a, b)$. For convenience, we will identify the vertices of C_{6t} and C'_{6t} with their labels. We have $f(C_{6t}) = G_1 + G_2 + G_3 + (6t - 2, 12t - 3, 0, 18t, 2)$, where

$$G_1 = \hat{P}(2(t - 1), 2, 4, 2, 14t + 2),$$

$$G_2 = \hat{P}(2t, 2, 4, 2t, 10t),$$

$$G_3 = \hat{P}(2(t - 1), 2, 4, 4t, 6t + 2).$$

By P3, the resulting edge label sets are:

$$\bar{f}(E(G_1)) = \{12t + 2 + 6i : 0 \leq i \leq t - 2\} \cup \{12t + 4 + 6i : 0 \leq i \leq t - 2\}$$

$$= \{\ell \equiv 2 \pmod{6} : 12t + 2 \leq \ell \leq 18t - 10\}$$

$$\cup \{\ell \equiv 4 \pmod{6} : 12t + 4 \leq \ell \leq 18t - 8\},$$

$$\bar{f}(E(G_2)) = \{6t + 6i : 0 \leq i \leq t - 1\} \cup \{6t + 2 + 6i : 0 \leq i \leq t - 1\}$$

$$= \{\ell \equiv 0 \pmod{6} : 6t \leq \ell \leq 12t - 6\}$$

$$\cup \{\ell \equiv 2 \pmod{6} : 6t + 2 \leq \ell \leq 12t - 4\},$$

$$\bar{f}(E(G_3)) = \{4 + 6i : 0 \leq i \leq t - 2\} \cup \{6 + 6i : 0 \leq i \leq t - 2\}$$

$$= \{\ell \equiv 4 \pmod{6} : 4 \leq \ell \leq 6t - 8\}$$

$$\cup \{\ell \equiv 0 \pmod{6} : 6 \leq \ell \leq 6t - 6\}.$$

Moreover, the edge labels $6t - 1, 12t - 3, 18t$, and $18t - 2$ occur on the path $(6t - 2, 12t - 3, 0, 18t, 2)$.

Similarly, we have $f(C'_{6t}) = G'_1 + G'_2 + G'_3 + (6t - 1, 18t - 3, 1)$, where

$$G'_1 = \hat{P}(2t, 2, 4, 1, 14t),$$

$$G'_2 = \hat{P}(2(t - 1), 2, 4, 2t + 1, 10t + 2),$$

$$G'_3 = \hat{P}(2t, 2, 4, 4t - 1, 6t).$$

By P3, the resulting edge label sets are:

$$\begin{aligned}\bar{f}(E(G'_1)) &= \{12t - 1 + 6i : 0 \leq i \leq t - 1\} \cup \{12t + 1 + 6i : 0 \leq i \leq t - 1\} \\ &= \{\ell \equiv 5 \pmod{6} : 12t - 1 \leq \ell \leq 18t - 7\} \\ &\quad \cup \{\ell \equiv 1 \pmod{6} : 12t + 1 \leq \ell \leq 18t - 5\},\end{aligned}$$

$$\begin{aligned}\bar{f}(E(G'_2)) &= \{6t + 3 + 6i : 0 \leq i \leq t - 2\} \cup \{6t + 5 + 6i : 0 \leq i \leq t - 2\} \\ &= \{\ell \equiv 3 \pmod{6} : 6t + 3 \leq \ell \leq 12t - 9\} \\ &\quad \cup \{\ell \equiv 5 \pmod{6} : 6t + 5 \leq \ell \leq 12t - 7\},\end{aligned}$$

$$\begin{aligned}\bar{f}(E(G'_3)) &= \{1 + 6i : 0 \leq i \leq t - 1\} \cup \{3 + 6i : 0 \leq i \leq t - 1\} \\ &= \{\ell \equiv 1 \pmod{6} : 1 \leq \ell \leq 6t - 5\} \\ &\quad \cup \{\ell \equiv 3 \pmod{6} : 3 \leq \ell \leq 6t - 3\}.\end{aligned}$$

Moreover, the edge labels $12t - 2$ and $18t - 4$ occur on the path $(6t - 1, 18t - 3, 1)$.

For each spoke $\{v_i, v'_i\}$, the edge label is given by $f(v_i) - f(v'_i)$ if i is even and by $f(v'_i) - f(v_i)$ if i is odd. Thus the labels on the spokes are given by

$$\bar{f}(\{v_i, v'_i\}) = \begin{cases} 18t - 3 & \text{for } i = 1, \\ 18t - 1 & \text{for } i = 2, \\ 18t - 3i + 3 & \text{for } 3 \leq i \leq 2t + 1, \\ 18t - 3i + 1 & \text{for } 2t + 2 \leq i \leq 4t, \\ 18t - 3i - 1 & \text{for } 4t + 1 \leq i \leq 6t - 1, \\ 6t - 2 & \text{for } i = 6t. \end{cases}$$

Thus the set of edge labels on the spokes is:

$$\begin{aligned}\bar{f}(E(F)) &= \{\ell \equiv 0 \pmod{3} : 12t \leq \ell \leq 18t - 6\} \\ &\quad \cup \{\ell \equiv 1 \pmod{3} : 6t + 1 \leq \ell \leq 12t - 5\} \\ &\quad \cup \{\ell \equiv 2 \pmod{3} : 2 \leq \ell \leq 6t - 4\} \\ &\quad \cup \{6t - 2, 18t - 3, 18t - 1\}.\end{aligned}$$

It is easy to verify that each label $\ell \in [0, 18t]$ occurs on exactly one edge in D_{6t} . Thus f is an α -labeling of D_{6t} with critical value $k = 6t - 1$. Now, let f' be the

complementary labeling of f . The critical value of f' is $k' = 18t - (k + 1) = 12t$. Thus, $k' > 2$. Moreover, since neither $18t - 1$ nor $6t + 1$ is a vertex label in $f(V(D_{6t}))$, neither 1 nor $k' - 1$ is a label in $f'(V(D_{6t}))$. Therefore f' is a free α -labeling of D_{6t} . Figure 3.12 shows the resulting free α -labeling of D_{12} .

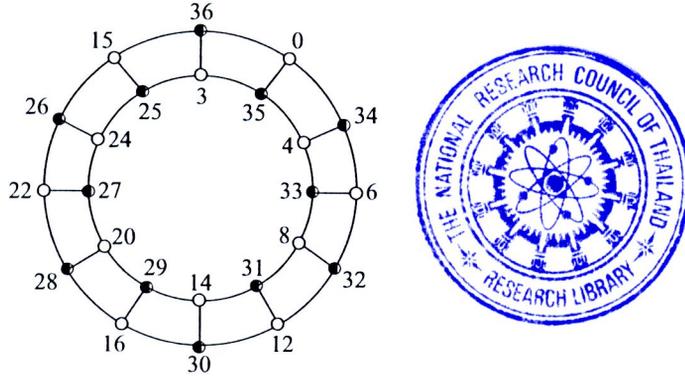


Figure 3.12: A free α -labeling of D_{12} .

Case 2: $n \equiv 2 \pmod{6}$.

Let $n = 6t + 2$. Thus, $|V(D_n)| = 12t + 4$ and $|E(D_n)| = 18t + 6$. If $t = 1$, we let $(0, 22, 10, 20, 14, 18, 3, 24, 0)$ denote the vertex labels of the outer cycle and let $(23, 5, 21, 12, 19, 16, 17, 4, 23)$ denote the vertex labels of the inner cycle with spokes $\{0, 23\}, \{22, 5\}, \dots, \{24, 4\}$. It is easy to verify that yields a free α -labeling of D_8 . For $t \geq 2$, we define a one-to-one function $f : V(D_{6t+2}) \rightarrow [0, 18t + 6]$ as

follows:

$$\begin{aligned}
f(v_i) &= i - 1, & v_i \in O &= \{v_i : i \text{ odd}, 1 \leq i < 6t + 2\}, \\
f(v_2) &= 18t + 6, \\
f(v_i) &= 18t - 2i + 7, & v_i \in W_1 &= \{v_i : i \text{ even}, 2 < i \leq 2t\}, \\
f(v_i) &= 18t - 2i + 4, & v_i \in W_2 &= \{v_i : i \text{ even}, 2t < i < 6t + 2\}, \\
f(v_{6t+2}) &= 18t + 3, \\
f(v'_1) &= 18t + 2, \\
f(v'_i) &= i - 1, & v_i \in W' &= \{v_i : i \text{ even}, 1 < i \leq 6t + 2\}, \\
f(v'_i) &= 18t - 2i + 7, & v_i \in O'_1 &= \{v_i : i \text{ odd}, 1 < i \leq 2t + 1\}, \\
f(v'_i) &= 18t - 2i + 4, & v_i \in O'_2 &= \{v_i : i \text{ odd}, 2t + 1 < i < 6t + 2\}.
\end{aligned}$$

Note that $W = \{v_2\} \cup W_1 \cup W_2 \cup \{v_{6t+2}\}$ and $O' = \{v'_1\} \cup O'_1 \cup O'_2$. Thus the domain of f is indeed $V(D_{6t+2})$.

Next, we confirm that f is one-to-one. We compute

$$\begin{aligned}
f(O) &= \{0, 2, \dots, 6t\}, \\
f(W_1) &= \{18t - 1, 18t - 5, \dots, 14t + 7\}, \\
f(W_2) &= \{14t, 14t - 4, \dots, 6t + 4\}, \\
f(W') &= \{1, 3, \dots, 6t + 1\}, \\
f(O'_1) &= \{18t + 1, 18t - 3, \dots, 14t + 5\}, \\
f(O'_2) &= \{14t - 2, 14t - 6, \dots, 6t + 2\}.
\end{aligned}$$

Note that f is piecewise strictly increasing by 2 or strictly decreasing by 4 and that all labels are distinct. Thus f is one-to-one. Moreover, $f(O \cup W') = [0, 6t + 1]$ and for $v_i \in W \cup O'$, $6t + 2 \leq f(v_i) \leq 18t + 6$.

To help compute the edge labels, we will describe $f(D_{6t+2})$ in terms of the paths $\hat{P}(2k, d_1, d_2, a, b)$. For convenience, we will identify the vertices of C_{6t+2} and C'_{6t+2} with their labels. We have $f(C_{6t+2}) = G_1 + G_2 + (6t, 18t + 3, 0, 18t + 6, 2)$, where

$$\begin{aligned}
G_1 &= \hat{P}(2(t - 1), 2, 4, 2, 14t + 7), \\
G_2 &= \hat{P}(2(2t), 2, 4, 2t, 6t + 4).
\end{aligned}$$

By P3, the resulting edge label sets are:

$$\begin{aligned}
\bar{f}(E(G_1)) &= \{12t + 7 + 6i : 0 \leq i \leq t - 2\} \cup \{12t + 9 + 6i : 0 \leq i \leq t - 2\} \\
&= \{\ell \equiv 1 \pmod{6} : 12t + 7 \leq \ell \leq 18t - 5\} \\
&\quad \cup \{\ell \equiv 3 \pmod{6} : 12t + 9 \leq \ell \leq 18t - 3\}, \\
\bar{f}(E(G_2)) &= \{4 + 6i : 0 \leq i \leq 2t - 1\} \cup \{6 + 6i : 0 \leq i \leq 2t - 1\} \\
&= \{\ell \equiv 4 \pmod{6} : 4 \leq \ell \leq 12t - 2\} \\
&\quad \cup \{\ell \equiv 0 \pmod{6} : 6 \leq \ell \leq 12t\}.
\end{aligned}$$

Moreover, the edge labels $12t + 3$, $18t + 3$, $18t + 6$ and $18t + 4$ occur on the path $(6t, 18t + 3, 0, 18t + 6, 2)$.

Similarly, we have $f(C'_{6t+2}) = G'_1 + G'_2 + (6t + 1, 18t + 2, 1)$, where

$$\begin{aligned}
G'_1 &= \hat{P}(2t, 2, 4, 1, 14t + 5), \\
G'_2 &= \hat{P}(2(2t), 2, 4, 2t + 1, 6t + 2).
\end{aligned}$$

By P3, the resulting edge label sets are:

$$\begin{aligned}
\bar{f}(E(G'_1)) &= \{12t + 4 + 6i : 0 \leq i \leq t - 1\} \cup \{12t + 6 + 6i : 0 \leq i \leq t - 1\} \\
&= \{\ell \equiv 4 \pmod{6} : 12t + 4 \leq \ell \leq 18t - 2\} \\
&\quad \cup \{\ell \equiv 0 \pmod{6} : 12t + 6 \leq \ell \leq 18t\}, \\
\bar{f}(E(G'_2)) &= \{1 + 6i : 0 \leq i \leq 2t - 1\} \cup \{3 + 6i : 0 \leq i \leq 2t - 1\} \\
&= \{\ell \equiv 1 \pmod{6} : 1 \leq \ell \leq 12t - 5\} \\
&\quad \cup \{\ell \equiv 3 \pmod{6} : 3 \leq \ell \leq 12t - 3\}.
\end{aligned}$$

Moreover, the edge labels $12t + 1$, and $18t + 1$ occur on the path $(6t + 1, 18t + 2, 1)$.

For each spoke $\{v_i, v'_i\}$, the edge label is given by $f(v_i) - f(v'_i)$ if i is even and by $f(v'_i) - f(v_i)$ if i is odd. Thus the labels on the spokes are given by

$$\bar{f}(\{v_i, v'_i\}) = \begin{cases} 18t + 2 & \text{for } i = 1, \\ 18t + 5 & \text{for } i = 2, \\ 18t - 3i + 8 & \text{for } 3 \leq i \leq 2t + 1, \\ 18t - 3i + 5 & \text{for } 2t + 2 \leq i \leq 6t + 1, \\ 12t + 2 & \text{for } i = 6t + 2. \end{cases}$$

Thus the set of edge labels on the spokes is:

$$\begin{aligned} \bar{f}(E(F)) &= \{\ell \equiv 2 \pmod{3} : 12t + 5 \leq \ell \leq 18t - 1\} \\ &\cup \{\ell \equiv 2 \pmod{3} : 2 \leq \ell \leq 12t - 1\} \\ &\cup \{12t + 2, 18t + 2, 18t + 5\}. \end{aligned}$$

It is easy to verify that each label $\ell \in [0, 18t + 6]$ occurs on exactly one edge in D_{6t+2} . Thus f is an α -labeling of D_{6t+2} . Although f is not free, it is easy to check that its complementary labeling f' is free. Figure 3.13 shows the resulting free α -labeling of D_{14} .

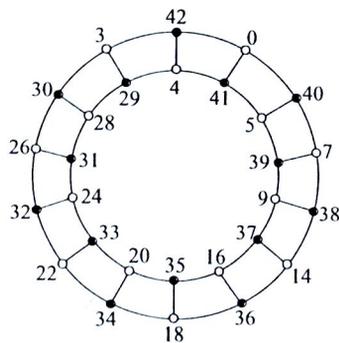


Figure 3.13: A free α -labeling of D_{14} .

Case 3: $n \equiv 4 \pmod{6}$.

Let $n = 6t - 2$. Thus, $|V(D_n)| = 12t - 4$ and $|E(D_n)| = 18t - 6$. A free α -labeling of D_4 is given in Table 3.1 (graph Bc2). For $t \geq 2$, define a one-to-one function

$f : V(D_{6t-2}) \rightarrow [0, 18t - 6]$ as follows:

$$\begin{aligned}
f(v_i) &= i - 1, & v_i \in O &= \{v_i : i \text{ odd}, 1 \leq i < 6t - 2\}, \\
f(v_i) &= 18t - 2i - 5, & v_i \in W_1 &= \{v_i : i \text{ even}, 1 < i \leq 2t - 2\}, \\
f(v_i) &= 18t - 2i - 8, & v_i \in W_2 &= \{v_i : i \text{ even}, 2t - 2 < i < 6t - 2\}, \\
f(v_{6t-2}) &= 18t - 8, \\
f(v'_1) &= 18t - 6, \\
f(v'_i) &= i - 1, & v_i \in W' &= \{v_i : i \text{ even}, 1 < i \leq 6t - 2\}, \\
f(v'_i) &= 18t - 2i - 5, & v_i \in O'_1 &= \{v_i : i \text{ odd}, 1 < i \leq 2t - 1\}, \\
f(v'_i) &= 18t - 2i - 8, & v_i \in O'_2 &= \{v_i : i \text{ odd}, 2t - 1 < i < 6t - 2\}.
\end{aligned}$$

Note that $W = W_1 \cup W_2 \cup \{v_{6t-2}\}$ and $O' = \{v'_1\} \cup O'_1 \cup O'_2$. Thus the domain of f is indeed $V(D_{6t-2})$.

Next, we confirm that f is one-to-one. We compute

$$\begin{aligned}
f(O) &= \{0, 2, \dots, 6t - 4\}, \\
f(W_1) &= \{18t - 9, 18t - 13, \dots, 14t - 1\}, \\
f(W_2) &= \{14t - 4, 14t - 8, \dots, 6t\}, \\
f(W') &= \{1, 3, \dots, 6t - 3\}, \\
f(O'_1) &= \{18t - 11, 18t - 15, \dots, 14t - 3\}, \\
f(O'_2) &= \{14t - 10, 14t - 14, \dots, 6t - 2\}.
\end{aligned}$$

Note that f is piecewise strictly increasing by 2 or strictly decreasing by 4 and that all labels are distinct. Thus f is one-to-one. Moreover, $f(O \cup W') = [0, 6t - 3]$ and for $v_i \in W \cup O'$, $6t - 2 \leq f(v_i) \leq 18t - 6$.

To help compute the edge labels, we will describe $f(D_{6t-2})$ in terms of the paths $\hat{P}(2k, d_1, d_2, a, b)$. For convenience, we will identify the vertices of C_{6t-2} and C'_{6t-2} with their labels. We have $f(C_{6t-2}) = G_1 + G_2 + (6t - 4, 18t - 8, 0)$, where

$$\begin{aligned}
G_1 &= \hat{P}(2(t - 1), 2, 4, 0, 14t - 1), \\
G_2 &= \hat{P}(2(2t - 1), 2, 4, 2t - 2, 6t).
\end{aligned}$$

By P3, the resulting edge label sets are:

$$\begin{aligned}
\bar{f}(E(G_1)) &= \{12t + 1 + 6i : 0 \leq i \leq t - 2\} \cup \{12t + 3 + 6i : 0 \leq i \leq t - 2\} \\
&= \{\ell \equiv 1 \pmod{6} : 12t + 1 \leq \ell \leq 18t - 11\} \\
&\quad \cup \{\ell \equiv 3 \pmod{6} : 12t + 3 \leq \ell \leq 18t - 9\}, \\
\bar{f}(E(G_2)) &= \{4 + 6i : 0 \leq i \leq 2t - 2\} \cup \{6 + 6i : 0 \leq i \leq 2t - 2\} \\
&= \{\ell \equiv 4 \pmod{6} : 4 \leq \ell \leq 12t - 8\} \\
&\quad \cup \{\ell \equiv 0 \pmod{6} : 6 \leq \ell \leq 12t - 6\}.
\end{aligned}$$

Moreover, the edge labels $12t - 4$ and $18t - 8$ occur on the path $(6t - 4, 18t - 8, 0)$.

Similarly, we have $f(C'_{6t-2}) = G'_1 + G'_2 + (6t - 3, 18t - 6, 1)$, where

$$\begin{aligned}
G'_1 &= \hat{P}(2(t - 1), 2, 4, 1, 14t - 3), \\
G'_2 &= \hat{P}(2(2t - 1), 2, 4, 2t - 1, 6t - 2).
\end{aligned}$$

By P3, the resulting edge label sets are:

$$\begin{aligned}
\bar{f}(E(G'_1)) &= \{12t - 2 + 6i : 0 \leq i \leq t - 2\} \cup \{12t + 6i : 0 \leq i \leq t - 2\} \\
&= \{\ell \equiv 2 \pmod{6} : 12t - 2 \leq \ell \leq 18t - 14\} \\
&\quad \cup \{\ell \equiv 0 \pmod{6} : 12t \leq \ell \leq 18t - 12\}, \\
\bar{f}(E(G'_2)) &= \{1 + 6i : 0 \leq i \leq 2t - 2\} \cup \{3 + 6i : 0 \leq i \leq 2t - 2\} \\
&= \{\ell \equiv 1 \pmod{6} : 1 \leq \ell \leq 12t - 11\} \\
&\quad \cup \{\ell \equiv 3 \pmod{6} : 3 \leq \ell \leq 12t - 9\}.
\end{aligned}$$

Moreover, the edge labels $12t - 3$ and $18t - 7$ occur on the path $(6t - 3, 18t - 6, 1)$.

For each spoke $\{v_i, v'_i\}$, the edge label is given by $f(v_i) - f(v'_i)$ if i is even and by $f(v'_i) - f(v_i)$ if i is odd. Thus the labels on the spokes are given by

$$\bar{f}(\{v_i, v'_i\}) = \begin{cases} 18t - 6 & \text{for } i = 1, \\ 18t - 3i - 4 & \text{for } 2 \leq i \leq 2t - 1, \\ 18t - 3i - 7 & \text{for } 2t \leq i \leq 6t - 3, \\ 12t - 5 & \text{for } i = 6t - 2. \end{cases}$$

Thus the set of edge labels on the spokes is:

$$\begin{aligned} \bar{f}(E(F)) = & \{\ell \equiv 2 \pmod{3} : 12t - 1 \leq \ell \leq 18t - 10\} \\ & \cup \{\ell \equiv 2 \pmod{3} : 2 \leq \ell \leq 12t - 7\} \\ & \cup \{12t - 5, 18t - 6\}. \end{aligned}$$

It is easy to verify that each label $\ell \in [0, 18t - 6]$ occurs on exactly one edge in D_{6t-2} . Thus f is an α -labeling of D_{6t-2} . Although f is not free, it is easy to check that its complementary labeling f' is free. Figure 3.14 shows the resulting free α -labeling of D_{10} . ■

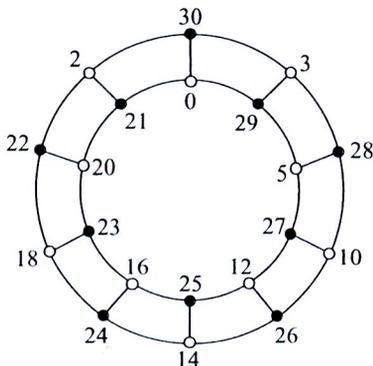
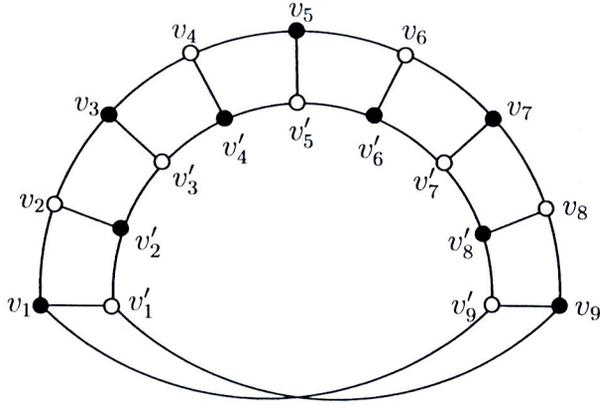


Figure 3.14: A free α -labeling of D_{10} .

3.2.2 Free α -labelings of Bipartite Möbius Ladders

For $n \geq 3$, let v_1, v_2, \dots, v_n and v'_1, v'_2, \dots, v'_n denote the consecutive vertices of two disjoint paths with n vertices. Recall that we obtain the Möbius ladder M_n by joining v_i to v'_i for $i \in \{1, 2, \dots, n\}$ and by joining v_1 to v'_n and v_n to v'_1 . For convenience, we let $M_n = P_n \cup P'_n \cup F \cup H$, where $P_n = (v_1, v_2, \dots, v_n)$, $P'_n = (v'_1, v'_2, \dots, v'_n)$, $F = \{\{v_i, v'_i\} : 1 \leq i \leq n\}$ and $H = \{\{v_1, v'_n\}, \{v_n, v'_1\}\}$. We shall refer to P_n as the *outer path*, to P'_n as the *inner path*, and to F as the *spokes*. Figure 3.15 shows the Möbius ladder M_9 . We note that M_{2n+1} , $n \geq 1$, is necessarily bipartite with bipartition $(O \cup W', W \cup O')$, where $O = \{v_{2i+1} : 1 \leq i \leq n\}$, $W' = \{v'_{2i} : 1 \leq i \leq n\}$, $W = \{v_{2i} : 1 \leq i \leq n\}$, and $O' = \{v'_{2i+1} : 1 \leq i \leq n\}$. We will show that M_n admits a free α -labeling for all odd integers $n \geq 3$.

Figure 3.15: The Möbius ladder M_9 .

Theorem 3.2.3 *The Möbius ladder M_n admits a free α -labeling for all odd $n \geq 3$.*

Proof. We separate the proof into 3 cases. In each case, we give an α -labeling f such that the complementary labeling of f is free.

Case 1: $n \equiv 1 \pmod{6}$.

Let $n = 6t + 1$. Thus, $|V(M_n)| = 12t + 2$ and $|E(M_n)| = 18t + 3$, where $t \geq 1$.

Define a one-to-one function $f : V(M_{6t+1}) \rightarrow [0, 18t + 3]$ as follows:

$$\begin{aligned}
 f(v_i) &= i - 1, & v_i \in O &= \{v_i : i \text{ odd}, 1 \leq i \leq 6t + 1\}, \\
 f(v_i) &= 18t - \frac{i}{2} + 1, & v_i \in W_1 &= \{v_i : i \text{ even}, 2 \leq i \leq 4t - 2\}, \\
 f(v_i) &= 18t - \frac{i}{2} - 2, & v_i \in W_2 &= \{v_i : i \text{ even}, 4t - 2 < i < 6t\}, \\
 f(v_{6t}) &= 6t + 1, \\
 f(v'_1) &= 18t + 3, \\
 f(v'_i) &= i - 1, & v_i \in W' &= \{v_i : i \text{ even}, 1 < i \leq 6t\}, \\
 f(v'_i) &= 9t - \frac{i-1}{2} + 2, & v_i \in O'_1 &= \{v_i : i \text{ odd}, 1 < i < 6t + 1\}, \\
 f(v'_{6t+1}) &= 18t + 1.
 \end{aligned}$$

Note that $W = W_1 \cup W_2 \cup \{v_{6t}\}$ and $O' = \{v'_1\} \cup O'_1 \cup \{v'_{6t+1}\}$. Thus the domain of f is indeed $V(M_{6t+1})$.

Next, we confirm that f is one-to-one. We compute

$$f(O) = \{0, 2, \dots, 6t\},$$

$$f(W_1) = \{18t, 18t - 1, \dots, 16t + 2\},$$

$$f(W_2) = \{16t - 2, 16t - 3, \dots, 15t - 1\},$$

$$f(W') = \{1, 3, \dots, 6t - 1\},$$

$$f(O'_1) = \{9t + 1, 9t, \dots, 6t + 3\}.$$

Note that f is piecewise strictly increasing by 2 or strictly decreasing by 1 and that all labels are distinct. Thus f is one-to-one. Moreover, $f(O \cup W') = [0, 6t]$ and for $v_i \in W \cup O'$, $6t + 1 \leq f(v_i) \leq 18t + 3$.

To help compute the edge labels, we will describe $f(M_{6t+1})$ in terms of the paths $\hat{P}(2k, d_1, d_2, a, b)$. For convenience, we will identify the vertices of P_{6t+1} and P'_{6t+1} with their labels. We have $f(P_{6t+1}) = G_1 + G_2 + (6t - 2, 6t + 1, 6t)$, where

$$G_1 = \hat{P}(2(2t - 1), 2, 1, 0, 16t + 2),$$

$$G_2 = \hat{P}(2t, 2, 1, 4t - 2, 15t - 1).$$

By P3, the resulting edge label sets are:

$$\begin{aligned} \bar{f}(E(G_1)) &= \{12t + 4 + 3i : 0 \leq i \leq 2t - 2\} \cup \{12t + 6 + 3i : 0 \leq i \leq 2t - 2\} \\ &= \{\ell \equiv 1 \pmod{3} : 12t + 4 \leq \ell \leq 18t - 2\} \\ &\quad \cup \{\ell \equiv 0 \pmod{3} : 12t + 6 \leq \ell \leq 18t\}, \end{aligned}$$

$$\begin{aligned} \bar{f}(E(G_2)) &= \{9t + 1 + 3i : 0 \leq i \leq t - 1\} \cup \{9t + 3 + 3i : 0 \leq i \leq t - 1\} \\ &= \{\ell \equiv 1 \pmod{3} : 9t + 1 \leq \ell \leq 12t - 2\} \\ &\quad \cup \{\ell \equiv 0 \pmod{3} : 9t + 3 \leq \ell \leq 12t\}. \end{aligned}$$

Moreover, the edge labels 3 and 1 occur on the path $(6t - 2, 6t + 1, 6t)$.

Similarly, we have $f(P'_{6t+1}) = (18t + 3, 1) + G'_1 + (6t - 1, 18t + 1)$, where

$$G'_1 = \hat{P}(2(3t - 1), 2, 1, 1, 6t + 3).$$

By P3, the resulting edge label sets are:

$$\begin{aligned}\bar{f}(E(G'_1)) &= \{3i + 4 : 0 \leq i \leq 3t - 2\} \cup \{3i + 6 : 0 \leq i \leq 3t - 2\} \\ &= \{\ell \equiv 1 \pmod{3} : 4 \leq \ell \leq 9t - 2\} \\ &\quad \cup \{\ell \equiv 0 \pmod{3} : 6 \leq \ell \leq 9t\}.\end{aligned}$$

Moreover, the edge labels $18t + 2$, and $12t + 2$ occur on the edge $\{18t + 3, 1\}$ and $\{6t - 1, 18t + 1\}$.

For each spoke $\{v_i, v'_i\}$, the edge label is given by $f(v_i) - f(v'_i)$ if i is even and by $f(v'_i) - f(v_i)$ if i is odd. Thus the labels on the spokes are given by

$$\bar{f}(\{v_i, v'_i\}) = \begin{cases} 18t + 3 & \text{for } i = 1, \\ 18t - \frac{3i}{2} + 2 & \text{for } i \text{ even, } 2 \leq i \leq 4t - 2, \\ 18t - \frac{3i}{2} - 1 & \text{for } i \text{ even, } 4t \leq i \leq 6t - 2, \\ 9t - \frac{3(i-1)}{2} + 2 & \text{for } i \text{ odd, } 3 \leq i \leq 6t - 1, \\ 2 & \text{for } i = 6t, \\ 12t + 1 & \text{for } i = 6t + 1. \end{cases}$$

Thus the set of edge labels on the spokes is:

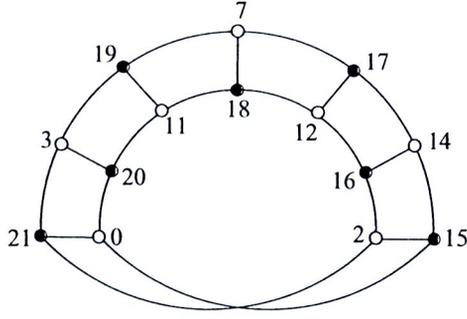
$$\begin{aligned}\bar{f}(E(F)) &= \{\ell \equiv 2 \pmod{3} : 12t + 5 \leq \ell \leq 18t - 1\} \\ &\quad \cup \{\ell \equiv 2 \pmod{3} : 9t + 2 \leq \ell \leq 12t - 1\} \\ &\quad \cup \{\ell \equiv 2 \pmod{3} : 5 \leq \ell \leq 9t - 1\} \\ &\quad \cup \{2, 12t + 1, 18t + 3\}.\end{aligned}$$

Moreover, the edge labels $18t + 1$, and $12t + 3$ occur on the edge $\{v_1, v'_{6t+1}\}$ and $\{v'_1, v_{6t+1}\}$.

It is easy to verify that each label $\ell \in [0, 18t + 3]$ occurs on exactly one edge in M_{6t+1} . Thus f is an α -labeling of M_{6t+1} . Although f is not free, it is easy to check that its complementary labeling f' is free. Figure 3.16 shows the resulting free α -labeling of M_7 .

Case 2: $n \equiv 3 \pmod{6}$.

Let $n = 6t - 3$. Thus, $|V(M_n)| = 12t - 6$ and $|E(M_n)| = 18t - 9$. A free α -labeling

Figure 3.16: A free α -labeling of M_7 .

of M_3 (which is isomorphic to $K_{3,3}$) is given in Table 3.1 (graph Bc1). For $t \geq 2$, define a one-to-one function $f : V(M_{6t-3}) \rightarrow [0, 18t - 9]$ as follows:

$$\begin{aligned}
 f(v_i) &= i - 1, & v_i \in O &= \{v_i : i \text{ odd}, 1 \leq i \leq 6t - 3\}, \\
 f(v_i) &= 18t - \frac{i}{2} - 10, & v_i \in W_1 &= \{v_i : i \text{ even}, 2 \leq i < 4t - 2\}, \\
 f(v_i) &= 18t - \frac{i}{2} - 12, & v_i \in W_2 &= \{v_i : i \text{ even}, 4t - 2 \leq i < 6t - 4\}, \\
 f(v_{6t-4}) &= 6t - 3, \\
 f(v'_1) &= 18t - 9, \\
 f(v'_i) &= i - 1, & v_i \in W' &= \{v_i : i \text{ even}, 2 \leq i \leq 6t - 4\}, \\
 f(v'_i) &= 9t - \frac{i-1}{2} - 2, & v_i \in O'_1 &= \{v_i : i \text{ odd}, 1 < i \leq 2t - 1\}, \\
 f(v'_i) &= 9t - \frac{i-1}{2} - 4, & v_i \in O'_2 &= \{v_i : i \text{ odd}, 2t - 1 < i < 6t - 3\}, \\
 f(v'_{6t-3}) &= 12t - 6.
 \end{aligned}$$

Note that $W = W_1 \cup W_2 \cup \{v_{6t-4}\}$ and $O' = \{v'_1\} \cup O'_1 \cup O'_2 \cup \{v'_{6t-3}\}$. Thus the domain of f is indeed $V(M_{6t-3})$.

Next, we confirm that f is one-to-one. We compute

$$\begin{aligned}
 f(O) &= \{0, 2, \dots, 6t - 4\}, \\
 f(W_1) &= \{18t - 11, 18t - 12, \dots, 16t - 8\}, \\
 f(W_2) &= \{16t - 11, 16t - 12, \dots, 15t - 9\}, \\
 f(W') &= \{1, 3, \dots, 6t - 5\}, \\
 f(O'_1) &= \{9t - 3, 9t - 4, \dots, 8t - 1\}, \\
 f(O'_2) &= \{8t - 4, 8t - 5, \dots, 6t - 1\}.
 \end{aligned}$$

Note that f is piecewise strictly increasing by 2 or strictly decreasing by 1 and that all labels are distinct. Thus f is one-to-one. Moreover, $f(O \cup W') = [0, 6t - 4]$ and for $v_i \in W \cup O'$, $6t - 3 \leq f(v_i) \leq 18t - 9$.

To help compute the edge labels, we will describe $f(M_{6t-3})$ in terms of the paths $\hat{P}(2k, d_1, d_2, a, b)$. For convenience, we will identify the vertices of P_{6t-3} and P'_{6t-3} with their labels. We have $f(P_{6t-3}) = G_1 + G_2 + (6t - 6, 6t - 3, 6t - 4)$, where

$$G_1 = \hat{P}(2(2t - 2), 2, 1, 0, 16t - 8),$$

$$G_2 = \hat{P}(2(t - 1), 2, 1, 4t - 4, 15t - 9).$$

By P3, the resulting edge label sets are:

$$\begin{aligned} \bar{f}(E(G_1)) &= \{12t - 4 + 3i : 0 \leq i \leq 2t - 3\} \\ &\quad \cup \{12t - 2 + 3i : 0 \leq i \leq 2t - 3\} \\ &= \{\ell \equiv 2 \pmod{3} : 12t - 4 \leq \ell \leq 18t - 13\} \\ &\quad \cup \{\ell \equiv 1 \pmod{3} : 12t - 2 \leq \ell \leq 18t - 11\}, \\ \bar{f}(E(G_2)) &= \{9t - 3 + 3i : 0 \leq i \leq t - 2\} \cup \{9t - 1 + 3i : 0 \leq i \leq t - 2\} \\ &= \{\ell \equiv 0 \pmod{3} : 9t - 3 \leq \ell \leq 12t - 9\} \\ &\quad \cup \{\ell \equiv 2 \pmod{3} : 9t - 1 \leq \ell \leq 12t - 7\}. \end{aligned}$$

Moreover, the edge labels 3 and 1 occur on the path $(6t - 6, 6t - 3, 6t - 4)$.

Similarly, we have $f(P'_{6t-3}) = \{18t - 9, 1\} + G'_1 + G'_2 + \{6t - 5, 12t - 6\}$, where

$$G'_1 = \hat{P}(2(t - 1), 2, 1, 1, 8t - 1),$$

$$G'_2 = \hat{P}(2(2t - 2), 2, 1, 2t - 1, 6t - 1).$$

By P3, the resulting edge label sets are:

$$\begin{aligned}
\bar{f}(E(G'_1)) &= \{6t + 3i : 0 \leq i \leq t - 2\} \cup \{6t + 2 + 3i : 0 \leq i \leq t - 2\} \\
&= \{\ell \equiv 0 \pmod{3} : 6t \leq \ell \leq 9t - 6\} \\
&\quad \cup \{\ell \equiv 2 \pmod{3} : 6t + 2 \leq \ell \leq 9t - 4\}, \\
\bar{f}(E(G'_2)) &= \{4 + 3i : 0 \leq i \leq 2t - 3\} \cup \{6 + 3i : 0 \leq i \leq 2t - 3\} \\
&= \{\ell \equiv 1 \pmod{3} : 4 \leq \ell \leq 6t - 5\} \\
&\quad \cup \{\ell \equiv 0 \pmod{3} : 6 \leq \ell \leq 6t - 3\}.
\end{aligned}$$

Moreover, the edge labels $18t - 10$, and $6t - 1$ occur on the edge $\{18t - 9, 1\}$ and $\{6t - 5, 12t - 6\}$.

For each spoke $\{v_i, v'_i\}$, the edge label is given by $f(v_i) - f(v'_i)$ if i is even and by $f(v'_i) - f(v_i)$ if i is odd. Thus the labels on the spokes are given by

$$\bar{f}(\{v_i, v'_i\}) = \begin{cases} 18t - 9 & \text{for } i = 1, \\ 18t - \frac{3i}{2} - 9 & \text{for } i \text{ even, } 2 \leq i \leq 4t - 4, \\ 18t - \frac{3i}{2} - 11 & \text{for } i \text{ even, } 4t - 2 \leq i \leq 6t - 6, \\ 9t - \frac{3(i-1)}{2} - 2 & \text{for } i \text{ odd, } 3 \leq i \leq 2t - 1, \\ 9t - \frac{3(i-1)}{2} - 4 & \text{for } i \text{ odd, } 2t + 1 \leq i \leq 6t - 5, \\ 2 & \text{for } i = 6t - 4, \\ 6t - 2 & \text{for } i = 6t - 3. \end{cases}$$

Thus the set of edge labels on the spokes is:

$$\begin{aligned}
\bar{f}(E(F)) &= \{\ell \equiv 0 \pmod{3} : 12t - 3 \leq \ell \leq 18t - 12\} \\
&\quad \cup \{\ell \equiv 1 \pmod{3} : 9t - 2 \leq \ell \leq 12t - 8\} \\
&\quad \cup \{\ell \equiv 1 \pmod{3} : 6t + 1 \leq \ell \leq 9t - 5\} \\
&\quad \cup \{\ell \equiv 2 \pmod{3} : 5 \leq \ell \leq 6t - 4\} \\
&\quad \cup \{2, 6t - 2, 18t - 9\}.
\end{aligned}$$

Moreover, the edge labels $12t - 6$, and $12t - 5$ occur on the edge $\{v_1, v'_{6t-3}\}$ and $\{v'_1, v_{6t-3}\}$.

It is easy to verify that each label $\ell \in [0, 18t - 9]$ occurs on exactly one edge in M_{6t-3} . Thus f is an α -labeling of M_{6t-3} . Although f is not free, it is easy to check that its complementary labeling f' is free. Figure 3.17 shows the resulting free α -labeling of M_9 .

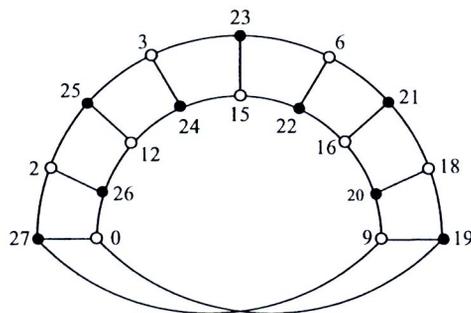


Figure 3.17: A free α -labeling of M_9 .

Case 3: $n \equiv 5 \pmod{6}$.

Let $n = 6t - 1$. Thus, $|V(M_n)| = 12t - 2$ and $|E(M_n)| = 18t - 3$. A free α -labeling of M_5 is given in Table 3.1 (graph Bc3). For $t \geq 2$, define a one-to-one function $f : V(M_{6t-1}) \rightarrow [0, 18t - 3]$ as follows:

$$f(v_i) = i - 1, \quad v_i \in O = \{v_i : i \text{ odd}, 1 \leq i \leq 6t - 1\},$$

$$f(v_2) = 18t - 3,$$

$$f(v_i) = 9t - \frac{i}{2} + 1, \quad v_i \in W_1 = \{v_i : i \text{ even}, 4 \leq i \leq 6t - 4\},$$

$$f(v_{6t-2}) = 6t - 1,$$

$$f(v'_i) = 18t - \frac{i-1}{2} - 7, \quad v_i \in O'_1 = \{v_i : i \text{ odd}, 1 \leq i \leq 4t - 3\},$$

$$f(v'_i) = 18t - \frac{i-1}{2} - 10, \quad v_i \in O'_2 = \{v_i : i \text{ odd}, 4t - 3 < i \leq 6t - 5\},$$

$$f(v'_i) = i - 1, \quad v_i \in W' = \{v_i : i \text{ even}, 2 \leq i \leq 6t - 2\},$$

$$f(v'_{6t-3}) = 6t + 1,$$

$$f(v'_{6t-1}) = 18t - 6.$$

Note that $W = \{v_2\} \cup W_1 \cup \{v_{6t-2}\}$ and $O' = O'_1 \cup O'_2 \cup \{v'_{6t-3}\} \cup \{v'_{6t-1}\}$. Thus the domain of f is indeed $V(M_{6t-1})$.

Next, we confirm that f is one-to-one. We compute

$$\begin{aligned} f(O) &= \{0, 2, \dots, 6t - 2\}, \\ f(W_1) &= \{9t - 1, 9t - 2, \dots, 6t + 3\}, \\ f(W') &= \{1, 3, \dots, 6t - 3\}, \\ f(O'_1) &= \{18t - 7, 18t - 8, \dots, 16t - 5\}, \\ f(O'_2) &= \{16t - 9, 16t - 10, \dots, 15t - 7\}. \end{aligned}$$

Note that f is piecewise strictly increasing by 2 or strictly decreasing by 1 and that all labels are distinct. Thus f is one-to-one. Moreover, $f(O \cup W') = [0, 6t - 2]$ and for $v_i \in W \cup O'$, $6t - 1 \leq f(v_i) \leq 18t - 3$.

To help compute the edge labels, we will describe $f(M_{6t-1})$ in terms of the paths $\hat{P}(2k, d_1, d_2, a, b)$. For convenience, we will identify the vertices of P_{6t-1} and P'_{6t-1} with their labels. We have $f(P_{6t-1}) = (0, 18t - 3, 2) + G_1 + (6t - 4, 6t - 1, 6t - 2)$, where

$$G_1 = \hat{P}(2(3t - 3), 2, 1, 2, 6t + 3).$$

By P3, the resulting edge label sets are:

$$\begin{aligned} \bar{f}(E(G_1)) &= \{7 + 3i : 0 \leq i \leq 3t - 4\} \cup \{9 + 3i : 0 \leq i \leq 3t - 4\} \\ &= \{\ell \equiv 1 \pmod{3} : 7 \leq \ell \leq 9t - 5\} \\ &\quad \cup \{\ell \equiv 0 \pmod{3} : 9 \leq \ell \leq 9t - 3\}. \end{aligned}$$

Moreover, the edge labels $18t - 3$ and $18t - 5$ occur on the path $(0, 18t - 3, 2)$ and the edge labels 3 and 1 occur on the path $(6t - 4, 6t - 1, 6t - 2)$.

Similarly, we have $f(P'_{6t-1}) = (18t - 7, 1) + G'_1 + G'_2 + (6t - 5, 6t + 1, 6t - 3, 18t - 6)$, where

$$\begin{aligned} G'_1 &= \hat{P}(2(2t - 2), 2, 1, 1, 16t - 5), \\ G'_2 &= \hat{P}(2(t - 1), 2, 1, 4t - 3, 15t - 7). \end{aligned}$$

By P3, the resulting edge label sets are:

$$\begin{aligned}
\bar{f}(E(G'_1)) &= \{12t - 2 + 3i : 0 \leq i \leq 2t - 3\} \cup \{12t + 3i : 0 \leq i \leq 2t - 3\} \\
&= \{\ell \equiv 1 \pmod{3} : 12t - 2 \leq \ell \leq 18t - 11\} \\
&\quad \cup \{\ell \equiv 0 \pmod{3} : 12t \leq \ell \leq 18t - 9\}, \\
\bar{f}(E(G'_2)) &= \{9t - 2 + 3i : 0 \leq i \leq t - 2\} \cup \{9t + 3i : 0 \leq i \leq t - 2\} \\
&= \{\ell \equiv 1 \pmod{3} : 9t - 2 \leq \ell \leq 12t - 8\} \\
&\quad \cup \{\ell \equiv 0 \pmod{3} : 9t \leq \ell \leq 12t - 6\}.
\end{aligned}$$

Moreover, the edge labels $18t - 8$ occur on the edge $\{18t - 9, 1\}$ and the edge labels $6, 4, 12t - 3$ occur on path $(6t - 5, 6t + 1, 6t - 3, 18t - 6)$.

For each spoke $\{v_i, v'_i\}$, the edge label is given by $f(v_i) - f(v'_i)$ if i is even and by $f(v'_i) - f(v_i)$ if i is odd. Thus the labels on the spokes are given by

$$\bar{f}(\{v_i, v'_i\}) = \begin{cases} 18t - \frac{3(i-1)}{2} - 7 & \text{for } i \text{ odd, } 1 \leq i \leq 4t - 3, \\ 18t - \frac{3(i-1)}{2} - 10 & \text{for } i \text{ odd, } 4t - 1 \leq i \leq 6t - 5, \\ 5 & \text{for } i = 6t - 3, \\ 12t - 4 & \text{for } i = 6t - 1, \\ 18t - 4 & \text{for } i = 2, \\ 9t - \frac{3i}{2} + 2 & \text{for } i \text{ even, } 4 \leq i \leq 6t - 4, \\ 2 & \text{for } i = 6t - 2. \end{cases}$$

Thus the set of edge labels on the spokes is:

$$\begin{aligned}
\bar{f}(E(F)) &= \{\ell \equiv 2 \pmod{3} : 12t - 1 \leq \ell \leq 18t - 7\} \\
&\quad \cup \{\ell \equiv 2 \pmod{3} : 9t - 1 \leq \ell \leq 12t - 7\} \\
&\quad \cup \{\ell \equiv 2 \pmod{3} : 8 \leq \ell \leq 9t - 4\} \cup \{2, 5, 12t - 4, 18t - 4\}.
\end{aligned}$$

Moreover, the edge labels $18t - 6$, and $12t - 5$ occur on the edge $\{v_1, v'_{6t-1}\}$ and $\{v'_1, v_{6t-1}\}$.

It is easy to verify that each label $\ell \in [0, 18t - 3]$ occurs on exactly one edge in M_{6t-1} . Thus f is an α -labeling of M_{6t-1} . Although f is not free, it is easy to check that its complementary labeling f' is free. Figure 3.18 shows the resulting free α -labeling of M_{11} . ■

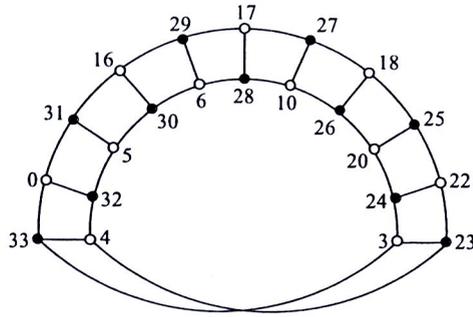


Figure 3.18: A free α -labeling of M_{11} .

3.2.3 Free α -labelings of Bipartite Cubic Graphs of Small Order

According to the book *An Atlas of Graphs* [39], there are 22 connected bipartite cubic graphs of order at most 14. Each of these graphs admits a free α -labeling (see Table 3.1). We referenced these graphs in the same way they are referenced in [39]. Thus we have the following.

Theorem 3.2.4 *Every bipartite cubic graph of order at most 14 admits a free α -labeling.*

Corollary 3.2.1 *Let G be a cubic bipartite graph such that each component of G is either a prism, a Möbius ladder, or has order at most 14. Then G admits a free α -labeling.*

3.2.4 Concluding Remarks

Based on our investigation, we believe that all bipartite cubic graphs admit free α -labelings.

Conjecture 3.2.1 *Every bipartite cubic graph admits a free α -labeling.*

Table 3.1: Connected bipartite cubic graphs: 6–14 vertices

<p>Bc1</p>	<p>Bc2</p>	<p>Bc3</p>
<p>Bc4</p>	<p>Bc5</p>	<p>Bc6</p>
<p>Bc7</p>	<p>Bc8</p>	<p>Bc9</p>
<p>Bc10</p>	<p>Bc11</p>	<p>Bc12</p>

Table 3.1 (cont.): Connected bipartite cubic graphs: 6–14 vertices

<p>Bc13</p>	<p>Bc14</p>	<p>Bc15</p>
<p>Bc16</p>	<p>Bc17</p>	<p>Bc18</p>
<p>Bc19</p>	<p>Bc20</p>	<p>Bc21</p>
<p>Bc22</p>		

3.3 ρ -tripartite Labelings of Some Cubic Graphs

Because the concept of a ρ -tripartite labeling is very recent, little has been done on this topic. It was shown in [14] that the union of two odd cycles admits a ρ -tripartite labeling if the total number of edges is congruent to 2 modulo 4 and admits what is called a σ -tripartite labeling otherwise. It was also shown in [14] that the Petersen graph admits a ρ -tripartite labeling. As stated earlier, a γ -labeling of an almost-bipartite graph G is necessarily a ρ -tripartite labeling of G . Known results on γ -labelings are stated in Theorem 2.3.6.

We will show that if G is an odd prism, an even Möbius ladder, or a connected cubic tripartite graph of order at most 10, then G admits a ρ -tripartite labeling. We conjecture that every connected tripartite cubic graph admits a ρ -tripartite labeling.

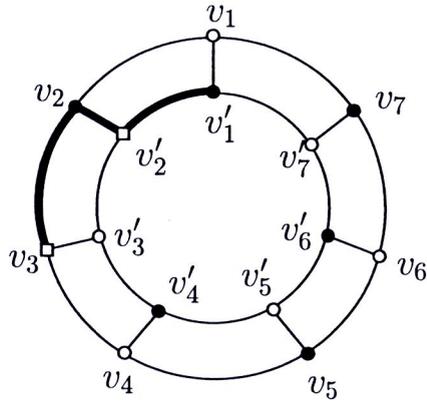
3.3.1 ρ -tripartite Labelings of Odd Prisms

Recall that the prism D_n ($n \geq 3$) is the cartesian product $C_n \times P_2$ of a cycle with n vertices and a path with 2 vertices. For convenience, we let $D_n = C_n \cup C'_n \cup F$, where $C_n = (v_1, v_2, \dots, v_n, v_1)$, $C'_n = (v'_1, v'_2, \dots, v'_n, v'_1)$, and $F = \{\{v_i, v'_i\} : 1 \leq i \leq n\}$. Recall that we refer to C_n as the outer cycle, to C'_n as the inner cycle, and to F as the spokes. We note that D_{2n+1} , $n \geq 1$, is necessarily tripartite with tripartition $\{A, B, C\}$ where $A = \{v_1\} \cup \{v_{2i} : 2 \leq i \leq n\} \cup \{v'_{2i+1} : 1 \leq i \leq n\}$, $B = \{v_2\} \cup \{v'_1\} \cup \{v_{2i+1} : 2 \leq i \leq n\} \cup \{v'_{2i} : 2 \leq i \leq n\}$, and $C = \{v_3, v'_2\}$. Figure 3.19 shows the prism D_7 . In this figure, the vertices in A are shown with open circles while the vertices in B are shown with filled circles and the vertices of C are shown with open squares. The edges between sets B and C are shown in thick lines. We will adopt this convention in all our figures. We will show that D_n admits a ρ -tripartite labeling for all odd integers $n \geq 3$.

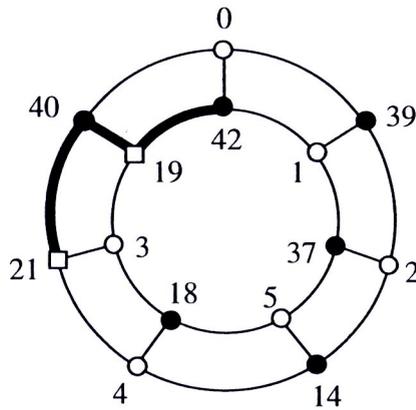
Theorem 3.3.1 *The prism D_n admits a ρ -tripartite labeling for all odd $n \geq 3$.*

Proof. We separate the proof into 3 cases.

Case 1: $n \equiv 1 \pmod{6}$.

Figure 3.19: The prism D_7 .

Let $n = 6t + 1$. Thus, $|V(D_n)| = 12t + 2$ and $|E(D_n)| = 18t + 3$. A ρ -tripartite labeling of D_7 is given in Figure 3.20.

Figure 3.20: A ρ -tripartite labeling of D_7 .

For $t \geq 2$, define a one-to-one function $f : V(D_{6t+1}) \rightarrow [0, 36t + 6]$ as follows:

$$f(v_1) = 0,$$

$$f(v_2) = 36t + 4,$$

$$f(v_3) = 18t + 3,$$

$$f(v_i) = i, \quad v_i \in A_1 = \{v_i : i \text{ even}, 4 \leq i \leq 6t - 2\},$$

$$f(v_i) = 18t - 2i + 8, \quad v_i \in B_1 = \{v_i : i \text{ odd}, 5 \leq i \leq 4t - 1\},$$

$$f(v_i) = 18t - 2i + 6, \quad v_i \in B_2 = \{v_i : i \text{ odd}, 4t - 1 < i \leq 6t - 1\},$$

$$f(v_{6t}) = 2,$$

$$f(v_{6t+1}) = 36t + 3,$$

$$f(v'_1) = 36t + 6,$$

$$f(v'_2) = 18t + 1,$$

$$f(v'_i) = i, \quad v'_i \in A'_1 = \{v'_i : i \text{ odd}, 3 \leq i \leq 6t - 1\},$$

$$f(v'_i) = 18t - 2i + 8, \quad v'_i \in B'_1 = \{v'_i : i \text{ even}, 4 \leq i \leq 4t\},$$

$$f(v'_i) = 18t - 2i + 6, \quad v'_i \in B'_2 = \{v'_i : i \text{ even}, 4t < i \leq 6t - 2\},$$

$$f(v'_{6t}) = 36t + 1,$$

$$f(v'_{6t+1}) = 1.$$

Note that $A = \{v_1, v_{6t}, v'_{6t+1}\} \cup A_1 \cup A'_1$, $B = \{v_2, v_{6t+1}, v'_1, v'_{6t}\} \cup B_1 \cup B_2 \cup B'_1 \cup B'_2$ and $C = \{v_3, v'_2\}$. Thus the domain of f is indeed $V(D_{6t+1})$. Next, we confirm that f is one-to-one. We compute

$$f(A_1) = \{4, 6, \dots, 6t - 2\},$$

$$f(A'_1) = \{3, 5, \dots, 6t - 1\},$$

$$f(B_1) = \{18t - 2, 18t - 6, \dots, 10t + 10\},$$

$$f(B_2) = \{10t + 4, 10t, \dots, 6t + 8\},$$

$$f(B'_1) = \{18t, 18t - 4, \dots, 10t + 8\},$$

$$f(B'_2) = \{10t + 2, 10t - 2, \dots, 6t + 10\}.$$

Note that f is piecewise strictly increasing by 2 or strictly decreasing by 4 and that all labels are distinct. Thus f is one-to-one. Moreover, $f(A) = [0, 6t - 1]$ and $f(B \cup C) \subseteq [6t, 36t + 6]$.

To help compute the edge labels, we will describe $f(V(D_{6t+1}))$ in terms of the paths $\hat{P}(2k, d_1, d_2, a, b)$. For convenience, we will identify the vertices of C_{6t+1} and C'_{6t+1} with their labels. We have $f(C_{6t+1}) = G_1 + G_2 + (6t - 2, 6t + 8, 2, 36t + 3, 0, 36t + 4, 18t + 3, 4)$, where

$$G_1 = \hat{P}(2(2t - 2), 2, 4, 4, 10t + 10),$$

$$G_2 = \hat{P}(2(t - 1), 2, 4, 4t, 6t + 12).$$

By P3, the resulting edge label sets are:

$$\begin{aligned} \bar{f}(E(G_1)) &= \{6t + 10 + 6i : 0 \leq i \leq 2t - 3\} \cup \{6t + 12 + 6i : 0 \leq i \leq 2t - 3\} \\ &= \{\ell \equiv 4 \pmod{6} : 6t + 10 \leq \ell \leq 18t - 8\} \\ &\quad \cup \{\ell \equiv 0 \pmod{6} : 6t + 12 \leq \ell \leq 18t - 6\}, \end{aligned}$$

$$\begin{aligned} \bar{f}(E(G_2)) &= \{14 + 6i : 0 \leq i \leq t - 2\} \cup \{16 + 6i : 0 \leq i \leq t - 2\} \\ &= \{\ell \equiv 2 \pmod{6} : 14 \leq \ell \leq 6t + 2\} \\ &\quad \cup \{\ell \equiv 4 \pmod{6} : 16 \leq \ell \leq 6t + 4\}. \end{aligned}$$

Moreover, edge labels $10, 6t + 6, 36t + 1, 36t + 3, 36t + 4, 18t + 1$ and $18t - 1$ occur on the path $(6t - 2, 6t + 8, 2, 36t + 3, 0, 36t + 4, 18t + 3, 4)$.

Similarly, we have $f(C'_{6t+1}) = G'_1 + G'_2 + (6t - 1, 36t + 1, 1, 36t + 6, 18t + 1, 3)$, where

$$G'_1 = \hat{P}(2(2t - 1), 2, 4, 3, 10t + 8),$$

$$G'_2 = \hat{P}(2(t - 1), 2, 4, 4t + 1, 6t + 10).$$

By P3, the resulting edge label sets are:

$$\begin{aligned} \bar{f}(E(G'_1)) &= \{6t + 7 + 6i : 0 \leq i \leq 2t - 2\} \cup \{6t + 9 + 6i : 0 \leq i \leq 2t - 2\} \\ &= \{\ell \equiv 1 \pmod{6} : 6t + 7 \leq \ell \leq 18t - 5\} \\ &\quad \cup \{\ell \equiv 3 \pmod{6} : 6t + 9 \leq \ell \leq 18t - 3\}, \end{aligned}$$

$$\begin{aligned} \bar{f}(E(G'_2)) &= \{11 + 6i : 0 \leq i \leq t - 2\} \cup \{13 + 6i : 0 \leq i \leq t - 2\} \\ &= \{\ell \equiv 5 \pmod{6} : 11 \leq \ell \leq 6t - 1\} \\ &\quad \cup \{\ell \equiv 1 \pmod{6} : 13 \leq \ell \leq 6t + 1\}. \end{aligned}$$

Moreover, edge labels $30t + 2, 36t, 36t + 5, 18t + 5$ and $18t - 2$ occur on the path $(6t - 1, 36t + 1, 1, 36t + 6, 18t + 1, 3)$.

For each spoke $\{v_i, v'_i\}$, the labels on the spokes are given by

$$\bar{f}(\{v_i, v'_i\}) = \begin{cases} 36t + 6 & \text{for } i = 1, \\ 18t + 3 & \text{for } i = 2, \\ 18t & \text{for } i = 3, \\ 18t - 3i + 8 & \text{for } 4 \leq i \leq 4t, \\ 18t - 3i + 6 & \text{for } 4t + 1 \leq i \leq 6t - 1, \\ 36t - 1 & \text{for } i = 6t, \\ 36t + 2 & \text{for } i = 6t + 1. \end{cases}$$

Thus the set of edge labels on the spokes is:

$$\begin{aligned} \bar{f}(E(F)) = & \{\ell \equiv 2 \pmod{3} : 6t + 8 \leq \ell \leq 18t - 4\} \\ & \cup \{\ell \equiv 0 \pmod{3} : 9 \leq \ell \leq 6t + 3\} \\ & \cup \{36t + 6, 18 + 3, 18t, 36t - 1, 36t + 2\}. \end{aligned}$$

It is easy to verify now that for each $\ell \in [1, 18t + 3]$ either ℓ or $36t + 7 - \ell$ occurs on exactly one edge in D_{6t+1} . Hence the defined labeling is a ρ -labeling, and condition (r1) for a ρ -tripartite labeling is satisfied. Condition (r2) also holds since $f(A) = [0, 6t - 1]$ and $f(B \cup C) \subseteq [6t, 36t + 6]$. Condition (r3) holds since $|f(v'_1) - f(v'_2)| + |f(v_2) - f(v_3)| = 36t + 6$ and $|f(v_2) - f(v'_2)| + |f(v_2) - f(v'_2)| = 36t + 6$, twice the number of edges of D_{6t+1} . Also $|f(b) - f(c)| = 36t + 6$, where $b \in B$ and $c \in C$, is impossible since all vertex labels are in $[0, 36t + 6]$ and $0 \in f(A)$. Thus condition (r4) holds, and we have a ρ -tripartite labeling of D_{6t+1} . Figure 3.21 shows a ρ -tripartite labeling of D_{13} .

Case 2: $n \equiv 3 \pmod{6}$.

Let $n = 6t - 3$. Thus, $|V(D_n)| = 12t - 6$ and $|E(D_n)| = 18t - 9$. A ρ -tripartite labeling of D_3 is given in Table 3.2 (graph C2). For $t \geq 2$, Define a one-to-one

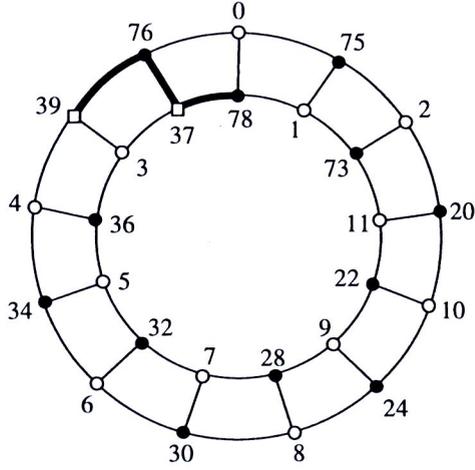


Figure 3.21: A ρ -tripartite labeling of D_{13} .

function $f : V(D_{6t-3}) \rightarrow [0, 36t - 18]$ as follows:

$$f(v_1) = 0,$$

$$f(v_2) = 36t - 20,$$

$$f(v_3) = 18t - 9,$$

$$f(v_i) = i, \quad v_i \in A_1 = \{v_i : i \text{ even}, 4 \leq i \leq 6t - 4\},$$

$$f(v_i) = 18t - 2i - 4, \quad v_i \in B_1 = \{v_i : i \text{ odd}, 5 \leq i \leq 4t - 3\},$$

$$f(v_i) = 18t - 2i - 10, \quad v_i \in B_2 = \{v_i : i \text{ odd}, 4t - 3 < i \leq 6t - 5\},$$

$$f(v_{6t-3}) = 36t - 19,$$

$$f(v'_1) = 36t - 18,$$

$$f(v'_2) = 18t - 11,$$

$$f(v'_i) = i, \quad v'_i \in A'_1 = \{v'_i : i \text{ odd}, 3 \leq i \leq 6t - 5\},$$

$$f(v'_i) = 18t - 2i - 4, \quad v'_i \in B'_1 = \{v'_i : i \text{ even}, 4 \leq i \leq 4t - 4\},$$

$$f(v'_i) = 18t - 2i - 10, \quad v'_i \in B'_2 = \{v'_i : i \text{ even}, 4t - 4 < i \leq 6t - 6\},$$

$$f(v'_{6t-4}) = 36t - 22,$$

$$f(v'_{6t-3}) = 30t - 21.$$

Note that $A = \{v_1, v'_{6t-3}\} \cup A_1 \cup A'_1$, $B = \{v_2, v'_1, v_{6t-3}, v'_{6t-4}\} \cup B_1 \cup B_2 \cup B'_1 \cup B'_2$ and $C = \{v_3, v'_2\}$. Thus the domain of f is indeed $V(D_{6t-3})$. Next, we confirm

that f is one-to-one. We compute

$$\begin{aligned} f(A_1) &= \{4, 6, \dots, 6t - 4\}, \\ f(A'_1) &= \{3, 5, \dots, 6t - 5\}, \\ f(B_1) &= \{18t - 14, 18t - 18, \dots, 10t + 2\}, \\ f(B_2) &= \{10t - 8, 10t - 12, \dots, 6t\}, \\ f(B'_1) &= \{18t - 12, 18t - 16, \dots, 10t + 4\}, \\ f(B'_2) &= \{10t - 6, 10t - 10, \dots, 6t + 2\}. \end{aligned}$$

Note that f is piecewise strictly increasing by 2 or strictly decreasing by 4 and that all labels are distinct. Thus f is one-to-one.

To help compute the edge labels, we will describe $f(D_{6t-3})$ in terms of the paths $\hat{P}(2k, d_1, d_2, a, b)$. For convenience, we will identify the vertices of C_{6t-3} and C'_{6t-3} with their labels. We have $f(C_{6t-3}) = G_1 + G_2 + (6t - 4, 36t - 19, 0, 36t - 20, 18t - 9, 4)$, where

$$\begin{aligned} G_1 &= \hat{P}(2(2t - 3), 2, 4, 4, 10t + 2), \\ G_2 &= \hat{P}(2(t - 1), 2, 4, 4t - 2, 6t). \end{aligned}$$

By P3, the resulting edge label sets are:

$$\begin{aligned} \bar{f}(E(G_1)) &= \{6t + 4 + 6i : 0 \leq i \leq 2t - 4\} \cup \{6t + 6 + 6i : 0 \leq i \leq 2t - 4\} \\ &= \{\ell \equiv 4 \pmod{6} : 6t + 4 \leq \ell \leq 18t - 20\} \\ &\quad \cup \{\ell \equiv 0 \pmod{6} : 6t + 6 \leq \ell \leq 18t - 18\}, \\ \bar{f}(E(G_2)) &= \{4 + 6i : 0 \leq i \leq t - 2\} \cup \{6 + 6i : 0 \leq i \leq t - 2\} \\ &= \{\ell \equiv 4 \pmod{6} : 4 \leq \ell \leq 6t - 8\} \\ &\quad \cup \{\ell \equiv 0 \pmod{6} : 6 \leq \ell \leq 6t - 6\}. \end{aligned}$$

Moreover, edge labels $30t - 15, 36t - 19, 36t - 20, 18t - 11$ and $18t - 13$ occur on the path $(6t - 4, 36t - 19, 0, 36t - 20, 18t - 9, 4)$.

Similarly, we have $f(C'_{6t-3}) = G'_1 + G'_2 + (6t - 5, 36t - 22, 30t - 21, 36t - 18, 18t - 11, 3)$, where

$$\begin{aligned} G'_1 &= \hat{P}(2(2t - 3), 2, 4, 3, 10t + 4), \\ G'_2 &= \hat{P}(2(t - 1), 2, 4, 4t - 3, 6t + 2). \end{aligned}$$

By P3, the resulting edge label sets are:

$$\begin{aligned}
\bar{f}(E(G'_1)) &= \{6t + 7 + 6i : 0 \leq i \leq 2t - 4\} \cup \{6t + 9 + 6i : 0 \leq i \leq 2t - 4\} \\
&= \{\ell \equiv 1 \pmod{6} : 6t + 7 \leq \ell \leq 18t - 17\} \\
&\quad \cup \{\ell \equiv 3 \pmod{6} : 6t + 9 \leq \ell \leq 18t - 15\}, \\
\bar{f}(E(G'_2)) &= \{7 + 6i : 0 \leq i \leq t - 2\} \cup \{9 + 6i : 0 \leq i \leq t - 2\}, \\
&= \{\ell \equiv 1 \pmod{6} : 7 \leq \ell \leq 6t - 5\} \\
&\quad \cup \{\ell \equiv 3 \pmod{6} : 9 \leq \ell \leq 6t - 3\}.
\end{aligned}$$

Moreover, edge labels $30t - 17, 6t - 1, 6t + 3, 18t - 7$ and $18t - 14$ occur on the path $(6t - 5, 36t - 22, 30t - 21, 36t - 18, 18t - 11, 3)$.

For each spoke $\{v_i, v'_i\}$, the labels on the spokes are given by

$$\bar{f}(\{v_i, v'_i\}) = \begin{cases} 36t - 18 & \text{for } i = 1, \\ 18t - 9 & \text{for } i = 2, \\ 18t - 12 & \text{for } i = 3, \\ 18t - 3i - 4 & \text{for } 4 \leq i \leq 4t - 3, \\ 18t - 3i - 10 & \text{for } 4t - 2 \leq i \leq 6t - 5, \\ 30t - 18 & \text{for } i = 6t - 4, \\ 6t + 2 & \text{for } i = 6t - 3. \end{cases}$$

Thus the set of edge labels on the spokes is:

$$\begin{aligned}
\bar{f}(E(F)) &= \{\ell \equiv 2 \pmod{3} : 6t + 5 \leq \ell \leq 18t - 16\} \\
&\quad \cup \{\ell \equiv 2 \pmod{3} : 5 \leq \ell \leq 6t - 4\} \\
&\quad \cup \{36t - 18, 18t - 9, 18t - 12, 30t - 18, 6t + 2\}.
\end{aligned}$$

It is easy to verify that for each $\ell \in [1, 18t - 9]$ either ℓ or $36t - 17 - \ell$ occurs on exactly one edge in D_{6t-3} . Hence the defined labeling is a ρ -labeling and condition (r1) for a ρ -tripartite labeling is satisfied. Condition (r2) also holds since $f(A) \setminus f(v'_{6t-3}) \subseteq [0, 6t - 4]$, $f(B \cup C) \subseteq [6t - 3, 36t - 18]$, and $f(v'_{6t-3}) < f(u)$ for $u \in \{v_{6t-3}, v'_1, v'_{6t-4}\}$. Condition (r3) holds since $|f(v'_1) - f(v'_2)| + |f(v_2) - f(v_3)| = 36t - 18$ and $|f(v_2) - f(v'_2)| + |f(v_2) - f(v'_2)| = 36t - 18$, twice the number of edges

of D_{6t-3} . Also $|f(b) - f(c)| = 36t - 18$, where $b \in B$ and $c \in C$, is impossible since all vertices are in $[0, 36t - 18]$ and $0 \in f(A)$. Thus condition (r4) holds, and we have a ρ -tripartite labeling of D_{6t-3} . Figure 3.22 shows a ρ -tripartite labeling of D_9 .

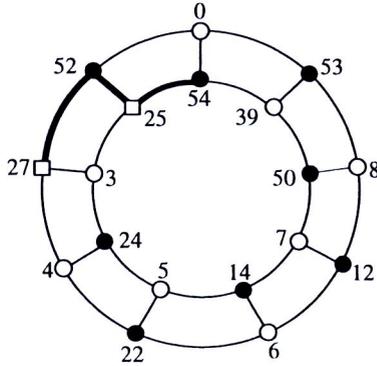


Figure 3.22: A ρ -tripartite labeling of D_9 .

Case 3: $n \equiv 5 \pmod{6}$.

Let $n = 6t - 1$. Thus, $|V(D_n)| = 12t - 2$ and $|E(D_n)| = 18t - 3$. A ρ -tripartite labeling of D_5 is given in Table 3.2 (graph C23). For $t \geq 2$, Define a one-to-one

function $f : V(D_{6t-1}) \rightarrow [0, 36t - 6]$ as follows:

$$\begin{aligned}
f(v_1) &= 0, \\
f(v_2) &= 36t - 8, \\
f(v_3) &= 18t - 3, \\
f(v_i) &= i, & v_i \in A_1 &= \{v_i : i \text{ even}, 4 \leq i \leq 6t - 2\}, \\
f(v_i) &= 18t - 2i + 2, & v_i \in B_1 &= \{v_i : i \text{ odd}, 5 \leq i \leq 4t - 1\}, \\
f(v_i) &= 18t - 2i, & v_i \in B_2 &= \{v_i : i \text{ odd}, 4t - 1 < i \leq 6t - 3\}, \\
f(v_{6t-1}) &= 36t - 9, \\
f(v'_1) &= 36t - 6, \\
f(v'_2) &= 18t - 5, \\
f(v'_i) &= i, & v'_i \in A'_1 &= \{v'_i : i \text{ odd}, 3 \leq i \leq 6t - 3\}, \\
f(v'_i) &= 18t - 2i + 2, & v'_i \in B'_1 &= \{v'_i : i \text{ even}, 4 \leq i \leq 4t - 2\}, \\
f(v'_i) &= 18t - 2i, & v'_i \in B'_2 &= \{v'_i : i \text{ even}, 4t - 2 < i \leq 6t - 2\}, \\
f(v'_{6t-1}) &= 1.
\end{aligned}$$

Note that $A = \{v_1, v'_{6t-1}\} \cup A_1 \cup A'_1$, $B = \{v_2, v_{6t-1}, v'_1\} \cup B_1 \cup B_2 \cup B'_1 \cup B'_2$ and $C = \{v_3, v'_2\}$. Thus the domain of f is indeed $V(D_{6t-1})$. Next, we confirm that f is one-to-one.

We compute

$$\begin{aligned}
f(A_1) &= \{4, 6, \dots, 6t - 2\}, \\
f(A'_1) &= \{3, 5, \dots, 6t - 3\}, \\
f(B_1) &= \{18t - 8, 18t - 12, \dots, 10t + 4\}, \\
f(B_2) &= \{10t - 2, 10t - 6, \dots, 6t + 6\}, \\
f(B'_1) &= \{18t - 6, 18t - 10, \dots, 10t + 6\}, \\
f(B'_2) &= \{10t, 10t - 4, \dots, 6t + 4\}.
\end{aligned}$$

Note that f is piecewise strictly increasing by 2 or strictly decreasing by 4 and that all labels are distinct. Thus f is one-to-one. Moreover, $f(A) \subseteq [0, 6t - 2]$ and $f(B \cup C) \subseteq [6t - 1, 36t - 6]$.

To help compute the edge labels, we will describe $f(D_{6t-1})$ in terms of the paths $\hat{P}(2k, d_1, d_2, a, b)$. For convenience, we will identify the vertices of C_{6t-1} and C'_{6t-1} with their labels. We have $f(C_{6t-1}) = G_1 + G_2 + (6t-2, 36t-9, 0, 36t-8, 18t-3, 4)$, where

$$G_1 = \hat{P}(2(2t-2), 2, 4, 4, 10t+4),$$

$$G_2 = \hat{P}(2(t-1), 2, 4, 4t, 6t+6).$$

By P3, the resulting edge label sets are:

$$\begin{aligned} \bar{f}(E(G_1)) &= \{6t+4+6i : 0 \leq i \leq 2t-3\} \cup \{6t+6+6i : 0 \leq i \leq 2t-3\} \\ &= \{\ell \equiv 4 \pmod{6} : 6t+4 \leq \ell \leq 18t-14\} \\ &\quad \cup \{\ell \equiv 0 \pmod{6} : 6t+6 \leq \ell \leq 18t-12\}, \\ \bar{f}(E(G_2)) &= \{8+6i : 0 \leq i \leq t-2\} \cup \{10+6i : 0 \leq i \leq t-2\} \\ &= \{\ell \equiv 2 \pmod{6} : 8 \leq \ell \leq 6t-4\} \\ &\quad \cup \{\ell \equiv 4 \pmod{6} : 10 \leq \ell \leq 6t-2\}. \end{aligned}$$

Moreover, edge labels $30t-7, 36t-9, 36t-8, 18t-5$ and $18t-7$ occur on the path $(6t-2, 36t-9, 0, 36t-8, 18t-3, 4)$.

Similarly, we have $f(C'_{6t-3}) = G'_1 + G'_2 + (6t-3, 6t+4, 1, 36t-6, 18t-5, 3)$, where

$$G'_1 = \hat{P}(2(2t-2), 2, 4, 3, 10t+6),$$

$$G'_2 = \hat{P}(2(t-1), 2, 4, 4t-1, 6t+8).$$

By P3, the resulting edge label sets are:

$$\begin{aligned} \bar{f}(E(G'_1)) &= \{6t+7+6i : 0 \leq i \leq 2t-3\} \cup \{6t+9+6i : 0 \leq i \leq 2t-3\} \\ &= \{\ell \equiv 1 \pmod{6} : 6t+7 \leq \ell \leq 18t-11\} \\ &\quad \cup \{\ell \equiv 3 \pmod{6} : 6t+9 \leq \ell \leq 18t-9\}, \\ \bar{f}(E(G'_2)) &= \{11+6i : 0 \leq i \leq t-2\} \cup \{13+6i : 0 \leq i \leq t-2\} \\ &= \{\ell \equiv 5 \pmod{6} : 11 \leq \ell \leq 6t-1\} \\ &\quad \cup \{\ell \equiv 1 \pmod{6} : 13 \leq \ell \leq 6t+1\}. \end{aligned}$$

Moreover, edge labels $7, 6t + 3, 36t - 7, 18t - 1$ and $18t - 8$ occur on the path $(6t - 3, 6t + 4, 1, 36t - 6, 18t - 5, 3)$.

For each spoke $\{v_i, v'_i\}$, the labels on the spokes are given by

$$\bar{f}(\{v_i, v'_i\}) = \begin{cases} 36t - 6 & \text{for } i = 1, \\ 18t - 3 & \text{for } i = 2, \\ 18t - 6 & \text{for } i = 3, \\ 18t - 3i + 2 & \text{for } 4 \leq i \leq 4t - 1, \\ 18t - 3i & \text{for } 4t - 1 < i \leq 6t - 2, \\ 36t - 10 & \text{for } i = 6t - 1. \end{cases}$$

Thus the set of edge labels on the spokes is:

$$\begin{aligned} \bar{f}(E(F)) = & \{\ell \equiv 2 \pmod{3} : 6t + 5 \leq \ell \leq 18t - 10\} \\ & \cup \{\ell \equiv 0 \pmod{3} : 6 \leq \ell \leq 6t\} \\ & \cup \{36t - 6, 18t - 3, 18t - 6, 36t - 10\}. \end{aligned}$$

It is easy to verify that for each $\ell \in [1, 18t - 3]$ either ℓ or $36t - 5 - \ell$ occurs on exactly one edge in D_{6t-1} . Hence the defined labeling is a ρ -labeling and condition (r1) for a ρ -tripartite labeling is satisfied. Condition (r2) also holds since $f(A) \subseteq [0, 6t - 2]$ and $f(B \cup C) \subseteq [6t - 1, 36t - 6]$. Condition (r3) holds since $|f(v'_1) - f(v'_2)| + |f(v_2) - f(v_3)| = 36t - 6$ and $|f(v_2) - f(v'_2)| + |f(v_2) - f(v'_2)| = 36t - 6$, twice the number of edges of D_{6t-1} . Also $|f(b) - f(c)| = 36t - 6$, where $b \in B$ and $c \in C$, is impossible since all vertices are in $[0, 36t - 6]$ and $0 \in f(A)$. Thus condition (r4) holds, and we have a ρ -tripartite labeling of D_{6t-1} . Figure 3.23 shows a ρ -tripartite labeling of D_{11} . ■

3.3.2 ρ -tripartite Labelings of Even Möbius Ladders

Recall that for $n \geq 3$, let v_1, v_2, \dots, v_n and v'_1, v'_2, \dots, v'_n denote the consecutive vertices of two disjoint paths with n vertices. The *Möbius ladder* is the graph M_n obtained by joining v_i to v'_i for $i = 1, 2, \dots, n$ and by joining v_1

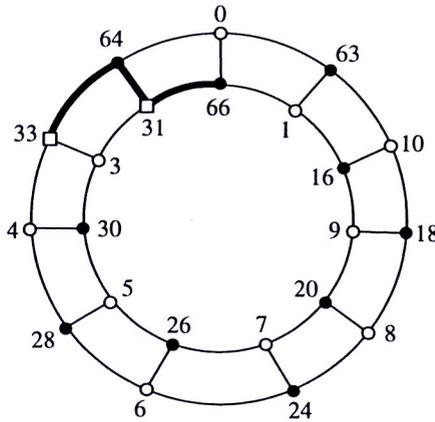


Figure 3.23: A ρ -tripartite labeling of D_{11} .

to v'_n and v_n to v'_1 . For convenience, we let $M_n = P_n \cup P'_n \cup F \cup H$, where $P_n = (v_1, v_2, \dots, v_n)$, $P'_n = (v'_1, v'_2, \dots, v'_n)$, $F = \{\{v_i, v'_i\} : 1 \leq i \leq n\}$ and $H = \{\{v_1, v'_n\}, \{v_n, v'_1\}\}$. We shall refer to P_n as the *outer path*, to P'_n as the *inner path*, and to F as the *spokes*. We note that M_{2n} , $n \geq 2$, is necessarily tripartite with tripartition $\{A, B, C\}$, where $A = \{v'_1, v_{2i-1} : 2 \leq i \leq n, v'_{2i} : 2 \leq i \leq n\}$, $B = \{v_{2i} : 1 \leq i \leq n, v'_{2i-1} : 2 \leq i \leq n\}$ and $C = \{v_1, v'_2\}$. Figure 3.24 shows the Möbius ladder M_{10} . We will show that M_n admits a ρ -tripartite labeling for all even integers $n \geq 4$.

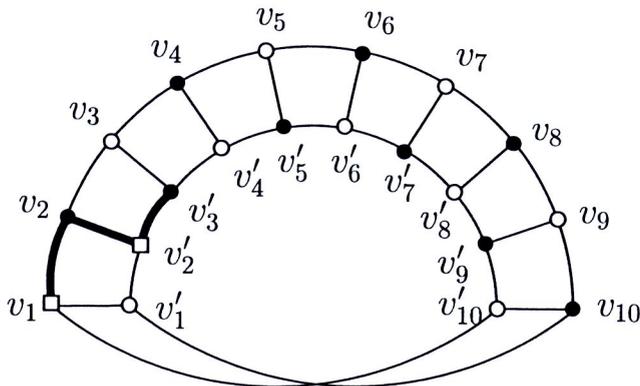


Figure 3.24: The Möbius ladder M_{10} .

Lemma 3.3.1 *The Möbius ladder M_n admits a ρ -tripartite labeling for all $n \in \{4, 6, 8, 10\}$.*

Proof. A ρ -tripartite labeling of M_4 is given in Table 3.2 (graph C7). We give ρ -tripartite labelings of the other three graphs in Figure 3.25.

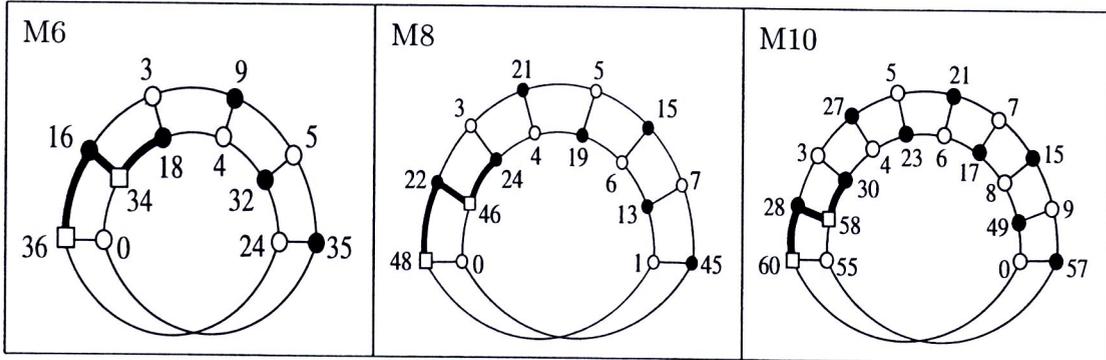


Figure 3.25: ρ -tripartite labelings of M_6 , M_8 and M_{10} .

Theorem 3.3.2 *The Möbius ladder M_n admits a ρ -tripartite labeling for all even $n \geq 4$.*

Proof. We separate the proof into 3 cases.

Case 1: $n \equiv 0 \pmod{6}$.

Let $n = 6t$. Thus, $|V(M_n)| = 12t$ and $|E(M_n)| = 18t$. A ρ -tripartite labeling of M_6 is given in Figure 3.25 (graph M6). For $t \geq 2$, define a one-to-one function

$f : V(M_{6t}) \rightarrow [0, 36t]$ as follows:

$$f(v_1) = 36t,$$

$$f(v_2) = 18t - 2,$$

$$f(v_i) = i, \quad v_i \in A_1 = \{v_i : i \text{ odd}, 3 \leq i \leq 6t - 1\},$$

$$f(v_i) = 18t - 2i + 5, \quad v_i \in B_1 = \{v_i : i \text{ even}, 4 \leq i \leq 4t - 2\},$$

$$f(v_i) = 18t - 2i - 1, \quad v_i \in B_2 = \{v_i : i \text{ even}, 4t - 2 < i \leq 6t - 2\},$$

$$f(v_{6t}) = 36t - 1,$$

$$f(v'_1) = 0,$$

$$f(v'_2) = 36t - 2,$$

$$f(v'_3) = 18t,$$

$$f(v'_i) = i, \quad v'_i \in A'_1 = \{v'_i : i \text{ even}, 4 \leq i \leq 6t - 2\},$$

$$f(v'_i) = 18t - 2i + 5, \quad v'_i \in B'_1 = \{v'_i : i \text{ odd}, 5 \leq i \leq 4t - 1\},$$

$$f(v'_i) = 18t - 2i - 1, \quad v'_i \in B'_2 = \{v'_i : i \text{ odd}, 4t - 1 < i \leq 6t - 3\},$$

$$f(v'_{6t-1}) = 12t + 4,$$

$$f(v'_{6t}) = 6t + 2.$$

Note that $A = \{v'_1\} \cup A_1 \cup A'_1 \cup \{v'_{6t}\}$ and $B = \{v_2, v_{6t}, v'_3, v'_{6t-1}\} \cup B_1 \cup B_2 \cup B'_1 \cup B'_2$ and $C = \{v_1, v'_2\}$. Thus the domain of f is indeed $V(M_{6t})$. Next, we confirm that f is one-to-one. We compute

$$f(A_1) = \{3, 5, \dots, 6t - 1\},$$

$$f(A'_1) = \{4, 6, \dots, 6t - 2\},$$

$$f(B_1) = \{18t - 3, 18t - 7, \dots, 10t + 9\},$$

$$f(B_2) = \{10t - 1, 10t - 5, \dots, 6t + 3\},$$

$$f(B'_1) = \{18t - 5, 18t - 9, \dots, 10t + 7\},$$

$$f(B'_2) = \{10t - 3, 10t - 7, \dots, 6t + 5\}.$$

Note that f is piecewise strictly increasing by 2 or strictly decreasing by 4 and that all labels are distinct. Thus f is one-to-one. Moreover, $f(A) \subseteq [0, 6t + 2]$ and $f(B \cup C) \subseteq [6t + 3, 36t]$.

To help compute the edge labels, we will describe $f(M_{6t})$ in terms of the paths $\hat{P}(2k, d_1, d_2, a, b)$. For convenience, we will identify the vertices of P_{6t} and P'_{6t} with their labels. We have $f(P_{6t}) = (36t, 18t - 2, 3) + G_1 + G_2 + (6t - 1, 36t - 1)$, where

$$G_1 = \hat{P}(2(2t - 2), 2, 4, 3, 10t + 9),$$

$$G_2 = \hat{P}(2t, 2, 4, 4t - 1, 6t + 3).$$

By P3, the resulting edge label sets are:

$$\begin{aligned} \bar{f}(E(G_1)) &= \{6t + 10 + 6i : 0 \leq i \leq 2t - 3\} \cup \{6t + 12 + 6i : 0 \leq i \leq 2t - 3\} \\ &= \{\ell \equiv 4 \pmod{6} : 6t + 10 \leq \ell \leq 18t - 8\} \\ &\quad \cup \{\ell \equiv 0 \pmod{6} : 6t + 12 \leq \ell \leq 18t - 6\}, \end{aligned}$$

$$\begin{aligned} \bar{f}(E(G_2)) &= \{4 + 6i : 0 \leq i \leq t - 1\} \cup \{6 + 6i : 0 \leq i \leq t - 1\} \\ &= \{\ell \equiv 4 \pmod{6} : 4 \leq \ell \leq 6t - 2\} \cup \{\ell \equiv 0 \pmod{6} : 6 \leq \ell \leq 6t\}. \end{aligned}$$

Moreover, edge labels $18t + 2$ and $18t - 5$ occur on the path $(36t, 18t - 2, 3)$ and the edge label $30t$ occurs on the edge $\{6t - 1, 36t - 1\}$.

Similarly, we have $f(P'_{6t}) = (0, 36t - 2, 18t, 4) + G'_1 + G'_2 + (6t - 2, 12t + 4, 6t + 2)$, where

$$G'_1 = \hat{P}(2(2t - 2), 2, 4, 4, 10t + 7),$$

$$G'_2 = \hat{P}(2(t - 1), 2, 4, 4t, 6t + 5).$$

By P3, the resulting edge label sets are:

$$\begin{aligned} \bar{f}(E(G'_1)) &= \{6t + 7 + 6i : 0 \leq i \leq 2t - 3\} \cup \{6t + 9 + 6i : 0 \leq i \leq 2t - 3\} \\ &= \{\ell \equiv 1 \pmod{6} : 6t + 7 \leq \ell \leq 18t - 11\} \\ &\quad \cup \{\ell \equiv 3 \pmod{6} : 6t + 7 \leq \ell \leq 18t - 9\}, \end{aligned}$$

$$\begin{aligned} \bar{f}(E(G'_2)) &= \{7 + 6i : 0 \leq i \leq t - 2\} \cup \{9 + 6i : 0 \leq i \leq t - 2\} \\ &= \{\ell \equiv 1 \pmod{6} : 7 \leq \ell \leq 6t - 5\} \\ &\quad \cup \{\ell \equiv 3 \pmod{6} : 9 \leq \ell \leq 6t - 3\}. \end{aligned}$$

Moreover, edge labels $36t - 2$, $18t - 2$ and $18t - 4$ occur on the path $(0, 36t - 2, 18t, 4)$ and edge labels $6t + 6$ and $6t + 2$ occur on the path $(6t - 2, 12t + 4, 6t + 2)$.

For each spoke $\{v_i, v'_i\}$, the labels on spokes are given by

$$\bar{f}(\{v_i, v'_i\}) = \begin{cases} 36t & \text{for } i = 1, \\ 18t & \text{for } i = 2, \\ 18t - 3 & \text{for } i = 3, \\ 18t - 3i + 5 & \text{for } 4 \leq i \leq 4t - 1, \\ 18t - 3i - 1 & \text{for } 4t - 1 < i \leq 6t - 2, \\ 6t + 5 & \text{for } i = 6t - 1, \\ 30t - 3 & \text{for } i = 6t. \end{cases}$$

Thus the set of edge labels on the spokes is:

$$\begin{aligned} \bar{f}(E(F)) = & \{\ell \equiv 2 \pmod{3} : 6t + 8 \leq \ell \leq 18t - 7\} \\ & \cup \{\ell \equiv 2 \pmod{3} : 5 \leq \ell \leq 6t - 1\} \\ & \cup \{36t, 18t, 18t - 3, 6t + 5, 30t - 3\}. \end{aligned}$$

Moreover, edge labels $30t - 2$ and $36t - 1$ occur on the edges $\{v_1, v'_{6t}\}$ and $\{v'_1, v_{6t}\}$.

It is easy to verify now that for each $\ell \in [1, 18t]$ either ℓ or $36t + 1 - \ell$ occurs on exactly one edge in M_{6t} . Hence the defined labeling is a ρ -labeling and condition (r1) for a ρ -tripartite labeling is satisfied. Condition (r2) also holds since $f(A) \subseteq [0, 6t + 2]$ and $f(B \cup C) \subseteq [6t + 3, 36t]$. Condition (r3) holds since $|f(v_1) - f(v_2)| + |f(v'_2) - f(v'_3)| = 36t$ and $|f(v_2) - f(v'_2)| + |f(v_2) - f(v'_2)| = 36t$, twice the number of edges of M_{6t} . Also $|f(b) - f(c)| = 36t$, where $b \in B$ and $c \in C$, is impossible since all vertices are in $[0, 36t]$ and $0 \in f(A)$. Thus condition (r4) holds, and we have a ρ -tripartite labeling of M_{6t} . Figure 3.26 shows a ρ -tripartite labeling of M_{12} .

Case 2: $n \equiv 2 \pmod{6}$.

Let $n = 6t + 2$. Thus, $|V(M_n)| = 12t + 4$ and $|E(M_n)| = 18t + 6$. A ρ -tripartite labeling of M_8 is given in Figure 3.25 (graph M8). For $t \geq 2$, define a one-to-one

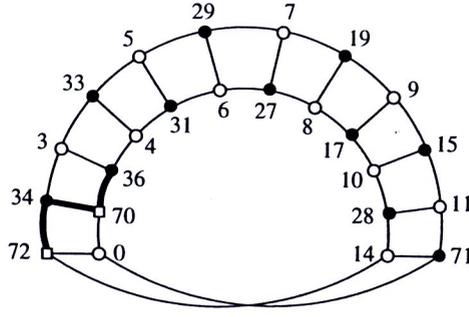


Figure 3.26: A ρ -tripartite labeling of M_{12} .

function $f : V(M_{6t+2}) \rightarrow [0, 36t + 12]$ as follows:

$$f(v_1) = 36t + 12,$$

$$f(v_2) = 18t + 4,$$

$$f(v_i) = i,$$

$$v_i \in A_1, A_1 = \{v_i : i \text{ odd}, 3 \leq i \leq 6t + 1\},$$

$$f(v_i) = 18t - 2i + 11,$$

$$v_i \in B_1, B_1 = \{v_i : i \text{ even}, 4 \leq i \leq 4t\},$$

$$f(v_i) = 18t - 2i + 9,$$

$$v_i \in B_2, B_2 = \{v_i : i \text{ even}, 4t < i \leq 6t\},$$

$$f(v_{6t+2}) = 36t + 9,$$

$$f(v'_1) = 0,$$

$$f(v'_2) = 36t + 10,$$

$$f(v'_3) = 18t + 6,$$

$$f(v'_i) = i,$$

$$v'_i \in A'_1, A'_1 = \{v'_i : i \text{ even}, 4 \leq i \leq 6t\},$$

$$f(v'_i) = 18t - 2i + 11,$$

$$v'_i \in B'_1, B'_1 = \{v'_i : i \text{ odd}, 5 \leq i \leq 4t + 1\},$$

$$f(v'_i) = 18t - 2i + 9,$$

$$v'_i \in B'_2, B'_2 = \{v'_i : i \text{ odd}, 4t + 1 < i \leq 6t + 1\},$$

$$f(v'_{6t+2}) = 1.$$

Note that $A = \{v'_1, v'_{6t+2}\} \cup A_1 \cup A'_1$ and $B = \{v_2, v_{6t+2}, v'_3\} \cup B_1 \cup B_2 \cup B'_1 \cup B'_2$ and $C = \{v_1, v'_2\}$. Thus the domain of f is indeed $V(M_{6t+2})$. Next, we confirm

that f is one-to-one. We compute

$$f(A_1) = \{3, 5, \dots, 6t + 1\},$$

$$f(A'_1) = \{4, 6, \dots, 6t\},$$

$$f(B_1) = \{18t + 3, 18t - 1, \dots, 10t + 11\},$$

$$f(B_2) = \{10t + 5, 10t + 1, \dots, 6t + 9\},$$

$$f(B'_1) = \{18t + 1, 18t - 3, \dots, 10t + 9\},$$

$$f(B'_2) = \{10t + 3, 10t - 1, \dots, 6t + 7\}.$$

Note that f is piecewise strictly increasing by 2 or strictly decreasing by 4 and that all labels are distinct. Thus f is one-to-one. Moreover, $f(A) \subseteq [0, 6t + 1]$ and $f(BUC) \subseteq [6t + 2, 36t + 12]$.

To help compute the edge labels, we will describe $f(M_{6t+2})$ in terms of the paths $\hat{P}(2k, d_1, d_2, a, b)$. For convenience, we will identify the vertices of P_{6t+2} and P'_{6t+2} with their labels. We have $f(P_{6t+2}) = (36t + 12, 18t + 4, 3) + G_1 + G_2 + (6t + 1, 36t + 9)$, where

$$G_1 = \hat{P}(2(2t - 1), 2, 4, 3, 10t + 11),$$

$$G_2 = \hat{P}(2t, 2, 4, 4t + 1, 6t + 9).$$

By P3, the resulting edge label sets are:

$$\bar{f}(E(G_1)) = \{6t + 10 + 6i : 0 \leq i \leq 2t - 2\} \cup \{6t + 12 + 6i : 0 \leq i \leq 2t - 2\}$$

$$= \{\ell \equiv 4 \pmod{6} : 6t + 10 \leq \ell \leq 18t - 2\}$$

$$\cup \{\ell \equiv 0 \pmod{6} : 6t + 12 \leq \ell \leq 18t\},$$

$$\bar{f}(E(G_2)) = \{8 + 6i : 0 \leq i \leq t - 1\} \cup \{10 + 6i : 0 \leq i \leq t - 1\}$$

$$= \{\ell \equiv 2 \pmod{6} : 8 \leq \ell \leq 6t + 2\}$$

$$\cup \{\ell \equiv 4 \pmod{6} : 10 \leq \ell \leq 6t + 4\}.$$

Moreover, edge labels $18t + 8$ and $18t + 1$ occur on the path $(36t + 12, 18t + 4, 3)$ and edge label $30t + 8$ occurs on the edge $\{6t + 1, 36t + 9\}$.

Similarly, we have $f(P'_{6t+2}) = (0, 36t + 10, 18t + 6, 4) + G'_1 + G'_2 + (6t, 6t + 7, 1)$,

where

$$G'_1 = \hat{P}(2(2t - 1), 2, 4, 4, 10t + 9),$$

$$G'_2 = \hat{P}(2(t - 1), 2, 4, 4t + 2, 6t + 11).$$

By P3, the resulting edge label sets are:

$$\begin{aligned} \bar{f}(E(G'_1)) &= \{6t + 7 + 6i : 0 \leq i \leq 2t - 2\} \cup \{6t + 9 + 6i : 0 \leq i \leq 2t - 2\} \\ &= \{\ell \equiv 1 \pmod{6} : 6t + 7 \leq \ell \leq 18t - 5\} \\ &\quad \cup \{\ell \equiv 3 \pmod{6} : 6t + 9 \leq \ell \leq 18t - 3\}, \end{aligned}$$

$$\begin{aligned} \bar{f}(E(G'_2)) &= \{11 + 6i : 0 \leq i \leq t - 2\} \cup \{13 + 6i : 0 \leq i \leq t - 2\} \\ &= \{\ell \equiv 5 \pmod{6} : 11 \leq \ell \leq 6t - 1\} \\ &\quad \cup \{\ell \equiv 1 \pmod{6} : 13 \leq \ell \leq 6t + 1\}. \end{aligned}$$

Moreover, edge labels $36t + 10, 18t + 4$ and $18t + 2$ occur on the path $(0, 36t + 10, 18t + 6, 4)$ and edge labels 7 and $6t + 6$ occur on the path $(6t, 6t + 7, 1)$.

For each spoke $\{v_i, v'_i\}$, the labels on the spokes are given by

$$\bar{f}(\{v_i, v'_i\}) = \begin{cases} 36t + 12 & \text{for } i = 1, \\ 18t + 6 & \text{for } i = 2, \\ 18t + 3 & \text{for } i = 3, \\ 18t - 3i + 11 & \text{for } 4 \leq i \leq 4t + 1, \\ 18t - 3i + 9 & \text{for } 4t + 1 < i \leq 6t + 1, \\ 36t + 8 & \text{for } i = 6t + 2. \end{cases}$$

Thus the set of edge labels on the spokes is:

$$\begin{aligned} \bar{f}(E(F)) &= \{\ell \equiv 2 \pmod{3} : 6t + 8 \leq \ell \leq 18t - 1\} \\ &\quad \cup \{\ell \equiv 0 \pmod{3} : 6 \leq \ell \leq 6t + 3\} \\ &\quad \cup \{36t + 12, 18t + 6, 18t + 3, 36t + 8\}. \end{aligned}$$

Moreover, edge labels $36t + 11$ and $36t + 9$ occur on the edges $\{v_1, v'_{6t+2}\}$ and $\{v'_1, v_{6t+2}\}$.

It is easy to verify now that for each $\ell \in [1, 18t + 6]$ either ℓ or $36t + 13 - \ell$ occurs on exactly one edge in M_{6t+2} . Hence the defined labeling is a ρ -labeling and condition (r1) for a ρ -tripartite labeling is satisfied. Condition (r2) also holds since $f(A) \subseteq [0, 6t + 1]$ and for $f(B \cup C) \subseteq [6t + 2, 36t + 12]$. Condition (r3) holds since $|f(v_1) - f(v_2)| + |f(v'_2) - f(v'_3)| = 36t + 12$ and $|f(v_2) - f(v'_2)| + |f(v_2) - f(v'_2)| = 36t + 12$, twice the number of edges of M_{6t+2} . Also $|f(b) - f(c)| = 36t + 12$, where $b \in B$ and $c \in C$, is impossible since all vertices are in $[0, 36t + 12]$ and $0 \in f(A)$. Thus condition (r4) holds, and we have a ρ -tripartite labeling of M_{6t+2} . Figure 3.27 shows a ρ -tripartite labeling of M_{14} .

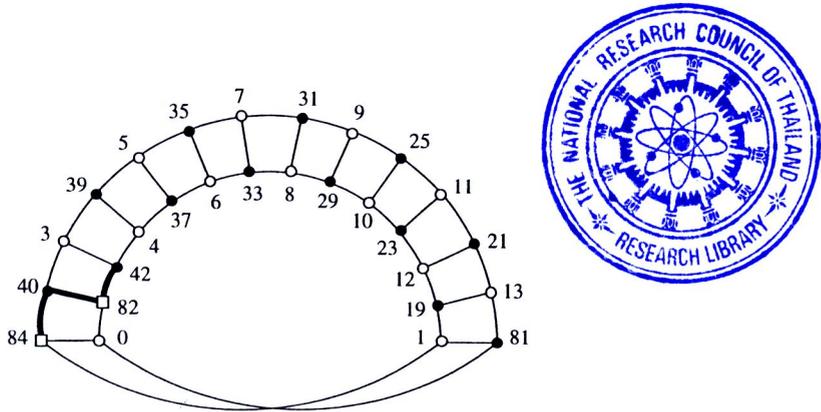


Figure 3.27: A ρ -tripartite labeling of M_{14} .

Case 3: $n \equiv 4 \pmod{6}$.

Let $n = 6t - 2$. Thus, $|V(M_n)| = 12t - 4$ and $|E(M_n)| = 18t - 6$. A ρ -tripartite labeling of M_4 and M_{10} is given in Table 3.2 and Figure 3.25 (graphs C7 and M10). For $t \geq 3$, define a one-to-one function $f : V(M_{6t-2}) \rightarrow [0, 36t - 12]$ as

follows:

$$f(v_1) = 36t - 12,$$

$$f(v_2) = 18t - 8,$$

$$f(v_i) = i, \quad v_i \in A_1, \quad A_1 = \{v_i : i \text{ odd}, 3 \leq i \leq 6t - 3\},$$

$$f(v_i) = 18t - 2i - 1, \quad v_i \in B_1, \quad B_1 = \{v_i : i \text{ even}, 4 \leq i \leq 2t\},$$

$$f(v_i) = 18t - 2i - 3, \quad v_i \in B_2, \quad B_2 = \{v_i : i \text{ even}, 2t < i \leq 4t - 2\},$$

$$f(v_i) = 18t - 2i - 5, \quad v_i \in B_3, \quad B_3 = \{v_i : i \text{ even}, 4t - 2 < i \leq 6t - 4\},$$

$$f(v_{6t-2}) = 36t - 15,$$

$$f(v'_1) = 36t - 17,$$

$$f(v'_2) = 36t - 14,$$

$$f(v'_3) = 18t - 6,$$

$$f(v'_i) = i, \quad v_i \in A'_1, \quad A'_1 = \{v_i : i \text{ even}, 4 \leq i \leq 6t - 4\},$$

$$f(v'_i) = 18t - 2i - 1, \quad v'_i \in B'_1, \quad B'_1 = \{v'_i : i \text{ odd}, 5 \leq i \leq 2t - 1\},$$

$$f(v'_i) = 18t - 2i - 3, \quad v'_i \in B'_2, \quad B'_2 = \{v'_i : i \text{ odd}, 2t - 1 < i \leq 4t - 3\},$$

$$f(v'_i) = 18t - 2i - 5, \quad v'_i \in B'_3, \quad B'_3 = \{v'_i : i \text{ odd}, 4t - 3 < i \leq 6t - 5\},$$

$$f(v'_{6t-3}) = 30t - 11,$$

$$f(v'_{6t-2}) = 0.$$

Note that $A = \{v'_1, v'_{6t-2}\} \cup A_1 \cup A'_1$ and $B = \{v_2, v_{6t-2}, v'_3, v'_{6t-3}\} \cup B_1 \cup B_2 \cup B_3 \cup B'_1 \cup B'_2 \cup B'_3$ and $C = \{v_1, v'_2\}$. Thus the domain of f is indeed $V(M_{6t-2})$. Next,

we confirm that f is one-to-one. We compute

$$f(A_1) = \{3, 5, \dots, 6t - 3\},$$

$$f(A'_1) = \{4, 6, \dots, 6t - 4\},$$

$$f(B_1) = \{18t - 9, 18t - 13, \dots, 14t - 1\},$$

$$f(B_2) = \{14t - 7, 14t - 11, \dots, 10t + 1\},$$

$$f(B_3) = \{10t - 5, 10t - 9, \dots, 6t + 3\},$$

$$f(B'_1) = \{18t - 11, 18t - 15, \dots, 14t + 1\},$$

$$f(B'_2) = \{14t - 5, 14t - 9, \dots, 10t + 3\},$$

$$f(B'_3) = \{10t - 3, 10t - 7, \dots, 6t + 5\}.$$

Note that f is piecewise strictly increasing by 2 or strictly decreasing by 4 and that all labels are distinct. Thus f is one-to-one.

To help compute the edge labels, we will describe $f(M_{6t-2})$ in terms of the paths $\hat{P}(2k, d_1, d_2, a, b)$. For convenience, we will identify the vertices of P_{6t-2} and P'_{6t-2} with their labels. We have $f(P_{6t-2}) = (36t - 12, 18t - 8, 3) + G_1 + G_2 + G_3 + (6t - 3, 36t - 15)$, where

$$G_1 = \hat{P}(2(t - 1), 2, 4, 3, 14t - 1),$$

$$G_2 = \hat{P}(2(t - 1), 2, 4, 2t + 1, 10t + 1),$$

$$G_3 = \hat{P}(2(t - 1), 2, 4, 4t - 1, 6t + 3).$$

By P3, the resulting edge label sets are:

$$\begin{aligned}
\bar{f}(E(G_1)) &= \{12t - 2 + 6i : 0 \leq i \leq t - 2\} \cup \{12t + 6i : 0 \leq i \leq t - 2\} \\
&= \{\ell \equiv 4 \pmod{6} : 12t - 2 \leq \ell \leq 18t - 14\} \\
&\quad \cup \{\ell \equiv 0 \pmod{6} : 12t \leq \ell \leq 18t - 12\}, \\
\bar{f}(E(G_2)) &= \{6t + 2 + 6i : 0 \leq i \leq t - 2\} \cup \{6t + 4 + 6i : 0 \leq i \leq t - 2\} \\
&= \{\ell \equiv 2 \pmod{6} : 6t + 2 \leq \ell \leq 12t - 10\} \\
&\quad \cup \{\ell \equiv 4 \pmod{6} : 6t + 4 \leq \ell \leq 12t - 8\}, \\
\bar{f}(E(G_3)) &= \{6 + 6i : 0 \leq i \leq t - 2\} \cup \{8 + 6i : 0 \leq i \leq t - 2\} \\
&= \{\ell \equiv 0 \pmod{6} : 6 \leq \ell \leq 6t - 6\} \\
&\quad \cup \{\ell \equiv 2 \pmod{6} : 8 \leq \ell \leq 6t - 4\}.
\end{aligned}$$

Moreover, edge labels $18t - 4$ and $18t - 11$ occur on the path $(36t - 12, 18t - 8, 3)$ and the edge label $30t - 12$ occur on the edge $\{6t - 3, 36t - 15\}$.

Similarly, we have $f(P'_{6t-2}) = (36t - 17, 36t - 14, 18t - 6, 4) + G'_1 + G'_2 + G'_3 + (6t - 4, 30t - 11, 0)$, where

$$\begin{aligned}
G'_1 &= \hat{P}(2(t - 2), 2, 4, 4, 14t + 1), \\
G'_2 &= \hat{P}(2(t - 1), 2, 4, 2t, 10t + 3), \\
G'_3 &= \hat{P}(2(t - 1), 2, 4, 4t - 2, 6t + 5).
\end{aligned}$$

By P3, the resulting edge label sets are:

$$\begin{aligned}
\bar{f}(E(G'_1)) &= \{12t + 1 + 6i : 0 \leq i \leq t - 3\} \cup \{12t + 3 + 6i : 0 \leq i \leq t - 3\} \\
&= \{\ell \equiv 1 \pmod{6} : 12t + 1 \leq \ell \leq 18t - 17\} \\
&\quad \cup \{\ell \equiv 3 \pmod{6} : 12t + 3 \leq \ell \leq 18t - 15\}, \\
\bar{f}(E(G'_2)) &= \{6t + 5 + 6i : 0 \leq i \leq t - 2\} \cup \{6t + 7 + 6i : 0 \leq i \leq t - 2\} \\
&= \{\ell \equiv 5 \pmod{6} : 6t + 5 \leq \ell \leq 12t - 7\} \\
&\quad \cup \{\ell \equiv 1 \pmod{6} : 6t + 7 \leq \ell \leq 12t - 5\}, \\
\bar{f}(E(G'_3)) &= \{9 + 6i : 0 \leq i \leq t - 2\} \cup \{11 + 6i : 0 \leq i \leq t - 2\} \\
&= \{\ell \equiv 3 \pmod{6} : 9 \leq \ell \leq 6t - 3\} \\
&\quad \cup \{\ell \equiv 5 \pmod{6} : 11 \leq \ell \leq 6t - 1\}.
\end{aligned}$$

Moreover, edge labels $3, 18t - 8$ and $18t - 10$ occur on the path $(36t - 17, 36t - 14, 18t - 6, 4)$ and edge labels $24t - 7$ and $30t - 11$ occur on the path $(6t - 4, 30t - 11, 0)$.

For each spoke $\{v_i, v'_i\}$, the labels on the spokes are given by

$$\bar{f}(\{v_i, v'_i\}) = \begin{cases} 5 & \text{for } i = 1, \\ 18t - 6 & \text{for } i = 2, \\ 18t - 9 & \text{for } i = 3, \\ 18t - 3i - 1 & \text{for } 4 \leq i \leq 2t, \\ 18t - 3i - 3 & \text{for } 2t < i \leq 4t - 2, \\ 18t - 3i - 5 & \text{for } 4t - 2 < i \leq 6t - 4, \\ 24t - 8 & \text{for } i = 6t - 3, \\ 36t - 15 & \text{for } i = 6t - 2. \end{cases}$$

Thus the set of edge labels on the spokes is:

$$\begin{aligned} \bar{f}(E(F)) = & \{\ell \equiv 2 \pmod{3} : 12t - 1 \leq \ell \leq 18t - 13\} \\ & \cup \{\ell \equiv 0 \pmod{3} : 6t + 3 \leq \ell \leq 12t - 6\} \\ & \cup \{\ell \equiv 1 \pmod{3} : 7 \leq \ell \leq 6t - 2\} \\ & \cup \{5, 18t - 6, 18t - 9, 24t - 8, 36t - 15\}. \end{aligned}$$

Moreover, the edge labels $36t - 12$, and 2 occur on the edge $\{v_1, v'_{6t-2}\}$ and $\{v'_1, v_{6t-2}\}$.

It is easy to verify now that for each $\ell \in [1, 18t - 6]$ either ℓ or $36t - 11 - \ell$ occurs on exactly one edge in M_{6t-2} . Hence the defined labeling is a ρ -labeling and condition (r1) for a ρ -tripartite labeling is satisfied. Condition (r2) also hold since $f(A) \setminus f(v'_1) \subseteq [0, 6t - 3]$, $f(B \cup C) \subseteq [6t - 2, 36t - 12]$ and $f(v'_1) < f(u)$ for $u \in \{v_1, v'_2, v_{6t-2}\}$. Condition (r3) holds since $|f(v_1) - f(v_2)| + |f(v'_2) - f(v'_3)| = 36t - 12$ and $|f(v_2) - f(v'_2)| + |f(v_2) - f(v'_2)| = 36t - 12$, twice the number of edges of M_{6t-2} . Also $|f(b) - f(c)| = 36t - 12$, where $b \in B$ and $c \in C$, is impossible since all vertices are in $[0, 36t - 12]$ and $0 \in f(A)$. Thus condition (r4) holds, and we have a ρ -tripartite labeling of M_{6t-2} . Figure 3.28 shows a ρ -tripartite labeling of M_{16} . ■

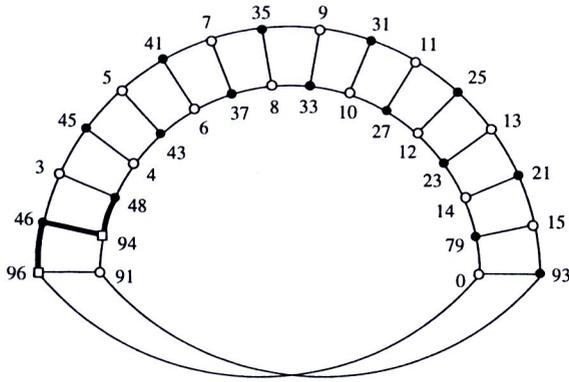


Figure 3.28: A ρ -tripartite labeling of M_{16} .

3.3.3 ρ -tripartite labelings of Tripartite Cubic Graphs of Small Order

According to the reference book *An Atlas of Graphs* [39], there are 27 connected cubic graphs of order at most 10. The complete graph K_4 is the only one of these graphs that is not tripartite. We give a ρ -tripartite labeling of each the remaining 26 graphs in Table 3.2. We referenced these graphs in the same way they are referenced in [39]. Four of these graphs (C3, C8, C24, and C26) are bipartite. The ρ -tripartite labeling we show on the four bipartite graphs is an α -labeling. Thus we have the following.

Theorem 3.3.3 *Every tripartite cubic graph of order at most 10 admits a ρ -tripartite labeling.*

3.3.4 Concluding Remarks

Because it is known that bipartite prisms and bipartite Möbius ladders admit α -labelings and in light of our results here, we have the following.

Corollary 3.3.1 *If G of size n is a prism, a Möbius ladder or a connected cubic graph of order at most 10, then there exists a cyclic G -decomposition of K_{2nt+1} for all positive integers t .*

We are currently investigating ρ -tripartite labelings of other cubic tripartite graphs such as cubic graphs of order 12 and generalized Petersen graphs. Based

on our investigation, we believe that all connected tripartite cubic graphs admit ρ -tripartite labelings.

Conjecture 3.3.1 *Every connected tripartite cubic graph admits a ρ -tripartite labeling.*

Table 3.2 (cont.): ρ -tripartite labelings of tripartite cubic graphs or order ≤ 10

<p>C11</p>	<p>C12</p>	<p>C13</p>
<p>C14</p>	<p>C15</p>	<p>C16</p>
<p>C17</p>	<p>C18</p>	<p>C19</p>

3.4 On Graceful Cubic Graphs

As stated earlier, β -labelings are better known as graceful labelings. Graphs that admit graceful labelings are called *graceful*. Investigating whether or not various classes of graphs admit graceful labelings is a very popular area of research. In fact, since Rosa's introduction of graph labelings [41] in 1967, hundreds of papers have dealt with variations of graceful labelings. We direct the interested reader to Gallian's dynamic survey on graph labelings [28] for the latest progress on this topic. The best-known conjecture on labelings is a conjecture by Kotzig, Ringel and Rosa that every tree is graceful. Despite the intense interest in the topic, this conjecture is far from being solved.

We list the known graceful labelings of cubic graphs. It is known that prisms and Möbius ladders are graceful (see [26], [27] and [47]) and they admit α -labelings when bipartite (see [26] and [38]). Vietri [42, 43] has shown that certain classes of generalized Petersen graphs are graceful. It is also known that rK_4 is graceful if and only if $r = 1$. However, there has not been a systemic study to determine which cubic graphs of small order are graceful.

We show that every cubic graph of order at most 12, other than $2K_4$ and $3K_4$, is graceful. We also conjecture that every connected cubic graph is graceful.

Theorem 3.4.1 *Every cubic graph of order at most 12, other than $2K_4$ and $3K_4$ is graceful.*

proof It is known that rK_4 is graceful if and only if $r = 1$ (see [28]). According to the reference book *An Atlas of Graphs* [39], there are 112 connected cubic graphs of order at most 12. Each of these graphs is graceful (see Table 1). There are an additional 12 disconnected cubic graphs of order at most 12. Ten of these 12 graphs are graceful (see Table 2). We referenced these graphs in the same way they are referenced in [39]. For convenience, we identified each vertex with its label. For the graphs that are bipartite, the given graceful labeling is an α -labeling.

We are currently investigating graceful labelings of the 540 cubic graphs of order 14. Based on our investigation, we believe that all connected cubic graphs are graceful.

Conjecture 3.4.1 *Every connected cubic graph is graceful.*

Table 3.3: Connected cubic graphs: 4–12 vertices

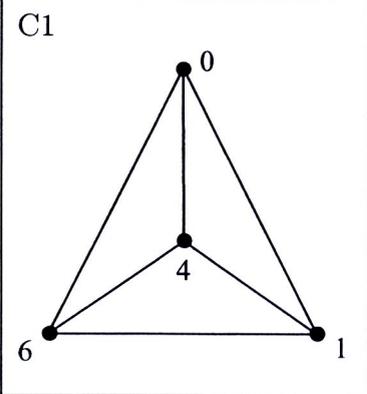
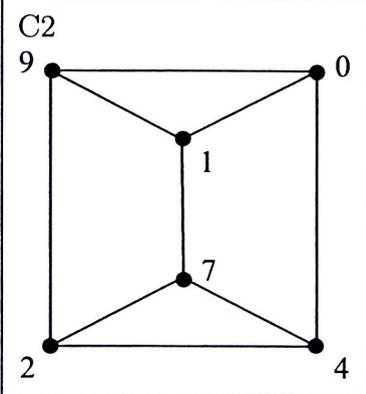
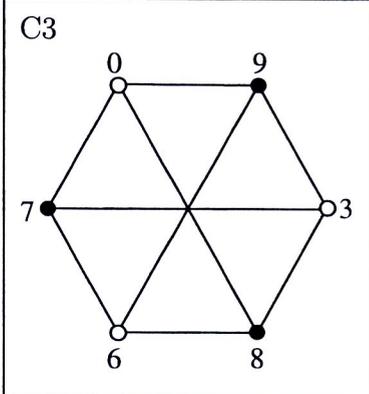
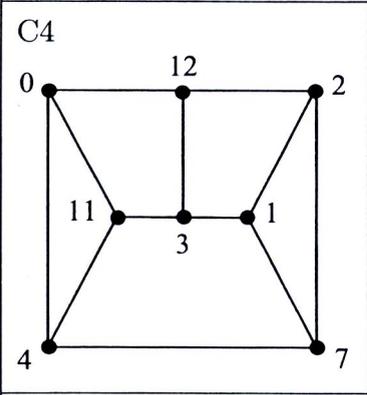
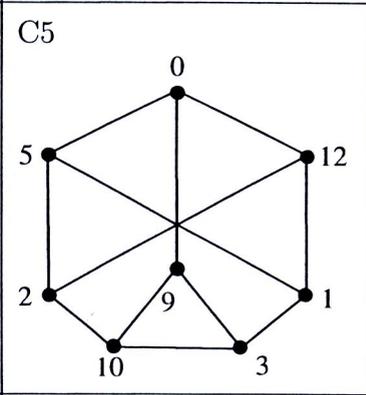
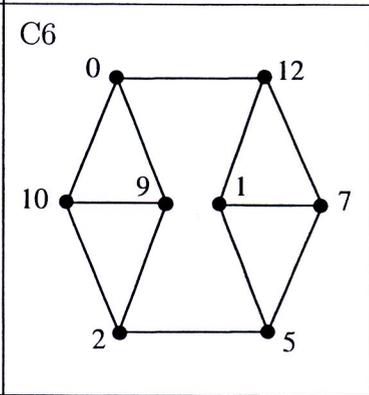
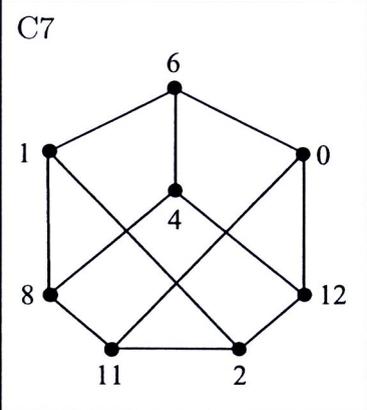
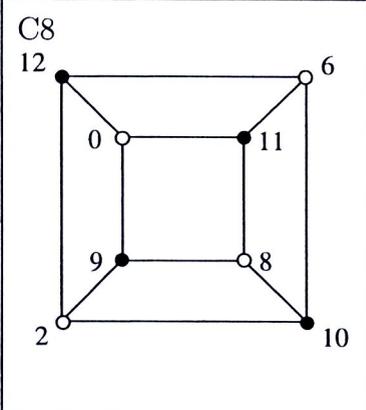
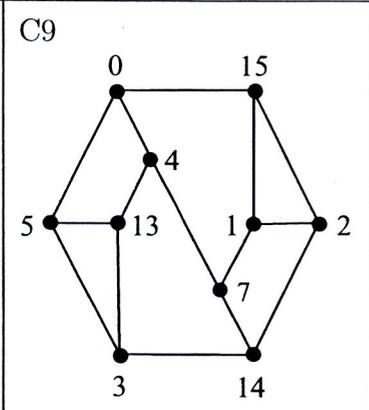
<p>C1</p> 	<p>C2</p> 	<p>C3</p> 
<p>C4</p> 	<p>C5</p> 	<p>C6</p> 
<p>C7</p> 	<p>C8</p> 	<p>C9</p> 

Table 3.3 (cont.): Connected cubic graphs with up to 12 vertices

<p>C10</p>	<p>C11</p>	<p>C12</p>
<p>C13</p>	<p>C14</p>	<p>C15</p>
<p>C16</p>	<p>C17</p>	<p>C18</p>
<p>C19</p>	<p>C20</p>	<p>C21</p>

Table 3.3 (cont.): Connected cubic graphs with up to 12 vertices

<p>C22</p>	<p>C23</p>	<p>C24</p>
<p>C25</p>	<p>C26</p>	<p>C27</p>
<p>C28</p>	<p>C29</p>	<p>C30</p>
<p>C31</p>	<p>C32</p>	<p>C33</p>

Table 3.3 (cont.): Connected cubic graphs with up to 12 vertices

<p>C34</p>	<p>C35</p>	<p>C36</p>
<p>C37</p>	<p>C38</p>	<p>C39</p>
<p>C40</p>	<p>C41</p>	<p>C42</p>
<p>C43</p>	<p>C44</p>	<p>C45</p>

Table 3.3 (cont.): Connected cubic graphs with up to 12 vertices

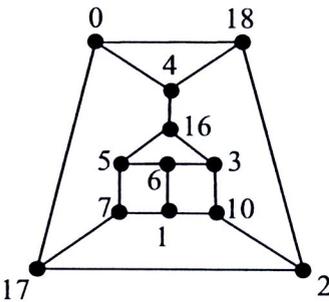
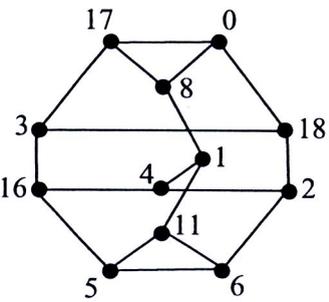
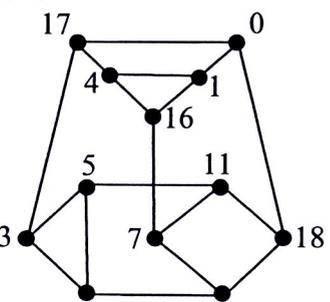
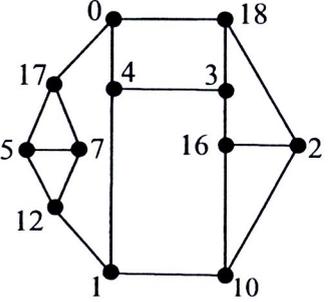
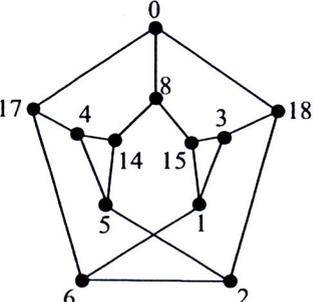
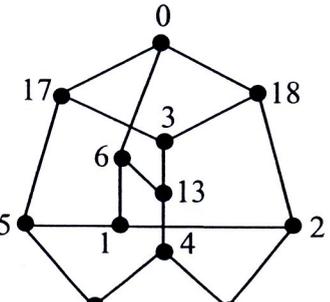
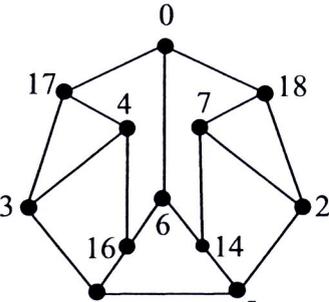
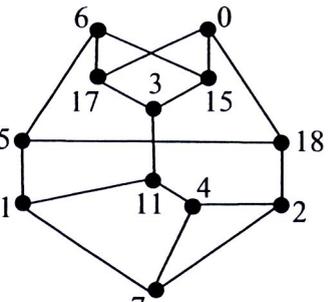
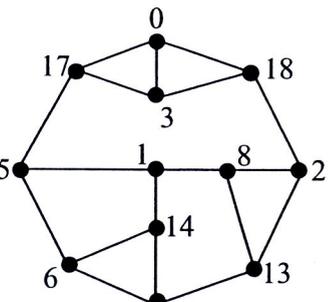
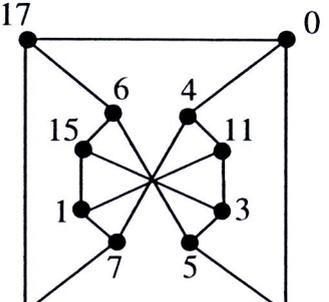
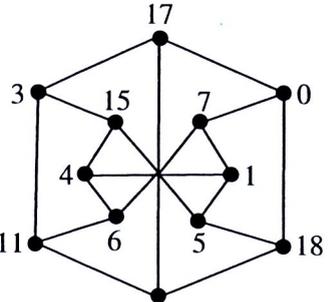
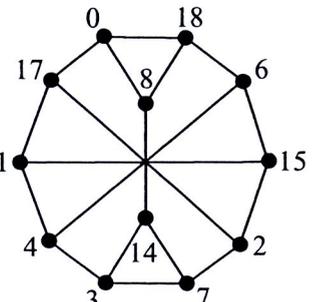
<p>C46</p> 	<p>C47</p> 	<p>C48</p> 
<p>C49</p> 	<p>C50</p> 	<p>C51</p> 
<p>C52</p> 	<p>C53</p> 	<p>C54</p> 
<p>C55</p> 	<p>C56</p> 	<p>C57</p> 

Table 3.3 (cont.): Connected cubic graphs with up to 12 vertices

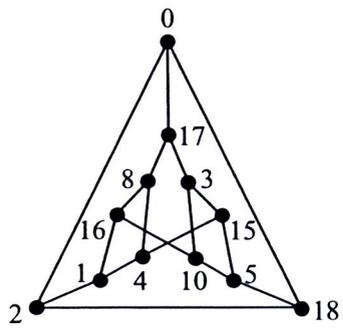
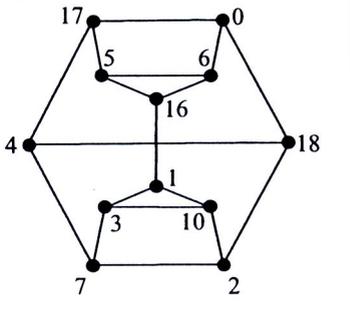
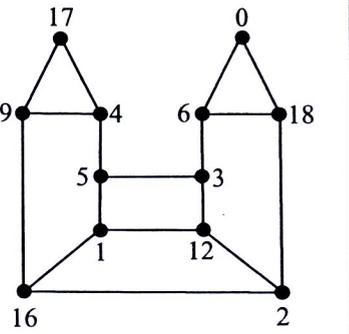
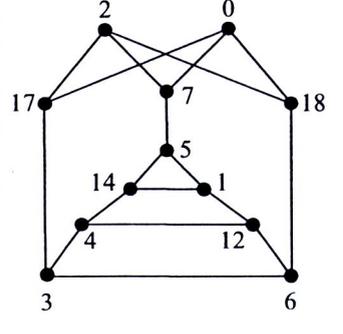
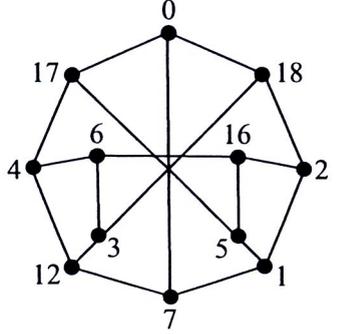
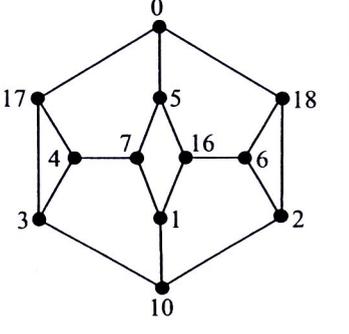
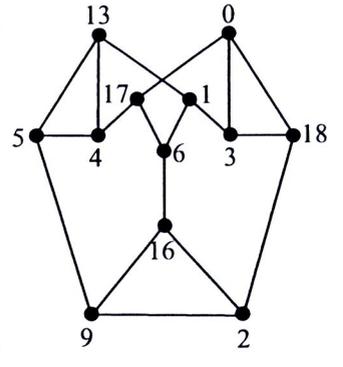
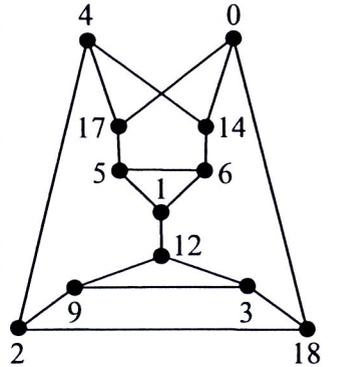
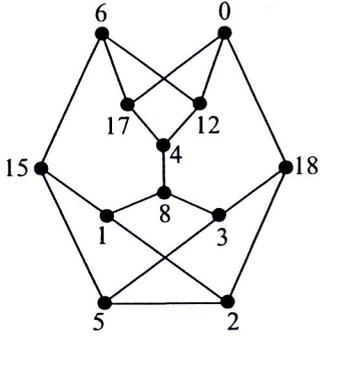
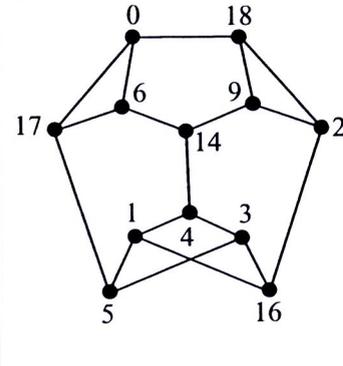
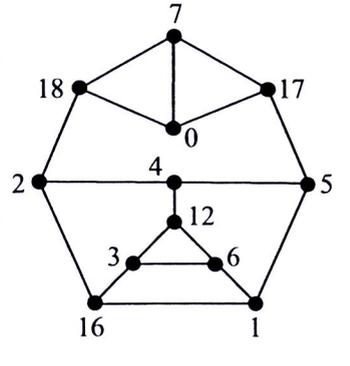
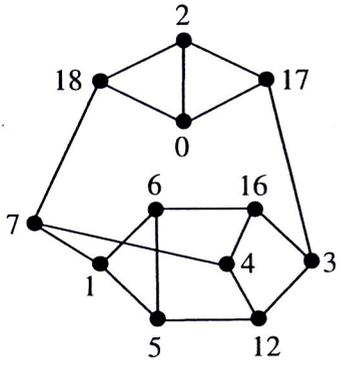
<p>C58</p> 	<p>C59</p> 	<p>C60</p> 
<p>C61</p> 	<p>C62</p> 	<p>C63</p> 
<p>C64</p> 	<p>C65</p> 	<p>C66</p> 
<p>C67</p> 	<p>C68</p> 	<p>C69</p> 

Table 3.3 (cont.): Connected cubic graphs with up to 12 vertices

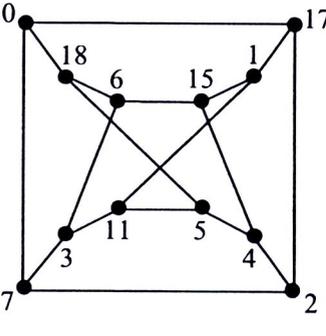
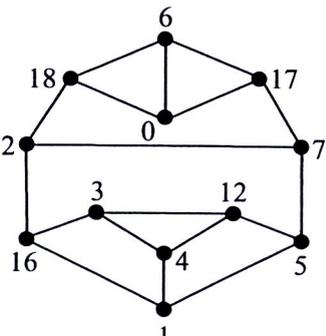
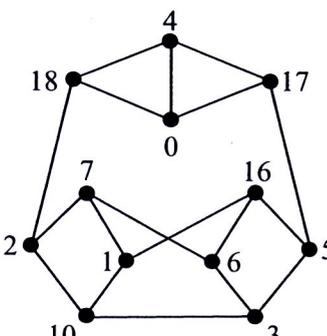
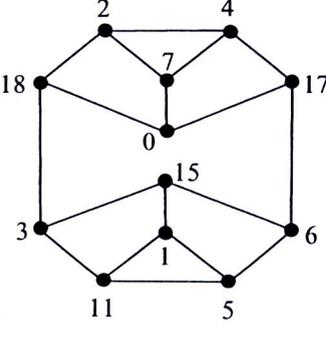
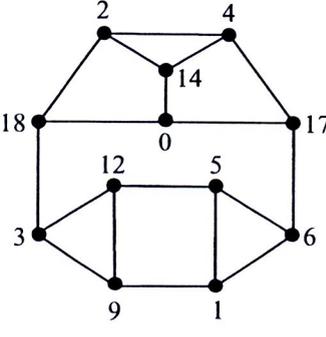
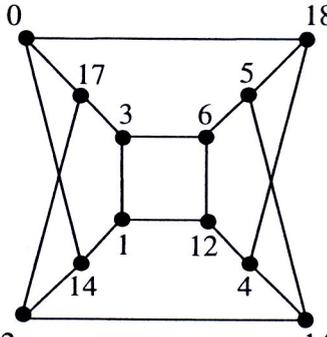
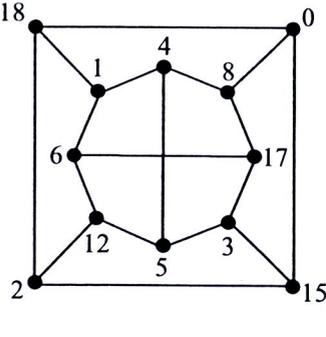
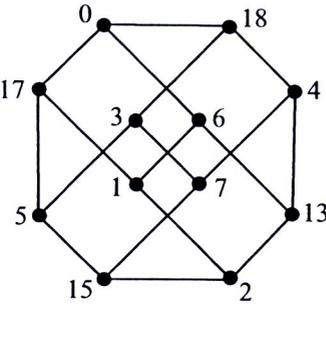
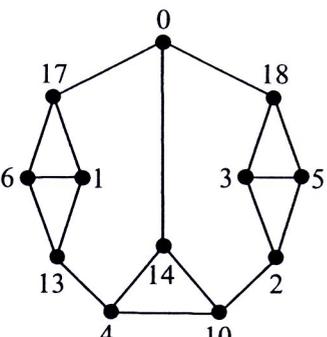
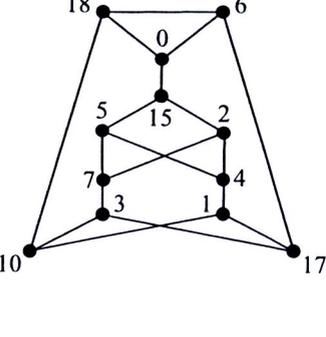
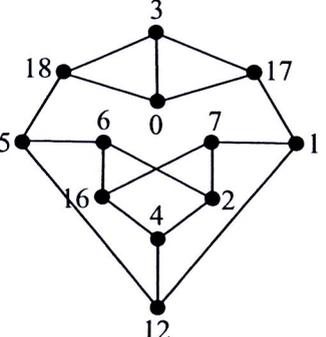
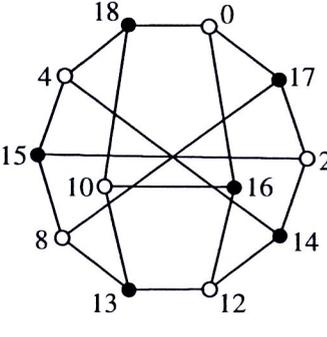
<p>C70</p> 	<p>C71</p> 	<p>C72</p> 
<p>C73</p> 	<p>C74</p> 	<p>C75</p> 
<p>C76</p> 	<p>C77</p> 	<p>C78</p> 
<p>C79</p> 	<p>C80</p> 	<p>C81</p> 

Table 3.3 (cont.): Connected cubic graphs with up to 12 vertices

<p>C82</p>	<p>C83</p>	<p>C84</p>
<p>C85</p>	<p>C86</p>	<p>C87</p>
<p>C88</p>	<p>C89</p>	<p>C90</p>
<p>C91</p>	<p>C92</p>	<p>C93</p>

Table 3.3 (cont.): Connected cubic graphs with up to 12 vertices

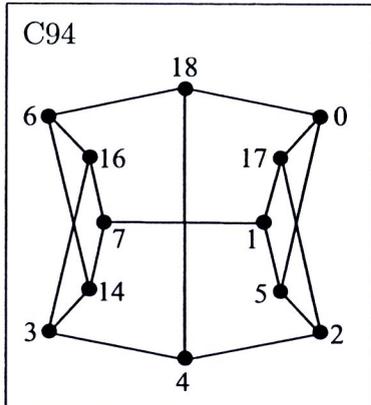
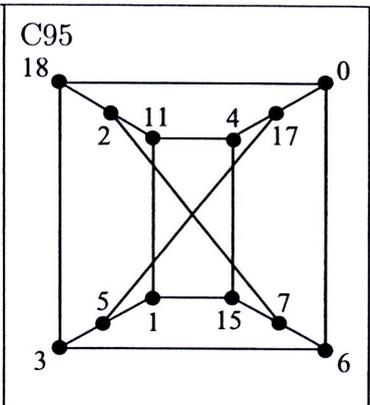
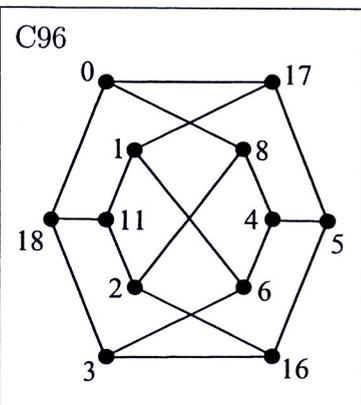
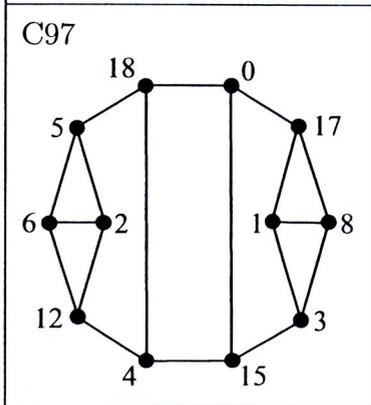
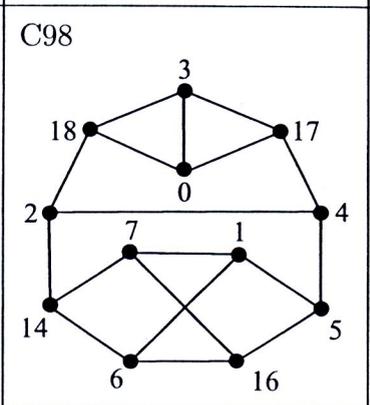
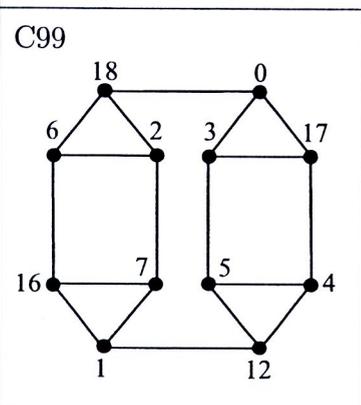
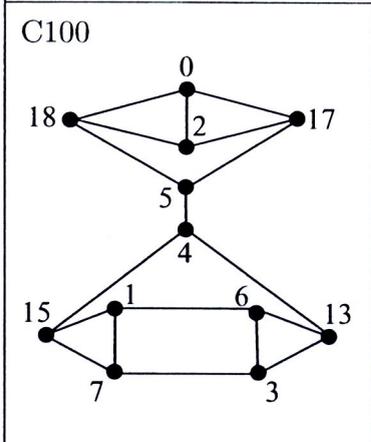
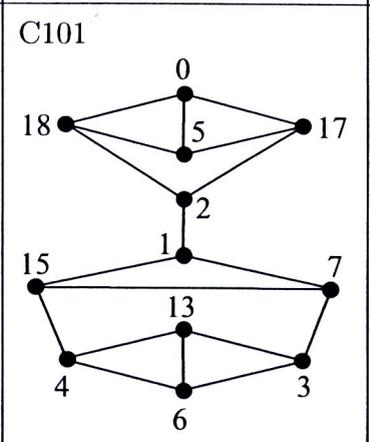
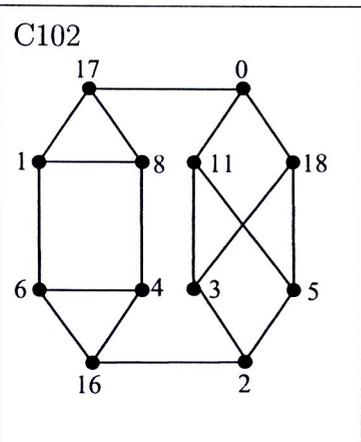
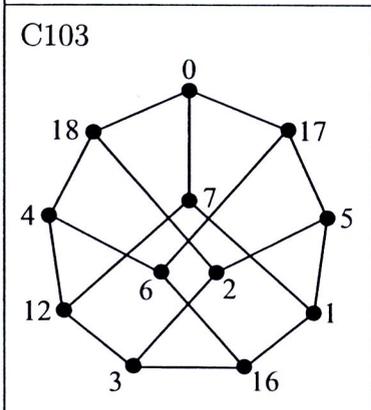
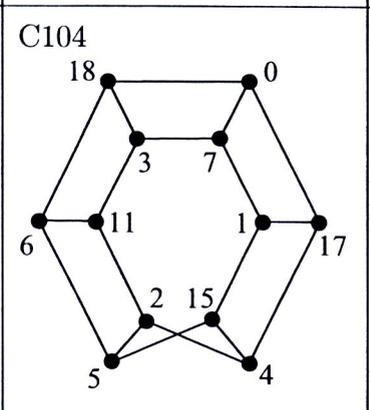
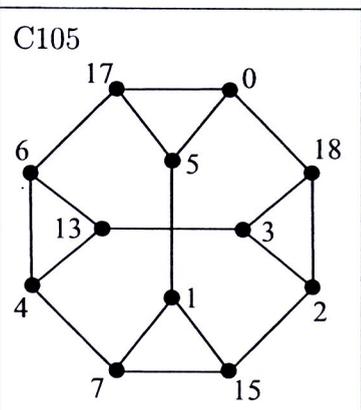
<p>C94</p> 	<p>C95</p> 	<p>C96</p> 
<p>C97</p> 	<p>C98</p> 	<p>C99</p> 
<p>C100</p> 	<p>C101</p> 	<p>C102</p> 
<p>C103</p> 	<p>C104</p> 	<p>C105</p> 

Table 3.3 (cont.): Connected cubic graphs with up to 12 vertices

<p>C106</p>	<p>C107</p>	<p>C108</p>	
<p>C109</p>	<p>C110</p>	<p>C111</p>	
<p>C112</p>			

Table 3.4: Disconnected cubic graphs with up to 12 vertices

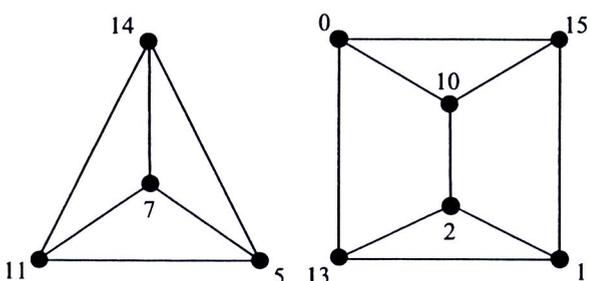
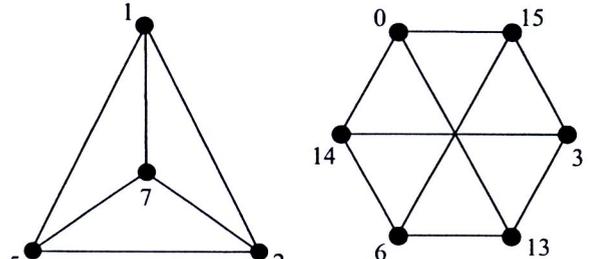
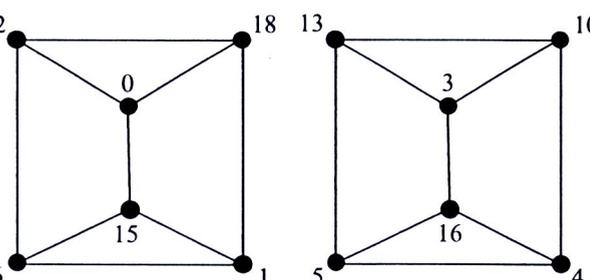
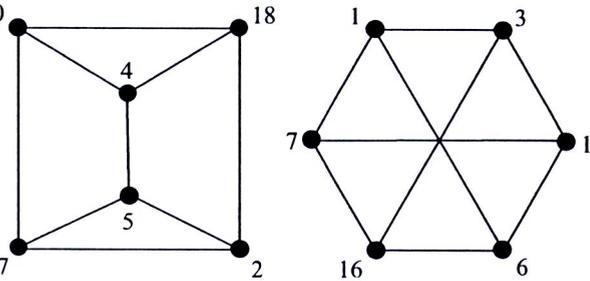
<p>$C1 \cup C2$</p> 
<p>$C1 \cup C3$</p> 
<p>$C2 \cup C2$</p> 
<p>$C2 \cup C3$</p> 

Table 3.4 (cont.): Disconnected cubic graphs with up to 12 vertices

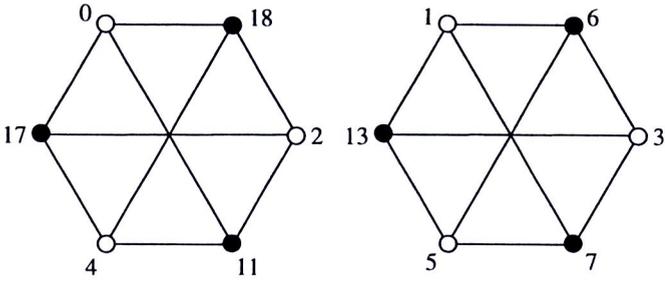
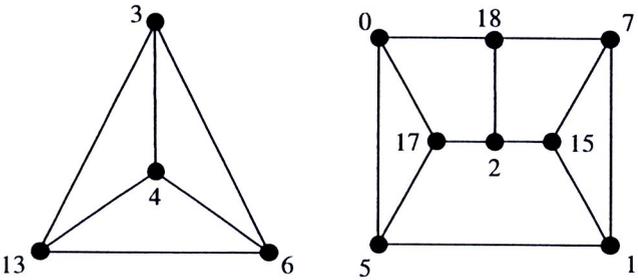
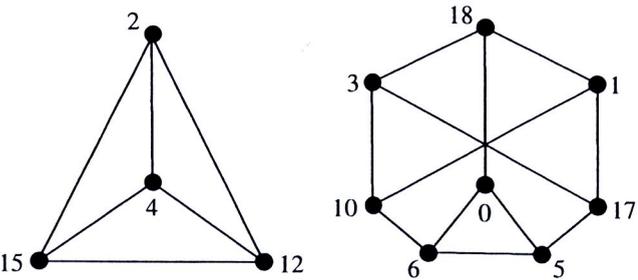
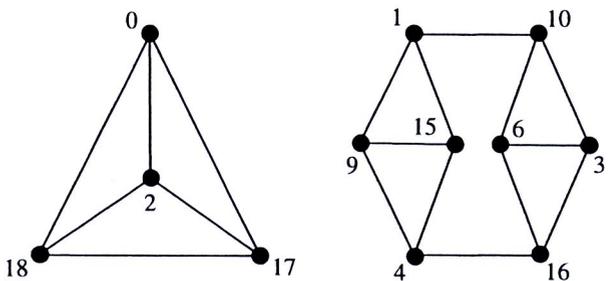
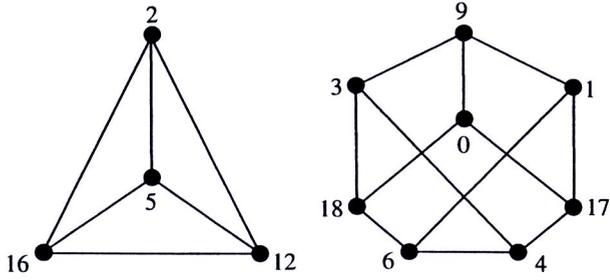
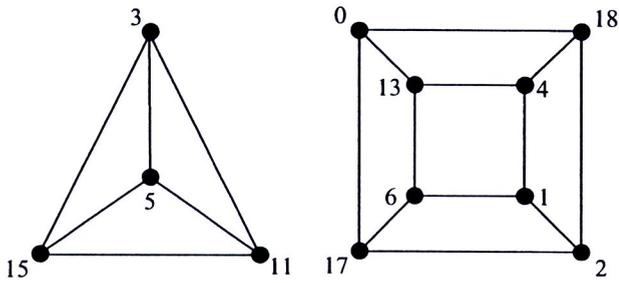
<p>$C_3 \cup C_3$</p> 	
<p>$C_1 \cup C_4$</p> 	
<p>$C_1 \cup C_5$</p> 	
<p>$C_1 \cup C_6$</p> 	

Table 3.4 (cont.): Disconnected cubic graphs with up to 12 vertices

 $C_1 \cup C_7$  $C_1 \cup C_8$ 

3.5 On 2-fold Rosa-Type Labelings

Rosa-type labelings can also be defined for multigraphs and can be used to yield cyclic G -decompositions of λ -fold complete multigraphs (see [15]). We will restrict our attention to decompositions of 2-fold complete multigraphs. For a finite set S , we let 2S denote the multiset that contains every element of S exactly two times. Similarly, for integers a and b with $a \leq b$, we let ${}^2[a, b]$ denote the multiset $\{a, a, a + 1, a + 1, \dots, b - 1, b - 1, b, b\}$. Also, let 2K_m denote the 2-fold complete multigraph of order m (i.e., 2K_m is the multigraph on m vertices with exactly two edges between every pair of vertices.)

The concept of edge length in 2K_m is defined in the same way as in K_m . If we let $V({}^2K_m) = \{0, 1, \dots, m - 1\}$, the *length* of an edge $\{i, j\}$ in 2K_m is defined to be $\min\{|i - j|, m - |i - j|\}$. Thus if the elements of $V({}^2K_m)$ are placed in order as vertices of an equisided m -gon, then the length of edge $\{i, j\}$ is the shortest distance around the polygon between i and j . Edge $\{i, j\}$ is a *wrap-around* edge if the length of $\{i, j\}$ is $m - |i - j|$. Note that if m is odd, then 2K_m consists of $2m$ edges of length i for $i = 1, 2, \dots, \frac{m-1}{2}$. If m is even, then 2K_m consists of $2m$ edges of length i for $i = 1, 2, \dots, \frac{m}{2} - 1$, and m edges of length $\frac{m}{2}$. Also, note that clicking an edge does not change its length.

The idea of G -decompositions of a graph extends in a natural way to decompositions of and by multigraphs. Note that a multigraph is not required to contain multiple edges. Thus a graph can be called a multigraph. If G and K are multigraphs with $V(G) \subseteq V(K)$ and such that every edge in G is also in K , then we shall refer to G as a *subgraph* of K . (We do this in order to avoid having to use terms such as “submultigraph”.)

Let K and G be multigraphs with G a subgraph of K . A G -decomposition of K is a set (or multiset) $\Delta = \{G_1, G_2, \dots, G_t\}$ of subgraphs of K each of which is isomorphic to G and such that each edge of K appears in exactly one G_i . The elements of Δ are called G -blocks. Such a G -decomposition Δ is *cyclic* (*purely cyclic*) if clicking is a permutation (t -cycle) of Δ . A G -decomposition of ${}^\lambda K_m$ is called a G -design of order m and index λ or a $({}^\lambda K_m, G)$ -design. A $({}^\lambda K_m, G)$ -design is *symmetric* if the number of G -blocks in the design equals m .

The following concept and the theorem that follows are introduced in [15]. Since [15] has not yet been published, we will sketch the proofs of any results we cite from that manuscript.

Definition 3.5.1 Let G be a subgraph of ${}^2K_{n+1}$ such that $|E(G)| = n$. A 2-fold ρ -labeling of G is a one-to-one function $f : V(G) \rightarrow \{0, 1, 2, \dots, n\}$ such that :

- (1) If n is even, then the multiset $\{\min\{|f(u) - f(v)|, n + 1 - |f(u) - f(v)|\} : \{u, v\} \in E(G)\} = {}^2[1, \frac{n}{2}]$.
- (2) If n is odd, then the multiset $\{\min\{|f(u) - f(v)|, n + 1 - |f(u) - f(v)|\} : \{u, v\} \in E(G)\} = {}^2[1, \frac{n-1}{2}] \cup \{\frac{n+1}{2}\}$.

Thus a 2-fold ρ -labeling of such a G is an embedding of G in ${}^2K_{n+1}$ so that:

- (1) There are two edges of G of length i for each $i \in [1, \frac{n}{2}]$ when n is even.
- (2) There are two edges of G of length i for each $i \in [1, \frac{n-1}{2}]$ and one edge of length $\frac{n+1}{2}$ when n is odd.

It is easy to see that the following extension of Theorem 2.1.1 must hold.

Theorem 3.5.1 Let G be a subgraph of ${}^2K_{n+1}$ such that $|E(G)| = n$. There exists a cyclic $({}^2K_{n+1}, G)$ -design if and only if G admits a 2-fold ρ -labeling.

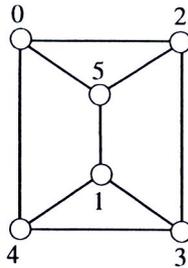
Proof. Let G be a subgraph of ${}^2K_{n+1}$ such that $|E(G)| = n$. We separate the proof into two cases depending on whether n is even or odd.

Case 1: n is even. Let $n = 2m$ and let G admit a 2-fold ρ -labeling. Then for each length $\ell \in [1, m]$, there exist two edges of length ℓ in G . Denote these edges by e'_ℓ and e''_ℓ . Let G' be the subgraph of G with $E(G') = \{e'_\ell : \ell \in [1, m]\}$. Similarly, let G'' be the subgraph with $E(G'') = \{e''_\ell : \ell \in [1, m]\}$. Then G' and G'' are edge disjoint and neither of them contains double-edges. Moreover, the 2-fold ρ -labeling of G induces simultaneously a ρ -labeling of G' and a ρ -labeling of G'' . Thus $\Delta_{G'} = \{G' + i : 0 \leq i \leq 2m\}$ is a cyclic (K_{2m+1}, G') -design. Similarly, $\Delta_{G''} = \{G'' + i : 0 \leq i \leq 2m\}$ is a cyclic (K_{2m+1}, G'') -design. Since $G = G' \cup G''$ and $2m + 1 = n + 1$, the set $\Delta_G = \{G + i : 0 \leq i \leq 2m\}$ is a cyclic $({}^2K_{n+1}, G)$ -design. Similarly, any G -block in a cyclic $({}^2K_{n+1}, G)$ -design, induces a 2-fold

ρ -labeling of G .

Case 2: n is odd. Let $n = 2m + 1$ and let G admit a 2-fold ρ -labeling. As in the previous case, let e'_ℓ and e''_ℓ denote the two edges in G of length ℓ for each $\ell \in [1, m]$. Also, let e_{m+1} be the edge of length $m + 1$. Let G' be the subgraph of G with $E(G') = \{e'_\ell : \ell \in [1, m]\}$, let G'' be the subgraph with $E(G'') = \{e''_\ell : \ell \in [1, m]\}$, and let G_{m+1} be the subgraph with $E(G_{m+1}) = \{e_{m+1}\}$. Let I be the 1-factor in K_{2m+2} induced by the edges of length $m + 1$. Since G' does not contain an edge of length $m + 1$, the set $\Delta_{G'} = \{G' + i : 0 \leq i \leq 2m + 1\}$ is a cyclic $(K_{2m+2} - I, G')$ -design. Similarly, the set $\Delta_{G''} = \{G'' + i : 0 \leq i \leq 2m + 1\}$ is a cyclic $(K_{2m+2} - I, G'')$ -design. Also, the set $\Delta_{G_{m+1}} = \{G_{m+1} + i : 0 \leq i \leq 2m + 1\}$ is the multigraph obtained by replacing each edge in I with a double edge. Since $G = G' \cup G'' \cup G_{m+1}$ and $2m + 2 = n + 1$, the set $\Delta_G = \{G + i : 0 \leq i \leq 2m + 1\}$ is a cyclic $({}^2K_{n+1}, G)$ -design. Conversely, any G -block in a cyclic $({}^2K_{n+1}, G)$ -design, induces a 2-fold ρ -labeling of G . ■

Example 3.5.1 An 2-fold ρ -labeling of D_3 that can be used for a cyclic D_3 -decomposition of ${}^2K_{10}$.



Let G of size n be a bipartite subgraph of ${}^2K_{n+1}$ with a bipartition (A, B) . A 2-fold ρ -labeling of G is *ordered* if $f(a) < f(b)$ for each edge $\{a, b\}$ with $a \in A$ and $b \in B$. An ordered 2-fold ρ -labeling is also called a 2-fold ρ^+ -labeling. Similarly, the 2-fold labeling is *uniformly-ordered* if $f(a) < f(b)$ for every $a \in A$ and $b \in B$. A uniformly-ordered ρ -labeling will also be called a 2-fold ρ^{++} -labeling.

First, we observe that a β^+ -labeling of a graph G with n edges is necessarily a 2-fold ρ^+ -labeling of G . Also, Theorem 2.2.3 has a corresponding 2-fold counterpart.

Theorem 3.5.2 *Let n be a positive integer and let G be a bipartite multigraph with n edges, at most $n + 1$ vertices, and edge multiplicity at most 2. If G admits a 2-fold ρ^+ -labeling, then a cyclic $({}^2K_{nx+1}, G)$ -design exists for every positive integer x .*

Proof. The case $x = 1$ is covered by Theorem 3.5.1. So we will assume $x \geq 2$. We separate the proof into two cases depending on whether n is even or odd.

Case 1: n is even. Let $n = 2m$ and let G admit a 2-fold ρ^+ -labeling. Then for each length $\ell \in [1, m]$, there exist two edges of length ℓ in G . Denote these edges by e'_ℓ and e''_ℓ . Let G' be the subgraph of G with $E(G') = \{e'_\ell : \ell \in [1, m]\}$. Similarly, let G'' be the subgraph with $E(G'') = \{e''_\ell : \ell \in [1, m]\}$. Then G' and G'' are edge disjoint and neither of them contains double-edges. Moreover, the 2-fold ρ^+ -labeling of G induces simultaneously a ρ^+ -labeling of G' and a ρ^+ -labeling of G'' . By Theorem 2.2.3, there exists a cyclic (K_{2mx+1}, G') -design $\Delta_{G'}$ and a cyclic (K_{2mx+1}, G'') -design $\Delta_{G''}$. Moreover, the $2mx + 1$ copies of G' in $\Delta_{G'}$ and the $2mx + 1$ copies of G'' in $\Delta_{G''}$ can be matched so that for each $i \in \mathbb{Z}_{2mx+1}$ the graph $(G' \cup G'') + i$ is isomorphic to $(G' + i) \cup (G'' + i)$. Thus, $\Delta_G = \{(G' \cup G'') + i : 0 \leq i \leq 2mx\}$ is a cyclic $({}^2K_{nx+1}, G)$ -design.

Case 2: n is odd. Let $n = 2m + 1$ and let G have vertex bipartition $\{A, B\}$. Let f be a 2-fold ρ^+ -labeling of G . Let $e = \{a, b\}$ be an edge in $E(G)$, where $a \in A$ and $b \in B$, so the label of e is $f(b) - f(a)$. For ease of notation we will denote the label of e as $\bar{f}(e)$.

Let B_1, B_2, \dots, B_x be x vertex-disjoint copies of B . The vertex in B_i that corresponds to $b \in B$ will be denoted b_i . For $1 \leq i \leq x$, let G_i be a copy of G with vertex bipartition $\{A, B_i\}$. Let $G' = G_1 \cup G_2 \cup \dots \cup G_x$. Thus, G' is a bipartite multigraph with nx edges, edge multiplicity at most 2, and vertex bipartition $\{A, \bigcup_{i=1}^x B_i\}$. We use E_e to denote the set of x copies of an edge $e \in E(G)$. Define a labeling $f': V(G') \rightarrow [0, nx]$ as follows:

$$f'(v) = \begin{cases} f(a) & \text{if } v = a \in A, \\ f(b) + (i - 1)n & \text{if } v = b_i \in B_i. \end{cases}$$

Thus, the labels of the x copies of e are $\bar{f}(e), \bar{f}(e) + n, \dots, \bar{f}(e) + (x-1)n$. These labels are clearly distinct and in the same congruence class modulo n .

First, we show that two edges in G with distinct lengths under f produce disjoint sets of lengths under f' . Let e_1 and e_2 be edges in $E(G)$ such that $(\bar{f}(e_1))^* \neq (\bar{f}(e_2))^*$. Hence, $\bar{f}(e_1) \neq \bar{f}(e_2)$ and $\bar{f}(e_1) \neq n+1 - \bar{f}(e_2)$. Since $\bar{f}(e_1), \bar{f}(e_2) \in [0, n]$, we have $\bar{f}(e_1) \not\equiv \bar{f}(e_2)$ and $\bar{f}(e_1) \not\equiv 1 - \bar{f}(e_2)$ modulo n . We now consider

$$(\bar{f}'(E_{e_1}))^* = \{\min\{\bar{f}(e_1) + (i-1)n, nx+1 - \bar{f}(e_1) - (i-1)n\} : 1 \leq i \leq x\}$$

and

$$(\bar{f}'(E_{e_2}))^* = \{\min\{\bar{f}(e_2) + (i-1)n, nx+1 - \bar{f}(e_2) - (i-1)n\} : 1 \leq i \leq x\}.$$

Assume there exists a length $\ell \in (\bar{f}'(E_{e_1}))^* \cap (\bar{f}'(E_{e_2}))^*$. Since $\ell \in (\bar{f}'(E_{e_1}))^*$, ℓ is congruent to $\bar{f}(e_1)$ or $1 - \bar{f}(e_1)$ modulo n . Since $\ell \in (\bar{f}'(E_{e_2}))^*$, ℓ is congruent to $\bar{f}(e_2)$ or $1 - \bar{f}(e_2)$ modulo n . Thus, either $\bar{f}(e_1) \equiv \bar{f}(e_2)$ or $\bar{f}(e_1) \equiv n+1 - \bar{f}(e_2)$ modulo n ; however, this contradicts the assumption that $(\bar{f}(e_1))^* \neq (\bar{f}(e_2))^*$. Therefore, $(\bar{f}'(E_{e_1}))^* \cap (\bar{f}'(E_{e_2}))^* = \emptyset$.

Second, we show that two of the x copies of an edge $e \in G$ have the same length in ${}^2K_{nx+1}$ under f' only when the label of e is $(n+1)/2$. Let i and j be distinct integers in $[1, x]$. Suppose the i^{th} copy of e and the j^{th} copy of e have the same length. Since it was shown that they cannot have the same label, $\bar{f}(e) + (i-1)n = nx+1 - [\bar{f}(e) + (j-1)n]$. Hence, $\bar{f}(e) \equiv 1 - \bar{f}(e) \pmod{n}$ which implies that $2\bar{f}(e) \equiv 1 \pmod{n}$. Thus, $2\bar{f}(e) = nt + 1$ for some integer t . Since $\bar{f}(e) \in [1, n]$, we must have $t = 1$, and thus $\bar{f}(e) = (n+1)/2$. For the remainder of this proof, let \hat{e} denote the single edge in $E(G)$ such that $\bar{f}(\hat{e}) = (n+1)/2$.

Third, we examine the lengths of the edges in $E_{\hat{e}}$ under f' . It was proven above that the lengths in $(\bar{f}'(E_{\hat{e}}))^*$ are distinct from all others in $(\bar{f}'(E(G')))^*$.

To determine the lengths in question, we first compute the following edge labels:

$$\begin{aligned}
\bar{f}'(E_{\hat{e}}) &= \left\{ \frac{n+1}{2}, \frac{n+1}{2} + n, \dots, \frac{n+1}{2} + \lfloor \frac{x-1}{2} \rfloor n, \right. \\
&\quad \left. \frac{n+1}{2} + \lfloor \frac{x+1}{2} \rfloor n, \dots, \frac{n+1}{2} + (x-2)n, \frac{n+1}{2} + (x-1)n \right\} \\
&= \left\{ \frac{n+1}{2}, \frac{n+1}{2} + n, \dots, \frac{n+1}{2} + \lfloor \frac{x-1}{2} \rfloor n \right\} \\
&\quad \cup \left\{ \frac{n+1}{2} + (x-1)n, \frac{n+1}{2} + (x-2)n, \dots, \frac{n+1}{2} + \lfloor \frac{x+1}{2} \rfloor n \right\} \\
&= \left\{ \frac{n+1}{2}, \frac{n+1}{2} + n, \dots, \frac{n+1}{2} + \lfloor \frac{x-1}{2} \rfloor n \right\} \\
&\quad \cup \left\{ nx + 1 - \left(\frac{n+1}{2} \right), nx + 1 - \left(\frac{n+1}{2} + n \right), \dots, \right. \\
&\quad \left. nx + 1 - \left(\frac{n+1}{2} + (x-1 - \lfloor \frac{x+1}{2} \rfloor)n \right) \right\}. \tag{3.1}
\end{aligned}$$

Since n is odd, the longest possible length in ${}^2K_{nx+1}$ is

$$\lfloor \frac{nx+1}{2} \rfloor = \frac{n+1}{2} + \lfloor \frac{x-1}{2} \rfloor n = \begin{cases} \frac{nx+1}{2} & \text{if } x \text{ is odd,} \\ \frac{nx+1-n}{2} & \text{if } x \text{ is even.} \end{cases}$$



Clearly, the first set of labels in the union of (3.1) would yield $\lfloor \frac{x+1}{2} \rfloor$ distinct lengths including the longest length, while the second set of labels in the union of (3.1) would yield $x - \lfloor \frac{x+1}{2} \rfloor$ distinct lengths. Therefore,

$$(\bar{f}'(E_{\hat{e}}))^* = \begin{cases} 2\left\{ \frac{n+1}{2}, \frac{n+1}{2} + n, \dots, \frac{nx+1}{2} - n \right\} \cup \left\{ \frac{nx+1}{2} \right\} & \text{if } x \text{ is odd,} \\ 2\left\{ \frac{n+1}{2}, \frac{n+1}{2} + n, \dots, \frac{nx+1-n}{2} \right\} & \text{if } x \text{ is even.} \end{cases} \tag{3.2}$$

Finally, we use a counting argument to show that we must have the correct multiplicity of each length in ${}^2K_{nx+1}$. Consider two subgraphs of G : H_1 and H_2 where each has exactly one edge of each length $\ell \in [1, (n-1)/2]$. Similarly, let H'_1 and H'_2 be the corresponding subgraphs of G' that together contain all edges of G' except those in $E_{\hat{e}}$. Then for $i \in \{1, 2\}$, $(\bar{f}'(E(H'_i)))^*$ is composed of $(n-1)/2$ disjoint sets of lengths in $[1, \lfloor \frac{nx+1}{2} \rfloor]$ each containing x distinct lengths and none containing a length that is congruent to $(n+1)/2 \pmod{n}$. Since there are exactly $(nx-x)/2$ such lengths, $(\bar{f}'(E(H'_1 \cup H'_2)))^*$ must contain exactly 2 edges of each length in $[1, \lfloor \frac{nx+1}{2} \rfloor]$ not congruent to $(n+1)/2 \pmod{n}$. Therefore, $(\bar{f}'(E(G')))^*$ contains the necessary multiplicity of each length in $[1, \lfloor \frac{nx+1}{2} \rfloor]$, and

thus f' is a 2-fold ρ -labeling of G' . Thus by Theorem 3.5.1, there exists a cyclic $({}^2K_{n \times n+1}, G')$ -design. Since there exists a (G', G) -design, the result follows. ■

Now we apply the above theorem to show that a certain class of cubic multigraphs admits 2-fold ρ^+ -labelings.

3.5.1 2-Fold ρ^+ -Labeling of Cycle with Double Edges

Let \tilde{C}_{2n} denote the bipartite multigraph with vertex bipartition (A, B) where $A = \{u_i : i \in [1, n]\}$ and $B = \{v_i : i \in [1, n]\}$ and edge set $S_1 \cup S_2$ where S_1 is the set of single edges $\{\{u_i, v_{i+1}\} : 1 \leq i \leq n-1\} \cup \{u_n, v_1\}$ and S_2 is the multiset of double edges ${}^2\{\{u_i, v_i\} : 1 \leq i \leq n\}$. We call such a bipartite multigraph a *cycle with double edges*. Note that \tilde{C}_{2n} is cubic and bipartite. The multigraph \tilde{C}_{12} is shown in Figure 3.29. We will show that \tilde{C}_{2n} admits a 2-fold ρ^+ -labeling for all positive integers $n \geq 2$.

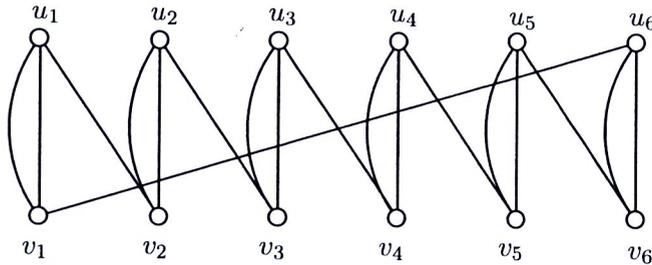


Figure 3.29: The 12-cycle with alternating double edges

Theorem 3.5.3 *The cycle with double edges \tilde{C}_{2n} admits a 2-fold ρ^+ -labeling for all $n \geq 2$.*

Proof. We separate the proof into 12 cases.

Case 1: $n \equiv 0 \pmod{12}$.

Let $n = 12t$. Then, $|V(\tilde{C}_{2n})| = 24t$ and $|E(\tilde{C}_{2n})| = 36t$. Define a one-to-one

function $f: V(\tilde{C}_{24t}) \rightarrow [0, 36t]$ as follows:

$$\begin{aligned}
 f(u_i) &= i - 1, & u_i \in A_1 &= \{u_i: 1 \leq i \leq 10t - 1\}, \\
 f(u_i) &= i, & u_i \in A_2 &= \{u_i: 10t \leq i \leq 12t\}, \\
 f(v_1) &= 30t - 1, \\
 f(v_i) &= 36t - 2i + 3, & v_i \in B_1 &= \{v_i: 2 \leq i \leq 2t + 1\}, \\
 f(v_i) &= 36t - 2i + 2, & v_i \in B_2 &= \{v_i: 2t + 2 \leq i \leq 6t\}, \\
 f(v_{6t+1}) &= 24t + 1, \\
 f(v_i) &= 36t - 2i + 1, & v_i \in B_3 &= \{v_i: 6t + 2 \leq i \leq 12t\}.
 \end{aligned}$$

Note that $A = A_1 \cup A_2$ and $B = B_1 \cup B_2 \cup B_3 \cup \{v_1, v_{6t+1}\}$. Thus, the domain of f is indeed $V(\tilde{C}_{24t})$. Next, we confirm that f is one-to-one. We compute

$$\begin{aligned}
 f(A_1) &= \{0, 1, \dots, 10t - 2\}, \\
 f(A_2) &= \{10t, 10t + 1, \dots, 12t\}, \\
 f(B_1) &= \{36t - 1, 36t - 3, \dots, 32t + 1\}, \\
 f(B_2) &= \{32t - 2, 32t - 4, \dots, 24t + 2\}, \\
 f(B_3) &= \{24t - 3, 24t - 5, \dots, 12t + 1\}.
 \end{aligned}$$

Note that f is piecewise strictly increasing by 1 or strictly decreasing by 2 and that all labels are distinct. Thus, f is one-to-one. Moreover, $f(A) \subseteq [0, 12t]$ and $f(B) \subseteq [12t + 1, 36t - 1]$.

Next, we compute the edge lengths. For $1 \leq i \leq 2t$, an edge $e = \{u_i, v_{i+1}\}$ in S_1 has label

$$|f(u_i) - f(v_{i+1})| = f(v_{i+1}) - f(u_i) = (36t - 2(i+1) + 3) - (i - 1) = 36t - 3i + 2.$$

Similarly, the other edges $e \in S_1$ have label:

$$\begin{aligned}
 36t - 3i + 1 & & \text{for } 2t + 1 \leq i \leq 6t - 1, \\
 18t + 2 & & \text{for } i = 6t, \\
 36t - 3i & & \text{for } 6t + 1 \leq i \leq 10t - 1, \\
 36t - 3i - 1 & & \text{for } 10t \leq i \leq 12t - 1, \\
 18t - 1 & & \text{for } e = \{u_n, v_1\}.
 \end{aligned}$$

Recall that the length of an edge $e \in E(\tilde{C}_{24t})$ is $(\bar{f}(e))^* = \min\{\bar{f}(e), |E(\tilde{C}_{24t})| + 1 - \bar{f}(e)\}$. Thus, we compute the following edge length sets that compose $(\bar{f}(S_1))^*$:

$$\begin{aligned} \{36t + 1 - (36t - 3i + 2) : 1 \leq i \leq 2t\} &= \{3i - 1 : 1 \leq i \leq 2t\} \\ &= \{\ell \equiv 2 \pmod{3} : 2 \leq \ell \leq 6t - 1\}, \\ \{36t + 1 - (36t - 3i + 1) : 2t + 1 \leq i \leq 6t - 1\} &= \{3i : 2t + 1 \leq i \leq 6t - 1\} \\ &= \{\ell \equiv 0 \pmod{3} : 6t + 3 \leq \ell \leq 18t - 3\}, \\ \{36t + 1 - (18t + 2)\} &= \{18t - 1\}, \\ \{36t - 3i : 6t + 1 \leq i \leq 10t - 1\} &= \{\ell \equiv 0 \pmod{3} : 6t + 3 \leq \ell \leq 18t - 3\}, \\ \{36t - 3i - 1 : 10t \leq i \leq 12t - 1\} &= \{\ell \equiv 2 \pmod{3} : 2 \leq \ell \leq 6t - 1\}, \\ \{18t - 1\} &= \{18t - 1\}. \end{aligned}$$

Also, an edge $\{u_i, v_i\} \in S_2$ has label

$$\begin{array}{ll} 30t - 1 & \text{for } i = 1, \\ 36t - 3i + 4 & \text{for } 2 \leq i \leq 2t + 1, \\ 36t - 3i + 3 & \text{for } 2t + 2 \leq i \leq 6t, \\ 18t + 1 & \text{for } i = 6t + 1, \\ 36t - 3i + 2 & \text{for } 6t + 2 \leq i \leq 10t - 1, \\ 36t - 3i + 1 & \text{for } 10t \leq i \leq 12t, \end{array}$$

yielding the following multisets that compose $(\bar{f}(S_2))^*$:

$$\begin{aligned} {}^2\{36t + 1 - (30t - 1)\} &= {}^2\{6t + 2\}, \\ {}^2\{36t + 1 - (36t - 3i + 4) : 2 \leq i \leq 2t + 1\} &= {}^2\{3i - 3 : 2 \leq i \leq 2t + 1\} \\ &= {}^2\{\ell \equiv 0 \pmod{3} : 3 \leq \ell \leq 6t\}, \\ {}^2\{36t + 1 - (36t - 3i + 3) : 2t + 2 \leq i \leq 6t\} &= {}^2\{3i - 2 : 2t + 2 \leq i \leq 6t\} \\ &= {}^2\{\ell \equiv 1 \pmod{3} : 6t + 4 \leq \ell \leq 18t - 2\}, \\ {}^2\{36t + 1 - (18t + 1)\} &= {}^2\{18t\}, \\ {}^2\{36t - 3i + 2 : 6t + 2 \leq i \leq 10t - 1\} \\ &= {}^2\{\ell \equiv 2 \pmod{3} : 6t + 5 \leq \ell \leq 18t - 4\}, \\ {}^2\{36t - 3i + 1 : 10t \leq i \leq 12t\} &= {}^2\{\ell \equiv 1 \pmod{3} : 1 \leq \ell \leq 6t + 1\}. \end{aligned}$$

Since every length $\ell \in [1, 18t]$ occurs on exactly two edges in \tilde{C}_{24t} and $f(V(\tilde{C}_{24t})) \subseteq [0, 36t]$, f is a 2-fold ρ -labeling. Also since $f(A) \subseteq [0, 12t]$ and $f(B) \subseteq [12t + 1, 36t - 1]$, we have a 2-fold ρ^+ -labeling of \tilde{C}_{24t} .

Case 2: $n \equiv 1 \pmod{12}$.

Let $n = 12t + 1$. Then, $|V(\tilde{C}_{2n})| = 24t + 2$ and $|E(\tilde{C}_{2n})| = 36t + 3$. Define a one-to-one function $f: V(\tilde{C}_{24t+2}) \rightarrow [0, 36t + 3]$ as follows:

$$\begin{aligned} f(u_i) &= i - 1, & u_i \in A_1 &= \{u_i : 1 \leq i \leq 2t + 1\}, \\ f(u_i) &= i, & u_i \in A_2 &= \{u_i : 2t + 2 \leq i \leq 12t + 1\}, \\ f(v_1) &= 30t + 1, \\ f(v_i) &= 36t - 2i + 6, & v_i \in B_1 &= \{v_i : 2 \leq i \leq 6t\}, \\ f(v_{6t+1}) &= 24t + 2, \\ f(v_i) &= 36t - 2i + 5, & v_i \in B_2 &= \{v_i : 6t + 2 \leq i \leq 10t\}, \\ f(v_i) &= 36t - 2i + 4, & v_i \in B_3 &= \{v_i : 10t + 1 \leq i \leq 12t + 1\}. \end{aligned}$$

Note that $A = A_1 \cup A_2$ and $B = B_1 \cup B_2 \cup B_3 \cup \{v_1, v_{6t+1}\}$. Thus, the domain of f is indeed $V(\tilde{C}_{24t+2})$. Next, we confirm that f is one-to-one. We compute

$$\begin{aligned} f(A_1) &= \{0, 1, \dots, 2t\}, \\ f(A_2) &= \{2t + 2, 2t + 3, \dots, 12t + 1\}, \\ f(B_1) &= \{36t + 2, 36t, \dots, 24t + 6\}, \\ f(B_2) &= \{24t + 1, 24t - 1, \dots, 16t + 5\}, \\ f(B_3) &= \{16t + 2, 16t, \dots, 12t + 2\}. \end{aligned}$$

Note that f is piecewise strictly increasing by 1 or strictly decreasing by 2 and that all labels are distinct. Thus, f is one-to-one. Moreover, $f(A) \subseteq [0, 12t + 1]$ and $f(B) \subseteq [12t + 2, 36t + 2]$.

Next, we compute the edge lengths. For $1 \leq i \leq 2t + 1$, an edge $e = \{u_i, v_{i+1}\}$ in S_1 has label

$$|f(u_i) - f(v_{i+1})| = f(v_{i+1}) - f(u_i) = (36t - 2(i + 1) + 6) - (i - 1) = 36t - 3i + 5.$$

Similarly, the other edges $e \in S_1$ have label:

$$\begin{array}{ll}
36t - 3i + 4 & \text{for } 2t + 2 \leq i \leq 6t - 1, \\
18t + 2 & \text{for } i = 6t, \\
36t - 3i + 3 & \text{for } 6t + 1 \leq i \leq 10t - 1, \\
36t - 3i + 2 & \text{for } 10t \leq i \leq 12t, \\
18t & \text{for } e = \{u_n, v_1\}.
\end{array}$$

Recall that the length of an edge $e \in E(\tilde{C}_{24t+2})$ is $(\bar{f}(e))^* = \min\{\bar{f}(e), |E(\tilde{C}_{24t+2})| + 1 - \bar{f}(e)\}$. Thus, we compute the following edge length sets that compose $(\bar{f}(S_1))^*$:

$$\begin{aligned}
& \{36t + 4 - (36t - 3i + 5) : 1 \leq i \leq 2t + 1\} = \{3i - 1 : 1 \leq i \leq 2t + 1\} \\
& \quad = \{\ell \equiv 2 \pmod{3} : 2 \leq \ell \leq 6t + 2\}, \\
& \{36t + 4 - (36t - 3i + 4) : 2t + 2 \leq i \leq 6t - 1\} = \{3i : 2t + 2 \leq i \leq 6t - 1\} \\
& \quad = \{\ell \equiv 0 \pmod{3} : 6t + 6 \leq \ell \leq 18t - 3\}, \\
& \{18t + 2\} = \{18t + 2\}, \\
& \{36t - 3i + 3 : 6t + 1 \leq i \leq 10t - 1\} \\
& \quad = \{\ell \equiv 0 \pmod{3} : 6t + 6 \leq \ell \leq 18t\}, \\
& \{36t - 3i + 2 : 10t \leq i \leq 12t\} \\
& \quad = \{\ell \equiv 2 \pmod{3} : 2 \leq \ell \leq 6t + 2\}, \\
& \{18t\} = \{18t\}.
\end{aligned}$$

Also, an edge $\{u_i, v_i\} \in S_2$ has label

$$\begin{array}{ll}
30t + 1 & \text{for } i = 1, \\
36t - 3i + 7 & \text{for } 2 \leq i \leq 2t + 1, \\
36t - 3i + 6 & \text{for } 2t + 2 \leq i \leq 6t, \\
18t + 1 & \text{for } i = 6t + 1, \\
36t - 3i + 5 & \text{for } 6t + 2 \leq i \leq 10t, \\
36t - 3i + 4 & \text{for } 10t + 1 \leq i \leq 12t + 1,
\end{array}$$

yielding the following multisets that compose $(\bar{f}(S_2))^*$:

$${}^2\{36t + 4 - (30t + 1)\} = {}^2\{6t + 3\},$$

$$\begin{aligned} {}^2\{36t + 4 - (36t - 3i + 7) : 2 \leq i \leq 2t + 1\} &= {}^2\{3i - 3 : 2 \leq i \leq 2t + 1\} \\ &= {}^2\{\ell \equiv 0 \pmod{3} : 3 \leq \ell \leq 6t\}, \end{aligned}$$

$$\begin{aligned} {}^2\{36t + 4 - (36t - 3i + 6) : 2t + 2 \leq i \leq 6t\} &= {}^2\{3i - 2 : 2t + 2 \leq i \leq 6t\} \\ &= {}^2\{\ell \equiv 1 \pmod{3} : 6t + 4 \leq \ell \leq 18t - 2\}, \end{aligned}$$

$${}^2\{18t + 1\} = {}^2\{18t + 1\},$$

$$\begin{aligned} {}^2\{36t - 3i + 5 : 6t + 2 \leq i \leq 10t\} \\ &= {}^2\{\ell \equiv 2 \pmod{3} : 6t + 5 \leq \ell \leq 18t - 1\}, \end{aligned}$$

$$\begin{aligned} {}^2\{36t - 3i + 4 : 10t + 1 \leq i \leq 12t + 1\} \\ &= {}^2\{\ell \equiv 1 \pmod{3} : 1 \leq \ell \leq 6t + 1\}. \end{aligned}$$

Since every length $\ell \in [1, 18t + 1]$ occurs on exactly two edges and length $18t + 2$ occurs on only one edge in \tilde{C}_{24t+2} and $f(V(\tilde{C}_{24t+2})) \subseteq [0, 36t + 3]$, f is a 2-fold ρ -labeling. Also since $f(A) \subseteq [0, 12t + 1]$ and $f(B) \subseteq [12t + 2, 36t + 2]$, we have a 2-fold ρ^+ -labeling of \tilde{C}_{24t+2} .

Case 3: $n \equiv 2 \pmod{12}$.

Let $n = 12t - 10$. Then, $|V(\tilde{C}_{2n})| = 24t - 20$ and $|E(\tilde{C}_{2n})| = 36t - 30$. A 2-fold ρ^+ -labeling of \tilde{C}_4 is given in Figure 3.30. For $t \geq 2$, define a one-to-one function

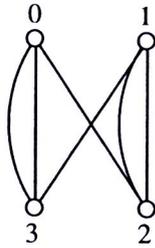


Figure 3.30: A 2-fold ρ^+ -labeling of \tilde{C}_4 .

$f: V(\tilde{C}_{24t-20}) \rightarrow [0, 36t - 30]$ as follows:

$$\begin{aligned}
f(u_i) &= i - 1, & u_i \in A_1 &= \{u_i: 1 \leq i \leq 12t - 10\}, \\
f(v_1) &= 30t - 27, \\
f(v_i) &= 36t - 2i - 27, & v_i \in B_1 &= \{v_i: 2 \leq i \leq 2t\}, \\
f(v_i) &= 36t - 2i - 28, & v_i \in B_2 &= \{v_i: 2t + 1 \leq i \leq 6t - 5\}, \\
f(v_{6t-4}) &= 24t - 19, \\
f(v_i) &= 36t - 2i - 29, & v_i \in B_3 &= \{v_i: 6t - 3 \leq i \leq 10t - 9\}, \\
f(v_i) &= 36t - 2i - 30, & v_i \in B_4 &= \{v_i: 10t - 8 \leq i \leq 12t - 10\}.
\end{aligned}$$

Note that $A = A_1$ and $B = B_1 \cup B_2 \cup B_3 \cup B_4 \cup \{v_1, v_{6t-4}\}$. Thus, the domain of f is indeed $V(\tilde{C}_{24t-20})$. Next, we confirm that f is one-to-one. We compute

$$\begin{aligned}
f(A_1) &= \{0, 1, \dots, 12t - 11\}, \\
f(B_1) &= \{36t - 31, 36t - 33, \dots, 32t - 27\}, \\
f(B_2) &= \{32t - 30, 32t - 32, \dots, 24t - 18\}, \\
f(B_3) &= \{24t - 23, 24t - 25, \dots, 16t - 11\}, \\
f(B_4) &= \{16t - 14, 16t - 16, \dots, 12t - 10\}.
\end{aligned}$$

Note that f is piecewise strictly increasing by 1 or strictly decreasing by 2 and that all labels are distinct. Thus, f is one-to-one. Moreover, $f(A) \subseteq [0, 12t - 11]$ and $f(B) \subseteq [12t - 10, 36t - 31]$.

Next, we compute the edge lengths. For $1 \leq i \leq 2t - 1$, an edge $e = \{u_i, v_{i+1}\}$ in S_1 has label

$$|f(u_i) - f(v_{i+1})| = f(v_{i+1}) - f(u_i) = (36t - 2(i+1) - 27) - (i-1) = 36t - 3i - 28.$$

Similarly, the other edges $e \in S_1$ have label:

$$\begin{aligned}
36t - 3i - 29 & & \text{for } 2t \leq i \leq 6t - 6, \\
18t - 13 & & \text{for } i = 6t - 5, \\
36t - 3i - 30 & & \text{for } 6t - 4 \leq i \leq 10t - 10, \\
36t - 3i - 31 & & \text{for } 10t - 9 \leq i \leq 12t - 11, \\
18t - 16 & & \text{for } e = \{u_n, v_1\}.
\end{aligned}$$

Recall that the length of an edge $e \in E(\tilde{C}_{24t-20})$ is $(\bar{f}(e))^* = \min\{\bar{f}(e), |E(\tilde{C}_{24t-20})| + 1 - \bar{f}(e)\}$. Thus, we compute the following edge length sets that compose $(\bar{f}(S_1))^*$:

$$\begin{aligned}
& \{36t - 29 - (36t - 3i - 28) : 1 \leq i \leq 2t - 1\} = \{3i - 1 : 1 \leq i \leq 2t - 1\} \\
& \quad = \{\ell \equiv 2 \pmod{3} : 2 \leq \ell \leq 6t - 4\}, \\
& \{36t - 29 - (36t - 3i - 29) : 2t \leq i \leq 6t - 6\} = \{3i : 2t \leq i \leq 6t - 6\} \\
& \quad = \{\ell \equiv 0 \pmod{3} : 6t \leq \ell \leq 18t - 18\}, \\
& \{36t - 29 - (18t - 13)\} = \{18t - 16\}, \\
& \{36t - 3i - 30 : 6t - 4 \leq i \leq 10t - 10\} \\
& \quad = \{\ell \equiv 0 \pmod{3} : 6t \leq \ell \leq 18t - 18\}, \\
& \{36t - 3i - 31 : 10t - 9 \leq i \leq 12t - 11\} \\
& \quad = \{\ell \equiv 2 \pmod{3} : 2 \leq \ell \leq 6t - 4\}, \\
& \{18t - 16\} = \{18t - 16\}.
\end{aligned}$$

Also, an edge $\{u_i, v_i\} \in S_2$ has label

$$\begin{array}{ll}
30t - 27 & \text{for } i = 1, \\
36t - 3i - 26 & \text{for } 2 \leq i \leq 2t, \\
36t - 3i - 27 & \text{for } 2t + 1 \leq i \leq 6t - 5, \\
18t - 14 & \text{for } i = 6t - 4, \\
36t - 3i - 28 & \text{for } 6t - 3 \leq i \leq 10t - 9, \\
36t - 3i - 29 & \text{for } 10t - 8 \leq i \leq 12t - 10,
\end{array}$$

yielding the following multisets that compose $(\bar{f}(S_2))^*$:

$$\begin{aligned}
& {}^2\{36t - 29 - (30t - 27)\} = {}^2\{6t - 2\}, \\
& {}^2\{36t - 29 - (36t - 3i - 26) : 2 \leq i \leq 2t\} = {}^2\{3i - 3 : 2 \leq i \leq 2t\} \\
& \quad = {}^2\{\ell \equiv 0 \pmod{3} : 3 \leq \ell \leq 6t - 3\}, \\
& {}^2\{36t - 29 - (36t - 3i - 27) : 2t + 1 \leq i \leq 6t - 5\} \\
& \quad = {}^2\{3i - 2 : 2t + 1 \leq i \leq 6t - 5\} \\
& \quad = {}^2\{\ell \equiv 1 \pmod{3} : 6t + 1 \leq \ell \leq 18t - 17\}, \\
& {}^2\{18t - 14\} = {}^2\{18t - 14\}, \\
& {}^2\{36t - 3i - 28 : 6t - 3 \leq i \leq 10t - 9\} \\
& \quad = {}^2\{\ell \equiv 2 \pmod{3} : 6t - 1 \leq \ell \leq 18t - 19\}, \\
& {}^2\{36t - 3i - 29 : 10t - 8 \leq i \leq 12t - 10\} \\
& \quad = {}^2\{\ell \equiv 1 \pmod{3} : 1 \leq \ell \leq 6t - 5\}.
\end{aligned}$$

Since every length $\ell \in [1, 18t - 14]$ occurs on exactly two edges in \tilde{C}_{24t-20} and $f(V(\tilde{C}_{24t-20})) \subseteq [0, 36t - 30]$, f is a 2-fold ρ -labeling. Also since $f(A) \subseteq [0, 12t - 11]$ and $f(B) \subseteq [12t - 10, 36t - 31]$, we have a 2-fold ρ^+ -labeling of \tilde{C}_{24t-20} .

Case 4: $n \equiv 3 \pmod{12}$.

Let $n = 12t - 9$. Then, $|V(\tilde{C}_{2n})| = 24t - 18$ and $|E(\tilde{C}_{2n})| = 36t - 27$. A 2-fold ρ^+ -labeling of \tilde{C}_6 is given in Figure 3.31. For $t \geq 2$, define a one-to-one function

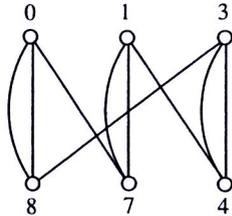


Figure 3.31: A 2-fold ρ^+ -labeling of \tilde{C}_6 .

$f: V(\tilde{C}_{24t-18}) \rightarrow [0, 36t - 27]$ as follows:

$$\begin{aligned}
 f(u_i) &= i - 1, & u_i \in A_1 &= \{u_i: 1 \leq i \leq 10t - 8\}, \\
 f(u_i) &= i, & u_i \in A_2 &= \{u_i: 10t - 7 \leq i \leq 12t - 9\}, \\
 f(v_1) &= 30t - 22, \\
 f(v_i) &= 36t - 2i - 24, & v_i \in B_1 &= \{v_i: 2 \leq i \leq 2t - 1\}, \\
 f(v_i) &= 36t - 2i - 25, & v_i \in B_2 &= \{v_i: 2t \leq i \leq 6t - 4\}, \\
 f(v_i) &= 36t - 2i - 26, & v_i \in B_3 &= \{v_i: 6t - 3 \leq i \leq 12t - 9\}.
 \end{aligned}$$

Note that $A = A_1 \cup A_2$ and $B = B_1 \cup B_2 \cup B_3 \cup \{v_1\}$. Thus, the domain of f is indeed $V(\tilde{C}_{24t-18})$. Next, we confirm that f is one-to-one. We compute

$$\begin{aligned}
 f(A_1) &= \{0, 1, \dots, 10t - 9\}, \\
 f(A_2) &= \{10t - 7, 10t - 6, \dots, 12t - 9\}, \\
 f(B_1) &= \{36t - 28, 36t - 30, \dots, 32t - 22\}, \\
 f(B_2) &= \{32t - 25, 32t - 27, \dots, 24t - 17\}, \\
 f(B_3) &= \{24t - 20, 24t - 22, \dots, 12t - 8\}.
 \end{aligned}$$

Note that f is piecewise strictly increasing by 1 or strictly decreasing by 2 and that all labels are distinct. Thus, f is one-to-one. Moreover, $f(A) \subseteq [0, 12t - 9]$ and $f(B) \subseteq [12t - 8, 36t - 28]$.

Next, we compute the edge lengths. For $1 \leq i \leq 2t - 2$, an edge $e = \{u_i, v_{i+1}\}$ in S_1 has label

$$|f(u_i) - f(v_{i+1})| = f(v_{i+1}) - f(u_i) = (36t - 2(i+1) - 24) - (i - 1) = 36t - 3i - 25.$$

Similarly, the other edges $e \in S_1$ have label:

$$\begin{aligned}
 36t - 3i - 26 & & \text{for } 2t - 1 \leq i \leq 6t - 5, \\
 36t - 3i - 27 & & \text{for } 6t - 4 \leq i \leq 10t - 8, \\
 36t - 3i - 28 & & \text{for } 10t - 7 \leq i \leq 12t - 10, \\
 18t - 13 & & \text{for } e = \{u_n, v_1\}.
 \end{aligned}$$

Recall that the length of an edge $e \in E(\tilde{C}_{24t-18})$ is $(\bar{f}(e))^* = \min\{\bar{f}(e), |E(\tilde{C}_{24t-18})| + 1 - \bar{f}(e)\}$. Thus, we compute the following edge length sets that compose $(\bar{f}(S_1))^*$:

$$\begin{aligned} \{36t - 26 - (36t - 3i - 25): 1 \leq i \leq 2t - 2\} &= \{3i - 1: 1 \leq i \leq 2t - 2\} \\ &= \{\ell \equiv 2 \pmod{3}: 2 \leq \ell \leq 6t - 7\}, \\ \{36t - 26 - (36t - 3i - 26): 2t - 1 \leq i \leq 6t - 5\} &= \{3i: 2t - 1 \leq i \leq 6t - 5\} \\ &= \{\ell \equiv 0 \pmod{3}: 6t - 3 \leq \ell \leq 18t - 15\}, \\ \{36t - 3i - 27: 6t - 4 \leq i \leq 10t - 8\} &= \{\ell \equiv 0 \pmod{3}: 6t - 3 \leq \ell \leq 18t - 15\}, \\ \{36t - 3i - 28: 10t - 7 \leq i \leq 12t - 10\} &= \{\ell \equiv 2 \pmod{3}: 2 \leq \ell \leq 6t - 7\}, \\ \{18t - 13\} &= \{18t - 13\}. \end{aligned}$$

Also, an edge $\{u_i, v_i\} \in S_2$ has label

$$\begin{array}{ll} 30t - 22 & \text{for } i = 1, \\ 36t - 3i - 23 & \text{for } 2 \leq i \leq 2t - 1, \\ 36t - 3i - 24 & \text{for } 2t \leq i \leq 6t - 4, \\ 36t - 3i - 25 & \text{for } 6t - 3 \leq i \leq 10t - 8, \\ 36t - 3i - 26 & \text{for } 10t - 7 \leq i \leq 12t - 9, \end{array}$$

yielding the following multisets that compose $(\bar{f}(S_2))^*$:

$$\begin{aligned} {}^2\{36t - 26 - (30t - 22)\} &= {}^2\{6t - 4\}, \\ {}^2\{36t - 26 - (36t - 3i - 23): 2 \leq i \leq 2t - 1\} &= {}^2\{3i - 3: 2 \leq i \leq 2t - 1\} \\ &= {}^2\{\ell \equiv 0 \pmod{3}: 3 \leq \ell \leq 6t - 6\}, \\ {}^2\{36t - 26 - (36t - 3i - 24): 2t \leq i \leq 6t - 4\} &= {}^2\{3i - 2: 2t \leq i \leq 6t - 4\} \\ &= {}^2\{\ell \equiv 1 \pmod{3}: 6t - 2 \leq \ell \leq 18t - 14\}, \\ {}^2\{36t - 3i - 25: 6t - 3 \leq i \leq 10t - 8\} \\ &= {}^2\{\ell \equiv 2 \pmod{3}: 6t - 1 \leq \ell \leq 18t - 16\}, \\ {}^2\{36t - 3i - 26: 10t - 7 \leq i \leq 12t - 9\} &= {}^2\{\ell \equiv 1 \pmod{3}: 1 \leq \ell \leq 6t - 5\}. \end{aligned}$$

Since every length $\ell \in [1, 18t - 14]$ occurs on exactly two edges and length $18t - 13$ occurs on only one edge in \tilde{C}_{24t-18} and $f(V(\tilde{C}_{24t-18})) \subseteq [0, 36t - 27]$, f is a 2-fold

ρ -labeling. Also since $f(A) \subseteq [0, 12t - 9]$ and $f(B) \subseteq [12t - 8, 36t - 28]$, we have a 2-fold ρ^+ -labeling of \tilde{C}_{24t-18} .

Case 5: $n \equiv 4 \pmod{12}$.

Let $n = 12t - 8$. Then, $|V(\tilde{C}_{2n})| = 24t - 16$ and $|E(\tilde{C}_{2n})| = 36t - 24$. A 2-fold ρ^+ -labeling of \tilde{C}_8 is given in Figure 3.32. For $t \geq 2$, define a one-to-one function

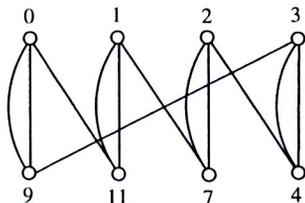


Figure 3.32: A 2-fold ρ^+ -labeling of \tilde{C}_8 .

$f: V(\tilde{C}_{24t-16}) \rightarrow [0, 36t - 24]$ as follows:

$$\begin{aligned} f(u_i) &= i - 1, & u_i \in A_1 &= \{u_i: 1 \leq i \leq 12t - 8\}, \\ f(v_1) &= 30t - 21, \\ f(v_i) &= 36t - 2i - 21, & v_i \in B_1 &= \{v_i: 2 \leq i \leq 2t\}, \\ f(v_i) &= 36t - 2i - 22, & v_i \in B_2 &= \{v_i: 2t + 1 \leq i \leq 6t - 4\}, \\ f(v_i) &= 36t - 2i - 23, & v_i \in B_3 &= \{v_i: 6t - 3 \leq i \leq 10t - 7\}, \\ f(v_i) &= 36t - 2i - 24, & v_i \in B_4 &= \{v_i: 10t - 6 \leq i \leq 12t - 8\}. \end{aligned}$$

Note that $A = A_1$ and $B = B_1 \cup B_2 \cup B_3 \cup B_4 \cup \{v_1\}$. Thus, the domain of f is indeed $V(\tilde{C}_{24t-16})$. Next, we confirm that f is one-to-one. We compute

$$\begin{aligned} f(A_1) &= \{0, 1, \dots, 12t - 9\}, \\ f(B_1) &= \{36t - 25, 36t - 27, \dots, 32t - 21\}, \\ f(B_2) &= \{32t - 24, 32t - 26, \dots, 24t - 14\}, \\ f(B_3) &= \{24t - 17, 24t - 19, \dots, 16t - 9\}, \\ f(B_4) &= \{16t - 12, 16t - 14, \dots, 12t - 8\}. \end{aligned}$$

Note that f is piecewise strictly increasing by 1 or strictly decreasing by 2 and that all labels are distinct. Thus, f is one-to-one. Moreover, $f(A) \subseteq [0, 12t - 9]$ and $f(B) \subseteq [12t - 8, 36t - 25]$.

Next, we compute the edge lengths. For $1 \leq i \leq 2t - 1$, an edge $e = \{u_i, v_{i+1}\}$ in S_1 has label

$$|f(u_i) - f(v_{i+1})| = f(v_{i+1}) - f(u_i) = (36t - 2(i+1) - 21) - (i-1) = 36t - 3i - 22.$$

Similarly, the other edges $e \in S_1$ have label:

$$\begin{aligned} 36t - 3i - 23 & \quad \text{for } 2t \leq i \leq 6t - 5, \\ 36t - 3i - 24 & \quad \text{for } 6t - 4 \leq i \leq 10t - 8, \\ 36t - 3i - 25 & \quad \text{for } 10t - 7 \leq i \leq 12t - 9, \\ 18t - 12 & \quad \text{for } e = \{u_n, v_1\}. \end{aligned}$$

Recall that the length of an edge $e \in E(\tilde{C}_{24t-16})$ is $(\bar{f}(e))^* = \min\{\bar{f}(e), |E(\tilde{C}_{24t-16})| + 1 - \bar{f}(e)\}$. Thus, we compute the following edge length sets that compose $(\bar{f}(S_1))^*$:

$$\begin{aligned} \{36t - 23 - (36t - 3i - 22) : 1 \leq i \leq 2t - 1\} &= \{3i - 1 : 1 \leq i \leq 2t - 1\} \\ &= \{\ell \equiv 2 \pmod{3} : 2 \leq \ell \leq 6t - 4\}, \\ \{36t - 23 - (36t - 3i - 23) : 2t \leq i \leq 6t - 5\} &= \{3i : 2t \leq i \leq 6t - 5\} \\ &= \{\ell \equiv 0 \pmod{3} : 6t \leq \ell \leq 18t - 15\}, \\ \{36t - 3i - 24 : 6t - 4 \leq i \leq 10t - 8\} &= \{\ell \equiv 0 \pmod{3} : 6t \leq \ell \leq 18t - 18\}, \\ \{36t - 3i - 25 : 10t - 7 \leq i \leq 12t - 9\} &= \{\ell \equiv 2 \pmod{3} : 2 \leq \ell \leq 6t - 4\}, \\ \{18t - 12\} &= \{18t - 12\}. \end{aligned}$$

Also, an edge $\{u_i, v_i\} \in S_2$ has label

$$\begin{aligned} 30t - 21 & \quad \text{for } i = 1, \\ 36t - 3i - 20 & \quad \text{for } 2 \leq i \leq 2t, \\ 36t - 3i - 21 & \quad \text{for } 2t + 1 \leq i \leq 6t - 4, \\ 36t - 3i - 22 & \quad \text{for } 6t - 3 \leq i \leq 10t - 7, \\ 36t - 3i - 23 & \quad \text{for } 10t - 6 \leq i \leq 12t - 8, \end{aligned}$$

yielding the following multisets that compose $(\bar{f}(S_2))^*$:

$$\begin{aligned}
& {}^2\{36t - 23 - (30t - 21)\} = {}^2\{6t - 2\}, \\
& {}^2\{36t - 23 - (36t - 3i - 20) : 2 \leq i \leq 2t\} = {}^2\{3i - 3 : 2 \leq i \leq 2t\} \\
& \quad = {}^2\{\ell \equiv 0 \pmod{3} : 3 \leq \ell \leq 6t - 3\}, \\
& {}^2\{36t - 23 - (36t - 3i - 21) : 2t + 1 \leq i \leq 6t - 4\} \\
& \quad = {}^2\{3i - 2 : 2t + 1 \leq i \leq 6t - 4\} \\
& \quad = {}^2\{\ell \equiv 1 \pmod{3} : 6t + 1 \leq \ell \leq 18t - 14\}, \\
& {}^2\{36t - 3i - 22 : 6t - 3 \leq i \leq 10t - 7\} \\
& \quad = {}^2\{\ell \equiv 2 \pmod{3} : 6t - 1 \leq \ell \leq 18t - 13\}, \\
& {}^2\{36t - 3i - 23 : 10t - 6 \leq i \leq 12t - 8\} = {}^2\{\ell \equiv 1 \pmod{3} : 1 \leq \ell \leq 6t - 5\}.
\end{aligned}$$

Since every length $\ell \in [1, 18t - 12]$ occurs on exactly two edges in \tilde{C}_{24t-16} and $f(V(\tilde{C}_{24t-16})) \subseteq [0, 36t - 24]$, f is a 2-fold ρ -labeling. Also since $f(A) \subseteq [0, 12t - 9]$ and $f(B) \subseteq [12t - 8, 36t - 25]$, we have a 2-fold ρ^+ -labeling of \tilde{C}_{24t-16} .

Case 6: $n \equiv 5 \pmod{12}$.

Let $n = 12t - 7$. Then, $|V(\tilde{C}_{2n})| = 24t - 14$ and $|E(\tilde{C}_{2n})| = 36t - 21$. A 2-fold ρ^+ -labeling of \tilde{C}_{10} is given in Figure 3.33. For $t \geq 2$, define a one-to-one function

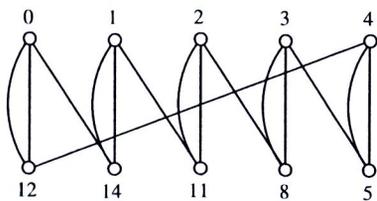


Figure 3.33: A 2-fold ρ^+ -labeling of \tilde{C}_{10} .

$f: V(\tilde{C}_{24t-14}) \rightarrow [0, 36t - 21]$ as follows:

$$\begin{aligned}
f(u_i) &= i - 1, & u_i \in A_1 &= \{u_i: 1 \leq i \leq 12t - 7\}, \\
f(v_1) &= 30t - 18, \\
f(v_i) &= 36t - 2i - 18, & v_i \in B_1 &= \{v_i: 2 \leq i \leq 2t\}, \\
f(v_i) &= 36t - 2i - 19, & v_i \in B_2 &= \{v_i: 2t + 1 \leq i \leq 6t - 3\}, \\
f(v_i) &= 36t - 2i - 20, & v_i \in B_3 &= \{v_i: 6t - 2 \leq i \leq 10t - 6\}, \\
f(v_i) &= 36t - 2i - 21, & v_i \in B_4 &= \{v_i: 10t - 5 \leq i \leq 12t - 7\}.
\end{aligned}$$

Note that $A = A_1$ and $B = B_1 \cup B_2 \cup B_3 \cup B_4 \cup \{v_1\}$. Thus, the domain of f is indeed $V(\tilde{C}_{24t-14})$. Next, we confirm that f is one-to-one. We compute

$$\begin{aligned}
f(A_1) &= \{0, 1, \dots, 12t - 8\}, \\
f(B_1) &= \{36t - 22, 36t - 24, \dots, 32t - 18\}, \\
f(B_2) &= \{32t - 21, 32t - 23, \dots, 24t - 13\}, \\
f(B_3) &= \{24t - 16, 24t - 18, \dots, 16t - 8\}, \\
f(B_4) &= \{16t - 11, 16t - 13, \dots, 12t - 7\}.
\end{aligned}$$

Note that f is piecewise strictly increasing by 1 or strictly decreasing by 2 and that all labels are distinct. Thus, f is one-to-one. Moreover, $f(A) \subseteq [0, 12t - 8]$ and $f(B) \subseteq [12t - 7, 36t - 22]$.

Next, we compute the edge lengths. For $1 \leq i \leq 2t - 1$, an edge $e = \{u_i, v_{i+1}\}$ in S_1 has label

$$|f(u_i) - f(v_{i+1})| = f(v_{i+1}) - f(u_i) = (36t - 2(i+1) - 18) - (i-1) = 36t - 3i - 19.$$

Similarly, the other edges $e \in S_1$ have label:

$$\begin{aligned}
36t - 3i - 20 & & \text{for } 2t \leq i \leq 6t - 4, \\
36t - 3i - 21 & & \text{for } 6t - 3 \leq i \leq 10t - 7, \\
36t - 3i - 22 & & \text{for } 10t - 6 \leq i \leq 12t - 8, \\
18t - 10 & & \text{for } e = \{u_n, v_1\}.
\end{aligned}$$

Recall that the length of an edge $e \in E(\tilde{C}_{24t-14})$ is $(\bar{f}(e))^* = \min\{\bar{f}(e), |E(\tilde{C}_{24t-14})| + 1 - \bar{f}(e)\}$. Thus, we compute the following edge length sets that compose $(\bar{f}(S_1))^*$:

$$\begin{aligned} \{36t - 20 - (36t - 3i - 19) : 1 \leq i \leq 2t - 1\} &= \{3i - 1 : 1 \leq i \leq 2t - 1\} \\ &= \{\ell \equiv 2 \pmod{3} : 2 \leq \ell \leq 6t - 4\}, \\ \{36t - 20 - (36t - 3i - 20) : 2t \leq i \leq 6t - 4\} &= \{3i : 2t \leq i \leq 6t - 4\} \\ &= \{\ell \equiv 0 \pmod{3} : 6t \leq \ell \leq 18t - 12\}, \\ \{36t - 3i - 21 : 6t - 3 \leq i \leq 10t - 7\} \\ &= \{\ell \equiv 0 \pmod{3} : 6t \leq \ell \leq 18t - 12\}, \\ \{36t - 3i - 22 : 10t - 6 \leq i \leq 12t - 8\} &= \{\ell \equiv 2 \pmod{3} : 2 \leq \ell \leq 6t - 4\}, \\ \{18t - 10\} &= \{18t - 10\}. \end{aligned}$$

Also, an edge $\{u_i, v_i\} \in S_2$ has label

$$\begin{aligned} 30t - 18 & \text{ for } i = 1, \\ 36t - 3i - 17 & \text{ for } 2 \leq i \leq 2t, \\ 36t - 3i - 18 & \text{ for } 2t + 1 \leq i \leq 6t - 3, \\ 36t - 3i - 19 & \text{ for } 6t - 2 \leq i \leq 10t - 6, \\ 36t - 3i - 20 & \text{ for } 10t - 5 \leq i \leq 12t - 7, \end{aligned}$$

yielding the following multisets that compose $(\bar{f}(S_2))^*$:

$$\begin{aligned} {}^2\{36t - 20 - (30t - 18)\} &= {}^2\{6t - 2\}, \\ {}^2\{36t - 20 - (36t - 3i - 17) : 2 \leq i \leq 2t\} &= {}^2\{3i - 3 : 2 \leq i \leq 2t\} \\ &= {}^2\{\ell \equiv 0 \pmod{3} : 3 \leq \ell \leq 6t - 3\}, \\ {}^2\{36t - 20 - (36t - 3i - 18) : 2t + 1 \leq i \leq 6t - 3\} \\ &= {}^2\{3i - 2 : 2t + 1 \leq i \leq 6t - 3\} \\ &= {}^2\{\ell \equiv 1 \pmod{3} : 6t + 1 \leq \ell \leq 18t - 11\}, \\ {}^2\{36t - 3i - 19 : 6t - 2 \leq i \leq 10t - 6\} \\ &= {}^2\{\ell \equiv 2 \pmod{3} : 6t - 1 \leq \ell \leq 18t - 13\}, \\ {}^2\{36t - 3i - 20 : 10t - 5 \leq i \leq 12t - 7\} \\ &= {}^2\{\ell \equiv 1 \pmod{3} : 1 \leq \ell \leq 6t - 5\}. \end{aligned}$$

Since every length $\ell \in [1, 18t - 11]$ occurs on exactly two edges and length $18t - 10$ occurs on only one edge in \tilde{C}_{24t-14} and $f(V(\tilde{C}_{24t-14})) \subseteq [0, 36t - 21]$, f is a 2-fold ρ -labeling. Also since $f(A) \subseteq [0, 12t - 8]$ and $f(B) \subseteq [12t - 7, 36t - 22]$, we have a 2-fold ρ^+ -labeling of \tilde{C}_{24t-14} .

Case 7: $n \equiv 6 \pmod{12}$.

Let $n = 12t - 6$. Then, $|V(\tilde{C}_{2n})| = 24t - 12$ and $|E(\tilde{C}_{2n})| = 36t - 18$. A 2-fold ρ^+ -labeling of \tilde{C}_{12} is given in Figure 3.34. For $t \geq 2$, define a one-to-one function

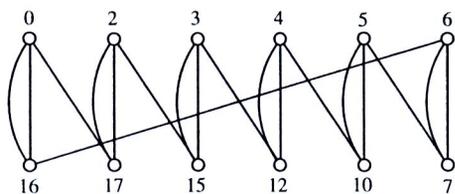


Figure 3.34: A 2-fold ρ^+ -labeling of \tilde{C}_{12} .

$f: V(\tilde{C}_{24t-12}) \rightarrow [0, 36t - 18]$ as follows:

$$\begin{aligned}
 f(u_i) &= i - 1, & u_i \in A_1 &= \{u_i : 1 \leq i \leq 2t - 1\}, \\
 f(u_i) &= i, & u_i \in A_2 &= \{u_i : 2t \leq i \leq 12t - 6\}, \\
 f(v_1) &= 30t - 14, \\
 f(v_i) &= 36t - 2i - 15, & v_i \in B_1 &= \{v_i : 2 \leq i \leq 6t - 3\}, \\
 f(v_i) &= 36t - 2i - 16, & v_i \in B_2 &= \{v_i : 6t - 2 \leq i \leq 10t - 5\}, \\
 f(v_i) &= 36t - 2i - 17, & v_i \in B_3 &= \{v_i : 10t - 4 \leq i \leq 12t - 6\}.
 \end{aligned}$$

Note that $A = A_1 \cup A_2$ and $B = B_1 \cup B_2 \cup B_3 \cup \{v_1\}$. Thus, the domain of f is indeed $V(\tilde{C}_{24t-12})$. Next, we confirm that f is one-to-one. We compute

$$\begin{aligned}
 f(A_1) &= \{0, 1, \dots, 2t - 2\}, \\
 f(A_2) &= \{2t, 2t + 1, \dots, 12t - 6\}, \\
 f(B_1) &= \{36t - 19, 36t - 21, \dots, 24t - 9\}, \\
 f(B_2) &= \{24t - 12, 24t - 14, \dots, 16t - 6\}, \\
 f(B_3) &= \{16t - 9, 16t - 11, \dots, 12t - 5\}.
 \end{aligned}$$

Note that f is piecewise strictly increasing by 1 or strictly decreasing by 2 and that all labels are distinct. Thus, f is one-to-one. Moreover, $f(A) \subseteq [0, 12t - 6]$ and $f(B) \subseteq [12t - 5, 36t - 19]$. Next, we compute the edge lengths. For $1 \leq i \leq 2t - 1$, an edge $e = \{u_i, v_{i+1}\}$ in S_1 has label

$$|f(u_i) - f(v_{i+1})| = f(v_{i+1}) - f(u_i) = (36t - 2(i+1) - 15) - (i-1) = 36t - 3i - 16.$$

Similarly, the other edges $e \in S_1$ have label:

$$\begin{array}{ll} 36t - 3i - 17 & \text{for } 2t \leq i \leq 6t - 4, \\ 36t - 3i - 18 & \text{for } 6t - 3 \leq i \leq 10t - 6, \\ 36t - 3i - 19 & \text{for } 10t - 5 \leq i \leq 12t - 7, \\ 18t - 8 & \text{for } e = \{u_n, v_1\}. \end{array}$$

Recall that the length of an edge $e \in E(\tilde{C}_{24t-12})$ is $(\bar{f}(e))^* = \min\{\bar{f}(e), |E(\tilde{C}_{24t-12})| + 1 - \bar{f}(e)\}$. Thus, we compute the following edge length sets that compose $(\bar{f}(S_1))^*$:

$$\begin{aligned} \{36t - 17 - (36t - 3i - 16) : 1 \leq i \leq 2t - 1\} &= \{3i - 1 : 1 \leq i \leq 2t - 1\} \\ &= \{\ell \equiv 2 \pmod{3} : 2 \leq \ell \leq 6t - 4\}, \\ \{36t - 17 - (36t - 3i - 17) : 2t \leq i \leq 6t - 4\} &= \{3i : 2t \leq i \leq 6t - 4\} \\ &= \{\ell \equiv 0 \pmod{3} : 6t \leq \ell \leq 18t - 12\}, \\ \{36t - 3i - 18 : 6t - 3 \leq i \leq 10t - 6\} &= \{\ell \equiv 0 \pmod{3} : 6t \leq \ell \leq 18t - 9\}, \\ \{36t - 3i - 19 : 10t - 5 \leq i \leq 12t - 7\} &= \{\ell \equiv 2 \pmod{3} : 2 \leq \ell \leq 6t - 4\}, \\ \{36t - 17 - (18t - 8)\} &= \{18t - 9\}. \end{aligned}$$

Also, an edge $\{u_i, v_i\} \in S_2$ has label

$$\begin{array}{ll} 30t - 14 & \text{for } i = 1, \\ 36t - 3i - 14 & \text{for } 2 \leq i \leq 2t - 1, \\ 36t - 3i - 15 & \text{for } 2t \leq i \leq 6t - 3, \\ 36t - 3i - 16 & \text{for } 6t - 2 \leq i \leq 10t - 5, \\ 36t - 3i - 17 & \text{for } 10t - 4 \leq i \leq 12t - 6. \end{array}$$

yielding the following multisets that compose $(\bar{f}(S_2))^*$:

$$\begin{aligned}
& {}^2\{36t - 17 - (30t - 14)\} = {}^2\{6t - 3\}, \\
& {}^2\{36t - 17 - (36t - 3i - 14) : 2 \leq i \leq 2t - 1\} = {}^2\{3i - 3 : 2 \leq i \leq 2t - 1\} \\
& \quad = {}^2\{\ell \equiv 0 \pmod{3} : 3 \leq \ell \leq 6t - 6\}, \\
& {}^2\{36t - 17 - (36t - 3i - 15) : 2t \leq i \leq 6t - 3\} = {}^2\{3i - 2 : 2t \leq i \leq 6t - 3\} \\
& \quad = {}^2\{\ell \equiv 1 \pmod{3} : 6t - 2 \leq \ell \leq 18t - 11\}, \\
& {}^2\{36t - 3i - 16 : 6t - 2 \leq i \leq 10t - 5\} \\
& \quad = {}^2\{\ell \equiv 2 \pmod{3} : 6t - 1 \leq \ell \leq 18t - 10\}, \\
& {}^2\{36t - 3i - 17 : 10t - 4 \leq i \leq 12t - 6\} = {}^2\{\ell \equiv 1 \pmod{3} : 1 \leq \ell \leq 6t - 5\}.
\end{aligned}$$

Since every length $\ell \in [1, 18t - 9]$ occurs on exactly two edges in \tilde{C}_{24t-12} and $f(V(\tilde{C}_{24t-12})) \subseteq [0, 36t - 18]$, f is a 2-fold ρ -labeling. Also since $f(A) \subseteq [0, 12t - 6]$ and $f(B) \subseteq [12t - 5, 36t - 19]$, we have a 2-fold ρ^+ -labeling of \tilde{C}_{24t-12} .

Case 8: $n \equiv 7 \pmod{12}$.

Let $n = 12t - 5$. Then, $|V(\tilde{C}_{2n})| = 24t - 10$ and $|E(\tilde{C}_{2n})| = 36t - 15$. A 2-fold ρ^+ -labeling of \tilde{C}_{14} is given in Figure 3.35. For $t \geq 2$, define a one-to-one function

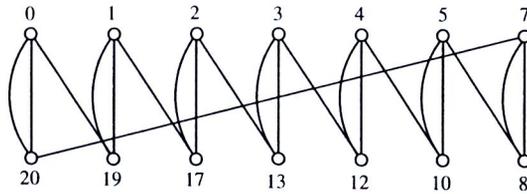


Figure 3.35: A 2-fold ρ^+ -labeling of \tilde{C}_{14} .

$f: V(\tilde{C}_{24t-10}) \rightarrow [0, 36t - 15]$ as follows:

$$\begin{aligned}
f(u_i) &= i - 1, & u_i \in A_1 &= \{u_i: 1 \leq i \leq 10t - 4\}, \\
f(u_i) &= i, & u_i \in A_2 &= \{u_i: 10t - 3 \leq i \leq 12t - 5\}, \\
f(v_1) &= 30t - 10, \\
f(v_i) &= 36t - 2i - 12, & v_i \in B_1 &= \{v_i: 2 \leq i \leq 2t - 1\}, \\
f(v_i) &= 36t - 2i - 13, & v_i \in B_2 &= \{v_i: 2t \leq i \leq 6t - 3\}, \\
f(v_{6t-2}) &= 24t - 11, \\
f(v_i) &= 36t - 2i - 14, & v_i \in B_3 &= \{v_i: 6t - 1 \leq i \leq 12t - 5\}.
\end{aligned}$$

Note that $A = A_1 \cup A_2$ and $B = B_1 \cup B_2 \cup B_3 \cup \{v_1, v_{6t-2}\}$. Thus, the domain of f is indeed $V(\tilde{C}_{24t-10})$. Next, we confirm that f is one-to-one. We compute

$$\begin{aligned}
f(A_1) &= \{0, 1, \dots, 10t - 5\}, \\
f(A_2) &= \{10t - 3, 10t - 2, \dots, 12t - 5\}, \\
f(B_1) &= \{36t - 16, 36t - 18, \dots, 32t - 10\}, \\
f(B_2) &= \{32t - 13, 32t - 15, \dots, 24t - 7\}, \\
f(B_3) &= \{24t - 12, 24t - 14, \dots, 12t - 4\}.
\end{aligned}$$

Note that f is piecewise strictly increasing by 1 or strictly decreasing by 2 and that all labels are distinct. Thus, f is one-to-one. Moreover, $f(A) \subseteq [0, 12t - 5]$ and $f(B) \subseteq [12t - 4, 36t - 16]$.

Next, we compute the edge lengths. For $1 \leq i \leq 2t - 2$, an edge $e = \{u_i, v_{i+1}\}$ in S_1 has label

$$|f(u_i) - f(v_{i+1})| = f(v_{i+1}) - f(u_i) = (36t - 2(i+1) - 12) - (i - 1) = 36t - 3i - 13.$$

Similarly, the other edges $e \in S_1$ have label:

$$\begin{aligned}
36t - 3i - 14 & & \text{for } 2t - 1 \leq i \leq 6t - 4, \\
18t - 7 & & \text{for } i = 6t - 3, \\
36t - 3i - 15 & & \text{for } 6t - 2 \leq i \leq 10t - 4, \\
36t - 3i - 16 & & \text{for } 10t - 3 \leq i \leq 12t - 6, \\
18t - 5 & & \text{for } e = \{u_n, v_1\}.
\end{aligned}$$

Recall that the length of an edge $e \in E(\tilde{C}_{24t-10})$ is $(\bar{f}(e))^* = \min\{\bar{f}(e), |E(\tilde{C}_{24t-10})| + 1 - \bar{f}(e)\}$. Thus, we compute the following edge length sets that compose $(\bar{f}(S_1))^*$:

$$\begin{aligned} \{36t - 14 - (36t - 3i - 13) : 1 \leq i \leq 2t - 2\} &= \{3i - 1 : 1 \leq i \leq 2t - 2\} \\ &= \{\ell \equiv 2 \pmod{3} : 2 \leq \ell \leq 6t - 5\}, \\ \{36t - 14 - (36t - 3i - 14) : 2t - 1 \leq i \leq 6t - 4\} &= \{3i : 2t - 1 \leq i \leq 6t - 4\} \\ &= \{\ell \equiv 0 \pmod{3} : 6t - 3 \leq \ell \leq 18t - 12\}, \\ \{18t - 7\} &= \{18t - 7\}, \\ \{36t - 3i - 15 : 6t - 2 \leq i \leq 10t - 4\} &= \{\ell \equiv 0 \pmod{3} : 6t - 3 \leq \ell \leq 18t - 9\}, \\ \{36t - 3i - 16 : 10t - 3 \leq i \leq 12t - 6\} &= \{\ell \equiv 2 \pmod{3} : 2 \leq \ell \leq 6t - 7\}, \\ \{36t - 14 - (18t - 5)\} &= \{18t - 9\}. \end{aligned}$$

Also, an edge $\{u_i, v_i\} \in S_2$ has label

$$\begin{array}{ll} 30t - 10 & \text{for } i = 1, \\ 36t - 3i - 11 & \text{for } 2 \leq i \leq 2t - 1, \\ 36t - 3i - 12 & \text{for } 2t \leq i \leq 6t - 3, \\ 18t - 8 & \text{for } i = 6t - 2, \\ 36t - 3i - 13 & \text{for } 6t - 1 \leq i \leq 10t - 4, \\ 36t - 3i - 14 & \text{for } 10t - 3 \leq i \leq 12t - 5. \end{array}$$

yielding the following multisets that compose $(\bar{f}(S_2))^*$:

$$\begin{aligned}
& {}^2\{36t - 14 - (30t - 10)\} = {}^2\{6t - 4\}, \\
& {}^2\{36t - 14 - (36t - 3i - 11) : 2 \leq i \leq 2t - 1\} = {}^2\{3i - 3 : 2 \leq i \leq 2t - 1\} \\
& \quad = {}^2\{\ell \equiv 0 \pmod{3} : 3 \leq \ell \leq 6t - 6\}, \\
& {}^2\{36t - 14 - (36t - 3i - 12) : 2t \leq i \leq 6t - 3\} = {}^2\{3i - 2 : 2t \leq i \leq 6t - 3\} \\
& \quad = {}^2\{\ell \equiv 1 \pmod{3} : 6t - 2 \leq \ell \leq 18t - 11\}, \\
& {}^2\{18t - 8\} = {}^2\{18t - 8\}, \\
& {}^2\{36t - 3i - 13 : 6t - 1 \leq i \leq 10t - 4\} \\
& \quad = {}^2\{\ell \equiv 2 \pmod{3} : 6t - 1 \leq \ell \leq 18t - 10\}, \\
& {}^2\{36t - 3i - 14 : 10t - 3 \leq i \leq 12t - 5\} \\
& \quad = {}^2\{\ell \equiv 1 \pmod{3} : 1 \leq \ell \leq 6t - 5\}.
\end{aligned}$$

Since every length $\ell \in [1, 18t - 8]$ occurs on exactly two edges and length $18t - 7$ occurs on only one edge in \tilde{C}_{24t-10} and $f(V(\tilde{C}_{24t-10})) \subseteq [0, 36t - 15]$, f is a 2-fold ρ -labeling. Also since $f(A) \subseteq [0, 12t - 5]$ and $f(B) \subseteq [12t - 4, 36t - 16]$, we have a 2-fold ρ^+ -labeling of \tilde{C}_{24t-10} .

Case 9: $n \equiv 8 \pmod{12}$.

Let $n = 12t - 4$. Then, $|V(\tilde{C}_{2n})| = 24t - 8$ and $|E(\tilde{C}_{2n})| = 36t - 12$. A 2-fold ρ^+ -labeling of \tilde{C}_{16} is given in Figure 3.36.

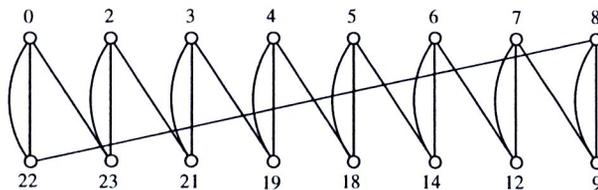


Figure 3.36: A 2-fold ρ^+ -labeling of \tilde{C}_{16} .

Recall that the length of an edge $e \in E(\tilde{C}_{24t-8})$ is $(\bar{f}(e))^* = \min\{\bar{f}(e), |E(\tilde{C}_{24t-8})| + 1 - \bar{f}(e)\}$. Thus, we compute the following edge length sets that compose $(\bar{f}(S_1))^*$:

$$\begin{aligned} \{36t - 11 - (36t - 3i - 10): 1 \leq i \leq 2t - 1\} &= \{3i - 1: 1 \leq i \leq 2t - 1\} \\ &= \{\ell \equiv 2 \pmod{3}: 2 \leq \ell \leq 6t - 4\}, \\ \{36t - 11 - (36t - 3i - 11): 2t \leq i \leq 6t - 3\} &= \{3i: 2t \leq i \leq 6t - 3\} \\ &= \{\ell \equiv 0 \pmod{3}: 6t \leq \ell \leq 18t - 9\}, \\ \{36t - 11 - (18t - 4)\} &= \{18t - 7\}, \\ \{36t - 3i - 12: 6t - 1 \leq i \leq 10t - 4\} &= \{\ell \equiv 0 \pmod{3}: 6t \leq \ell \leq 18t - 9\}, \\ \{36t - 3i - 13: 10t - 3 \leq i \leq 12t - 5\} &= \{\ell \equiv 2 \pmod{3}: 2 \leq \ell \leq 6t - 4\}, \\ \{36t - 11 - (18t - 4)\} &= \{18t - 7\}. \end{aligned}$$

Also, an edge $\{u_i, v_i\} \in S_2$ has label

$$\begin{aligned} 30t - 8 & \quad \text{for } i = 1, \\ 36t - 3i - 8 & \quad \text{for } 2 \leq i \leq 2t - 1, \\ 36t - 3i - 9 & \quad \text{for } 2t \leq i \leq 6t - 2, \\ 18t - 5 & \quad \text{for } i = 6t - 1, \\ 36t - 3i - 10 & \quad \text{for } 6t \leq i \leq 10t - 3, \\ 36t - 3i - 11 & \quad \text{for } 10t - 2 \leq i \leq 12t - 4. \end{aligned}$$

yielding the following multisets that compose $(\bar{f}(S_2))^*$:

$$\begin{aligned} {}^2\{36t - 11 - (30t - 8)\} &= {}^2\{6t - 3\}, \\ {}^2\{36t - 11 - (36t - 3i - 8): 2 \leq i \leq 2t - 1\} &= {}^2\{3i - 3: 2 \leq i \leq 2t - 1\} \\ &= {}^2\{\ell \equiv 0 \pmod{3}: 3 \leq \ell \leq 6t - 6\}, \\ {}^2\{36t - 11 - (36t - 3i - 9): 2t \leq i \leq 6t - 2\} &= {}^2\{3i - 2: 2t \leq i \leq 6t - 3\} \\ &= {}^2\{\ell \equiv 1 \pmod{3}: 6t - 2 \leq \ell \leq 18t - 11\}, \\ {}^2\{36t - 11 - (18t - 5)\} &= {}^2\{18t - 6\}, \\ {}^2\{36t - 3i - 10: 6t \leq i \leq 10t - 3\} \\ &= {}^2\{\ell \equiv 2 \pmod{3}: 6t - 1 \leq \ell \leq 18t - 10\}, \\ {}^2\{36t - 3i - 11: 10t - 2 \leq i \leq 12t - 4\} &= {}^2\{\ell \equiv 1 \pmod{3}: 1 \leq \ell \leq 6t - 5\}. \end{aligned}$$

For $t \geq 2$, define a one-to-one function $f: V(\tilde{C}_{24t-8}) \rightarrow [0, 36t - 12]$ as follows:

$$f(u_i) = i - 1, \quad u_i \in A_1 = \{u_i : 1 \leq i \leq 2t - 1\},$$

$$f(u_i) = i, \quad u_i \in A_2 = \{u_i : 2t \leq i \leq 12t - 4\},$$

$$f(v_1) = 30t - 8,$$

$$f(v_i) = 36t - 2i - 9, \quad v_i \in B_1 = \{v_i : 2 \leq i \leq 6t - 2\},$$

$$f(v_{6t-1}) = 24t - 6,$$

$$f(v_i) = 36t - 2i - 10, \quad v_i \in B_2 = \{v_i : 6t \leq i \leq 10t - 3\},$$

$$f(v_i) = 36t - 2i - 11, \quad v_i \in B_3 = \{v_i : 10t - 2 \leq i \leq 12t - 4\}.$$

Note that $A = A_1 \cup A_2$ and $B = B_1 \cup B_2 \cup B_3 \cup \{v_1, v_{6t-1}\}$. Thus, the domain of f is indeed $V(\tilde{C}_{24t-8})$. Next, we confirm that f is one-to-one. We compute

$$f(A_1) = \{0, 1, \dots, 2t - 2\},$$

$$f(A_2) = \{2t, 2t + 1, \dots, 12t - 4\},$$

$$f(B_1) = \{36t - 13, 36t - 15, \dots, 24t - 5\},$$

$$f(B_2) = \{24t - 10, 24t - 12, \dots, 16t - 4\},$$

$$f(B_3) = \{16t - 7, 16t - 9, \dots, 12t - 3\}.$$

Note that f is piecewise strictly increasing by 1 or strictly decreasing by 2 and that all labels are distinct. Thus, f is one-to-one. Moreover, $f(A) \subseteq [0, 12t - 4]$ and $f(B) \subseteq [12t - 3, 36t - 13]$.

Next, we compute the edge lengths. For $1 \leq i \leq 2t - 1$, an edge $e = \{u_i, v_{i+1}\}$ in S_1 has label

$$|f(u_i) - f(v_{i+1})| = f(v_{i+1}) - f(u_i) = (36t - 2(i+1) - 9) - (i - 1) = 36t - 3i - 10.$$

Similarly, the other edges $e \in S_1$ have label:

$$36t - 3i - 11 \quad \text{for } 2t \leq i \leq 6t - 3,$$

$$18t - 4 \quad \text{for } i = 6t - 2,$$

$$36t - 3i - 12 \quad \text{for } 6t - 1 \leq i \leq 10t - 4,$$

$$36t - 3i - 13 \quad \text{for } 10t - 3 \leq i \leq 12t - 5,$$

$$18t - 4 \quad \text{for } e = \{u_n, v_1\}.$$

Since every length $\ell \in [1, 18t - 6]$ occurs on exactly two edges in \tilde{C}_{24t-8} and $f(V(\tilde{C}_{24t-8})) \subseteq [0, 36t - 12]$, f is a 2-fold ρ -labeling. Also since $f(A) \subseteq [0, 12t - 4]$ and $f(B) \subseteq [12t - 3, 36t - 13]$, we have a 2-fold ρ^+ -labeling of \tilde{C}_{24t-8} .

Case 10: $n \equiv 9 \pmod{12}$.

Let $n = 12t - 3$. Then, $|V(\tilde{C}_{2n})| = 24t - 6$ and $|E(\tilde{C}_{2n})| = 36t - 9$. A 2-fold ρ^+ -labeling of \tilde{C}_{18} is given in Figure 3.37. For $t \geq 2$, define a one-to-one function

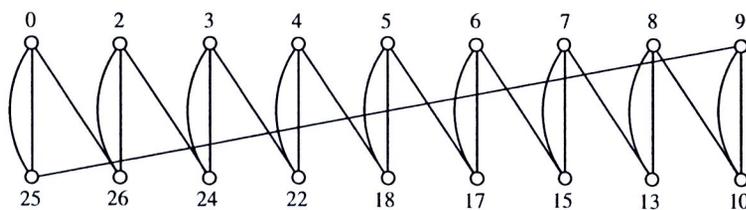


Figure 3.37: A 2-fold ρ^+ -labeling of \tilde{C}_{18} .

$f: V(\tilde{C}_{24t-6}) \rightarrow [0, 36t - 9]$ as follows:

$$\begin{aligned} f(u_i) &= i - 1, & u_i \in A_1 &= \{u_i : 1 \leq i \leq 2t - 1\}, \\ f(u_i) &= i, & u_i \in A_2 &= \{u_i : 2t \leq i \leq 12t - 3\}, \\ f(v_1) &= 30t - 5, \\ f(v_i) &= 36t - 2i - 6, & v_i \in B_1 &= \{v_i : 2 \leq i \leq 6t - 2\}, \\ f(v_{6t-1}) &= 24t - 6, \\ f(v_i) &= 36t - 2i - 7, & v_i \in B_2 &= \{v_i : 6t \leq i \leq 10t - 2\}, \\ f(v_i) &= 36t - 2i - 8, & v_i \in B_3 &= \{v_i : 10t - 1 \leq i \leq 12t - 3\}. \end{aligned}$$

Note that $A = A_1 \cup A_2$ and $B = B_1 \cup B_2 \cup B_3 \cup \{v_1, v_{6t-1}\}$. Thus, the domain of f is indeed $V(\tilde{C}_{24t-6})$. Next, we confirm that f is one-to-one. We compute

$$\begin{aligned} f(A_1) &= \{0, 1, \dots, 2t - 2\}, \\ f(A_2) &= \{2t, 2t + 1, \dots, 12t - 3\}, \\ f(B_1) &= \{36t - 10, 36t - 12, \dots, 24t - 2\}, \\ f(B_2) &= \{24t - 7, 24t - 9, \dots, 16t - 3\}, \\ f(B_3) &= \{16t - 6, 16t - 8, \dots, 12t - 2\}. \end{aligned}$$

Note that f is piecewise strictly increasing by 1 or strictly decreasing by 2 and that all labels are distinct. Thus, f is one-to-one. Moreover, $f(A) \subseteq [0, 12t - 3]$ and $f(B) \subseteq [12t - 2, 36t - 10]$.

Next, we compute the edge lengths. For $1 \leq i \leq 2t - 1$, an edge $e = \{u_i, v_{i+1}\}$ in S_1 has label

$$|f(u_i) - f(v_{i+1})| = f(v_{i+1}) - f(u_i) = (36t - 2(i+1) - 6) - (i - 1) = 36t - 3i - 7.$$

Similarly, the other edges $e \in S_1$ have label:

$$\begin{aligned} 36t - 3i - 8 & \quad \text{for } 2t \leq i \leq 6t - 3, \\ 18t - 4 & \quad \text{for } i = 6t - 2, \\ 36t - 3i - 9 & \quad \text{for } 6t - 1 \leq i \leq 10t - 3, \\ 36t - 3i - 10 & \quad \text{for } 10t - 2 \leq i \leq 12t - 4, \\ 18t - 2 & \quad \text{for } e = \{u_n, v_1\}. \end{aligned}$$

Recall that the length of an edge $e \in E(\tilde{C}_{24t-6})$ is $(\bar{f}(e))^* = \min\{\bar{f}(e), |E(\tilde{C}_{24t-6})| + 1 - \bar{f}(e)\}$. Thus, we compute the following edge length sets that compose $(\bar{f}(S_1))^*$:

$$\begin{aligned} \{36t - 8 - (36t - 3i - 7) : 1 \leq i \leq 2t - 1\} &= \{3i - 1 : 1 \leq i \leq 2t - 1\} \\ &= \{\ell \equiv 2 \pmod{3} : 2 \leq \ell \leq 6t - 4\}, \\ \{36t - 8 - (36t - 3i - 8) : 2t \leq i \leq 6t - 3\} &= \{3i : 2t \leq i \leq 6t - 3\} \\ &= \{\ell \equiv 0 \pmod{3} : 6t \leq \ell \leq 18t - 9\}, \\ \{18t - 4\} &= \{18t - 4\}, \\ \{36t - 3i - 12 : 6t - 1 \leq i \leq 10t - 3\} &= \{\ell \equiv 0 \pmod{3} : 6t \leq \ell \leq 18t - 6\}, \\ \{36t - 3i - 13 : 10t - 2 \leq i \leq 12t - 4\} &= \{\ell \equiv 2 \pmod{3} : 2 \leq \ell \leq 6t - 4\}, \\ \{36t - 8 - (18t - 2)\} &= \{18t - 6\}. \end{aligned}$$

Also, an edge $\{u_i, v_i\} \in S_2$ has label

$$\begin{array}{ll}
30t - 5 & \text{for } i = 1, \\
36t - 3i - 5 & \text{for } 2 \leq i \leq 2t - 1, \\
36t - 3i - 6 & \text{for } 2t \leq i \leq 6t - 2, \\
18t - 5 & \text{for } i = 6t - 1, \\
36t - 3i - 7 & \text{for } 6t \leq i \leq 10t - 2, \\
36t - 3i - 8 & \text{for } 10t - 1 \leq i \leq 12t - 3.
\end{array}$$

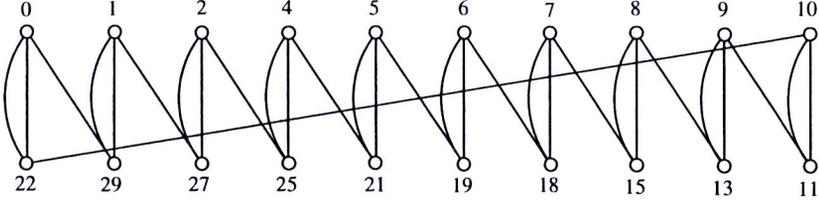
yielding the following multisets that compose $(\bar{f}(S_2))^*$:

$$\begin{aligned}
& {}^2\{36t - 8 - (30t - 5)\} = {}^2\{6t - 3\}, \\
& {}^2\{36t - 8 - (36t - 3i - 5) : 2 \leq i \leq 2t - 1\} = {}^2\{3i - 3 : 2 \leq i \leq 2t - 1\} \\
& \quad = {}^2\{\ell \equiv 0 \pmod{3} : 3 \leq \ell \leq 6t - 6\}, \\
& {}^2\{36t - 8 - (36t - 3i - 6) : 2t \leq i \leq 6t - 2\} = {}^2\{3i - 2 : 2t \leq i \leq 6t - 2\} \\
& \quad = {}^2\{\ell \equiv 1 \pmod{3} : 6t - 2 \leq \ell \leq 18t - 8\}, \\
& {}^2\{18t - 5\} = {}^2\{18t - 5\}, \\
& {}^2\{36t - 3i - 7 : 6t \leq i \leq 10t - 2\} \\
& \quad = {}^2\{\ell \equiv 2 \pmod{3} : 6t - 1 \leq \ell \leq 18t - 7\}, \\
& {}^2\{36t - 3i - 8 : 10t - 1 \leq i \leq 12t - 3\} = {}^2\{\ell \equiv 1 \pmod{3} : 1 \leq \ell \leq 6t - 5\}.
\end{aligned}$$

Since every length $\ell \in [1, 18t - 3]$ occurs on exactly two edges and length $18t - 4$ occurs on only one edge in \tilde{C}_{24t-6} and $f(V(\tilde{C}_{24t-6})) \subseteq [0, 36t - 9]$, f is a 2-fold ρ -labeling. Also since $f(A) \subseteq [0, 12t - 3]$ and $f(B) \subseteq [12t - 2, 36t - 10]$, we have a 2-fold ρ^+ -labeling of \tilde{C}_{24t-6} .

Case 11: $n \equiv 10 \pmod{12}$.

Let $n = 12t - 2$. Then, $|V(\tilde{C}_{2n})| = 24t - 4$ and $|E(\tilde{C}_{2n})| = 36t - 6$. A 2-fold ρ^+ -labeling of \tilde{C}_{20} is given in Figure 3.38. For $t \geq 2$, define a one-to-one function

Figure 3.38: A 2-fold ρ^+ -labeling of \tilde{C}_{20} .

$f: V(\tilde{C}_{24t-4}) \rightarrow [0, 36t - 6]$ as follows:

$$\begin{aligned}
 f(u_i) &= i - 1, & u_i \in A_1 &= \{u_i: 1 \leq i \leq 2t + 1\}, \\
 f(u_i) &= i, & u_i \in A_2 &= \{u_i: 2t + 2 \leq i \leq 12t - 2\}, \\
 f(v_1) &= 30t - 8, \\
 f(v_i) &= 36t - 2i - 3, & v_i \in B_1 &= \{v_i: 2 \leq i \leq 6t - 2\}, \\
 f(v_{6t-1}) &= 24t - 3, \\
 f(v_{6t}) &= 24t - 5, \\
 f(v_i) &= 36t - 2i - 4, & v_i \in B_2 &= \{v_i: 6t + 1 \leq i \leq 10t - 3\}, \\
 f(v_i) &= 36t - 2i - 5, & v_i \in B_3 &= \{v_i: 10t - 2 \leq i \leq 12t - 2\}.
 \end{aligned}$$

Note that $A = A_1 \cup A_2$ and $B = B_1 \cup B_2 \cup B_3 \cup \{v_1, v_{6t-1}, v_{6t}\}$. Thus, the domain of f is indeed $V(\tilde{C}_{24t-4})$. Next, we confirm that f is one-to-one. We compute

$$\begin{aligned}
 f(A_1) &= \{0, 1, \dots, 2t\}, \\
 f(A_2) &= \{2t + 2, 2t + 3, \dots, 12t - 2\}, \\
 f(B_1) &= \{36t - 7, 36t - 9, \dots, 24t + 1\}, \\
 f(B_2) &= \{24t - 6, 24t - 8, \dots, 16t + 3\}, \\
 f(B_3) &= \{16t - 1, 16t - 3, \dots, 12t - 1\}.
 \end{aligned}$$

Note that f is piecewise strictly increasing by 1 or strictly decreasing by 2 and that all labels are distinct. Thus, f is one-to-one. Moreover, $f(A) \subseteq [0, 12t - 2]$ and $f(B) \subseteq [12t - 1, 36t - 7]$.

Next, we compute the edge lengths. For $1 \leq i \leq 2t + 1$, an edge $e = \{u_i, v_{i+1}\}$ in S_1 has label

$$|f(u_i) - f(v_{i+1})| = f(v_{i+1}) - f(u_i) = (36t - 2(i+1) - 3) - (i - 1) = 36t - 3i - 4.$$

Similarly, the other edges $e \in S_1$ have label:

$$\begin{array}{ll}
36t - 3i - 5 & \text{for } 2t + 2 \leq i \leq 6t - 3, \\
18t - 1 & \text{for } i = 6t - 2, \\
18t - 4 & \text{for } i = 6t - 1, \\
36t - 3i - 6 & \text{for } 6t \leq i \leq 10t - 4, \\
36t - 3i - 7 & \text{for } 10t - 3 \leq i \leq 12t - 3, \\
18t - 6 & \text{for } e = \{u_n, v_1\}.
\end{array}$$

Recall that the length of an edge $e \in E(\tilde{C}_{24t-4})$ is $(\bar{f}(e))^* = \min\{\bar{f}(e), |E(\tilde{C}_{24t-4})| + 1 - \bar{f}(e)\}$. Thus, we compute the following edge length sets that compose $(\bar{f}(S_1))^*$:

$$\begin{aligned}
& \{36t - 5 - (36t - 3i - 4) : 1 \leq i \leq 2t + 1\} = \{3i - 1 : 1 \leq i \leq 2t + 1\} \\
& \quad = \{\ell \equiv 2 \pmod{3} : 2 \leq \ell \leq 6t + 2\}, \\
& \{36t - 5 - (36t - 3i - 5) : 2t + 2 \leq i \leq 6t - 3\} = \{3i : 2t + 2 \leq i \leq 6t - 3\} \\
& \quad = \{\ell \equiv 0 \pmod{3} : 6t + 6 \leq \ell \leq 18t - 9\}, \\
& \{36t - 5 - (18t - 1)\} = \{18t - 4\}, \\
& \{18t - 4\} = \{18t - 4\}, \\
& \{36t - 3i - 6 : 6t \leq i \leq 10t - 4\} = \{\ell \equiv 0 \pmod{3} : 6t + 6 \leq \ell \leq 18t - 6\}, \\
& \{36t - 3i - 7 : 10t - 3 \leq i \leq 12t - 3\} = \{\ell \equiv 2 \pmod{3} : 2 \leq \ell \leq 6t + 2\}, \\
& \{18t - 6\} = \{18t - 6\}.
\end{aligned}$$

Also, an edge $\{u_i, v_i\} \in S_2$ has label

$$\begin{array}{ll}
30t - 8 & \text{for } i = 1, \\
36t - 3i - 2 & \text{for } 2 \leq i \leq 2t + 1, \\
36t - 3i - 3 & \text{for } 2t + 2 \leq i \leq 6t - 2, \\
18t - 2 & \text{for } i = 6t - 1, \\
18t - 5 & \text{for } i = 6t, \\
36t - 3i - 4 & \text{for } 6t + 1 \leq i \leq 10t - 3, \\
36t - 3i - 5 & \text{for } 10t - 2 \leq i \leq 12t - 2.
\end{array}$$

yielding the following multisets that compose $(\bar{f}(S_2))^*$:

$$\begin{aligned}
& {}^2\{36t - 5 - (30t - 8)\} = {}^2\{6t - 3\}, \\
& {}^2\{36t - 5 - (36t - 3i - 2): 2 \leq i \leq 2t + 1\} = {}^2\{3i - 3: 2 \leq i \leq 2t + 1\} \\
& \quad = {}^2\{\ell \equiv 0 \pmod{3}: 3 \leq \ell \leq 6t\}, \\
& {}^2\{36t - 5 - (36t - 3i - 3): 2t + 2 \leq i \leq 6t - 2\} \\
& \quad = {}^2\{3i - 2: 2t + 2 \leq i \leq 6t - 2\} \\
& \quad = {}^2\{\ell \equiv 1 \pmod{3}: 6t + 4 \leq \ell \leq 18t - 8\}, \\
& {}^2\{36t - 5 - (18t - 2)\} = {}^2\{18t - 3\}, \\
& {}^2\{18t - 5\} = {}^2\{18t - 5\}, \\
& {}^2\{36t - 3i - 4: 6t + 1 \leq i \leq 10t - 3\} \\
& \quad = {}^2\{\ell \equiv 2 \pmod{3}: 6t + 5 \leq \ell \leq 18t - 7\}, \\
& {}^2\{36t - 3i - 5: 10t - 2 \leq i \leq 12t - 2\} = {}^2\{\ell \equiv 1 \pmod{3}: 1 \leq \ell \leq 6t + 1\}.
\end{aligned}$$

Since every length $\ell \in [1, 18t - 3]$ occurs on exactly two edges in \tilde{C}_{24t-4} and $f(V(\tilde{C}_{24t-4})) \subseteq [0, 36t - 6]$, f is a 2-fold ρ -labeling. Also since $f(A) \subseteq [0, 12t - 2]$ and $f(B) \subseteq [12t - 1, 36t - 7]$, we have a 2-fold ρ^+ -labeling of \tilde{C}_{24t-4} .

Case 12: $n \equiv 11 \pmod{12}$.

Let $n = 12t - 1$. Then, $|V(\tilde{C}_{2n})| = 24t - 2$ and $|E(\tilde{C}_{2n})| = 36t - 3$. Define a one-to-one function $f: V(\tilde{C}_{24t-2}) \rightarrow [0, 36t - 3]$ as follows:

$$\begin{aligned}
f(u_i) &= i - 1, & u_i \in A_1 &= \{u_i: 1 \leq i \leq 10t - 2\}, \\
f(u_i) &= i, & u_i \in A_2 &= \{u_i: 10t - 1 \leq i \leq 12t - 1\}, \\
f(v_1) &= 30t - 4, \\
f(v_i) &= 36t - 2i, & v_i \in B_1 &= \{v_i: 2 \leq i \leq 2t + 1\}, \\
f(v_i) &= 36t - 2i - 1, & v_i \in B_2 &= \{v_i: 2t + 2 \leq i \leq 6t - 1\}, \\
f(v_{6t}) &= 24t - 3, \\
f(v_i) &= 36t - 2i - 2, & v_i \in B_3 &= \{v_i: 6t + 1 \leq i \leq 12t - 1\}.
\end{aligned}$$

Note that $A = A_1 \cup A_2$ and $B = B_1 \cup B_2 \cup B_3 \cup \{v_1, v_{6t}\}$. Thus, the domain of f is indeed $V(\tilde{C}_{24t-2})$. Next, we confirm that f is one-to-one. We compute

$$f(A_1) = \{0, 1, \dots, 10t - 3\},$$

$$f(A_2) = \{10t - 1, 10t, \dots, 12t - 1\},$$

$$f(B_1) = \{36t - 4, 36t - 6, \dots, 32t - 2\},$$

$$f(B_2) = \{32t - 5, 32t - 7, \dots, 24t + 1\},$$

$$f(B_3) = \{24t - 4, 24t - 6, \dots, 12t\}.$$

Note that f is piecewise strictly increasing by 1 or strictly decreasing by 2 and that all labels are distinct. Thus, f is one-to-one. Moreover, $f(A) \subseteq [0, 12t - 1]$ and $f(B) \subseteq [12t, 36t - 4]$.

Next, we compute the edge lengths. For $1 \leq i \leq 2t$, an edge $e = \{u_i, v_{i+1}\}$ in S_1 has label

$$|f(u_i) - f(v_{i+1})| = f(v_{i+1}) - f(u_i) = (36t - 2(i + 1)) - (i - 1) = 36t - 3i - 1.$$

Similarly, the other edges $e \in S_1$ have label:

$$36t - 3i - 2 \quad \text{for } 2t + 1 \leq i \leq 6t - 2,$$

$$18t - 1 \quad \text{for } i = 6t - 1,$$

$$36t - 3i - 3 \quad \text{for } 6t \leq i \leq 10t - 2,$$

$$36t - 3i - 4 \quad \text{for } 10t - 1 \leq i \leq 12t - 2,$$

$$18t - 3 \quad \text{for } e = \{u_n, v_1\}.$$

Recall that the length of an edge $e \in E(\tilde{C}_{24t-2})$ is $(\bar{f}(e))^* = \min\{\bar{f}(e), |E(\tilde{C}_{24t-2})| + 1 - \bar{f}(e)\}$. Thus, we compute the following edge length sets that compose $(\bar{f}(S_1))^*$:

$$\begin{aligned} \{36t - 2 - (36t - 3i - 1) : 1 \leq i \leq 2t\} &= \{3i - 1 : 1 \leq i \leq 2t\} \\ &= \{\ell \equiv 2 \pmod{3} : 2 \leq \ell \leq 6t - 1\}, \\ \{36t - 2 - (36t - 3i - 2) : 2t + 1 \leq i \leq 6t - 2\} &= \{3i : 2t + 1 \leq i \leq 6t - 2\} \\ &= \{\ell \equiv 0 \pmod{3} : 6t + 3 \leq \ell \leq 18t - 6\}, \\ \{18t - 1\} &= \{18t - 1\}, \\ \{36t - 3i - 3 : 6t \leq i \leq 10t - 2\} &= \{\ell \equiv 0 \pmod{3} : 6t - 3 \leq \ell \leq 18t - 3\}, \\ \{36t - 3i - 4 : 10t - 1 \leq i \leq 12t - 2\} &= \{\ell \equiv 2 \pmod{3} : 2 \leq \ell \leq 6t - 1\}, \\ \{18t - 3\} &= \{18t - 3\}. \end{aligned}$$

Also, an edge $\{u_i, v_i\} \in S_2$ has label

$$\begin{array}{ll} 30t - 4 & \text{for } i = 1, \\ 36t - 3i + 1 & \text{for } 2 \leq i \leq 2t + 1, \\ 36t - 3i & \text{for } 2t + 2 \leq i \leq 6t - 1, \\ 18t - 2 & \text{for } i = 6t, \\ 36t - 3i - 1 & \text{for } 6t + 1 \leq i \leq 10t - 2, \\ 36t - 3i - 2 & \text{for } 10t - 1 \leq i \leq 12t - 1. \end{array}$$

yielding the following multisets that compose $(\bar{f}(S_2))^*$:

$$\begin{aligned} {}^2\{36t - 2 - (30t - 4)\} &= {}^2\{6t + 2\}, \\ {}^2\{36t - 2 - (36t - 3i + 1) : 2 \leq i \leq 2t + 1\} &= {}^2\{3i - 3 : 2 \leq i \leq 2t + 1\} \\ &= {}^2\{\ell \equiv 0 \pmod{3} : 3 \leq \ell \leq 6t\}, \\ {}^2\{36t - 2 - (36t - 3i) : 2t + 2 \leq i \leq 6t - 1\} &= {}^2\{3i - 2 : 2t + 2 \leq i \leq 6t - 1\} \\ &= {}^2\{\ell \equiv 1 \pmod{3} : 6t + 4 \leq \ell \leq 18t - 5\}, \\ {}^2\{18t - 2\} &= {}^2\{18t - 2\}, \\ {}^2\{36t - 3i - 1 : 6t + 1 \leq i \leq 10t - 2\} &= {}^2\{\ell \equiv 2 \pmod{3} : 6t + 5 \leq \ell \leq 18t - 4\}, \\ {}^2\{36t - 3i - 2 : 10t - 1 \leq i \leq 12t - 1\} &= {}^2\{\ell \equiv 1 \pmod{3} : 1 \leq \ell \leq 6t + 1\}. \end{aligned}$$

Since every length $\ell \in [1, 18t - 2]$ occurs on exactly two edges and length $18t - 1$ occurs on only one edge in \tilde{C}_{24t-2} and $f(V(\tilde{C}_{24t-2})) \subseteq [0, 36t - 3]$, f is a 2-fold ρ -labeling. Also since $f(A) \subseteq [0, 12t - 1]$ and $f(B) \subseteq [12t, 36t - 4]$, we have a 2-fold ρ^+ -labeling of \tilde{C}_{24t-2} . ■

Since \tilde{C}_{2n} admits a 2-fold ρ^+ -labeling for every integer $n \geq 2$, we have the following.

Corollary 3.5.1 *Let $n \geq 2$ be an integer. There exists a cyclic $({}^2K_{3nx+1}, \tilde{C}_{2n})$ -design for every positive integer x .*