Chapter 2

Review of the Literature

In this chapter, we review the relevant works on graph labelings that lead to cyclic graph designs.

2.1 Rosa's Original Labelings

Recall that for a graph G, a function $h\colon V(G) \to \mathbb{N}$ is called a *labeling* of G. In [41], Rosa introduced a hierarchy of graph labelings (he called them graph *valuations*) seemingly with the purpose of using them to obtain cyclic graph decompositions. Rosa's original labelings are: ρ, σ, β and α.

2.1.1 ρ-labelings

Let G be a graph with n edges. A *ρ-labeling* of G is a one-to-one function $f\colon V(G) \to \{0, 1, \ldots, 2n\}$ such that $\{\min\{|f(u) - f(v)|, 2n + 1 - |f(u) - f(v)|\} : \{u, v\} \in E(G)\} = \{1, 2, \ldots, n\}$. Thus a ρ-labeling of G is an embedding of G in K_{2n+1} with $V(K_{2n+1}) = \{0, 1, \ldots, 2n\}$ so that there is exactly one edge of G of length i for $i = 1, 2, \ldots, n$. Rosa [41] proved the following.

Theorem 2.1.1 *Let G be a graph with n edges. There exists a purely cyclic G-decomposition of K_{2n+1} if and only if G has a ρ-labeling.*

It is easy to see how Theorem 2.1.1 works. If $\ell \in [1, n]$, and $e = \{a, b\}$ is any edge of length ℓ in K_{2n+1}, then each of the $2n + 1$ edges of length ℓ in K_{2n+1} occurs exactly once in the set $\Delta_e = \{\{a + i, b + i\}\colon 0 \le i \le 2n\}$, where addition is done modulo $2n + 1$. Note that Δ_e consists of e and the $2n$ successive clickings of e. Since a ρ-labeling of G induces one edge of each length in K_{2n+1}, the set $\Delta_G = \{G + i\colon 0 \le i \le 2n\}$ consisting of G and the $2n$ successive clickings of

G is a cyclic G-decomposition of K_{2n+1}. Similarly, any G-block in a cyclic G-decomposition of K_{2n+1}, where $V(K_{2n+1}) = \mathbb{Z}_{2n+1}$, must contain exactly one edge of each length $\ell \in [1, n]$ and is thus a ρ-labeling of G.

Theorem 2.1.1 does not necessarily extend to G-decompositions of K_{2nx+1}. Also, if G is bipartite, then a ρ-labeling of G does not necessarily yield a G-decomposition of $K_{n,n}$.

Figure 2.9 shows a ρ-labeling of D_3, the 3-prism. To obtain a cyclic (K_{19}, D_3)-design, we take $\Delta_{D_3} = \{D_3 + i\colon 0 \le i \le 18\}$. We note that according to [3], all but 18 of the 12,345 non-empty graphs of order at most 8 admit ρ-labelings. El-Zanati and Vanden Eynden have conjectured that all bipartite graphs and all 2-regular graphs admit ρ-labelings (see [23]).

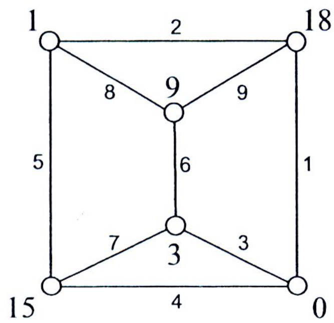


Figure 2.9: A ρ-labeling of the 3-prism.

As for cubic graphs, it is known that $2K_4$ does not admit a ρ-labeling (although a $(K_{25}, 2K_4)$-design does exist). At this time, $2K_4$ is the only cubic graph that is known not to admit a ρ-labeling.

2.1.2 σ-labelings

A lesser-known labeling introduced by Rosa [41] is the σ-labeling. A ρ-labeling f of a graph G with n edges is a *σ-labeling* if $\{|f(u) - f(v)|\colon \{u, v\} \in E(G)\} = \{1, 2, \ldots, n\}$. Thus in a σ-labeling there are no wrap-around edges.

Suppose G with n edges has a σ-labeling. Then G can be embedded in K_{2n+2} so that there is exactly one edge of G of length i for $i = 1, 2, \ldots, n$. Since the edges of length $n+1$ constitute a 1-factor in K_{2n+2}, we have the following.

Theorem 2.1.2 *If G with n edges admits a σ-labeling, then there exists a cyclic G-decomposition of $K_{2n+2} - I$, where I is a 1-factor of K_{2n+2}.*

As with ρ-labelings, a σ-labeling of G with n edges does not necessarily yield G-decompositions of K_{2nx+1} nor does it necessarily yield a G-decomposition of $K_{n,n}$.

Let G be a graph with n edges and suppose every vertex of G has even degree. If G has a σ-labeling f, then the sum of edge labels in G is $\sum_{\{u,v\}\in E(G)} |f(u)-f(v)|$ which is necessarily even (since every vertex has even degree) and equals $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. Thus we must have $4|n(n+1)$ and hence n must satisfy $n \equiv 0$ or $3 \pmod 4$. This was shown in Rosa's original article [41] and is known as the *parity condition.*

Theorem 2.1.3 *Let G be a graph of size n and suppose every vertex of G has even degree. If G admits a σ-labeling then $n \equiv 0$ or $3 \pmod 4$.*

We note that there is no known bipartite graph that satisfies the parity condition and has no σ-labeling.

2.1.3 β-labelings

The best-known labeling introduced by Rosa is β-labeling. Let G be a graph with n edges. A one-to-one function $f\colon V(G) \to \{0, 1, \ldots, n\}$ is a *β-labeling* of G if $\{|f(u) - f(v)| : \{u, v\} \in E(G)\} = \{1, 2, \ldots, n\}$.

Five years after Rosa's paper, Golomb [29] introduced what Rosa had called a β-valuation as a *graceful* labeling. Martin Gardner is credited with popularizing graceful labelings by writing about them in his *Scientific American* columns (see [28]).

A graph that admits a graceful labeling is called *graceful.* Note that a β-labeling is necessarily a σ-labeling, which in turn is a ρ-labeling. Thus Theorems 2.1.1 and 2.1.2 also apply to "graceful" graphs. Unfortunately, as far as Conjectures 1.3.2 and 1.3.3 are concerned, a graceful labeling, which is far more restrictive than a ρ-labeling, offers little additional advantage.

Theorem 2.1.4 *Let G be a graph with n edges that has a β-labeling. Then there exists a purely cyclic G-decomposition of the complete graph K_{2n+1}.*

Again, Theorem 2.1.4 does not necessarily extend to G-decompositions of K_{2nx+1} nor does it necessarily yield a G-decomposition of $K_{n,n}$ when G is bipartite.

A β-labeling does however lead to some decomposition results that cannot be obtained from ρ or σ-labelings. Recall that for a graph G, we let rG denote the vertex-disjoint union of r copies of G.

Theorem 2.1.5 *Let G be a graph of size n that has a β-labeling. Then there exists a cyclic $2G$-decomposition of $K_{2n+2} - I$, where I is a 1-factor.*

We illustrate how Theorem 2.1.5 works. Take two vertex-disjoint copies of G. Keep the same labels in one copy and add $n + 1$ to each label in the second copy. Clicking the new graph n times yields the desired result. The 1-factor is the set edges of length $n + 1$. For example, if we let $[a, b, c, d]$ denote the complete graph K_4 with $V(K_4) = \{a, b, c, d\}$. Then $[0, 2, 4, 6]$ is a β-labeling of K_4 and $\{([0, 2, 4, 6] \cup [7, 9, 11, 13]) + i : i \in \mathbb{Z}_{14}\}$ gives a cyclic K_4-decomposition of $K_{14} - I$, where I is the 1-factor consisting of the set of edges of length 7.

Of course the most famous conjecture on labelings is the conjecture that every tree is graceful.

Conjecture 2.1.1 *Every tree has a β-labeling.*

Conjecture 2.1.1 is known as the *graceful tree conjecture* and is often credited to Kotzig and Ringel or to Rosa. Its first appearance in the literature is in [41]. In spite of many partial results, the conjecture remains open.

A necessary condition for a graph G to be graceful is $|E(G)| \geq |V(G)| - 1$. This need not hold for σ- and ρ-labelings of G. There are bipartite graphs that satisfy the parity condition as well as the condition $|E(G)| \geq |V(G)| - 1$ and yet fail to be graceful. The vertex-disjoint union of a C_4 and K_2 is one such example.

2.1.4 α-labelings

The last valuation introduced by Rosa is the α-labeling. It is restricted to bipartite graphs.

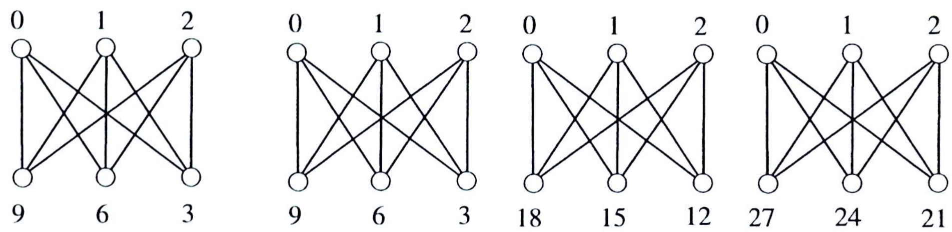
A β-labeling f of a bipartite graph G with n edges and bipartition (A, B) is an *α-labeling* if there exists an integer k such that $f(a) \leq k$ for every $a \in A$ and $f(b) > k$ for every $b \in B$.

An α-labeling of a graph G with n edges yields broader graph decomposition applications than do the previous labelings. Rosa [41] showed the following.

Theorem 2.1.6 *If G with n edges admits an α-labeling, then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x.*

It is not difficult to see how Theorem 2.1.6 works. Let f be an α-labeling of a graph G with n edges and bipartition (A, B). Let $A = \{u_1, u_2, \ldots, u_r\}$ and $B = \{v_1, v_2, \ldots, v_s\}$. Let x be a positive integer. For $1 \leq i \leq x$, let G_i be a copy of G with bipartition (A, B_i) where $B_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,s}\}$ and $v_{i,j}$ corresponds to v_j in B. Let $G(x) = G_1 \cup G_2 \cup \ldots \cup G_x$. Thus $G(x)$ is bipartite with bipartition $(A, B_1 \cup B_2 \cup \ldots \cup B_x)$. Define a labeling f' of $G(x)$ as follows: $f'(u_j) = f(u_j)$ for each $u_j \in A$ and $f'(v_{i,j}) = f(j) + (i-1)n$ for $1 \leq i \leq x$ and $1 \leq j \leq s$. It is easy to see that f' is an α-labeling of $G(x)$ and thus Theorem 2.1.1 applies.

Example 2.1.1 *An α-labeling of $K_{3,3}$ and three $K_{3,3}$-blocks that can be used for a cyclic $K_{3,3}$-decomposition of K_{55}.*

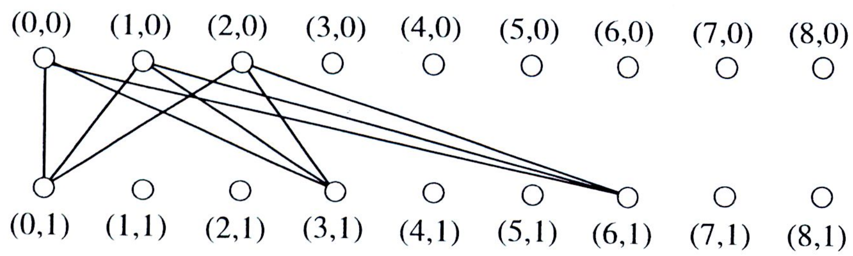


Theorem 2.1.6 offers a substantial advantage over Theorem 2.1.1. Moreover, α-labelings can be used to obtain cyclic decompositions of complete bipartite graphs (see [36]).

Theorem 2.1.7 *If a bipartite graph G of size n has an α-labeling, then there exists a purely cyclic G-decomposition of $K_{n,n}$.*

We give a brief description of how Theorem 2.1.7 works (see Example 2.1.2). Let f be an α-labeling of a graph G with n edges and bipartition (A, B). Let $A = \{u_1, u_2, \ldots, u_r\}$ and $B = \{v_1, v_2, \ldots, v_s\}$. Thus if $\{u_i, v_j\} \in E(G)$, then $f(u_i) < f(v_j)$. Without loss of generality, assume $\{u_1, v_s\} \in E(G)$ with $f(u_1) = 0$ and $f(v_s) = n$. Let $V(K_{n,n}) = \{0, 1, \ldots, n-1\} \times \mathbb{Z}_2$. Embed G in $K_{n,n}$ so that u_i coincides with $(f(u_i), 0)$ for $1 \leq i \leq r$ and v_j coincides with $(f(v_j), 1)$ for $0 \leq j \leq s-1$. Finally, place v_s at $(0, 1)$. This embedding of G ensures that we have exactly one edge of G of length i for $1 \leq i \leq n-1$. Clicking the embedded copy of our graph $n-1$ times gives the desired result.

Example 2.1.2 *An embedding of $K_{3,3}$ in $K_{9,9}$ with one edge of each length i for $0 \leq i \leq 8$.*



Unlike ρ, σ, or β-labelings, it cannot be conjectured that every tree has an α-labeling. In his original article [41], Rosa points out that trees of diameter four that contain the comet $S_{3,2}$ as a subtree do not admit α-labelings. The *comet* $S_{k,m}$ is the graph obtained from the k-star $K_{1,k}$ by replacing each edge by a path with m edges. We note that not every comet $S_{k,m}$ with $k > 2$ and $m > 1$ fails to admit an α-labeling (see [20]).

2.2 Other Rosa-Type Labelings

From a graph decomposition point of view, if a graph G is bipartite, then the most desirable labeling of G would be an α-labeling. The fact that many graphs, including some infinite classes of trees, do not admit α-labelings has lead to the introduction of labelings that produce α-like results but are less restrictive than α-labelings. Ordered labelings are examples of such Rosa-type labelings of bipartite graphs.

2.2.1 Ordered Labelings

The concept of an ordered labeling was developed by El-Zanati and Vanden Eynden and introduced through a series of articles (see for example, [20], [24] and [9]).

A β-, σ-, or ρ-labeling f of a bipartite graph G with n edges and bipartition (A, B) is *ordered* if $f(a) < f(b)$ for each edge $\{a, b\}$ with $a \in A$ and $b \in B$. Ordered β-, σ- and ρ-labelings are called β^+-, σ^+- and ρ^+-labelings, respectively. The labeling is *uniformly-ordered* if $f(a) < f(b)$ for every $a \in A$ and $b \in B$. Uniformly-ordered β-, σ- and ρ-labelings are called β^{++}-, σ^{++}- and ρ^{++}-labelings, respectively.

We note that a uniformly-ordered β-labeling (i.e., a β^{++}-labeling) is the same as an α-labeling, as introduced by Rosa. A β^+-labeling is called a *near α-labeling* in [20] and a *gracious* labeling in [30].

In [20], it was shown that a β^+-labeling of a bipartite graph G of size n leads to cyclic G-decompositions of $K_{n,n}$ and of K_{2nx+1}.

Theorem 2.2.1 *If a bipartite graph G of size n has an β^+-labeling, then there exists a purely cyclic G-decomposition of $K_{n,n}$.*

Theorem 2.2.2 *If G with n edges admits an β^+-labeling, then there exists a cyclic G-decomposition of K_{2nx+1} for every positive integer x.*

Theorems 2.2.1 and 2.2.2 work in an almost identical way to Theorems 2.1.7 and 2.1.6, respectively. Unlike with α-labelings, it can be conjectured that every tree admits a β^+-labeling (see [20] and [30]). In [20], it is shown that the comets $S_{k,2}$, which are known not to admit α-labelings when $k \geq 3$, admit β^+-labelings.

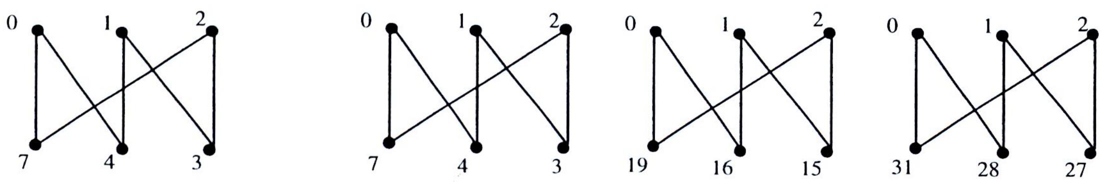
Conjecture 2.2.1 *Every tree admits a β^+-labeling.*

The ordered labeling with the least restrictions is a ρ^+-labeling. The following is proved in [24].

Theorem 2.2.3 *Let G be a bipartite graph with n edges. If G admits a ρ^+-labeling, then there exists a cyclic (K_{2nx+1}, G)-design for all positive integers x.*

Theorem 2.2.3 works in a similar way to Theorems 2.1.6 and 2.2.2. The set up is exactly the same as that for Theorem 2.1.6, except that rather than having $f'(v_{i,j}) = f(j) + (i-1)n$ for $1 \leq i \leq x$ and $1 \leq j \leq s$, we now have $f'(v_{i,j}) = f(j) + (i-1)2n$.

Example 2.2.1 *A ρ^+-labeling of C_6 and three C_6-blocks that can be used for a cyclic C_6-decomposition of K_{37}.*



There is no known example of a bipartite graph that does not admit a ρ^+-labeling. El-Zanati and Vanden Eynden conjecture that every bipartite graph admits a ρ^+-labeling (see [23]).

2.2.2 γ-labelings

If G with n edges is not bipartite, then the best that could be obtained up until recently from a Rosa-type labeling was a cyclic G-decomposition of K_{2n+1}. A non-bipartite graph G is *almost-bipartite* if G contains an edge e whose removal renders the remaining graph bipartite (for example, odd cycles are almost-bipartite). In [10], Blinco, El-Zanati, and Vanden Eynden introduced a variation of a ρ-labeling of an almost-bipartite graph G of size n that yields cyclic G-decompositions of K_{2nx+1}. They called this labeling a γ-labeling.

Let G be a graph with n edges and h a labeling of the vertices of G. We call h a *γ-labeling* of G if the following conditions hold.

g1 The function h is a ρ-labeling of G.

g2 The graph G is tripartite with vertex tripartition $\{A, B, C\}$ with $C = \{c\}$ and $\bar{b} \in B$ such that $\{\bar{b}, c\}$ is the unique edge joining an element of B to c.

g3 If $\{a, v\}$ is an edge of G with $a \in A$, then $h(a) < h(v)$.

g4 We have $h(c) - h(\bar{b}) = n$.

Note that if a nonbipartite graph G has a γ-labeling, then it is almost-bipartite as defined earlier. In this case, removing the edge $\{c, \bar{b}\}$ from G produces a bipartite graph.

Theorem 2.2.4 *Let G be a graph with n edges having a γ-labeling. Then there exists a cyclic (K_{2nx+1}, G)-design for all positive integers x.*

The γ-labeling concept was subsumed by a recent breakthrough on labelings of tripartite graphs. We summarize the new labeling in the next section.

2.2.3 ρ-tripartite Labelings

In [14], Bunge, Chantasartrassmee, El-Zanati, and Vanden Eynden generalized the concept of a γ-labeling and removed the requirement that the graph be almost-bipartite. However, they require the graph to be tripartite (i.e., the graph is required to have chromatic number at most 3). They introduced two new labelings (one of them subsuming γ-labelings) and showed that if a tripartite graph G with n edges has one of these labelings, then there exists a cyclic G-decomposition of K_{2nx+1} for every positive integer x. We will present only the labeling that is relevant to our study.

Let G be a tripartite graph with n edges having the vertex tripartition $\{A, B, C\}$. A *ρ-tripartite* labeling of G is a one-to-one function $h\colon V(G) \to [0, 2n]$ that satisfies

(r1) h is a ρ-labeling of G.

(r2) If $\{a, v\} \in E(G)$ with $a \in A$, then $h(a) < h(v)$.

(r3) If $e = \{b, c\} \in E(G)$ with $b \in B$ and $c \in C$, then there exists an edge $e' = \{b', c'\} \in E(G)$ with $b' \in B$ and $c' \in C$ such that $|h(c') - h(b')| + |h(c) - h(b)| = 2n$.

(r4) If $b \in B$ and $c \in C$, then $|h(b) - h(c)| \neq 2n$.

Note that if G is bipartite, then we can take the set C in the tripartition to be empty. In this case, a ρ^+-labeling (and hence an α-labeling) of G is a ρ-tripartite labeling. Also note that a γ-labeling of an almost-bipartite graph G is necessarily a ρ-tripartite labeling of G.

The following theorem from [14] shows that a ρ-tripartite labeling yields results similar to α- and ρ^+-labelings:

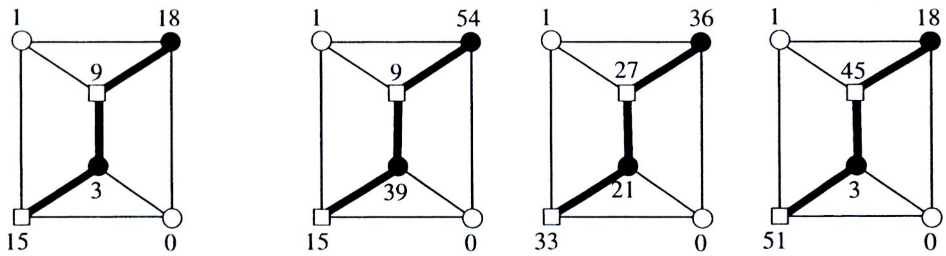
Theorem 2.2.5 *If a tripartite graph G with n edges has a ρ-tripartite labeling, then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x.*

We illustrate how Theorem 2.2.5 works. Let G have n edges and let h be a ρ-tripartite labeling for G, with A, B, and C as in the above definition. Let $B_1, B_2, \ldots, B_x$ be x vertex-disjoint copies of B, and let $C_1, C_2, \ldots, C_x$ be x vertex-disjoint copies of C. The vertex in B_i corresponding to $b \in B$ will be called b_i. Similarly, the vertex in C_i corresponding to $c \in C$ will be called c_i. Let $B^* = \bigcup_{i=1}^{x} B_i$ and $C^* = \bigcup_{i=1}^{x} C_i$. We define a new graph G^* with vertex set $A \cup B^* \cup C^*$ and edges $\{a, v_i\}$, $1 \leq i \leq x$, whenever $a \in A$ and $\{a, v\}$ is an edge of G, and $\{b_i, c_i\}$, $1 \leq i \leq x$, whenever $\{b, c\}$ is an edge of G with $b \in B$ and $c \in C$. Clearly G^* has nx edges and G divides G^*. Define a labeling h^* on G^* by

$$h^*(v) = \begin{cases} h(v) & v \in A, \\ h(b) + (i-1)2n & v = b_i \in B_i, \\ h(c) + (x-i)2n & v = c_i \in C_i. \end{cases}$$

The labeling h^* is a ρ-labeling of G^* and the result follows by Theorem 2.1.1.

Example 2.2.2 *A ρ-tripartite labeling of D_3 and three D_3-blocks that can be used for a cyclic D_3-decomposition of K_{55}.*



Because the concept of a ρ-tripartite labeling is very recent, little is known about which graphs admit ρ-tripartite labelings. In [14], the authors investigate ρ-tripartite labelings of the vertex-disjoint union of two cycles. They also show that the Petersen graph admits a ρ-tripartite labeling.

2.3 Some of the Known Results on Labelings

We present a small number of results on classes of graphs that admit various Rosa-type labelings. For comprehensive surveys of the topic, we direct the reader to the dynamic survey by Gallian [28].

Theorem 2.3.1 *The following graphs admit α-labelings.*

(1) *Caterpillars, complete bipartite graphs, $4k$-cycles for all $k \geq 1$* [41].

(2) *Cubes* [37].

(3) *Bipartite 2-regular graphs that satisfy the parity condition and have at most 3 components, except for the graph $3C_4$* (see [25]).

(4) rC_4, $r \neq 3$ (see [1]).

Theorem 2.3.2 *The following graphs are graceful.*

(1) *Trees with at most 27 vertices* [7].

(2) *Trees with at most 4 end-vertices* [35].

(3) *Trees of diameter at most 5* [34].

(4) *K_n if and only if $n \leq 4$* (see [28]).

(5) *2-regular graphs that satisfy the parity condition and have at most two components* [2].

Theorem 2.3.3 *The following graphs admit the indicated ordered labelings.*

(1) *Trees with up to 20 edges have β^+-labelings* [30].

(2) $S_{k,2}$ *admits a β^+-labeling* [20].

(3) *2-regular bipartite graphs have σ^+-labelings if the parity condition is satisfied, and ρ^+-labelings, otherwise* [9].

(4) *The union of vertex-disjoint graphs that have α-labelings has a σ^+-labeling* [24].

Theorem 2.3.4 [3] *The following 18 graphs do not admit ρ-labelings:* $K_6 - K_2$, $K_7 - K_{3,3}$, $K_7 - K_{1,5}$, $K_7 - K_2$, K_7, $K_8 - K_{4,4}$, $K_8 - K_{3,4}$, $K_8 - K_{2,6}$, $K_8 - K_{1,6}$, $K_8 - K_{1,5}$, $K_8 - K_{2,2}$, $K_8 - (K_3 \cup K_2)$, $K_8 - 4K_2$, $K_8 - K_3$, $K_8 - 3K_2$, $K_8 - 2K_2$, *and* $K_8 - K_2$. *We note that the* K_3 *and* K_2 *in the graph* $K_8 - (K_3 \cup K_2)$ *are vertex-disjoint.*

Theorem 2.3.5 *The following graphs admit ρ-labelings, and σ-labelings if indicated.*

(1) *Graphs with at most 8 vertices, except for the ones in Theorem 2.3.4* [3].

(2) *Graphs with at most 11 edges* [3].

(3) *Trees with at most 55 vertices admit σ-labelings* [12].

(4) K_{p^t+1} *where p is prime and* $t \geq 1$ (see [41]).

(5) *2-regular graphs with at most 3 components. These admit σ-labelings if the parity condition is satisfied* [5, 6].

(6) *The union of vertex-disjoint triangles* [19]. *These admit σ-labelings if the parity condition is satisfied.*

(7) *Graphs in which one component has a* β*-labeling and every other component has an* α*-labeling admit* σ*-labelings* [33].

(8) *Graphs with graceful bases admit* σ*-labelings* (see [11]).

Theorem 2.3.6 *The following almost-bipartite graphs admit* γ*-labelings.*

(1) *2-regular graphs with exactly one odd component, except for* C_3 *and* $C_3 \cup C_4$ (in [17]).

(2) $K_{m,n} + e$, *where* $n \geq 3$ *and* e *is an edge joining two vertices in the* n*-vertex part of* $V(K_{m,n})$ (in [21]).

(3) $C_{2m} + e$, *where* $m \geq 3$ *and* e *is an edge joining two vertices in the same part in the bipartition of* $V(C_{2m})$ (in [8]).

(4) $P_m + e$, *where* $m \geq 4$ *and* e *is an edge joining two vertices in the same part in the bipartition of* $V(P_m)$ (in [16]).