Chapter 1
Introduction

1.1 Prologue

Combinatorial design theory is one of the fastest growing research areas in discrete mathematics. In particular, the area of graph designs has experienced tremendous growth in the last few decades. Some of the best known open problems in design theory are in this popular area. For example, the problem of finding necessary and sufficient conditions for the existence of affine and projective planes (which *are* graph designs) would be ranked as some of the most important open problems in mathematics today.

A graph design is basically a decomposition of some graph into copies of another graph having certain properties. If H and G are graphs with G a subgraph of H, an *(H, G)-design* is a partition of the edge set of H into subgraphs isomorphic to G. The most studied designs are the ones where H is a complete graph. Cyclic (K_k, G)-designs are of particular interest.

Let G be a graph with n edges. A primary question in the study of graph designs is: "*For what values of k does there exist a (K_k, G)-design?*" For most studied graphs G, it is often the case that if $k \equiv 1 \pmod{2n}$, then there exists a (K_k, G)-design. A common approach to finding these designs is through the use of graph labelings.

Graph labelings were introduced in 1967 by Rosa [41] as means of attacking the conjecture of Ringel [40] that every tree with n edges decomposes the complete graph K_{2n+1}. In fact, Rosa showed that for a graph G with n edges, there exists a cyclic (K_{2n+1}, G)-design if and only if G admits what he called a ρ-labeling. Rosa also showed that if a graph G with n edges admits what he called an α-labeling, then there exists a cyclic (K_{2nx+1}, G)-design for every positive integer x.

Unfortunately, α-labelings are restricted to bipartite graphs and numerous classes of bipartite graphs, including infinite classes of trees, do not admit such labelings.

Several graph labelings that lead to cyclic (K_{2nx+1}, G)-designs have since been introduced. These include ordered ρ-labelings (introduced by El-Zanati, Punnim, and Vanden Eynden [24]) which can be conjectured to exist for all bipartite graphs G. Blinco, El-Zanati, and Vanden Eynden introduced γ-labelings [10] for almost-bipartite graphs. Similar labelings for tripartite graphs were recently introduced by Bunge, Chantasartrassmee, El-Zanati, and Vanden Eynden [14].

The above labelings have been investigated for various classes of regular graphs. The bulk of these investigations have focused on 2-regular graphs in particular. Little is known about labelings of 3-regular (i.e., cubic) graphs. In this study, we propose to contribute to the state of knowledge on labelings of cubic graphs and the corresponding cyclic designs.

1.2 Basic Notation and Terminology

In this dissertation, we consider finite graphs only. For the most part, our notation and terminology follow the textbook by Chartrand and Lesniak [18].

A *graph* G is a nonempty finite set of objects called *vertices* together with a (possibly empty) set of unordered pairs of distinct vertices of G called *edges*. The *vertex set* of G is denoted by $V(G)$, while the *edge set* is denoted by $E(G)$. The cardinality of the vertex set of a graph G is called the *order* of G, while the cardinality of its edge set is the *size* of G.

If $e = \{u, v\}$ is an edge of a graph G, we say that u and v are the *endvertices* of e and that u and v are *adjacent*. In this case, we also say that u and e are *incident*, as are v and e. Furthermore, if e_1 and e_2 are distinct edges of G incident with a common vertex, then e_1 and e_2 are *adjacent* edges. It is often convenient to denote an edge by uv or vu rather than by $\{u, v\}$. The *degree* of a vertex v in a graph G is the number of edges in G that are incident with v, which is denoted by $\deg_G v$ or simply by $\deg v$ if G is clear from the context. A vertex of degree 0 is called an *isolated* vertex in G.

It is customary to define or describe a graph G by means of a diagram in which each vertex of G is represented by a point (often drawn as a small circle or some similar object) and each edge $e = \{u, v\}$ of G is represented by a line segment or curve that joins the points corresponding to u and v. We then refer to this diagram as the graph G itself. There are occasions when we are only interested in the structure of a graph defined by a diagram and the vertex set of the graph is irrelevant. In this case, we refer to the graph as an *unlabeled* graph. The four graphs in Figure 1.2 are examples of such unlabeled graphs.

Two graphs often have the same structure, differing only in the way their vertices and edges are labeled or in the way they are drawn. To make this idea more precise, the concept of isomorphism is introduced. An *isomorphism* from a graph G to a graph H is a bijection $f\colon V(G) \to V(H)$ such that $\{u, v\} \in E(G)$ if and only if $\{f(u), f(v)\} \in E(H)$. We say G is *isomorphic to* H, written $G \cong H$, if there is an isomorphism from G to H.

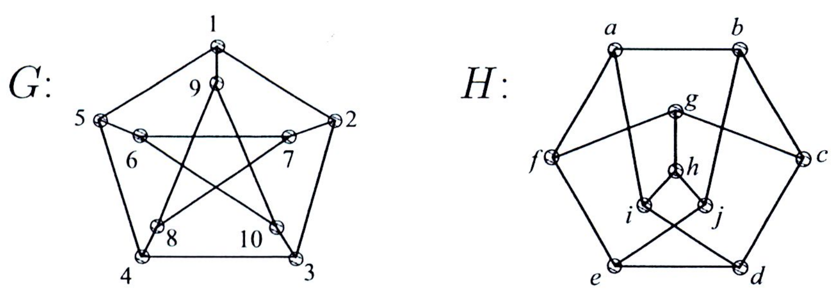

Figure 1.1: Two isomorphic graphs.

The graphs G and H of Figure 1.1 both have order 10 and size 15. Define the function $\phi\colon V(G) \to V(H)$ by $\phi(1) = g, \phi(2) = h, \phi(3) = j, \phi(4) = e, \phi(5) = f, \phi(6) = a, \phi(7) = i, \phi(8) = d, \phi(9) = c$, and $\phi(10) = b$. It is easy to verify that ϕ preserves both adjacency and non-adjacency, so G is isomorphic to H.

A graph G is a *subgraph* of a graph H if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$; in such a case, we also say that H contains G as a subgraph. Whenever a subgraph G of a graph H has the same order as H, then G is called a *spanning subgraph* of H. The *complement* $\bar{G}$ of a graph G is the graph with vertex set $V(G)$ defined by $\{u, v\} \in E(\bar{G})$ if and only if $\{u, v\} \notin E(G)$.

A graph G is *regular of degree* r if $\deg\ v = r$ for each vertex v of G. Such graphs are called *r-regular*. A 3-regular graph is also called a *cubic graph*. A 4-regular graph is called a *quartic graph*. A graph is *complete* if every two of its vertices are adjacent. A complete graph of order n is therefore $(n-1)$-regular and has size $\binom{n}{2}$. We denote this graph by K_n. In Figure 1.2, we show all (nonisomorphic) regular graphs with order 4, including the complete graph $G_4 \cong K_4$.

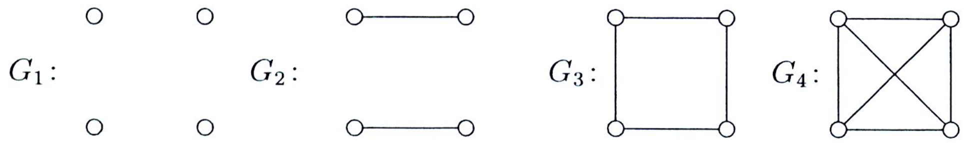

Figure 1.2: The regular graphs of order 4.

Any spanning subgraph of a graph G is referred to as a *factor* of G. A k-regular factor is called a *k-factor*. Of course, if a graph H has a 1-factor, then H has even order.

A *path* is a graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A path is *empty* if it contains only one vertex and thus no edges. Note that a nonempty path starts with a vertex of degree 1 and ends with a vertex of degree 1. All other vertices between the first and the last vertex of a path have degree 2. If the first vertex in a path G is u and the last vertex is v, then G is called a *u-v path* or a *path from u to v*. A path with n vertices is often denoted by P_n.

A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. The number of vertices in a cycle is called its *length*. The cycle with n vertices is denoted by C_n. A cycle is *even* if its length is even; otherwise, it is *odd*. A graph is *acyclic* it does not contain a cycle as a subgraph.

A vertex u is said to be *connected* to a vertex v in a graph G if there exists a u-v path in G. A graph G is *connected* if every pair of its vertices is connected. A graph that is not connected is *disconnected*. The relation "is connected to" is an equivalence relation on $V(G)$. The subgraphs of G induced by the resulting equivalence classes are called the *components* of G.

A graph G is *k-partite*, $k \geq 1$, if $V(G)$ can be partitioned into into k subsets $V_1, V_2, \ldots, V_k$ (called *partite sets*) such that every element of $E(G)$ joins a vertex of V_i to a vertex of V_j, $i \neq j$. Note that every graph is k-partite for some k; indeed, if G has order n, then G is n-partite. If G is a 1-partite graph of order n, then $G = \bar{K}_n$. For $k = 2$, such graphs are called *bipartite* graphs, and for $k = 3$ they are are called *tripartite* graphs. The graph G in Figure 1.3 is bipartite, while the graph H is tripartite.

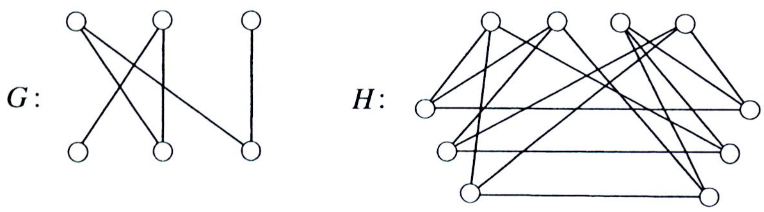

Figure 1.3: A bipartite graph G and a tripartite graph H.

A *complete k-partite graph* G is a k-partite graph with partite sets $V_1, V_2, \ldots, V_k$ having the added property that if $u \in V_i$ and $v \in V_j, i \neq j$, then $\{u, v\} \in E(G)$. If $|V_i| = n_i$, then this graph is denoted by $K(n_1, n_2, \ldots, n_k)$ or $K_{n_1, n_2, \ldots, n_k}$. (The order in which the numbers $n_1, n_2, \ldots, n_k$ are written is not important.) Note that a complete k-partite graph is complete if and only if $n_i = 1$ for all i, in which case it is K_k. A *complete bipartite graph* with partite sets V_1 and V_2, where $|V_1| = r$ and $|V_2| = s$, is then denoted by $K(r, s)$ or more commonly $K_{r,s}$. The complete bipartite graph $K_{3,3}$ and the complete tripartite graph $K_{4,3,3}$ are shown in Figure 1.4.

A *tree* is a connected acyclic graph. A tree of order n necessarily has size $n - 1$. Trees are bipartite and every nontrivial tree contains at least two vertices of degree 1.

The *base* of a graph G is the graph obtained when the vertices of degree 1 are removed from G. A *caterpillar* is a tree whose base is a path. A *lobster* is a tree whose base is a caterpillar.

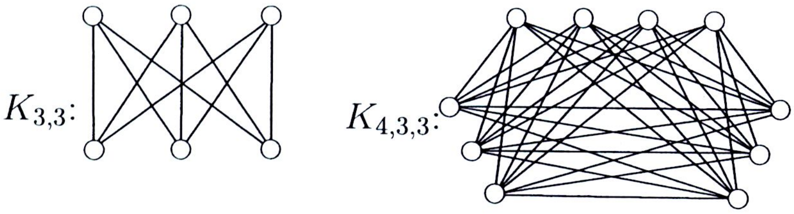

Figure 1.4: The graphs $K_{3,3}$ and $K_{4,3,3}$.

If $S \subseteq V(G)$, then $G - S$ denotes the subgraph with vertex set $V(G) - S$ and whose edges are all those of G not incident with a vertex of S. If $S = \{v\}$, we simply write $G - v$ for $G - \{v\}$. If $F \subseteq E(G)$, we can analogously define a graph $G - F$ as the subgraph having vertex set $V(G)$ and edge set $E(G) - F$. Also we simply write $G - e$ for $G - \{e\}$. Furthermore, if u and v are nonadjacent vertices of a graph G, then $G + e$, where $e = \{u, v\}$, denotes the graph with vertex set $V(G)$ and edge set $E(G) \cup \{e\}$. Clearly, G is a subgraph of $G + e$.

There are many ways of combining graphs to produce new graphs. The *union* of two graphs G_1 and G_2, written $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. This concept is illustrated in Figure 1.2. If a graph G consists of k (≥ 2) vertex-disjoint copies of a graph H, then we write $G = kH$.

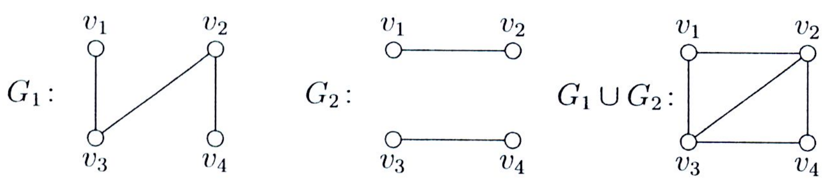

Figure 1.5: The graph $G_1 \cup G_2$.

Let G_1 and G_2 be vertex-disjoint graphs. The *Cartesian product* $G = G_1 \times G_2$ has $V(G) = V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either $u_1 = v_1$ and $\{u_2, v_2\} \in E(G_2)$ or $u_2 = v_2$ and $\{u_1, v_1\} \in E(G_1)$.

The *prism* D_n ($n \geq 3$) is the cartesian product $C_n \times P_2$ of a cycle with n vertices and a path with 2 vertices (D_7 is shown in Figure 1.6).

The *Möbius ladder* is the graph M_n obtained from $P_n \times P_2$ by adding two

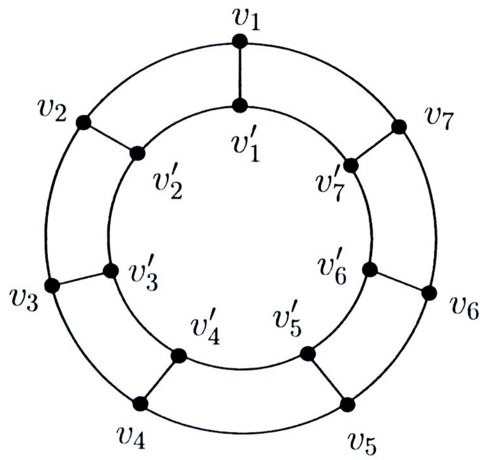

Figure 1.6: The Prism D_7.

edges that join the non-corresponding endvertices of the two paths (M_{10} is shown in Figure 1.7).

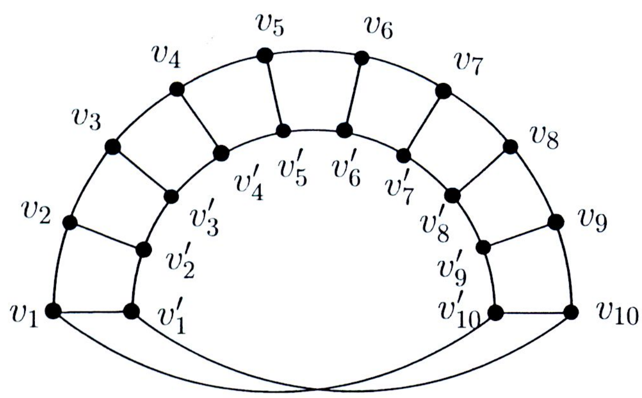

Figure 1.7: The Möbius ladder M_{10}.

There are occasions when the standard definition of a graph does not serve our purposes. For example, we may wish to impose an order to the endvertices of an edge $\{u, v\}$ and use the ordered pair (u, v) or (v, u) instead. This gives rise to the concept of directed graphs. Similarly, one may wish for an edge in a graph G to be a k-element subset of $V(G)$, where $k > 2$. This leads to the study of hypergraphs. Another generalization would allow for multiple edges between a pair of vertices or even for edges with endvertices that are not distinct. This in turn leads to the concept of a multigraph. In this dissertation, we will not make use of directed graphs or hypergraphs, but we will investigate labelings of a certain class of multigraphs.

If one allows more than one edge (but yet a finite number) between the same pair of vertices in a graph, the resulting structure is a *multigraph*. Such edges are called *parallel edges*. A *loop* is an edge that joins a vertex to itself. Although loops are permitted to occur in multigraphs, these will occur rarely. Loops are not part of our investigation. The maximum number of parallel edges between a pair of vertices in a multigraph G is called the *edge multiplicity* of the multigraph and is denoted by $\mu(G)$. Most concepts from graphs extend naturally to multigraphs. Figure 1.8 shows a cubic multigraph of order 18 and edge multiplicity 2.

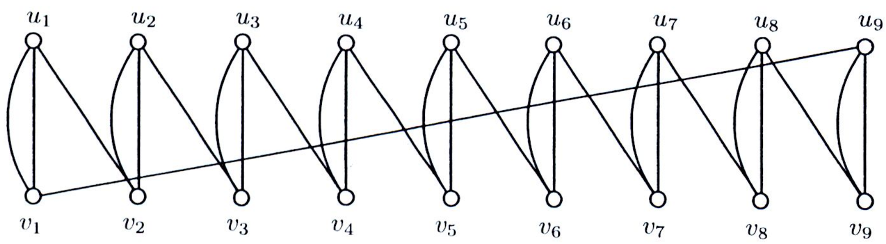

Figure 1.8: The cycle with alternating double edges $\tilde{C}_{18}$

1.3 Graph Decompositions and Graph Designs

If a and b are integers with $a \leq b$ we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$. Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_m$ the group of integers modulo m. If we consider K_m to have the vertex set $\mathbb{Z}_m$ and let G be a subgraph of K_m, By *clicking* G, we mean applying the permutation $i \mapsto i+1$ to $V(G)$. Moreover in this case, if $j \in \mathbb{Z}_m$, then $G + j$ is the graph obtained from G by successively clicking G a total of j times. Likewise if we consider $K_{m,m}$ to have the vertex set $\mathbb{Z}_m \times \mathbb{Z}_2$, with the obvious vertex bipartition, and if we let G be a subgraph of $K_{m,m}$, by *clicking* G we mean applying the isomorphism $(i, j) \mapsto (i+1, j)$ to

$V(G)$.

Let $V(K_m) = \{0, 1, \ldots, m-1\}$. The *label* of an edge $\{i, j\}$ in K_m is $|i-j|$ while the *length* of $\{i, j\}$ is $\min\{|i-j|, m-|i-j|\}$. Thus if the elements of $V(K_m)$ are placed in order as vertices of an equisided m-gon, then the length of edge $\{i, j\}$ is the shortest distance around the polygon between i and j. Edge $\{i, j\}$ is a *wrap-around* edge if the length of $\{i, j\}$ is not the same as its label. Note that clicking an edge does not change its length. Also note that if m is odd, then K_m consists of m edges of length i for $i = 1, 2, \ldots, \frac{m-1}{2}$. If m is even, then K_m consists of m edges of length i for $i = 1, 2, \ldots, \frac{m}{2} - 1$, and $\frac{m}{2}$ edges of length $\frac{m}{2}$; moreover, in this case, the edges of length $\frac{m}{2}$ constitute a 1-factor in K_m.

Let $V(K_{m,m}) = \{0, 1, \ldots, m-1\} \times \mathbb{Z}_2$. The *length* of an edge $e = \{(i, 0), (j, 1)\}$ in $K_{m,m}$ is $j - i$ if $j \geq i$ and $m + j - i$ otherwise. If $i > j$, then e is a *wrap-around* edge. As with K_m, note that clicking an edge in $K_{m,m}$ does not change its length. Also note that $K_{m,m}$ consists of m edges of length i for $i \in \{0, 1, \ldots, m-1\}$.

Let H and G be graphs such that G is a subgraph of H. A *G-decomposition of H* is a set $\Delta = \{G_1, G_2, \ldots, G_r\}$ of pairwise edge-disjoint subgraphs of H, each of which is isomorphic to G, and such that $E(H) = \bigcup_{i=1}^{r} E(G_i)$. The elements of Δ are called *G-blocks*. A G-decomposition of H is also known as a *(H, G)-design*. If a (H, G)-design exists, then we say *G divides H*. A (K_m, G)-design Δ is *symmetric* if the number of G-blocks in Δ is m. A (K_m, G)-design Δ is *cyclic* (*purely cyclic*) if clicking is a permutation (m-cycle) of Δ. For surveys on G-designs, see [4] and [13].

Note that a graph G with n edges can always be embedded in K_t for $t \geq 2n$. It is simple however to find graphs with n edges that do not decompose K_{2n} (for example, there is no (K_6, K_3)-design). Thus it is natural to ask the following:

Question 1.3.1 *For which graphs G with n edges does there exist a (K_{2n+1}, G)-design?*

Note that such a design is necessarily symmetric. Similarly, if G of size n is bipartite, then G can be embedded in $K_{t,t}$ for all $t \geq n$. Here one can ask:

Question 1.3.2 *For which bipartite graphs G with n edges does there exist a G-decomposition of $K_{n,n}$?*

If G has size n, then the simplest way to find (K_{2n+1}, G)-designs (or $(K_{n,n}, G)$-designs if G is bipartite) is to embed G in K_{2n+1} (or in $K_{n,n}$) so that there is exactly one edge of G of length i for $1 \leq i \leq n$ (or $0 \leq i \leq n-1$ in $K_{n,n}$). Then clicking G the appropriate number of times ($2n$ in K_{2n+1} and $n-1$ times in $K_{n,n}$) would yield the desired designs cyclically. That is exactly the idea behind graph labelings.

A *labeling* or *valuation* of a graph G is a function from $V(G)$ into $\mathbb{N}$. In 1967, Rosa [41], with certain input from Kotzig, introduced several types of graph labelings seemingly as tools for decomposing complete graphs into isomorphic subgraphs. These labelings are particularly useful in attacking the following conjectures.

Conjecture 1.3.1 *Every tree with n edges divides the complete graph K_{2n+1}.*

Conjecture 1.3.2 *Every tree with n edges divides the complete bipartite graph $K_{n,n}$.*

Since every tree with n edges divides a tree with nx edges for all positive integers x, Conjecture 1.3.1 implies the following.

Conjecture 1.3.3 *Every tree with n edges divides K_p for all $p \equiv 1 \pmod{2n}$.*

Conjecture 1.3.1 is known as *Ringel's Conjecture.* Ringel proposed it at a meeting in Smolenice, Slovakia, in 1963. Conjecture 1.3.2 is part of the folklore of the subject and cannot necessarily be credited to anyone in particular. Both Conjectures 1.3.1 and 1.3.2 are special cases of conjectures of Graham and Häggkvist (see [31]).

Conjecture 1.3.4 *Every tree on m edges decomposes every $2m$-regular graph.*

Conjecture 1.3.5 *Every tree on m edges decomposes every m-regular bipartite graph.*

Labelings that in a straightforward way lead to graph decompositions will be called *Rosa-type* labelings because of the influence of Rosa's original article [41] on the subject. In Chapter 2.3, we survey some of the known Rosa-type labelings and some of the related results and conjectures. For a comprehensive look at all varieties of graph labelings we direct the reader to the dynamic survey on the topic by Gallian [28]. Before proceeding, we give some additional information on graph designs.

It seems certain that Ringel's conjecture was the original motivator for Rosa's (and Kotzig's) introduction of labelings. Rosa [41] credits Kotzig with the following strengthening of Ringel's conjecture.

Conjecture 1.3.6 *If T is a given tree with n edges, then there exists a purely cyclic T-decomposition of K_{2n+1}.*

In design terms, Conjecture 1.3.6 proposes that there exists a cyclic (K_{2n+1}, G)-design for every tree G of size n. Such designs need not be restricted to trees and may or may not be cyclic. Recall that Question 1.3.1 asks for graphs G of size n for which there exists a (K_{2n+1}, G)-design.

Call a graph G with n edges *Ringelian* if there exists a (K_{2n+1}, G)-design. We note incidentally that if such a G is the complete graph of order k, then a (K_{2n+1}, G)-design is a (K_{k^2-k+1}, K_k)-design, which is necessarily a projective plane of order $k-1$ (see [11]). Thus there are graphs which are not Ringelian. For example, there is no (K_{43}, K_7)-design. Until recently, it was unknown whether or not there is a non-complete graph G which is not Ringelian. This was settled in [32], where it is shown that there is no $(K_{29}, K_6 - e)$-design. It is also known that there exist Ringelian graphs G of size n for which there does not exist a cyclic (K_{2n+1}, G)-design. The smallest such graph is $K_7 - K_{3,3}$, the graph consisting of two copies of K_4 sharing a vertex.

As for Question 1.3.2, it has been conjectured by El-Zanati and Vanden Eynden (see [23]) that every bipartite graph G of size n decomposes $K_{n,n}$. At this point, there is no known counterexample. It is easy however to find bipartite graphs G of size n that do not decompose $K_{n,n}$ cyclically. The graph consisting of two vertex-disjoint copies of K_2 is the smallest such graph.