EXISTENCE AND ALGORITHMS CONSTRUCTION FOR FIXED POINT APPROACH TO VARIATIONAL INEQUALITIES WITH APPLICATION TO OPTIMIZATION PROBLEM

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Abstract

The first purpose of this dissertation is to introduce the existence of common fixed point theorems in fuzzy metric spaces and abstract metric spaces. Also, we give examples to validate our main results. The second purpose is to construct several new iterative algorithms for approximating a common solution of several mathematical problems which consist of hierarchical fixed point problems, hierarchical variational inequality problems and hierarchical optimization problems in Hilbert spaces. The third purpose is to establish the existence theorems of solutions of the system of hierarchical variational inequality problems and hierarchical variational inclusion problem and new iterative algorithms for approximating solution of the system of hierarchical variational inequality problems and hierarchical variational inclusion problem. The final purpose is to prove the new existence of common fixed point theorems, the new existence theorems of solutions of the system of hierarchical variational inequality problems and the system of hierarchical variational inclusion problem, and the new convergence theorems of the generated iterative algorithms.

Keywords : Abstract Metric Space / Fuzzy Metric Space / Hierarchical Fixed Point Problem / Hierarchical Optimization Problem / Hierarchical Variational Inequality Problem / Hilbert Space

หัวข้อวิทยานิพนธ์	การมีอยู่จริงและการสร้างเครื่องมือของการทำซ้ำสำหรับจุดตรึงที่เกี่ยวข้องกับอสมการการแปรผันและประยุกต์สู่ปัญหาการหาค่าเหมาะสม
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บทคัดย่อ

วัตถุประสงค์แรกในการทำวิทยานิพนธ์นี้คือ เพื่อนำเสนอทฤษฎีการมีอยู่ของจุดตรึงร่วมในปริภูมิอิงระยะทางคลุมเครือ และในปริภูมิอิงระยะทางนามธรรม พร้อมทั้งแสดงตัวอย่างที่สอดคล้องกับทฤษฎี วัตถุประสงค์ที่สองคือเพื่อสร้างระเบียบวิธีการทำซ้ำใหม่หลายวิธีสำหรับประมาณค่าคำตอบร่วมของปัญหาทางคณิตศาสตร์ที่หลากหลาย ซึ่งประกอบไปด้วยปัญหาจุดตรึงเชิงลำดับชั้น ปัญหาอสมการการแปรผันเชิงลำดับชั้น และปัญหาการหาค่าเหมาะสมเชิงลำดับชั้นในปริภูมิฮิลเบิร์ต วัตถุ-ประสงค์ที่สามคือ เพื่อแสดงให้เห็นทฤษฎีการมีคำตอบของปัญหาระบบอสมการการแปรผันเชิงลำดับชั้น และของปัญหาระบบการเป็นเซตย่อยกึ่งการแปรผันเชิงลำดับชั้น และแสดงให้เห็นระเบียบวิธีการทำซ้ำ สำหรับประมาณค่าผลเฉลยของปัญหาระบบอสมการการแปรผันเชิงลำดับชั้น และของปัญหาระบบการเป็นเซตย่อยกึ่งการแปรผันเชิงลำดับชั้น วัตถุประสงค์สุดท้ายคือ เพื่อพิสูจน์ทฤษฎีใหม่ของการมีอยู่ของจุดตรึงร่วม พิสูจน์ทฤษฎีใหม่ของการมีคำตอบของปัญหาระบบอสมการการแปรผันเชิงลำดับชั้นและของปัญหาระบบการเป็นเซตย่อยการแปรผันเชิงลำดับชั้น และพิสูจน์ทฤษฎีใหม่ของการลู่เข้าของระเบียบวิธีการทำซ้ำที่ได้สร้างขึ้น

คำสำคัญ : ปริภูมิอิงระยะทางคลุมเครือ / ปริภูมิอิงระยะทางนามธรรม / ปริภูมิฮิลเบิร์ต / ปัญหาการหาค่าเหมาะสมเชิงลำดับชั้น / ปัญหาจุดตรึงเชิงลำดับชั้น / ปัญหาอสมการการแปรผันเชิงลำดับชั้น

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CHAPTER 1 INTRODUCTION

1.1 Background

1.1.1 Fixed Point Theory and Iteration

Fixed point theory is one of the most powerful instrument of modern mathematics. Fixed point theory concern the existence and properties of fixed points. Fixed point theory is a gorgeous fusion of analysis, topology and geometry. Fixed point theory has been applied in such field as engineering, physics, economics, biology, chemistry game theory,optimization theory and approximation theory etc. In 1886, Poincare [1] was the first to work in in the field of fixed point theory. Afterward Brouwer [2] proved fixed point theorem for the solution of the equation $f(x) = x$, a square, a sphere and their n-dimensional counter parts which was further extended by Kakutani [3]. In 1922, Banach [4] proved that a contraction mapping in the complete metric space maintain a unique fixed point. Thereafter it was extended by Kannan [5]. The fixed point theory as well as Banach contraction principle, has been studied and generalized in different spaces and various fixed point theorem were developed.

In 1953, Mann [10] introduced the well-known iteration process, called *Mann iteration*, for approximating fixed point of a mapping T , which start from $x_0 \in E$ and defined the sequence $\{x_n\}_{n=0}^{\infty}$ iteratively by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad \forall n \geq 0, \tag{1.1.1}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0, 1]$ satisfy the appropriate conditions.

In 1974, Ishikawa [11] introduced the iteration for approximating fixed point of a mapping T as follows: the sequences $\{x_n\}$ defined by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)T x_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T y_n, \quad \forall n \geq 0, \end{cases} \tag{1.1.2}$$

where $x_0 \in C$ is arbitrary and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0, 1]$. After the work of Mann [10] and Ishikawa [11] a new direction took place in the field of fixed point theory for approximating fixed point and convergence of iterative of sequences.

In 1976, Korpelevich [21] introduced the well-known iteration process, called *extragradient method*, which the sequence $\{x_n\}$ was defined as follows:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \iota T x_n), \\ x_{n+1} = P_C(x_n - \iota T y_n), \quad \forall n \geq 0, \end{cases} \tag{1.1.3}$$

where $\iota \in (0, \frac{1}{L})$ and T is a monotone and L-Lipschitz continuous mapping. He proved that, if the set of solutions of variational inequality problem is nonempty, then the sequences $\{x_n\}$ converges to an element in the set of solutions of variational inequality problems.

In 2000, Moudafi [24] introduced the viscosity approximation method for nonexpansive mappings. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \sigma_n f(x_n) + (1 - \sigma_n) T x_n, \quad n \geq 0 \ , \tag{1.1.4}$$

where $\{\sigma_n\}$ is a sequence in $(0, 1)$. It is proved that under certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ strongly converges to the unique solution x^* in C of the variational inequality

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \quad x \in C. \tag{1.1.5}$$

In 2006, Marino and Xu [22] introduced a general iterative method for nonexpansive mapping. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \epsilon_n \gamma f(x_n) + (I - \epsilon_n A) T x_n, \quad n \geq 0. \tag{1.1.6}$$

They proved that if the sequence $\{\epsilon_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.1.6) strongly converges to the unique solution $\tilde{x} = P_{F(T)}(I - A + \gamma f)\tilde{x}$ of the variational inequality

$$\langle (A - \gamma f)\tilde{x}, x - \tilde{x} \rangle \geq 0, \quad \forall x \in F(T), \tag{1.1.7}$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(T)} \frac{1}{2} \langle Ax, x \rangle - h(x), \tag{1.1.8}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

In 2007, Moudafi [55] intoduced the following Krasnoselski-Mann algorithm:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(\beta_n S x_n + (1 - \beta_n)T x_n), \tag{1.1.9}$$

where $S, T : C \to C$ are two nonexpansive mappings, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$. Then he showed that $\{x_n\}$ converges weakly to a fixed point of T which is a solution of a hierarchical fixed point problem: Find $x^* \in F(T)$ such that

$$\langle x^* - Sx^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T). \tag{1.1.10}$$

For obtaining a strong convergence result, Mainge and Moudafi in [56] introduced the following algorithm:

$$x_{n+1} = (1 - \alpha_n)f(x_n) + \alpha_n(\beta_n S x_n + (1 - \beta_n)T x_n), \tag{1.1.11}$$

where $f : C \to C$ is a contraction mapping, S and $T : C \to C$ are two nonexpansive mappings, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$. Then they showed that $\{x_n\}$ converges strongly to a fixed point of T which is a solution of problem (1.1.10).

In 2009, Iiduka [7] introduced an iterative algorithm for the following triple hierarchical constrained optimization problem, the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_1 \in H$ is chosen arbitrarily,

$$\begin{cases} y_n = T(x_n - \lambda_n A_1 x_n), \\ x_{n+1} = y_n - \mu \alpha_n A_2 y_n, \ \forall n \geq 0, \end{cases} \tag{1.1.12}$$

where $\alpha_n \in (0, 1]$ and $\lambda_n \in (0, 2\alpha]$ satisfies certain conditions. Let $A_1 : H \to H$ be an inverse-strongly monotone, $A_2 : H \to H$ be a strongly monotone and Lipschitz continuous and $T : H \to H$ be a nonexpansive mapping, then the sequence converge strongly o the set solution of the triple hierarchical constrained optimization problem.

On the other hand, Cianciaruso et al. [53] introduced a two step algorithm as follows:

$$\begin{cases} y_n = \beta_n S x_n + (1 - \beta_n)x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T y_n, \end{cases} \tag{1.1.13}$$

where $f : C \to C$ is a contraction mapping, S and $T : C \to C$ are two nonexpansive mappings, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$. Under some certain restrictions on parameters, the authors proved the sequence $\{x_n\}$ converges strongly to $x^* \in F(T)$, which is a unique solution of the following variational inequality:

$$\langle (I-f)x^*, x-x^*\rangle \geq 0, \quad \forall x \in F(T). \tag{1.1.14}$$

By changing the restrictions on parameters, the sequence $\{x_n\}$ converges strongly to $x^* \in F(T)$, which is a unique solution of the following variational inequality:

$$\langle \frac{1}{\tau}(I-f)x^* + (I-S)x^*, x-x^*\rangle \geq 0, \quad \forall x \in F(T), \tag{1.1.15}$$

where $\tau \in (0,\infty)$ is a constant.

In 2010, Yao et al.[8] modified the two step algorithm (1.1.13) to extend Range of f from C to H by using the metric projection of H onto C. They introduced the following iterative scheme:

$$\begin{cases} y_n = \beta_n S x_n + (1-\beta_n)x_n, \\ x_{n+1} = P_C[\alpha_n f(x_n) + (1-\alpha_n)Ty_n], \end{cases} \tag{1.1.16}$$

where $f : C \to H$ is a contraction mapping, S and $T : C \to C$ are two nonexpansive mappings, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$. The authors proved the sequence $\{x_n\}$ generated by (1.1.16) converges strongly to $x^* \in F(T)$, which is a unique solution of one of the variational inequalities (1.1.14) and (1.1.15).

On the other hand, Tian [23] considered the following iterative method for a nonexpansive mapping $T : H \to H$ with $F(T) \neq \emptyset$,

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n F)Tx_n, \quad \forall n \geq 1, \tag{1.1.17}$$

where F is a k Lipschitzian and η-strongly monotone operator on H. He proved that the sequence $\{x_n\}$ generated by (1.1.17) converges to a fixed point q in $F(T)$, which is the unique solution of the variational inequality

$$\langle (\gamma f - \mu F)q, p-q\rangle \leq 0, \quad \forall p \in F(T).$$

Meanwhile, Maingè [25] proposed the viscosity approximation scheme for quasi-nonexpansive mappings in Hilbert spaces as follows:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_w x_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\} \subset (0,1)$ and T_w was generated by $T_w = (1-w)I + wT$ for all $w \in (0,1)$. He also proved the convergence theorem under the suitable conditions.

In 2011, Yao et.al [9] studied new algorithms. For $x_0 \in C$ is chosen arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by

$$x_{n+1} = \beta_n x_n + (1-\beta_n)TP_C[I - \alpha_n(A - \gamma f)]x_n, \ \forall n \geq 0,$$

where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$. Then $\{x_n\}$ converges strongly to $x^* \in F(T)$ which is the unique solution of the variational inequality:

$$\text{Find } x^* \in F(T) \text{ such that } \langle (A-\gamma f)x^*, x - x^* \rangle \geq 0, \ \forall x \in F(T). \tag{1.1.18}$$

At the same time, Gu et al. [54] introduced the following iterative algorithm:

$$\begin{cases} y_n = P_C[\beta_n S x_n + (1-\beta_n)x_n], \\ x_{n+1} = P_C[\alpha_n f(x_n) + \sum\limits_{i=1}^{n}(\alpha_{i-1} - \alpha_i)T_i y_n], \quad \forall n \geq 1, \end{cases} \tag{1.1.19}$$

where $f : C \to H$ is a contraction mapping, $S : C \to H$ is a nonexpansive mapping, $\{T_i\}_{i=1}^{\infty} : C \to C$ is a countable family of nonexpansive mappings, $\alpha_0 = 1$, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$. The authors proved the sequence $\{x_n\}$ converges strongly to a common fixed point of a countable family of nonexpansive mappings which is a solution of a hierarchical fixed point problem.

Meanwhile, Yu and Liang [51] introduced the following iterative algorithm:

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n \Sigma_{m=1}^{r} \delta_{(m,n)} P_C(\hat{\lambda}_m B_m x_n - \lambda_m A_m x_n), \ \forall n \geq 1,$$

where C is a nonempty closed and convex subset of a real Hilbert space H, $A_m : C \to H$ is a relaxed (η_m, ρ_m)-cocoercive and ν_m-Lipschitz continuous mapping, $B_m : C \to H$ is a relaxed $(\hat{\eta}_m, \hat{\rho}_m)$-cocoercive and $\hat{\nu}_m$-Lipschitz continuous mapping for each $1 \leq m \leq r$, $u \in C$ is a fixed point, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_{(1,n)}\}, \{\delta_{(2,n)}\}, ..., \{\delta_{(r,n)}\}$ are sequences in $(0,1)$, They proved the sequence $\{x_n\}$ strongly to a common element $\tilde{x} \in \cap_{m=1}^{r} GVI(C, B_m, A_m)$, which is the unique solution of the following:

$$\langle u - \tilde{x}, x - \tilde{x} \rangle \leq 0, \quad \forall x \in \cap_{m=1}^{r} GVI(C, B_m, A_m). \tag{1.1.20}$$

Later, He and Du[52] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in C, \\ u_n^i = T_{r_n}^i x_n, \qquad \forall i = 1, 2, \ldots, l, \\ z_n = \frac{u_n^1 + u_n^2 + \cdots + u_n^l}{l}, \\ y_n = (1-\lambda)x_n + \lambda T z_n, \\ x_{n+1} = \alpha_n f(x_n) + (I - \alpha_n) y_n, \end{cases} \tag{1.1.21}$$

where $T_{r_n}^i x = \{z \in C : F_i(z, y) + \frac{1}{r_n}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$, $\alpha_n \subset (0,1)$ and $r_n \subset (0, +\infty)$. They proved that if the sequence $\{\alpha_n\}$ and $\{r_n\}$ of parameters satisfies appropriate conditions and $\Omega = (\cap_{i=1}^l EP(F_i)) \cap F(T) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.1.21) strongly converges to the unique solution $c = P_\Omega f(c) \in \Omega$.

In 2012, Kraikaew and Saejung[58] introduced the sequence $\{x_n\}$ and $\{y_n\}$ generated by

$$\begin{cases} x_0, y_0, z_0 \in H, \\ x_{n+1} = (1-\alpha_n)T_1 x_n + \alpha_n f_1(T_2 y_n), \\ y_{n+1} = (1-\alpha_n)T_2 y_n + \alpha_n f_2(T_1 x_n), \qquad n = 0, 1, 2, \ldots, \end{cases} \tag{1.1.22}$$

where $T_1, T_2 : H \to H$ are quasi-nonexpansive mappings, $f_1, f_2 : H \to H$ are contraction mappings and $\{\alpha_n\}$ is a sequence in $(0,1)$. They proved that the sequence $\{x_n\}$ and $\{y_n\}$ converge to x^* and y^*, which is the unique solution of bi-level hierarchical optimization problems, i.e, find $(x^*, y^*) \in F(T_1) \times F(T_2)$ such that for given positive real numbers ρ and η, the following inequalities hold:

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^*\rangle \geq 0 \quad \forall x \in F(T_1), \\ \langle \eta F(x^*) + y^* - x^*, y - y^*\rangle \geq 0 \quad \forall y \in F(T_2), \end{cases} \tag{1.1.23}$$

Recently, Chang et al. [59] introduced *bi-level hierarchical variational inclusion problems*, i.e, find $(x^*, y^*) \in \Omega_1 \times \Omega_2$ such that for given positive real numbers ρ and η, the following inequalities hold:

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^*\rangle \geq 0, \quad \forall x \in \Omega_1, \\ \langle \eta F(x^*) + y^* - x^*, y - y^*\rangle \geq 0, \quad \forall y \in \Omega_2, \end{cases} \tag{1.1.24}$$

where $F, A_1, A_2 : H \to H$ are mappings and $M_1, M_2 : H \to 2^H$ are multi-valued mappings, Ω_i is the set of solutions to variational inclusion problem with $A = A_i, M = M_i$ for $i = 1, 2$. They solved the convex programming problems and quadratic minimization problems by using Maingés scheme.

1.1.2 Fixed Point Theory in Fuzzy Metric Spaces

The concept of fuzzy sets was coined by Zadeh [115] in his seminal in 1965. Many authors have introduced the concept of fuzzy metric in different ways; see, e.g., [96, 104]. Thereafter, Kramosil and Michalek [105] introduced the concept of fuzzy metric spaces which could be considered as a reformulation, in the fuzzy context, of the notion of probabilistic metric space due to Menger [106].

The concept of fixed point theory in fuzzy metric spaces was introduced by Heilpern [98]. He introduced the concept of fuzzy mappings and proved some fixed point theorems for fuzzy contraction mappings in metric linear spaces, which is a fuzzy extension of the Banach contraction principle. Afterward, George and Veeramani [96, 97] gave the notion of fuzzy metric spaces which constitutes a modification of the one due to Kramosil and Michalek. From now on, by fuzzy metric we mean a fuzzy metric in the sense of George and Veeramani. Many authors have contributed to the development of this theory and apply to fixed point theory, for instance [93, 95, 99, 112, 113].

On the other hand, Jungk [100] introduced the notion of commuting mappings. In 1982, Sessa [114] gave the notion of weakly commuting mappings. Thereafter, Jungck [101] defined the notion of compatible mappings to generalize the concept of weak commutativity and showed that weakly commuting mappings are compatible but the converse is not true. Afterward, a number of fixed point theorems have been obtained by various authors utilizing this notion (see [91, 92, 94, 103, 109]). In 1997, Pathak et al. [108] introduced the concept of R-weakly commuting of type (A_g).

In 2002, Aamri and El Moutawakil [90] introduced the concept of E.A. property in metric spaces. Afterward, Mihet [107] proved two common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces by using E.A. property. Thereafter, Sintunavarat and Kumam in [111] obtained the results of Mihet [107] require some special condition. However, some case is not satisfying this condition (see [111] for more details). So the results of Mihet [107] can not be used for this case. They introduced the concept of *the common limit in the range property* for solve this problem and also established existence of a common fixed point theorems for generalize contractive mappings satisfy this property in fuzzy metric spaces. Recently, Sintunavarat and Kumam [117] gave the concept of R-weakly commuting of type (A_g) in fuzzy metric spaces and establish the existence of common fixed point theorems by using the common limit in the range property.

1.1.3 Fixed Point Theory in Abstract Metric Spaces

The concept of K-metric spaces was reintroduced by Huang and Zhang under the name of cone metric spaces [74] which is the generalization of a metric space. The idea of cone metric spaces is to replace the codomain of metric from the set of real numbers to an ordered Banach space. They reintroduce the definitions of convergent and Cauchy sequences in sense of interior point of the underlying cone. They also continued with results concerned with the normal cones only. One of the main results of Huang and Zhang in [74] is fixed point theorems for contractive mappings in normal cone spaces. In fact, the fixed point theorem in cone metric spaces is appropriate only in the case when the underlying cone is non-normal and its interior is nonempty. Janković et al. [80] studied this topic and gave some examples showing that theorems from ordinary metric spaces cannot be applied in the setting of non-normal cone metric spaces. Many works for fixed point theorems in cone metric spaces were appeared in [61, 62, 64, 65, 73, 63, 66, 68, 82, 81, 84, 86, 88, 75, 76, 77, 78].

In 2011, Abbas et al. [61] introduced the concept of w-compatible mappings and obtained a coupled coincidence point and a coupled point of coincidence for such mappings satisfying a contractive condition in cone metric spaces. Very recently, Aydi et al. [67] introduced the concept of W-compatible mappings for mappings $F : X \times X \times X \to X$ and $g : X \to X$, where (X, d) is an abstract metric space and established tripled coincidence point and common tripled fixed point theorems in these spaces.

On the other hand, Sintunavarat and Kumam [85] coined the idea of common limit range property for mappings $F : X \to X$ and $g : X \to X$, where (X, d) is metric space (and fuzzy metric spaces) and proved the common fixed point theorems by using this property. Afterward, Jain et al. [79] extend this property for mappings $F : X \times X \to X$ and $g : X \to X$, where (X, d) is metric space (and fuzzy metric spaces) and established coupled fixed point theorem for mappings satisfy this property. Several common fixed point theorems have been proved by many researcher in framework of many spaces via common limit range property (see [70, 71, 72, 87] and references therein).

Starting from the background of the coupled fixed points; the concept of tripled fixed points was introduced by Samet and Vetro [83] and Berinde and Borcut [69] which motivated by the fact that, through the coupled fixed point technique we can not solve the solution of some problems in nonlinear analysis such as a system with following form:

$$\begin{cases} x^2 + 2yz - 6x + 3 = 0 \\ y^2 + 2xz - 6y + 3 = 0 \\ z^2 + 2yx - 6z + 3 = 0. \end{cases}$$

1.2 Summary of Dissertation

Motivated and inspired by the above works, the purposes of this dissertation is as follows:

(1) We study new property, tools, and procedure for prove the existence of tripled fixed point theorems. Moreover, we generalize and extend the existence of tripled fixed point theorems in fuzzy metric spaces and abstract metric spaces and give illustrate examples to validate the some results in this thesis.

(2) We study the iterative procedure for approximating common solutions of problems involving variational inequality and fixed point for nonlinear operators. Moreover, we extend and improve the previous mentioned iterative procedure for approximating common solutions of problems involving variational inequality and fixed point for nonlinear operators and Construct the convergence theorems for the generated iterative procedure and apply its results to another kind of problems as its applications.

(3) We establish the new existence theorems of solutions of the system of hierarchical variational inequality problems and hierarchical variational inclusion problem and new iterative algorithms for approximating solution of the system of hierarchical variational inequality problems and hierarchical variational inclusion problem.

We studied and followed the above objectives and already reached all of our main goal. Throughout this dissertation, we summarize and divide this literature into 6 Chapters as shown in the following:

In Chapter 1, we review the background of this thesis for fixed point theorems and fixed point iteration

In Chapter 2, we give the necessary notations, definitions, some useful lemmas and the previous related theorems which will be used in the later chapter.

In Chapter 3, we introduce new property and new condition of contraction mappings for prove the existence of tripled fixed point theorems in fuzzy metric spaces and abstract metric spaces. Moreover, we establish and prove the new existence of tripled fixed point theorems in fuzzy metric spaces and abstract metric spaces. We also give some examples to validate some results in this section.

In Chapter 4, we present the new iterative algorithms for approximating a common solution of several mathematical problems which consist of hierarchical fixed point problems, hierarchical variational inequality problems and hierarchical optimization problems. Exactly, its convergence theorem of the new iterative algorithms are also proved and presented.

In Chapter 5, we show the new existence theorems of solutions of the system of hierarchical variational inequality problems and the system of hierarchical variational inclusion problem and new iterative algorithms for approximating solution of the system of hierarchical variational inequality problems and the system of hierarchical variational inclusion problem. Exactly, we prove the new existence theorems and convergence theorem of the new iterative algorithms.

Finally, in Chapter 6, we give the summary of all the results and the conclusion of this dissertation.

CHAPTER 2 PRELIMINARIES

In this chapter, we give some basic concepts including with definitions, notations and some useful lemmas which are all necessary to the later chapters. Throughout this dissertation, let $\mathbb{R}$ and $\mathbb{N}$ stand for the set of all real numbers and the set of all natural numbers, respectively.

2.1 Fundamental Definitions

Definition 2.1.1. Let X be a nonempty set. Assume that, for any $x \in X$ and $\alpha \in \mathbb{R}$, there exists a unique element $\alpha \cdot x$, which is called the *scalar multiplication*. Also, assume that, for any $x, y \in X$, there exists a unique element $x + y$, which is called the *addition*. The system $(X, \cdot, +)$ is called a *linear space* over $\mathbb{R}$ or a *vector space* over $\mathbb{R}$ if the following conditions are satisfied: for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$.

(1) $x + y = y + x$;

(2) $x + (y + z) = (x + y) + z$;

(3) $\alpha(x + y) = \alpha x + \alpha y$;

(4) $x + y = x + z \Rightarrow y = z$;

(5) $(\alpha + \beta)x = \alpha x + \beta x$;

(6) $(\alpha\beta)x = \alpha(\beta x)$;

(7) $1x = x$.

Definition 2.1.2. Let X be a nonempty set. A *metric* on X is a real function $d : X \to \mathbb{R}$ satisfying the following conditions:

(1) $d(x, y) \geq 0$ for all $x, y \in X$;

(2) $d(x, y) = 0 \iff x = y$ for all $x, y \in X$;

(3) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(4) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

A set X with a metric d is called a *metric space*. The elements of X are called the *points* of the metric space (X, d).

Definition 2.1.3. Let X be a linear space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A function $\|\cdot\| : X \to \mathbb{R}$ is called a *norm* on X if the following conditions are satisfied:

(1) $\|x\| \geq 0$ for all $x \in X$;

(2) $\|x\| = 0 \Longleftrightarrow x = 0$;

(3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$;

(4) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and $\alpha \in \mathbb{K}$.

The distance induced by its norm such that $d(x, y) = \|x - y\|$ for all $x, y \in X$. A linear space X equipped with the norm $\|\cdot\|$ is called a *normed linear space* or a *normed space*.

Definition 2.1.4. A sequence $\{x_n\}$ in a normed space is said to be *strongly convergent* to a point $x \in X$ if $\lim_{n\to\infty} \|x_n - x\| = 0$. That is, if, for any $\epsilon > 0$, there exists a positive integer N such that $\|x_n - x\| < \epsilon$ for all $n \geq N$. We denote the strong convergence by the notation $x_n \to x$.

Definition 2.1.5. A sequence $\{x_n\}$ in a normed space is called a *Cauchy sequence* in X if $\lim_{m,n\to\infty} \|x_m - x_n\| = 0$. That is, if for any $\epsilon > 0$ there exists a positive integer N such that $\|x_m - x_n\| < \epsilon$for all $m, n \geq N$.

Theorem 2.1.6. [33] *Let $\{x_n\}$ be a sequence in a normed linear space, then $x_n \to x \in X$ if and only if, for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exist a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converging to x.*

Definition 2.1.7. Let X and Y be linear spaces over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$).

(1) A mapping $L : Y \to X$ is called a *linear operator* if $L(x + y) = Lx + Ly$ and $L(\alpha x) = \alpha Lx$ for all $x, y \in Y$ and $\alpha \in \mathbb{K}$;

(2) A linear operator $L : Y \to \mathbb{K}$ is called a *linear functional* on Y;

(3) A mapping $L : Y \to X$ be a continuous at $x_0 \in Y$ if, for any $\epsilon > 0$, there exists $\delta > 0$ such that $\|L(x) - L(x_0)\| < \epsilon$ whenever $\|x - x_0\| < \delta$. If L is continuous at each $x \in Y$, then L is called *continuous* on Y;

(4) A linear operator $L : X \to Y$ is said to be *bounded* on X if there exists a real number $M \geq 0$ such that $\|L(x)\| \leq M\|x\|$ for all $x \in X$.

Definition 2.1.8. A sequence $\{x_n\}$ in a normed space is said to be *convergent weakly* to a point $x \in X$ if $\lim_{n\to\infty} f(x_n) = f(x)$ holds for every continuous linear functional f. We denote the weak convergence by the notation $x_n \rightharpoonup x$.

Definition 2.1.9. A sequence $\{x_n\}$ in a normed linear space X is said to be *bounded* if there exists a real number $M > 0$ such that $\|x_n\| \leq M$ for all $n \geq 1$.

Definition 2.1.10. A normed space X is said to be *complete* if every Cauchy sequences in X converge to an element in X.

Definition 2.1.11. A complete normed linear space over the field $\mathbb{K}$ is called a *Banach space* over the field $\mathbb{K}$.

Definition 2.1.12. A subset C of a normed linear space X is called a *closed* subset in X if $\{x_n\}$ is a sequence in C and $x_n \to x$ then $x \in C$.

Definition 2.1.13. A subset C of a normed linear space X is said to be *convex* in X if $\lambda x + (1-\lambda)y \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$.

2.2 Hilbert Spaces

Definition 2.2.1. The real-value function of two variables $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ is called the *inner product* on a real vector space X if the following conditions are satisfied: for any $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$

(1) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$;

(2) $\langle x, y \rangle = \langle y, x \rangle$;

(3) $\langle x, x \rangle \geq 0$;

(4) $\langle x, x \rangle = 0 \Longleftrightarrow x = 0$.

A real vector space X equipped with an inner product $\langle \cdot, \cdot \rangle$ is called a *real inner product space.*

Definition 2.2.2. A *Hilbert space* is an inner product space which is complete under the norm induced by its inner product such that $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in X$.

Lemma 2.2.3. [33] (The Schwarz inequality) *If x and y are any two vectors in an inner product space X, then*

$$|\langle x, y \rangle| \leq \|x\| \|y\|.$$

Lemma 2.2.4. [33] *Let H be a real Hilbert space. Then the following inequalities are satisfied:*

(H1) $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$;

(H2) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$;

(H3) $\|x+y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$;

(H4) $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$ for all $\lambda \in [0,1]$;

(H5) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$.

Definition 2.2.5. Let H be a Hilbert space and C be a nonempty closed and convex subset of H. Let f be a function of C into $(-\infty, \infty]$, where $(-\infty, \infty] = \mathbb{R} \cup \{\infty\}$. Then f is called *lower semi-continuous* if, for any $a \in \mathbb{R}$, the set $\{x \in C : f(x) \leq a\}$ is closed.

Lemma 2.2.6. [33] *Let H be an inner product space and $\{x_n\}$ be a bounded sequence of H such that $x_n \rightharpoonup x$. Then the following inequality holds:*

$$\|x\| \leq \lim_{n\to\infty} \inf \|x_n\|.$$

Lemma 2.2.7. [34] *Let X be an inner product space. Then, for any $x, y, z \in X$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\begin{aligned} &\|\alpha x + \beta y + \gamma z\|^2 \\ &= \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2. \end{aligned}$$

Lemma 2.2.8. [35] *Each Hilbert space H satisfies Opial's condition, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$,*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in H$ with $y \neq x$.

Lemma 2.2.9. [36],[37] *Each Hilbert space H satisfies the Kadec-Klee property, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $\|x_n - x\| \to 0$.*

Definition 2.2.10. A mapping $P_C : H \to C$ is called the *metric projection* from H onto C if, for all $x \in H$, there exists the unique nearest point in C, denoted by $P_C x$ satisfying the property

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

Lemma 2.2.11. [33] *Let C be a closed and convex subset of a real Hilbert space H and $x \in H$, $y \in C$. Then the following inequalities are satisfied:*

(P1) $z = P_C x \Longleftrightarrow \langle z - x, y - z\rangle \geq 0$ for all $y \in C$;

(P2) $\|P_C x - P_C y\| \leq \|x - y\|$ for all $x, y \in H$;

(P3) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$ for all $x, y \in H$;

(P4) $\langle x - P_C x, y - P_C x\rangle \leq 0$ for all $x \in H$ and $y \in C$;

(P5) $\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2$ for all $x \in H$ and $y \in C$;

(P6) $\|(x - y) - (P_C x - P_C y)\|^2 \leq \|x - y\|^2 - \|P_C x - P_C y\|^2$ for all $x, y \in H$.

Lemma 2.2.12. [38] *Let C be a closed and convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is $\omega_w(x_n) \subset C$ and satisfies the condition $\|x_n - u\| \leq \|u - q\|$ for all $n \geq 1$. Then $x_n \to q$.*

Lemma 2.2.13. [39] *Let C be a closed and convex subset of a real Hilbert space H. Let $\{x_n\}$ be a bounded sequence in H. Assume that*

(1) The weak ω-limit set $\omega_w(x_n) \subset C$;

(2) For each $z \in C$, $\lim_{n\to\infty} \|x_n - z\|$ exists.

Then the sequence $\{x_n\}$ is weakly convergent to a point in C.

2.3 Banach Spaces

Definition 2.3.1. [37] Let E be a Banach space and E^* be the dual space of E. For each $x \in E$, we associate the set

$$J(x) = \{f \in E^* | \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

where $\langle \cdot, \cdot\rangle$ denotes the duality pairing between E and E^*. The multivalued operator $J : E \to E^*$ is called the *normalized duality mapping* or the *duality mapping* of E.

Theorem 2.3.2. [37] *Let E be a Banach space and J be the duality mapping of E. Then we have the following:*

(1) *For all $x \in E, J(x)$ is nonempty bounded closed and convex;*

(2) $J(0) = \{0\}$;

(3) *For all $x \in E$ and $\alpha \in \mathbb{R}$, $J(\alpha x) = \alpha J(x)$;*

(4) *For all* $x, y \in E, f \in J(x)$ *and* $g \in J(y)$, $\langle x - y, f - g\rangle \geq 0$*;*

(5) *For all*$x, y \in E$ *and* $f \in J(y)$, $\|x\|^2 - \|y\|^2 \geq 2\langle x - y, f\rangle$.

Remark 2.3.3. If, E is a Hilbert space, then $J = I$, where I is the identity mapping.

Definition 2.3.4. [40] Let E be a normed space. Suppose that, for each $x \in E$, there corresponds a unique bounded linear functional $g_x \in E^{**}$ given by $g_x(f) = f(x), f \in E^*$. A mapping $h : E \to E^{**}$ defined by $x \mapsto g_x$ is called the *canonical mapping*.

Definition 2.3.5. [40] A normed space E is said to be *reflexive* if the canonical mapping $h : E \to E^{**}$ is surjective.

Definition 2.3.6. [40] Let x be an element and $\{x_n\}$ be a sequence in a normed space E. Then $\{x_n\}$ *converges strongly* to $x \in E$, written by $x_n \to x$, if

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

Definition 2.3.7. [40] Let x be an element and $\{x_n\}$ a sequence in a normed space E. Then $\{x_n\}$ *converges weakly* to x, written by $x_n \rightharpoonup x$, if $f(x_n) \to f(x)$ wherever $f \in E^*$.

Definition 2.3.8. [40] The *weak* convergence* of a sequence $\{x_n^*\}$ to x^*, written by $x_n^* \rightharpoonup *x^*$, where $x^* \in E^*$ and $\{x_n^*\}$ is a sequence in E^*.

Theorem 2.3.9. [40] *A normed space E is* reflexive *if and only if each of its bounded sequence has a weakly convergence subsequence.*

Definition 2.3.10. [40] A nonempty subset C of a Banach space E is said to be *weakly sequentially compact* if every sequence $\{x_n\}$ in C has a subsequence converging to a point of E in the weak topology.

Theorem 2.3.11. [37] *Let E be a reflexive Banach space. Then a nonempty subset C of X is weakly sequentially compact if and only if C is bounded.*

Definition 2.3.12. [37] A Banach space E is said to be:

(1) *strictly convex* if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S(E) = \{x \in E : \|x\| = 1\}$ and $x \neq y$,

(2) *uniformly convex* if, for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|\frac{x+y}{2}\| < 1 - \delta$ whenever $x, y \in S(E)$ and $\|x - y\| \geq \epsilon$.

Definition 2.3.13. [37] The *modulus of convexity* of E is the function $\tilde{\delta} : [0, 2] \to [0, 1]$ defined by

$$\tilde{\delta}(\epsilon) = \inf\{1 - \|\frac{x+y}{2}\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon\}.$$

Note that a Banach space E is uniformly convex if and only if $\tilde{\delta}(\epsilon) > 0$ for all $\epsilon \in (0, 2]$.

Definition 2.3.14. [37] Let E be a Banach space and let $S(E) = \{x \in E : \|x\| = 1\}$. A Banach space is said to be *smooth* provided the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.3.1}$$

exists for each $x, y \in S(E)$. In this case, the norm of E is said to be *Gâteaux differentiable*. The space E is said to have a uniformly *Gâteaux differentiable* norm if for each $y \in S(E)$, the limit (2.3.1) is attained uniformly for $x \in S(E)$. The norm of E is said to be a *Fréchet differentiable* norm if, for each $x \in S(E)$, the limit (2.3.1) is attained uniformly for $y \in S(E)$. The norm of E is said to be *uniformly Fréchet differentiable* (E is said to be *uniformly smooth*) if the limit (2.3.1)($*$) is attained uniformly for $(x, y) \in S(E) \times S(E)$.

Remark 2.3.15. The following basic properties were proposed by Cioranescu [41]:

(1) If E is a strictly convex, then J is strictly monotone;

(2) If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E;

(3) If E is a reflexive and strictly convex Banach space, then J^{-1} is norm-weak*-continuous;

(4) If E is a reflexive smooth and strictly convex, then the normalized duality mapping J is single valued, one-to-one and onto;

(5) If E is a reflexive strictly convex and smooth Banach space and J is the duality mapping from E into E^*, then J^{-1} is also single valued, bijective and is also duality mapping from E^* into E and thus $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$;

(6) If E is a uniformly smooth, then E is a smooth and reflexive;

(7) E is a uniformly smooth if and only if E^* is uniformly convex.

Definition 2.3.16. [37] Let E be a linear space and C be a convex subset of E. A function $f : C \to (-\infty, \infty]$ is said to be *convex* on C if, for any $x, y \in C$ and $t \in (0, 1)$,

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y).$$

Definition 2.3.17. [33] A Banach space E is said to satisfy *Opial's condition* if $x_n \to x$ weakly as $n \to \infty$ and $x \ne y$ imply that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

Definition 2.3.18. [33] A Banach space E is said to have the *Kadec-Klee property* if, for every sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $\|x_n - x\| \to 0$.

Remark 2.3.19. Each uniformly convex Banach space E has the *Kadec-Klee property*, that is, for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.

2.4 Fuzzy Metric Spaces

Definition 2.4.1. [110] A *continuous t-norm* is a binary operation $* : [0,1]\times[0,1] \to [0,1]$ satisfying the following conditions:

(i) $*$ is commutative and associative;

(ii) $a * 1 = a$ for all $a \in [0, 1]$;

(iii) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ $(a, b, c, d \in [0, 1])$;

(iv) $* : [0, 1] \times [0, 1] \to [0, 1]$ is continuous.

Example 2.4.2. The following examples are classical examples of a continuous t-norms.

(TL) (The Lukasiewicz t-norm) A mapping $T_L : [0, 1] \times [0, 1] \to [0, 1]$ which defined through

$$T_L(a, b) = \max\{a + b - 1, 0\}.$$

(TP) (The product t-norm) A mapping $T_P : [0,1] \times [0,1] \to [0,1]$ which defined through

$$T_P(a,b) = ab.$$

(TM) (The minimum t-norm) A mapping $T_M : [0,1] \times [0,1] \to [0,1]$ which defined through

$$T_M(a,b) = \min\{a,b\}.$$

Definition 2.4.3. [96, 97] A *fuzzy metric space* is a triple $(X, M, *)$ where X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$ and the following conditions are satisfied for all $x, y \in X$ and $t, s > 0$:

(GV-1) $M(x,y,t) > 0$;

(GV-2) $M(x,y,t) = 1 \iff x = y$;

(GV-3) $M(x,y,t) = M(y,x,t)$;

(GV-4) $M(x,y,\cdot) : (0,\infty) \to [0,1]$ is continuous;

(GV-5) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$.

And $M(x,y,t)$ denote the degree of nearness between x and y with respect to t.

Example 2.4.4. Let (X,d) be a metric space, $a * b = ab$ and the mapping $M : X \times X \times (0,\infty)$ define by for all $x, y \in X$ and $t > 0$,

$$M(x,y,t) = \frac{t}{t + d(x,y)}.$$

Then $(X, M, *)$ is a fuzzy metric space, called a standard fuzzy metric space induced by (X,d). If we take $a * b = T_M(a,b)$ also is a fuzzy metric space.

Definition 2.4.5. Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be *convergent* to $x \in X$ if

$$\lim_{n\to\infty} M(x_n, x, t) = 1$$

for all $t > 0$.

Definition 2.4.6. Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be *Cauchy sequence* if

$$\lim_{m\to\infty}\lim_{n\to\infty} M(x_n, x_m, t) = 1$$

for all $t > 0$ and $m, n \in \mathbb{N}$.

Definition 2.4.7. A fuzzy metric space $(X, M, *)$ is called *complete* if every Cauchy sequence converge to a point in X.

Definition 2.4.8. [100] Let X be a nonempty set. Two mappings $f, g : X \to X$ are said to be *commuting* if $fgx = gfx$ for all $x \in X$.

Definition 2.4.9. [101] Let (X, d) be a metric space. Two mappings $f, g : X \to X$ are said to be *compatible* if

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z,$$

for some $z \in X$.

Definition 2.4.10. [102] Let X be a nonempty set. Two mappings $f, g : X \to X$ are said to be *weakly compatible* if $fgx = gfx$ for all x which $fx = gx$.

Definition 2.4.11. [90] Let (X, d) be a metric space and $f, g : X \to X$. Two mappings f and g are said to satisfy *E.A. property* if there exists a sequence $\{x_n\}$ in X such that

$$\mathit{lim}_{n\to\infty} fx_n = \mathit{lim}_{n\to\infty} gx_n = t$$

for some $t \in X$.

The class of E.A. mappings contains the class of noncompatible mappings. In a similar mode, it is said that two self-mappings of f and g of a fuzzy metric space $(X, M, *)$ satisfy E.A. property, if there exist a sequence $\{x_n\}$ in X such that fx_n and gx_n converge to t for some $t \in X$ in the sense of Definition 2.4.5.

Definition 2.4.12. [111] Let (X, d) be a metric space and $f, g : X \to X$. Two mappings f and g are said to be satisfy the *common limit in the range of g property* if there exists a sequence $\{x_n\}$ in X such that

$$\mathit{lim}_{n\to\infty} fx_n = \mathit{lim}_{n\to\infty} gx_n = gx \tag{2.4.1}$$

for some $x \in X$.

In what follows, the common limit in the range of g property will be denote by (CLRg) property.

Example 2.4.13. Let $X = [0, \infty)$ be the usual metric space. Define $f, g : X \to X$ by $fx = \frac{x}{2}$ and $gx = \frac{2x}{3}$ for all $x \in X$. We consider the sequence $x_n = \frac{1}{n}$. Since

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0 = g0. \tag{2.4.2}$$

Therefore f and g satisfy the (CLRg) property.

In a similar mode, two self-mappings f and g of a fuzzy metric space $(X, M, *)$ satisfy the (CLRg) property, if there exist a sequence $\{x_n\}$ in X such that fx_n and gx_n converge to gx for some $x \in X$ in the sense of Definition 2.4.5.

Definition 2.4.14. Let $(X, M, *)$ be a fuzzy metric space and $f, g : X \to X$. A pair of (f, g) is said to be

(i) weakly commuting [116] if

$$M(fgx, gfx, t) \geq M(fx, gx, t), \quad \text{forall } x \in X \text{ and } t > 0.$$

(ii) R-weakly commuting [116] if there exists $R > 0$ such that

$$M(fgx, gfx, t) \geq M(fx, gx, \frac{t}{R}), \quad \text{forall } x \in X \text{ and } t > 0.$$

(iii) R-weakly commuting of type (A_g) if there exists $R > 0$ such that

$$M(ffx, gfx, t) \geq M(fx, gx, \frac{t}{R}), \quad \text{forall } x \in X \text{ and } t > 0.$$

(iv) R-weakly commuting of type (A_f) if there exists $R > 0$ such that

$$M(fgx, ggx, t) \geq M(fx, gx, \frac{t}{R}), \quad \text{forall } x \in X \text{ and } t > 0.$$

(v) R-weakly commuting of type (P) if there exists $R > 0$ such that

$$M(ffx, ggx, t) \geq M(fx, gx, \frac{t}{R}), \quad \text{forall } x \in X \text{ and } t > 0.$$

Notice that

- If (f, g) is weakly commuting then (f, g) is R-weakly commuting with $R = 1$.

- (iii) and (iv) are inspired by Pathak et al. [108] whereas (v) seems to be unreported.

Example 2.4.15. Let $X = [1, \infty)$ with the usual metric d and $a * b = ab$. Let M be a usual fuzzy metric space on $(X, M, *)$ which is define by $M(x, y, t) = \dfrac{t}{t + d(x, y)}$ for all $x, y \in X$ and $t > 0$. Let $f, g : X \to X$ define by $fx = \sqrt[3]{x}$ and $gx = x^3$. Next, we will show that (f, g) is R-weakly commuting.

For $x \in [1, \infty)$ and $R, t > 0$, we have

$$M(fgx, gfx, t) = M(fx^3, g\sqrt[3]{x}, t) = M(x, x, t) = 1 \geq M(fx, gx, \frac{t}{R}).$$

Therefore, (f, g) is R-weakly commuting for all $R > 0$.

Example 2.4.16. [117] Let $X = [1, \infty)$ with the usual metric d and $a * b = ab$. Let M be a usual fuzzy metric space on $(X, M, *)$ which is define by $M(x, y, t) = \dfrac{t}{t + d(x, y)}$ for all $x, y \in X$ and $t > 0$. Let $f, g : X \to X$ define by

$$fx = \begin{cases} 1 & ; x \in \{1, 5\}; \\ 5 & ; x \in (1, 5) \cup (5, \infty) \end{cases}$$

and

$$gx = \begin{cases} 1 & ; x \in \{1, 5\}; \\ x + 4 & ; x \in (1, 5) \cup (5, \infty). \end{cases}$$

Then, (f, g) is R-weakly commuting of type (A_g) for all $R > 0$.

Example 2.4.17. Let $X = [0, 1]$ with the usual metric d and $a * b = ab$. Let M be a usual fuzzy metric space on $(X, M, *)$ which is define by $M(x, y, t) = \dfrac{t}{t + d(x, y)}$ for all $x, y \in X$ and $t > 0$. Let $f, g : X \to X$ define by

$$fx = \begin{cases} 0 & ; x \in [0, \frac{1}{100}] \cup [\frac{1}{10}, 1]; \\ \frac{x}{2} & ; x \in (\frac{1}{100}, \frac{1}{10}) \end{cases}$$

and

$$gx = \begin{cases} 0 & ; x \in [0, \frac{1}{100}] \cup [\frac{1}{10}, 1]; \\ 10x & ; x \in (\frac{1}{100}, \frac{1}{10}). \end{cases}$$

Next, we will show that (f, g) is R-weakly commuting of type (A_f).

For $x \in [0, \frac{1}{100}] \cup [\frac{1}{10}, 1]$ and $R, t > 0$, we have

$$M(fgx, ggx, t) = M(f0, g0, t) = M(0, 0, t) = 1 \geq M(fx, gx, \frac{t}{R}).$$

For $x \in (\frac{1}{100}, \frac{1}{10})$ and $R, t > 0$, we get

$$M(fgx, ggx, t) = M(f10x, g10x, t) = M(0, 0, t) = 1 \geq M(fx, gx, \frac{t}{R}).$$

Now, we have

$$M(ffx, gfx, t) \geq M(fx, gx, \frac{t}{R})$$

for all $x \in X$ and $R, t > 0$.

Therefore, (f, g) is R-weakly commuting of type (A_f) for all $R > 0$.

Example 2.4.18. Let $X = [0, 1]$ with the usual metric d and $a * b = ab$. Let M be a usual fuzzy metric space on $(X, M, *)$ which is define by $M(x, y, t) = \dfrac{t}{t + d(x, y)}$ for all $x, y \in X$ and $t > 0$. Let $f, g : X \to X$ define by

$$fx = \begin{cases} 0 & ; x \in [0, \frac{1}{4}] \cup [\frac{1}{2}, 1]; \\ \frac{x}{4} & ; x \in (\frac{1}{4}, \frac{1}{2}) \end{cases}$$

and

$$gx = \begin{cases} 0 & ; x \in [0, \frac{1}{4}] \cup [\frac{1}{2}, 1]; \\ \frac{x}{2} & ; x \in (\frac{1}{4}, \frac{1}{2}). \end{cases}$$

Next, we will show that (f, g) is R-weakly commuting of type (P).

For $x \in [0, \frac{1}{4}] \cup [\frac{1}{2}, 1]$ and $R, t > 0$, we have

$$M(ffx, ggx, t) = M(f0, g0, t) = M(0, 0, t) = 1 = M(fx, gx, \frac{t}{R}).$$

For $x \in (\frac{1}{4}, \frac{1}{2})$ and $R, t > 0$, we get

$$M(ffx, ggx, t) = M(f\frac{x}{4}, g\frac{x}{2}, t) = M(0, 0, t) = 1 \geq M(fx, gx, \frac{t}{R}).$$

Now, we have

$$M(ffx, gfx, t) \geq M(fx, gx, \frac{t}{R})$$

for all $x \in X$ and $R, t > 0$.

Therefore, (f, g) is R-weakly commuting of type (P) for all $R > 0$.

It is well known (Imdad and Ali [**?**]) the independence of R-weakly commutativity of type (A_g) with R-weakly commutativity of type (A_f) or (P) and the independence of R-weakly commutativity of type (A_f) with R-weakly commutativity of type (P).

2.5 Cone Metric Spaces

In this dissertation, we denote from now on $\underbrace{X \times X \cdots X \times X}_{k \text{ terms}}$ by X^k where $k \in \mathbb{N}$ and X is a non-empty set. The following definitions and results will be needed in the sequel.

Definition 2.5.1. Let E be a real Banach space and 0_E be the zero element in E. A subset P of E is called a cone if satisfy the following conditions:

(a) P is closed, non-empty and $P \neq \{0_E\}$,

(b) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P$ imply that $ax + by \in P$,

(c) $P \cap (-P) = \{0_E\}$.

Given a cone P of real Banach space E, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \ll y$ for $y - x \in \text{Int}(P)$, where $\text{Int}(P)$ stands for interior of P. Also we will use $x \prec y$ to indicate that $x \preceq y$ and $x \neq y$.

The cone P in normed space $(E, \|\cdot\|)$ is called normal whenever there is a number $k > 0$ such that for all $x, y \in E$, $0_E \preceq x \preceq y$ implies $\|x\| \leq k\|y\|$. The least positive number satisfying this norm inequality is called the normal constant of P. In 2008, Rezapour and Hamlbarani [82] showed that there are no normal cones with normal constant $k < 1$.

In what follows we always suppose that E is a real Banach space with cone P satisfying $\text{Int}(P) \neq \emptyset$ (such cones are called *solid*).

Definition 2.5.2 ([74, 89])**.** Let X be a non-empty set. Suppose that $d : X \times X \to E$ satisfies the following conditions:

(d1) $0_E \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0_E$ if and only if $x = y$,

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(d3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric or K-metric on X and (X, d) is called a cone metric space or $K-$metric space.

Remark 2.5.3. A concept of a $K-$metric space is more general than a concept of a metric space, because each metric space is a $K-$metric space where $X = \mathbb{R}$ with usual norm and cone $P = [0, \infty)$.

Definition 2.5.4 ([74])**.** Let X be a K-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is

$(C1)$ a Cauchy sequence if and only if for each $c \in E$ with $c \gg 0_E$ there is some $k \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq k$,

$(C2)$ a convergent sequence if and only if for each $c \in E$ with $c \gg 0_E$ there is some $k \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n \geq k$, where $x \in X$. This limit is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Remark 2.5.5. Every convergent sequence in a K-metric space X is a Cauchy sequence but the converse is not true.

Definition 2.5.6. A K-metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Definition 2.5.7 ([83])**.** Let X be a nonempty set. An element $(x, y, z) \in X^3$ is called a tripled fixed point of a given mapping $F : X^3 \to X$ if $x = F(x, y, z)$, $y = F(y, z, x)$ and $z = F(z, x, y)$.

Berinde and Borcut [69] defined differently the notion of a tripled fixed point in the case of ordered sets in order to keep true the mixed monotone property.

Definition 2.5.8 ([67])**.** Let X be a non-empty set. An element $(x, y, z) \in X^3$ is called

(i) a tripled coincidence point of mappings $F : X^3 \to X$ and $g : X \to X$ if $gx = F(x, y, z)$, $gy = F(y, x, z)$ and $gz = F(z, x, y)$. In this case (gx, gy, gz) is called a tripled point of coincidence;

(ii) a common tripled fixed point of mappings $F : X^3 \to X$ and $g : X \to X$ if $x = gx = F(x, y, z)$, $y = gy = F(y, z, x)$ and $z = gz = F(z, x, y)$.

Example 2.5.9. Let $X = \mathbb{R}$. We define $F : X^3 \to X$ and $g : X \to X$ as follows

$$F(x, y, z) = \left(\frac{2x + 2y}{\pi}\right) \sin(2z) \quad \text{and} \quad gx = 1 + \pi - 4x$$

for all $x, y, z \in X$. Then $(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$ is a tripled coincidence point of F and g, and $(1, 1, 1)$ is a tripled point of coincidence.

Definition 2.5.10 ([67])**.** Let X be a non-empty set. Mappings $F : X^3 \to X$ and $g : X \to X$ are called W-compatible if

$$F(gx, gy, gz) = g(F(x, y, z))$$

whenever $F(x, y, z) = gx$, $F(y, z, x) = gy$ and $F(z, y, x) = gz$.

Example 2.5.11. Let $X = [0, 1)$. Define $F : X^3 \to X$ and $g : X \to X$ as follows

$$F(x, y, z) = \frac{x^2 + y^2 + z^2}{12} \quad \text{and} \quad gx = \frac{x}{4}$$

for all $x, y, z \in X$. One can show that (x, y, z) is a tripled coincidence point of F and g if and only if $x = y = z = 0$. Since $F(g0, g0, g0) = g(F(0, 0, 0))$, we get F and g are W-compatible.

2.6 Interesting Problems

Convex Feasibility Problem

The convex feasibility problem (CFP) is the problem of finding a point in the intersection of finitely many closed convex sets in a real Hilbert spaces H. That is, finding an $x \in \cap_{m=1}^{r} C_m$, where $r \geq 1$ is an integer and each C_m is a nonempty closed and convex subset of H. Many problems in mathematics, for example in physical sciences, in engineering and in real-world applications of various technological innovations can be modeled as CFP. There is a considerable investigation on CFP in the setting of Hilbert spaces which captures applications in various disciplines such as image restoration [57, 27] computer tomography [28] and radiation therapy treatment planning [29].

Variational Inequality Problem

The variational inequality problem is the problem of finding a point $u \in C$ such that

$$\langle Ax, x - y \rangle \geq 0, \qquad \forall y \in C. \tag{2.6.1}$$

where $A : H \to H$ is nonlinear mappings. We use $VI(C, A)$ to denote the set of solutions of the variational inequality (2.6.1). It is easy to see that an element $x \in C$ is a solution to the variational inequality problem if and only if x is a fixed point of the mapping $P_C(I-\lambda A)$, that is $x = P_C(I-\lambda A))x \Leftrightarrow x \in VI(C,A)$. Therefore, fixed point algorithms can be applied to solve $VI(C,A)$. It is well known that the variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral, and equilibrium problems; which arise in several branches of pure and applied sciences in a unified and general framework. Several numerical methods have been developed for solving variational inequalities and related optimization problems, see [12, 13, 49, 21, 6, 18, 19, 20, 14, 16, 17] and the references therein.

Generalized Variational Inequality Problem

the generalized variational inequality problem is the problem of finding $x \in C$ such that

$$\langle x - \hat{\lambda}Bx + \lambda Ax, x - y\rangle \geq 0, \forall y \in C,$$

where $A, B : C \to H$, $\hat{\lambda}$ and λ are two positive constants. We use $GVI(C,B,A)$ to denote *the set of solutions of the generalized variational inequality.* It is easy to see that an element $x \in C$ is a solution to the generalized variational inequality problem if and only if x is a fixed point of the mapping $P_C(\hat{\lambda}B - \lambda A)$, that is $x \in F(P_C(\hat{\lambda}B - \lambda A)) \Leftrightarrow x \in GVI(C,B,A)$. Therefore, fixed point algorithms can be applied to solve $GVI(C,B,A)$. If $B = I$ and $\hat{\lambda} = 1$, then the generalized variational inequality problem is reduced to the variational inequality problem.

Hierarchical Fixed Point Problem

The hierarchical fixed point problem is the problem of finding a point $x^* \in F(T)$ such that

$$\langle Fx^*, x - x^*\rangle \geq 0, \qquad \forall x \in F(T),$$

where $F : H \to H$ is nonlinear mappings and $T : C \to C$ is nonlinear mappings. It is well known that the iterative methods for finding hierarchical fixed points of nonexpansive mappings can also be used to solve a convex minimization problem; see, for example, [50, 26] and the references therein.

Hierarchical Variational Inequality problem

The hierarchical variational inequality problem (HVIP) is the problem of finding a point $x^* \in VI(C,A)$ such that

$$\langle Fx^*, x - x^* \rangle \geq 0, \qquad \forall x \in VI(C,A), \tag{2.6.2}$$

where $F, A : H \to H$ are nonlinear mappings. Many problems in mathematics, for example the signal recovery [30], the power control problem [31] and the beamforming problem [32] can be considered in the framework of this kind of the hierarchical variational inequality problems.

Hierarchical Generalized Variational Inequality Problem

The hierarchical generalized variational inequality problem (HGVIP) is the problem of finding a point $\tilde{x} \in GVI(C,B,A)$ such that

$$\langle F\tilde{x}, x - \tilde{x} \rangle \leq 0, \quad \forall x \in GVI(C,B,A), \tag{2.6.3}$$

where $GVI(C,B,A)$ is the solution set of the generalized variational inequality. If If the set $GVI(C,B,A)$ is replaced by the set $VI(C,A)$, the solution set of the variational inequality, then the HGVIP is called a hierarchical variational inequality problems (HVIP). Many problems in mathematics, for example the signal recovery[57], the power control problem[7] and the beamforming problem[32] can be modeled as HGVIP.

Equilibrium Problem

The equilibrium problem for finding $x \in H$ such that $F(x,y) \geq 0, \quad \forall y \in H$. where $F : H \times H \longrightarrow \mathbb{R}$ is a bifunction. The set of solutions of the equilibrium problem is denoted by $EP(F)$, that is, $EP(F) = \left\{ \ x \in H : F(x,y) \geq 0, \quad \forall H \in C \ \right\}$. Numerous problems in physics, optimization, and economics reduce to find a solution of the equilibrium problem see, for example, [120, 60, 121, 118, 122] and the references therein. If $F(x,y) = \langle Ax, y - x \rangle$ for all $x, y \in C$. Then, $z \in EP(F)$ if and only if $\langle Az, y - z \rangle \geq 0$ for all $y \in C$, that is, z is a solution of the variational inequality.

Variational Inclusion Problem

The *variational inclusion problem* is as follows: Find $x \in H$ such that

$$\theta \in A(x) + M(x), \tag{2.6.4}$$

where $A : H \to H$ is a single-valued nonlinear mapping, $M : H \to 2^H$ is a set-valued mapping and θ is the zero vector in H. We denote the set of solution of this problem by $I(A, M)$.

Remark 2.6.1. (1) If $M = \partial\phi : H \to 2^H$ in (2.6.4), where $\phi : H \to \mathbb{R}$ is proper convex lower semi-continuous and $\partial\phi$ is the sub-differential of ϕ, then the variational inclusion (2.6.4) is equivalent to the following problem: Find $x \in H$ such that

$$\langle Ax, v - x\rangle + \phi(y) - \phi(x) \geq 0, \quad \forall v, y \in H,$$

which is called the *mixed quasi-variational inequality* in Noor [43].

(2) Let $M = \partial\delta_C$ in (2.6.4), where C is a nonempty closed convex subset of H and $\delta_C : H \to [0, \infty)$ is the *indicator function* of C, i.e.,

$$\delta_C(x) = \begin{cases} 0, & x \in C; \\ +\infty, & \text{otherwise.} \end{cases}$$

Then the variational inclusion (2.6.4) is equivalent to the following problem: Find $x \in H$ such that

$$\langle Ax, v - x\rangle \geq 0, \quad \forall v \in H,$$

which is called *Hartman-Stampacchia's variational inequality.*

Remark 2.6.2. (1) If $H = \mathbb{R}^m$, then the problem (2.6.4) becomes the generalized equation introduced by Robinson [44];

(2) If $A = 0$, then the problem (2.6.4) becomes the inclusion problem introduced by Rockafellar [45].

The problem (2.6.4) is the most widely use for the study of optimal solutions in many related areas including mathematical programming, complementarity, variational inequalities, optimal control and many other fields. Many kinds of variational inclusions problems have been improved, extended and generalized in recent years by many authors.

The System of Hierarchical Variational Inequality Problem

The system of hierarchical variational inequality problem: find $(x^*, y^*, z^*) \in VI(C, A_1)\times$

$VI(C, A_2) \times VI(C, A_3)$ such that for given positive real numbers ρ, η and ξ, the following inequalities hold:

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in VI(C, A_1), \\ \langle \eta F(z^*) + y^* - z^*, y - y^* \rangle \geq 0, & \forall y \in VI(C, A_2), \\ \langle \xi F(x^*) + z^* - x^*, z - z^* \rangle \geq 0, & \forall z \in VI(C, A_3). \end{cases} \tag{2.6.5}$$

where $F, A_1, A_2, A_3 : H \to H$ are mappings.

Some special cases of the system of hierarchical variational inequality problem (2.6.5):

(*I*) If $A_i = I - T_i$, where $T_i : H \to H$ is a nonlinear mapping for each $i = 1, 2, 3$, in (2.6.5), then $VI(C, A_i) = F(T_i)$ and the system of hierarchical variational inequality problem (2.6.5) reduces to the following *a system of hierarchical optimization problem*: finding $(x^*, y^*, z^*) \in F(T_1) \times F(T_2) \times F(T_3)$ such that

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in F(T_1), \\ \langle \eta F(z^*) + y^* - z^*, y - y^* \rangle \geq 0, & \forall y \in F(T_2), \\ \langle \xi F(x^*) + z^* - x^*, z - z^* \rangle \geq 0, & \forall z \in F(T_3). \end{cases} \tag{2.6.6}$$

(*II*) If $T_i = P_{K_i}$ for each $i = 1, 2, 3$, where P_{K_i} is the metric projection from H onto a nonempty closed convex subset K_i in (2.6.6), then it is clear that the $VI(C, A_i) = F(T_i) = K_i$ and the system of hierarchical optimization problem(2.6.6) reduces to the following problem: finding $(x^*, y^*, z^*) \in K_1 \times K_2 \times K_3$ such that

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in K_1, \\ \langle \eta F(z^*) + y^* - z^*, y - y^* \rangle \geq 0, & \forall y \in K_2, \\ \langle \xi F(x^*) + z^* - x^*, z - z^* \rangle \geq 0, & \forall z \in K_3. \end{cases} \tag{2.6.7}$$

(*III*) If $K_1 = K_2 = K_3$, then the system of optimization problem(2.6.7) reduces to the following *a system of variational inequality problem*: finding $(x^*, y^*, z^*) \in K_1 \times K_1 \times K_1$ such that

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in K_1, \\ \langle \eta F(z^*) + y^* - z^*, y - y^* \rangle \geq 0, & \forall y \in K_1, \\ \langle \xi F(x^*) + z^* - x^*, z - z^* \rangle \geq 0, & \forall z \in K_1. \end{cases} \tag{2.6.8}$$

(IV) If $\xi = 0, \rho, \eta > 0, VI(C, A_1) = VI(C, A_3)$ and $x^* = z^*$ in (2.6.5) then the system of hierarchical variational inequality problem (2.6.5) reduces to the following *bi-level hierarchical variational inequality problem*: finding $(x^*, y^*) \in VI(C, A_1) \times VI(C, A_2)$ such that

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in VI(C, A_1), \\ \langle \eta F(x^*) + y^* - x^*, y - y^* \rangle \geq 0, & \forall y \in VI(C, A_2) \end{cases} \tag{2.6.9}$$

(V) In (2.6.9), if $A_i = I - T_i$, for each $i = 1, 2$, then bi-level hierarchical variational inequality problem (2.6.9) reduces to the following *bi-level hierarchical optimization problem*: finding $(x^*, y^*) \in F(T_1) \times F(T_2)$ such that

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in F(T_1), \\ \langle \eta F(x^*) + y^* - x^*, y - y^* \rangle \geq 0, & \forall y \in F(T_2) \end{cases} \tag{2.6.10}$$

which was Kraikaew and Saejung[58].

(VI) In (2.6.10), if $T_i = P_{K_i}$ for each $i = 1, 2,$ then bi-level hierarchical optimization problem (2.6.10) reduces to the following problem : finding $(x^*, y^*) \in K_1 \times K_2$ such that

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in K_1, \\ \langle \eta F(x^*) + y^* - x^*, y - y^* \rangle \geq 0, & \forall y \in K_2 \end{cases} \tag{2.6.11}$$

(VII) In (2.6.11), If $K_1 = K_2$ then the bi-level optimization problem (2.6.11) reduces to the following *bi-level variational inequality problem*: finding $(x^*, y^*) \in K_1 \times K_1$ such that

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in K_1, \\ \langle \eta F(x^*) + y^* - x^*, y - y^* \rangle \geq 0, & \forall y \in K_1 \end{cases} \tag{2.6.12}$$

$(VIII)$ In (2.6.5), If $\xi = \eta = 0, \rho > 0, VI(C, A_1) = VI(C, A_2) = VI(C, A_3)$ and $x^* = y^* = z^*$ then the system of hierarchical variational inequality problem (2.6.5) reduces to the following *a hierarchical variational inequality problem*: finding $x^* \in VI(C, A_1)$ such that

$$\langle F(y^*), x - x^* \rangle \geq 0, \qquad \forall x \in VI(C, A_1), \tag{2.6.13}$$

(IX) In (2.6.13), if $A_i = I - T_1$ then the hierarchical variational inequality problem (2.6.13) reduces to the following *a hierarchical fixed point problem*: finding $x^* \in F(T_1)$ such that

$$\langle F(y^*), x - x^* \rangle \geq 0, \qquad \forall x \in F(T_1), \tag{2.6.14}$$

(X) In (2.6.14), if $T_1 = P_{K_1}$ then the hierarchical fixed point problem (2.6.14) reduces to the following *a classic variational inequality problem*: finding $x^* \in K_1$ such that

$$\langle F(y^*), x - x^* \rangle \geq 0, \qquad \forall x \in K_1, \tag{2.6.15}$$

The System of Hierarchical Variational Inclusion Problem

The system of hierarchical variational inclusion problem: find $(x^*, y^*, z^*) \in \Omega_1 \times \Omega_2 \times \Omega_3$ such that for given positive real numbers ρ, η and ξ, the following inequalities hold:

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in \Omega_1, \\ \langle \eta F(z^*) + y^* - z^*, y - y^* \rangle \geq 0, & \forall y \in \Omega_2, \\ \langle \xi F(x^*) + z^* - x^*, z - z^* \rangle \geq 0, & \forall z \in \Omega_3, \end{cases} \tag{2.6.16}$$

where Ω_i is a solution set of the variational inclusion problem $I(A_i, M_i)$, for each $i = 1, 2, 3$.

Some special cases of the system of hierarchical variational inclusion problem (2.6.16) as follows:

(I) If $\xi = 0, \rho, \eta > 0, \Omega_1 = \Omega_3$ and $x^* = z^*$ in (2.6.16) then the system of hierarchical variational inclusion problem (2.6.16) reduces to the following *bi-level hierarchical variational inclusion problem*: finding $(x^*, y^*) \in \Omega_1 \times \Omega_2$ such that

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in \Omega_1, \\ \langle \eta F(x^*) + y^* - x^*, y - y^* \rangle \geq 0, & \forall y \in \Omega_2, \end{cases} \tag{2.6.17}$$

which was studied by Chang et al. [59].

(II) In (2.6.16), If $\xi = \eta = 0, \rho > 0, \Omega_1 = \Omega_2 = \Omega_3$ and $x^* = y^* = z^*$ then the system of hierarchical variational inclusion problem (2.6.16) reduces to the following *a hierarchical variational inclusion problem*: finding $x^* \in \Omega_1$ such that

$$\langle F(y^*), x - x^* \rangle \geq 0, \qquad \forall x \in \Omega_1. \tag{2.6.18}$$

2.7 Useful Lemmas

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$ and C is a nonempty closed convex subset of H. We use $F(T)$ to denote the set of *fixed points* of $T : H \to H$, that is, $F(T) = \{x \in H : Tx = x\}$.
Let $A, T : H \to H$ be a nonlinear mappings. Recall the following definitions:

(a) A is said to be monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \ \ \forall x, y \in H.$$

(b) A is said to be ρ-strongly monotone if there exists a positive real number $\rho > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \rho \|x - y\|^2, \ \ \forall x, y \in H.$$

(c) A is said to be η-cocoercive or η-inverse strongly monotone if there exists a positive real number $\eta > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \eta \|Ax - Ay\|^2, \ \ \forall x, y \in H.$$

(d) A is said to be relaxed η-cocoercive if there exists a positive real number $\eta > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq (-\eta) \|Ax - Ay\|^2, \ \ \forall x, y \in H.$$

(e) A is said to be relaxed (η, ρ)-cocoercive if there exists a positive real number $\eta, \rho > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq (-\eta) \|Ax - Ay\|^2 + \rho \|x - y\|^2, \ \ \forall x, y \in H.$$

(f) G is said to be L-Lipschitzian on C if there exists a positive real number $L > 0$ such that

$$\|A(x) - A(y)\| \leq L \|x - y\|, \ \ \forall x, y \in H.$$

(g) A is said to be k-contraction if there exists a positive real number $k \in (0, 1)$ such that

$$\|A(x) - A(y)\| \leq k \|x - y\|, \ \ \forall x, y \in H.$$

(*h*) A mapping T is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \qquad \forall x, y \in H.$$

It is well known that $F(T)$ is a closed convex set, if T is *nonexpansive*.

(*i*) A mapping T is called *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\|, \qquad \forall x \in H, p \in F(T).$$

It should be noted that T is quasi-nonexpansive if and only if $\forall x \in H, p \in F(T)$

$$\langle x - Tx, x - p\rangle \geq \frac{1}{2}\|x - Tx\|^2.$$

(*j*) A mapping T is called *strongly quasi-nonexpansive* if T is quasi-nonexpansive and $x_n - Tx_n \to 0$,
whenever $\{x_n\}$ is a bounded sequence in H and $\|x_n - p\| - \|Tx_n - p\| \to 0$ for some $p \in F(T)$.

(*j*) A mapping T is called *k-strict pseudo-contraction* if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \;\; \forall x, y \in C.$$

(*l*) A mapping A is said to be *strongly positive* if there exists a constant $\mu > 0$ such that

$$\langle Ax, x\rangle \geq \mu\|x\|^2, \quad \forall x \in H. \tag{2.7.1}$$

Lemma 2.7.1. *[47] Let H be a Hilbert space, C a closed convex subset of H and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$; in particular, if $y = 0$ then $x \in F(T)$.*

Lemma 2.7.2. *[22] Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \alpha \leq \|A\|^{-1}$. Then $\|I-\alpha A\| \leq 1-\alpha\bar{\gamma}$.*

Lemma 2.7.3. *[23] Let $F : C \to C$ be a η-strongly monotone and L-Lipschitzian operator with $L > 0, \eta > 0$. Assume that $0 < \mu < 2\eta/L^2, \tau = \mu(\eta - \mu L^2/2)$ and $0 < t < 1$. Then $\|(I - \mu tF)x - (I - \mu tF)y\| \leq (1 - t\tau)\|x - y\|$.*

Lemma 2.7.4. *[23] Let H be a real Hilbert space, $f : H \to H$ a contraction with coefficient $0 < k < 1$, and $G : H \to H$ a L-Lipschitzian continuous operator and ξ-strongly monotone operator with $L > 0$, $\xi > 0$. Then for $0 < \gamma < \mu\xi/k$,*

$$\langle x - y, (\mu G - \gamma f)x - (\mu G - \gamma f)y\rangle \geq (\mu\xi - \gamma k)\|x - y\|^2, \forall x, y \in H.$$

That is, $\mu G - \gamma f$ is strongly monotone with coefficient $\mu\xi - \gamma k$.

Lemma 2.7.5. [54] *Let H be a Hilbert space and C be a nonempty closed and convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Then*

$$\|Tx - x\|^2 \leq 2\langle x - Tx, x - x'\rangle, \quad \forall x' \in F(T), \forall x \in C.$$

Lemma 2.7.6. *[48] Assume that $\{a_n\}$ is a sequence of nonnegative numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

1. $\sum_{n=1}^{\infty} \gamma_n = \infty$,

2. $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7.7. *[26] Let $B : H \to H$ be β-strongly monotone and L-Lipschitz continuous and let $\mu \in (0, \frac{2\beta}{L^2})$. For $\lambda \in [0, 1]$, define $T_\lambda : H \to H$ by $T_\lambda(x) := x - \lambda\mu B(x)$ for all $x \in H$. Then, for all $x, y \in H$,*

$$\|T_\lambda(x) - T_\lambda(y)\| \leq (1 - \lambda\tau)\|x - y\|$$

hold, where $\tau := 1 - \sqrt{1 - \mu(2\beta - \mu L^2)} \in (0, 1]$.

Definition 2.7.8. Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping. Then the mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), u \in H$$

is called *the resolvent operator associated with M*, where λ is any positive number and I is the identity mapping.

Proposition 2.7.9. *[119] Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping, and let $A : H \to H$ be an α-inverse-strongly monotone mapping. Then the following conclusions hold.*

1. *The resolvent operator $J_{M,\lambda}$ associated with M is single-valued and nonexpansive for all $\lambda > 0$.*

2. *The resolvent operator $J_{M,\lambda}$ is 1-inverse-strongly monotone, i.e.,*

$$\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\|^2 \leq \langle x - y, J_{M,\lambda}(x) - J_{M,\lambda}(y)\rangle, \forall x, y \in H.$$

3. *$u \in H$ is a solution of the variational inclusion (2.6.4) if and only if $u = J_{M,\lambda}(u - \lambda Au), \forall \lambda > 0$, i.e., u is a fixed point of the mapping $J_{M,\lambda}(I - \lambda A)$. Therefore we have*

$$\Omega = F(J_{M,\lambda}(I - \lambda A)), \forall \lambda > 0,$$

where Ω is the set of solutions of variational inclusion problem (2.6.4).

4. *If $\lambda \in (0, 2\alpha]$, then Ω is a closed convex subset in H.*

Lemma 2.7.10. *[59] Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping, $A : H \to H$ be an α-inverse-strongly monotone mapping and let Ω be the set of solutions of variational inclusion problem and $\Omega \neq \emptyset$. Then the following statements hold.*

1. *If $\lambda \in (0, 2\alpha]$, then the mapping $K : H \to H$ defined by*

$$K := J_{M,\lambda}(I - \lambda A)$$

is quasi-nonexpansive, where I is the identity mapping and $J_{M,\lambda}$ is the resolvent operator associated with M.

2. *The mapping $I - K : H \to H$ is demiclosed at zero, i.e., for any sequence $\{x_n\} \subset H$, if $x_n \rightharpoonup x$ and $(I - K)x_n \to 0$, then $x = Kx$.*

3. *For any $\beta \in (0, 1)$, the mapping K_β defined by*

$$K_\beta = (1 - \beta)I + \beta K$$

is a strongly quasi-nonexpansive mapping and $F(K_\beta) = F(K)$.

4. $I - K_\beta, \beta \in (0,1)$ is demiclosed at zero.

Lemma 2.7.11. *[22]) Let H be a Hilbert space, C a closed convex subset of H, $f : C \to H$ be a contraction with coefficient $0 < \rho < 1$, $T : C \to C$ be nonexpansive mapping. Let A be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$. Then, for $0 < \gamma < \bar{\gamma}/\rho$, for $x, y \in C$,*

1. *the mapping $(I - f)$ is strongly monotone with coefficient $(1 - \rho)$ that is*

$$\langle x - y, (I - f)x - (I - f)y \rangle \geq (1 - \rho)\|x - y\|^2.$$

2. *the mapping $(I - T)$ is monotone, that is*

$$\langle x - y, (I - T)x - (I - T)y \rangle \geq 0.$$

3. *the mapping $(A - \gamma f)$ is strongly monotone with coefficient $\bar{\gamma} - \gamma\rho$ that is*

$$\langle x - y, (A - \gamma f)x - (A - \gamma f)y \rangle \geq (\bar{\gamma} - \gamma\rho)\|x - y\|^2.$$

Lemma 2.7.12. *[119] Let $A : H \to H$ be an α-inverse-strongly monotone mapping. Then*

1. *A is an $\frac{1}{\alpha}$-Lipschitz continuous and monotone mapping;*
2. *For any constant $\lambda > 0$, we have*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2;$$

3. *If $\lambda \in (0, 2\alpha]$, then $I - \lambda A$ is a nonexpansive mapping, where I is the identity mapping on H.*

Lemma 2.7.13. *Let $x \in H$ and $z \in C$ be any points. Then we have the following:*

1. *That $z = P_C[x]$ if and only if there holds the relation:*

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in C.$$

2. *That $z = P_C[x]$ if and only if there holds the relation:*

$$\|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2, \quad \forall y \in C.$$

3. *There holds the relation:*

$$\langle P_C[x] - P_C[y], x - y\rangle \geq \|P_C[x] - P_C[y]\|^2, \quad \forall x, y \in H.$$

Consequently, P_C is nonexpansive and monotone.

4. $u \in VI(C, A) \Leftrightarrow u \in F(P_C(I - \lambda A)) \quad , \forall\lambda > 0.$

Proposition 2.7.14. *[123] Let C be a bounded closed convex subset of a real Hilbert space H and let A be an α inverse strongly-monotone mapping of H into H. Then, $VI(C, A)$ is nonempty.*

Lemma 2.7.15. *[33] For $x, y \in H$ and $\omega \in (0, 1)$, the following statements hold:*

1. $\|\lambda x+(1-\lambda)y\|^2 = \lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)\|x-y\|^2, \forall x, y \in H,\ \lambda \in [0, 1];$

2. $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle;$

3. $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle,\ \forall x, y \in H.$

4. $\|(1 - \omega)x + \omega y\|^2 = (1 - \omega)\|x\|^2 + \omega\|y\|^2 - \omega(1 - \omega)\|x - y\|^2.$

Lemma 2.7.16. *[25] Let $\{a_n\}$ be a sequence of real numbers, and there exists a subsequence $\{a_{m_j}\}$ of $\{a_n\}$ such that $a_{m_j} < a_{m_j+1}$ for all $j \in N$, where N is the set of all positive integers. Then there exists a nondecreasing sequence $\{n_k\}$ of N such that $\lim_{k\to\infty} n_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in N$:*

$$a_{n_k} \leq a_{n_k+1} \quad \text{and} \quad a_k \leq a_{n_k+1}.$$

In fact, n_k is the largest number n in the set $\{1, 2, \ldots, k\}$ such that $a_n < a_{n+1}$ holds.

Lemma 2.7.17. *[58] Let $\{a_n\} \subset [0, \infty), \{\alpha_n\} \subset [0, 1), \{b_n\} \subset (-\infty, +\infty), \hat{\alpha} \in [0, 1)$ be such that*

1. *$\{a_n\}$ is a bounded sequence;*

2. $a_{n+1} \leq (1 - \alpha_n)^2 a_n \hat{\alpha}\sqrt{a_n}\sqrt{a_{n+1}} + \alpha_n b_n, \forall n \geq 1;$

3. *whenever $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ satisfying*

$$\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0,$$

it follows that $\limsup_{k\to\infty} b_{n_k} \leq 0$;

4. $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7.18. *Let $A : H \to H$ be an α-inverse-strongly monotone mapping and let $VI(C, A)$ be the set of solutions of variational inequality problem (2.6.1). Then the following statements hold.*

1. *If $\lambda \in (0, 2\alpha]$, then the mapping $K : H \to C$ defined by*

$$K := P_C(I - \lambda A)$$

is quasi-nonexpansive, where I is the identity mapping.

2. *The mapping $I - K : H \to H$ is demiclosed at zero, i.e., for any sequence $\{x_n\} \subset H$, if $x_n \rightharpoonup x$ and $(I - K)x_n \to 0$, then $x = Kx$.*

3. *For any $\beta \in (0, 1)$, the mapping K_β defined by*

$$K_\beta = (1 - \beta)I + \beta K \tag{2.7.2}$$

is a strongly quasi-nonexpansive mapping and $F(K_\beta) = F(K)$.

4. *$I - K_\beta, \beta \in (0, 1)$ is demiclosed at zero.*

Proof. 1. By Lemma 2.7.13 and Proposition 2.7.14, the mapping K is nonexpansive and $VI(C, A) = F(K) \neq \emptyset$. This implies that K is quasi-nonexpansive.

2. Since K is a nonexpansive mapping on C, $I - K$ is demiclosed at zero.

3. It obvious that $F(K_\beta) = F(K)$.
Next we prove that $K_\beta, \beta \in (0, 1)$ is a strongly quasi-nonexpansive mapping. Let $\{x_n\}$ be any bounded sequence in H and let $p \in K_\beta$ be a given point such that

$$\|x_n - p\| - \|K_\beta x_n - p\| \to 0. \tag{2.7.3}$$

First, we prove that $K_\beta, \beta \in (0, 1)$ is a quasi-nonexpansive mapping.
By (2.7.2) and K is quasi-nonexpansive, we have

$$\begin{aligned}
\|K_\beta x - p\| &= \|(1 - \beta)[x - p] + \beta(Kx - p)\| \\
&\leq (1 - \beta)\|x - p\| + \beta\|Kx - p\| \\
&\leq \|x - p\|, \quad \forall x \in C.
\end{aligned}$$

Therefore, K_β is a quasi-nonexpansive mapping.

Next we prove that $\|K_\beta x_n - x_n\| \to 0$.

In fact, it follows from (2.7.2) that

$$\begin{aligned} \|K_\beta x_n - p\|^2 &= \|x_n - p - \beta(x_n - Kx_n)\|^2 \\ &= \|x_n - p\|^2 - 2\beta\langle x_n - p, x_n - Kx_n\rangle + \beta^2\|x_n - Kx_n\|^2 \\ &\leq \|x_n - p\|^2 - \beta(1-\beta)\|x_n - Kx_n\|^2. \end{aligned}$$

By (2.7.3), we have

$$\beta(1-\beta)\|x_n - Kx_n\|^2 \leq \|x_n - p\|^2 - \|K_\beta x_n - p\|^2 \to 0.$$

Since $\beta(1-\beta) > 0$ then $\|x_n - Kx_n\| \to 0$. Hence

$$\|x_n - K_\beta x_n\| = \beta\|x_n - Kx_n\| \to 0.$$

4. Since $I - K_\beta = \beta(I - K)$ and IK is demiclosed at zero, hence $I - K_\beta$ is demiclosed at zero. This completes the proof.

□

Lemma 2.7.19. *[46] Let C be a closed convex subset of a strictly convex Banach space E. Let $T_m : C \to C$ be a nonexpansive mappings for each $1 \leq m \leq r$, where r is some integer. Suppose that $\cap_{m=1}^{r} F(T_m)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{m=1}^{r} \lambda_n = 1$. Then the mapping $S : C \to C$ defined by*

$$Sx = \sum_{m=1}^{r} \lambda_m T_m x, \quad \forall x \in C,$$

is well defined, nonexpansive and $F(S) = \cap_{m=1}^{r} F(T_m)$ holds.

Lemma 2.7.20. *[15] Let C be a nonempty closed convex subset of H and let $r > 0$ and $x \in H$. Let $F : C \times C \longrightarrow \mathbb{R}$ satisfying*

(A1) *$F(x, x) = 0$ for all $x \in C$;*

(A2) *F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;*

(A3) *for each $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$;*

(A4) *for each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.*

Then, there exists $z \in C$ such that

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.7.21. *[42] Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r^F : H \to C$ as follows:*

$$T_r^F(x) = \left\{ \; z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C \; \right\}$$

for all $x \in H$. Then, the following hold:

1. *T_r^F is single- valued;*
2. *T_r^F is firmly nonexpansive, i.e., $\forall x, y \in H$,*

$$\left\| T_r^F x - T_r^F y \right\|^2 \leq \left\langle T_r^F x - T_r^F y, x - y \right\rangle;$$

3. *$F(T_r^F) = EP(F)$; and*
4. *$EP(F)$ is closed and convex.*

Lemma 2.7.22. *[52] Let H be a real Hilbert space. Then for any $x_1, x_2, ..., x_k \in H$ and $a_1, a_2, ..., a_k \in [0,1]$ with $\sum_{i=1}^{k} a_i = 1, k \in \aleph$, we have*

$$\left\| \sum_{i=1}^{k} a_i x_i \right\|^2 = \sum_{i=1}^{k} a_i \|x_i\|^2 - \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} a_i a_j \|x_i - x_j\|^2.$$

CHAPTER 3 COMMON FIXED POINT AND COMMON TRIPLED FIXED POINT THEOREMS

3.1 Common Fixed Point Theorems in Fuzzy Metric Spaces

In this section, we establish the existence of common fixed point theorems for R-weakly commuting in fuzzy metric spaces by using the common limit in the range property. We also give the example to validate our main results.

Let Θ denote the class of those functions $\theta : (0,1]^5 \to [0,1]$ such that θ is continuous and

$$\theta(x,1,1,x,x) = x.$$

There are examples of $\theta \in \Theta$:

1. $\theta_1(x_1,x_2,x_3,x_4,x_5) = \min\{x_1,x_2,x_3,x_4,x_5\}$;
2. $\theta_2(x_1,x_2,x_3,x_4,x_5) = \dfrac{x_1(x_1+x_2+x_3+x_4+x_5)}{x_1+x_4+x_5+2}$;
3. $\theta_3(x_1,x_2,x_3,x_4,x_5) = \sqrt[3]{x_1x_2x_3x_4x_5}$;
4. $\theta_4(x_1,x_2,x_3,x_4,x_5) = \dfrac{x_1x_2x_3+x_2x_3x_4+x_2x_3x_5}{3}$;
5. $\theta_5(x_1,x_2,x_3,x_4,x_5) = \dfrac{x_1x_2x_3+x_4x_5}{x_1+1}$.

Now we prove our main results.

3.1.1 Existence Results

Theorem 3.1.1. *Let $(X,M,*)$ be a fuzzy metric space and let f,g be self-mappings of X such that (f,g) is any one of the following:*

(a) R-weakly commuting,

(b) R-weakly commuting of type (A_g),

(c) *R-weakly commuting of type* (A_f),

(d) *R-weakly commuting of type* (P).

If the following holds: (i) f *and* g *satisfy the (CLRg) property;*
(ii) $\int_0^{M(fx,fy,t)} \psi(s)ds \geq \int_0^{M(gx,gy,t)} \psi(s)ds$ *for* $x, y \in X$;
(iii) $\int_0^{M(fx,ffx,t)} \psi(s)ds > \int_0^{\eta(x)} \psi(s)ds$ *for* $fx \neq ffx$ *and*

$$\eta(x) = \theta(M(gx, gfx, t), M(fx, gx, t), M(ffx, gfx, t), M(fx, gfx, t), M(gx, ffx, t))$$

for some $\theta \in \Theta$,

whenever $\psi : \mathbb{R}_+ \to \mathbb{R}$ *is a Lebesgue integrable mapping which is summable, nonnegative and such that*

$$\int_0^{\epsilon} \psi(s)ds > 0$$

for each $\epsilon > 0$, *then* f *and* g *have a common fixed point.*

Proof. Since f and g satisfy the (CLRg) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx \tag{3.1.1}$$

for some $x \in X$. Let t be a continuity point of $(X, M, *)$. By (ii), we have

$$\int_0^{M(fx_n,fx,t)} \psi(s)ds \geq \int_0^{M(gx_n,gx,t)} \psi(s)ds$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$, we have

$$\int_0^{M(gx,fx,t)} \psi(s)ds \geq \int_0^{M(gx,gx,t)} \psi(s)ds,$$

which implies that $fx = gx$.

In case (f, g) is R-weakly commuting, we have

$$M(fgx, gfx, t) \geq M(fx, gx, \frac{t}{R}) = 1$$

that is $fgx = gfx$. Therefore, we obtain that

$$ffx = fgx = gfx = ggx.$$

In case (f,g) is R-weakly commuting of type (A_g), we can see in [117], we have

$$M(gfx, ffx, t) \geq M(fx, gx, \frac{t}{R}) = 1$$

that is $gfx = ffx$. Therefore, we obtain that

$$ggx = gfx = ffx = fgx.$$

In case (f,g) is R-weakly commuting of type (A_f), we have

$$M(fgx, ggx, t) \geq M(fx, gx, \frac{t}{R}) = 1$$

that is $fgx = ggx$. Therefore, we obtain that

$$ffx = fgx = ggx = gfx.$$

In case (f,g) is R-weakly commuting of type (P), we have

$$M(ffx, ggx, t) \geq M(fx, gx, \frac{t}{R}) = 1$$

that is $ffx = ggx$. Therefore, we obtain that

$$fgx = ffx = ggx = gfx.$$

Next, we prove that $ffx = fx$. In deed, we assume that $ffx \neq fx$. By the inequality (iii), we get

$$\begin{aligned}
\int_0^{M(fx,ffx,t)} \psi(s)ds &> \int_0^{\eta(x)} \psi(s)ds \\
&= \int_0^{\theta(M(gx,gfx,t),M(fx,gx,t),M(ffx,gfx,t),M(fx,gfx,t),M(gx,ffx,t))} \psi(s)ds \\
&= \int_0^{\theta(M(fx,ffx,t),M(fx,fx,t),M(ffx,ffx,t),M(fx,ffx,t),M(fx,ffx,t))} \psi(s)ds \\
&= \int_0^{\theta(M(fx,ffx,t),1,1,M(fx,ffx,t),M(fx,ffx,t))} \psi(s)ds \\
&= \int_0^{M(fx,ffx,t)} \psi(s)ds,
\end{aligned}$$

which is a contradiction. Thus $ffx = fx$ and then $fx = ffx = gfx$. So fx is a common fixed point of f and g. □

Corollary 3.1.2. *Let $(X, M, *)$ be a fuzzy metric space and let f, g be self-mappings of X such that (f, g) is any one of the following:*

(a) R-weakly commuting,

(b) R-weakly commuting of type (A_g),

(c) R-weakly commuting of type (A_f),

(d) R-weakly commuting of type (P).

If the following holds:

(i) f and g satisfy the (CLRg) property;
(ii) $\int_0^{M(fx,fy,t)} \psi(s)ds \geq \int_0^{M(gx,gy,t)} \psi(s)ds$ for $x, y \in X$;
(iii) $\int_0^{M(fx,ffx,t)} \psi(s)ds > \int_0^{N} \psi(s)ds$ for $fx \neq ffx$,
whenever

$$N \in \{M(gx, gfx, t), M(fx, gx, t), M(ffx, gfx, t), M(fx, gfx, t), M(gx, ffx, t)\}$$

and $\psi : \mathbb{R}_+ \to \mathbb{R}$ is a Lebesgue integrable mapping which is summable, nonnegative and such that

$$\int_0^{\epsilon} \psi(s)ds > 0$$

for each $\epsilon > 0$, then f and g have a common fixed point.

Proof. We obtain that

$$\begin{aligned} N \quad \geq \quad & \min\{M(gx, gfx, t), M(fx, gx, t), M(ffx, gfx, t) \\ & , M(fx, gfx, t), M(gx, ffx, t), \} \end{aligned} \tag{3.1.2}$$

for

$$N \in \{M(gx, gfx, t), M(fx, gx, t), M(ffx, gfx, t), M(fx, gfx, t), M(gx, ffx, t)\}.$$

By (iii) and (3.1.2), we have

$$\int_0^{M(fx,ffx,t)} \psi(s)ds$$

$$> \int_0^N \psi(s)ds$$

$$\geq \int_0^{\min\{M(gx,gfx,t),M(fx,gx,t),M(ffx,gfx,t),M(fx,gfx,t),M(gx,ffx,t)\}} \psi(s)ds$$

for $x, y \in X$. By taking $\theta = \theta_1$ in Theorem 3.1.1, we get f and g have a common fixed point. □

Theorem 3.1.3. *Let $(X, M, *)$ be a fuzzy metric space and let f, g be self-mappings of X such that (f, g) is any one of the following:*

(a) *R-weakly commuting,*

(b) *R-weakly commuting of type (A_g),*

(c) *R-weakly commuting of type (A_f),*

(d) *R-weakly commuting of type (P).*

If the following holds:

(i) *f and g satisfy E.A. property and gX is a closed subspace of X;*
(ii) $\int_0^{M(fx,fy,t)} \psi(s)ds \geq \int_0^{M(gx,gy,t)} \psi(s)ds$ *for $x, y \in X$;*
(iii) $\int_0^{M(fx,ffx,t)} \psi(s)ds > \int_0^{\eta(x)} \psi(s)ds$ *for $fx \neq ffx$ and*

$$\eta(x) = \theta(M(gx, gfx, t), M(fx, gx, t), M(ffx, gfx, t), M(fx, gfx, t), M(gx, ffx, t))$$

for some $\theta \in \Theta$,

whenever $\psi : \mathbb{R}_+ \to \mathbb{R}$ is a Lebesgue integrable mapping which is summable, nonnegative and such that

$$\int_0^{\epsilon} \psi(s)ds > 0$$

for each $\epsilon > 0$, then f and g have a common fixed point.

Proof. Since f and g satisfy E.A. property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u \tag{3.1.3}$$

for some $u \in X$. It follows from gX is a closed subspace of X that $u = gx$ for some $x \in X$ and then f and g satisfy the (CLRg) property. By Theorem 3.1.1, we get f and g have a common fixed point. □

Since the pair of noncompatible mappings implies to the pair satisfying E.A. property, we get the following corollary.

Corollary 3.1.4. *Let $(X, M, *)$ be a fuzzy metric space and let f, g be self-mappings of X such that (f, g) is any one of the following:*

(a) *R-weakly commuting,*

(b) *R-weakly commuting of type (A_g),*

(c) *R-weakly commuting of type (A_f),*

(d) *R-weakly commuting of type (P).*

If the following holds:

(i) *f and g are noncompatible mappings and gX is a closed subspace of X;*
(ii) $\int_0^{M(fx,fy,t)} \psi(s)ds \geq \int_0^{M(gx,gy,t)} \psi(s)ds$ *for* $x, y \in X$;
(iii) $\int_0^{M(fx,ffx,t)} \psi(s)ds > \int_0^{\eta(x)} \psi(s)ds$ *for* $fx \neq ffx$ *and*

$$\eta(x) = \theta(M(gx, gfx, t), M(fx, gx, t), M(ffx, gfx, t), M(fx, gfx, t), M(gx, ffx, t))$$

for some $\theta \in \Theta$,

whenever $\psi : \mathbb{R}_+ \to \mathbb{R}$ is a Lebesgue integrable mapping which is summable, nonnegative and such that

$$\int_0^{\epsilon} \psi(s)ds > 0$$

for each $\epsilon > 0$, then f and g have a common fixed point.

Let Δ denote the class of those functions $\delta : (0, 1]^4 \to [0, 1]$ such that δ is continuous and

$$\delta(x, 1, x, 1) = x.$$

There are examples of $\delta \in \Delta$:

1. $\delta_1(x_1, x_2, x_3, x_4) = \min\{x_1, x_2, x_3, x_4\}$;

2. $\delta_2(x_1, x_2, x_3, x_4) = \sqrt{x_1 x_2 x_3 x_4}$;

3. $\delta_3(x_1, x_2, x_3, x_4) = \min\{\sqrt{x_1 x_3}, \sqrt{x_2, x_4}\}$;

4. $\delta_4(x_1, x_2, x_3, x_4) = \dfrac{x_1x_2 + x_3x_4}{2}$;

5. $\delta_5(x_1, x_2, x_3, x_4) = \dfrac{x_1x_2 + x_1x_3 + x_1x_4}{2 + x_1}$.

Theorem 3.1.5. *Let $(X, M, *)$ be a fuzzy metric space and let f, g be self-mappings of X such that (f, g) is any one of the following:*

(a) *R-weakly commuting,*

(b) *R-weakly commuting of type (A_g),*

(c) *R-weakly commuting of type (A_f),*

(d) *R-weakly commuting of type (P).*

If the following holds:

(i) *f and g satisfy the (CLRg) property;*
(ii) $\int_0^{M(fx,fy,t)} \psi(s)ds \geq \int_0^{M(gx,gy,t)} \psi(s)ds$ *for $x, y \in X$;*
(iii) $\int_0^{M(fx,ffx,t)} \psi(s)ds > \int_0^{\eta(x)} \psi(s)ds$ *for $fx \neq ffx$ and*

$$\eta(x) = \delta(M(gx, gfx, t), M(fx, gx, t), M(fx, gfx, t), M(ffx, gfx, t))$$

for some $\delta \in \Delta$,

whenever $\psi : \mathbb{R}_+ \to \mathbb{R}$ is a Lebesgue integrable mapping which is summable, nonnegative and such that

$$\int_0^{\epsilon} \psi(s)ds > 0$$

for each $\epsilon > 0$, then f and g have a common fixed point.

Proof. Since f and g satisfy the (CLRg) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx \tag{3.1.4}$$

for some $x \in X$. Let t be a continuity point of $(X, M, *)$. By (ii), we have

$$\int_0^{M(fx_n,fx,t)} \psi(s)ds \geq \int_0^{M(gx_n,gx,t)} \psi(s)ds$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$, we have

$$\int_0^{M(gx,fx,t)} \psi(s)ds \geq \int_0^{M(gx,gx,t)} \psi(s)ds,$$

which implies that $fx = gx$.

In case (f, g) is R-weakly commuting, we have

$$M(fgx, gfx, t) \geq M(fx, gx, \frac{t}{R}) = 1$$

that is $fgx = gfx$. Therefore, we obtain that

$$ffx = fgx = gfx = ggx.$$

In case (f, g) is R-weakly commuting of type (A_g), we can see [117], we have

$$M(gfx, ffx, t) \geq M(fx, gx, \frac{t}{R}) = 1$$

that is $gfx = ffx$. Therefore, we obtain that

$$ggx = gfx = ffx = fgx.$$

In case (f, g) is R-weakly commuting of type (A_f), we have

$$M(fgx, ggx, t) \geq M(fx, gx, \frac{t}{R}) = 1$$

that is $fgx = ggx$. Therefore, we obtain that

$$ffx = fgx = ggx = gfx.$$

In case (f, g) is R-weakly commuting of type (P), we have

$$M(ffx, ggx, t) \geq M(fx, gx, \frac{t}{R}) = 1$$

that is $ffx = ggx$. Therefore, we obtain that

$$fgx = ffx = ggx = gfx.$$

Next, we prove that $ffx = fx$. We may suppose that $ffx \neq fx$. By inequality (iii), we get

$$\begin{aligned}
\int_0^{M(fx,ffx,t)} \psi(s)ds &> \int_0^{\eta(x)} \psi(s)ds \\
&= \int_0^{\delta(M(gx,gfx,t),M(fx,gx,t),M(fx,gfx,t),M(ffx,gfx,t))} \psi(s)ds \\
&= \int_0^{\delta(M(fx,ffx,t),M(fx,fx,t),M(fx,ffx,t),M(ffx,ffx,t))} \psi(s)ds \\
&= \int_0^{\delta(M(fx,ffx,t),1,M(fx,ffx,t),1)} \psi(s)ds \\
&= \int_0^{M(fx,ffx,t)} \psi(s)ds,
\end{aligned}$$

which is a contradiction. Thus $ffx = fx$ and then $fx = ffx = gfx$. So fx is a common fixed point of f and g. □

Theorem 3.1.6. *Let $(X, M, *)$ be a fuzzy metric space and let f, g be self-mappings of X such that (f, g) is any one of the following:*

(a) R-weakly commuting,

(b) R-weakly commuting of type (A_g),

(c) R-weakly commuting of type (A_f),

(d) R-weakly commuting of type (P).

If the following holds:

(i) f and g satisfy the (E.A.) property and gX is a closed subspace of X;
(ii) $\int_0^{M(fx,fy,t)} \psi(s)ds \geq \int_0^{M(gx,gy,t)} \psi(s)ds$ for $x, y \in X$;
(iii) $\int_0^{M(fx,ffx,t)} \psi(s)ds > \int_0^{\eta(x)} \psi(s)ds$ for $fx \neq ffx$ and

$$\eta(x) = \delta(M(gx, gfx, t), M(fx, gx, t), M(fx, gfx, t), M(ffx, gfx, t))$$

for some $\delta \in \Delta$,

whenever $\psi : \mathbb{R}_+ \to \mathbb{R}$ is a Lebesgue integrable mapping which is summable, nonnegative and such that

$$\int_0^{\epsilon} \psi(s)ds > 0$$

for each $\epsilon > 0$, then f and g have a common fixed point.

Proof. Since f and g satisfy E.A. property and gX is a closed subspace of X, we get there exists a sequence $\{x_n\}$ in X and some $u \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx \tag{3.1.5}$$

This show that f and g satisfy the (CLRg) property. By Theorem 3.1.5, we conclude that f and g have a common fixed point. □

Corollary 3.1.7. *Let $(X, M, *)$ be a fuzzy metric space and let f, g be self-mappings of X such that (f, g) is any one of the following:*

(a) *R-weakly commuting,*

(b) *R-weakly commuting of type (A_g),*

(c) *R-weakly commuting of type (A_f),*

(d) *R-weakly commuting of type (P).*

If the following holds:

(i) *f and g are noncompatible mappings and gX is a closed subspace of X;*
(ii) $\int_0^{M(fx,fy,t)} \psi(s)ds \geq \int_0^{M(gx,gy,t)} \psi(s)ds$ *for* $x, y \in X$;
(iii) $\int_0^{M(fx,ffx,t)} \psi(s)ds > \int_0^{\delta(M(gx,gfx,t),M(fx,gx,t),M(fx,gfx,t),M(ffx,gfx,t))} \psi(s)ds$ *for* $fx \neq ffx$ *and*
some $\delta \in \Delta$,

whenever $\psi : \mathbb{R}_+ \to \mathbb{R}$ is a Lebesgue integrable mapping which is summable, nonnegative and such that

$$\int_0^{\epsilon} \psi(s)ds > 0$$

for each $\epsilon > 0$, then f and g have a common fixed point.

Proof. Since f and g are noncompatible mappings, we get f and g satisfy E.A. property. Therefore, we apply Theorem 3.1.6 for conclude that f and g have a common fixed point. □

3.1.2 Examples

Example 3.1.8. Let $X = [1, \infty)$ with the usual metric d and $a * b = ab$. Let M be a usual fuzzy metric space on $(X, M, *)$ which is define by $M(x, y, t) = \dfrac{t}{t + d(x, y)}$ for all $x, y \in X$ and $t > 0$. Define $fx = \sqrt[3]{x}$ and $gx = x^3$. Let Lebesgue integrable $\psi : \mathbb{R}_+ \to \mathbb{R}$ define by $\psi(s) = e^s$. Now, we show that all hypothesis of Theorem 3.1.1 holds.

- Let us prove that f and g satisfy the (CLRg) property.

  Consider the sequence $\{x_n\}$ in X which is define by

  $$x_n = 1 + \frac{1}{n} \quad ; n = 1, 2, 3, \ldots$$

  Since

  $$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 1 = g1.$$

  Thus f and g satisfy the (CLRg) property with this sequence.

- From Example 2.4.15, we conclude that (f, g) is R-weakly commuting for all $R > 0$.

- We prove that

  $$\int_0^{M(fx,fy,t)} \psi(s)ds \geq \int_0^{M(gx,gy,t)} \psi(s)ds$$

  for $x, y \in X$, we have

  $$\begin{aligned}
  \int_0^{M(fx,fy,t)} \psi(s)ds &= \int_0^{M(\sqrt[3]{x}, \sqrt[3]{y}, t)} e^s ds \\
  &= \int_0^{\frac{t}{t+|\sqrt[3]{x}-\sqrt[3]{y}|}} e^s ds \\
  &= e^{\frac{t}{t+|\sqrt[3]{x}-\sqrt[3]{y}|}} - 1 \\
  &\geq e^{\frac{t}{t+|x^3-y^3|}} - 1 \\
  &= \int_0^{\frac{t}{t+|x^3-y^3|}} e^s ds \\
  &= \int_0^{M(x^3,y^3,t)} e^s ds \\
  &= \int_0^{M(gx,gy,t)} \psi(s)ds.
  \end{aligned}$$

  Therefore, the condition (ii) in Theorem 3.1.1 holds.

- We show that
$$\int_0^{M(fx,ffx,t)} \psi(s)ds > \int_0^{\eta(x)} \psi(s)ds$$
for $fx \neq ffx$ and
$$\eta(x) = \theta(M(gx,gfx,t), M(fx,gx,t), M(ffx,gfx,t), M(fx,gfx,t), M(gx,ffx,t))$$
for some $\theta \in \Theta$.

  Now, let $\theta : (0,1]^5 \to [0,1]$ define by
$$\theta(x_1,x_2,x_3,x_4,x_5) = \min\{x_1,x_2,x_3,x_4,x_5\}.$$
If $fx \neq ffx$, then we get $x \in (1,\infty)$. Thus we only show for this case.
We obtain that
$$\begin{aligned}
&\int_0^{M(fx,ffx,t)} \psi(s)ds \\
&= \int_0^{M(\sqrt[3]{x},\sqrt[9]{x},t)} e^s ds \\
&= \int_0^{\frac{t}{t+\sqrt[3]{x}-\sqrt[9]{x}}} e^s ds \\
&= e^{\frac{t}{t+\sqrt[3]{x}-\sqrt[9]{x}}} - 1 \\
&> e^{\frac{t}{t+x^3-\sqrt[9]{x}}} - 1 \\
&= \int_0^{\frac{t}{t+x^3-\sqrt[9]{x}}} e^s ds \\
&= \int_0^{M(x^3,\sqrt[9]{x},t)} e^s ds \\
&= \int_0^{M(gx,ffx,t)} \psi(s)ds \\
&\geq \int_0^{\min\{M(gx,gfx,t),M(fx,gx,t),M(ffx,gfx,t),M(fx,gfx,t),M(gx,ffx,t)\}} \psi(s)ds \\
&= \int_0^{\theta(M(gx,gfx,t),M(fx,gx,t),M(ffx,gfx,t),M(fx,gfx,t),M(gx,ffx,t))} \psi(s)ds.
\end{aligned}$$
Therefore, we get condition (iii) of Theorem 3.1.1 holds.

Now, all the required hypotheses of Theorem 3.1.1 are satisfied. Thus we deduce the existence of a common fixed point of f and g. Here, a point 1 is a common fixed point of f and g.

Example 3.1.9. Let $X = [0,1]$ with the usual metric d and $a * b = ab$. Let M be a usual fuzzy metric space on $(X, M, *)$ which is define by $M(x,y,t) = \dfrac{t}{t+d(x,y)}$

for all $x, y \in X$ and $t > 0$. Let $f, g : X \to X$ define by

$$f(x) = \begin{cases} 0 & ; x \in [0, \frac{1}{100}] \cup [\frac{1}{10}, 1]; \\ \frac{x}{2} & ; x \in (\frac{1}{100}, \frac{1}{10}) \end{cases}$$

and

$$g(x) = \begin{cases} 0 & ; x \in [0, \frac{1}{100}] \cup [\frac{1}{10}, 1]; \\ 10x & ; x \in (\frac{1}{4}, \frac{1}{2}). \end{cases}$$

Let Lebesgue integrable $\psi : \mathbb{R}_+ \to \mathbb{R}$ define by $\psi(s) = e^s$. Now, we show that all hypothesis of Theorem 3.1.1 holds.

- Let us prove that f and g satisfy the (CLRg) property.

  Consider the sequence $\{x_n\}$ in X which is define by

  $$x_n = \frac{n}{n+1} \; ; n = 1, 2, 3 \ldots$$

  Since

  $$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0 = g1.$$

  Thus f and g satisfy the (CLRg) property with this sequence.

- From Example 2.4.17, we conclude that (f, g) is R-weakly commuting of type (A_f) for all $R > 0$.

- We prove that

  $$\int_0^{M(fx,fy,t)} \psi(s)ds \geq \int_0^{M(gx,gy,t)} \psi(s)ds$$

  for $x, y \in X$. We distinguish the following cases.

  Case 1: $x, y \in [0, \frac{1}{100}] \cup [\frac{1}{10}, 1]$.

  In this case, we have

  $$\int_0^{M(fx,fy,t)} \psi(s)ds = \int_0^{M(gx,gy,t)} \psi(s)ds.$$

  Case 2: $x \in [0, \frac{1}{100}] \cup [\frac{1}{10}, 1]$ and $y \in (\frac{1}{100}, \frac{1}{10})$.

  In this case, we have

  $$\begin{aligned} \int_0^{M(fx,fy,t)} \psi(s)ds &= \int_0^{M(0,\frac{y}{2},t)} e^s ds \\ &= \int_0^{\frac{t}{t+\frac{y}{2}}} e^s ds \end{aligned}$$

$$
\begin{aligned}
&= e^{\frac{t}{t+\frac{y}{2}}} - 1 \\
&> e^{\frac{t}{t+10y}} - 1 \\
&= \int_0^{\frac{t}{t+10y}} e^s ds \\
&= \int_0^{M(0,10y,t)} e^s ds \\
&= \int_0^{M(gx,gy,t)} \psi(s) ds.
\end{aligned}
$$

Case 3: $x \in (\frac{1}{100}, \frac{1}{10})$ and $y \in [0, \frac{1}{100}] \cup [\frac{1}{10}, 1]$.

In this case, we have

$$
\begin{aligned}
\int_0^{M(fx,fy,t)} \psi(s) ds &= \int_0^{M(\frac{x}{2},0,t)} e^s ds \\
&= \int_0^{\frac{t}{t+\frac{x}{2}}} e^s ds \\
&= e^{\frac{t}{t+\frac{x}{2}}} - 1 \\
&> e^{\frac{t}{t+10x}} - 1 \\
&= \int_0^{\frac{t}{t+10x}} e^s ds \\
&= \int_0^{M(10x,0,t)} e^s ds \\
&= \int_0^{M(gx,gy,t)} \psi(s) ds.
\end{aligned}
$$

Case 4: $x, y \in (\frac{1}{100}, \frac{1}{10})$.

In this case, we have

$$
\begin{aligned}
\int_0^{M(fx,fy,t)} \psi(s) ds &= \int_0^{M(\frac{x}{2},\frac{y}{2},t)} e^s ds \\
&= \int_0^{\frac{t}{t+|\frac{x}{2}-\frac{y}{2}|}} e^s ds \\
&= e^{\frac{t}{t+|\frac{x}{2}-\frac{y}{2}|}} - 1 \\
&= e^{\frac{t}{t+\frac{|x-y|}{2}}} - 1 \\
&\geq e^{\frac{t}{t+10|x-y|}} - 1 \\
&= e^{\frac{t}{t+|10x-10y|}} - 1 \\
&= \int_0^{M(10x,10y,t)} e^s ds \\
&= \int_0^{M(gx,gy,t)} \psi(s) ds.
\end{aligned}
$$

Therefore, we can conclude that

$$\int_0^{M(fx,fy,t)} \psi(s)ds \geq \int_0^{M(gx,gy,t)} \psi(s)ds$$

for $x, y \in X$. So the condition (ii) in Theorem 3.1.1 holds.

- We show that

$$\int_0^{M(fx,ffx,t)} \psi(s)ds > \int_0^{\eta(x)} \psi(s)ds$$

for $fx \neq ffx$ and

$$\eta(x) = \theta(M(gx,gfx,t), M(fx,gx,t), M(ffx,gfx,t), M(fx,gfx,t), M(gx,ffx,t))$$

for some $\theta \in \Theta$.

Now, let $\theta : (0,1]^5 \to [0,1]$ define by

$$\theta(x_1, x_2, x_3, x_4, x_5) = \min\{x_1, x_2, x_3, x_4, x_5\}.$$

If $fx \neq ffx$, then we get $x \in (\frac{1}{100}, \frac{1}{10})$. We distinguish the following cases.

Case 1: $x \in (\frac{1}{100}, \frac{2}{100}]$.

In this case, we have $ffx = 0 \neq \frac{x}{2} = fx$ and

$$\begin{aligned}
\int_0^{M(fx,ffx,t)} \psi(s)ds
&= \int_0^{M(\frac{x}{2},0,t)} e^s ds \\
&= \int_0^{\frac{t}{t+\frac{x}{2}}} e^s ds \\
&= e^{\frac{t}{t+\frac{x}{2}}} - 1 \\
&> e^{\frac{t}{t+10x}} - 1 \\
&= \int_0^{M(10x,0,t)} e^s ds \\
&= \int_0^{M(gx,ffx,t)} \psi(s)ds \\
&\geq \int_0^{\min\{M(gx,gfx,t),M(fx,gx,t),M(ffx,gfx,t),M(fx,gfx,t),M(gx,ffx,t)\}} \psi(s)ds \\
&= \int_0^{\theta(M(gx,gfx,t),M(fx,gx,t),M(ffx,gfx,t),M(fx,gfx,t),M(gx,ffx,t))} \psi(s)ds.
\end{aligned}$$

Case 2: $x \in (\frac{2}{100}, \frac{1}{10})$.

In this case, we have $ffx = \frac{x}{4} \neq \frac{x}{2} = fx$ and

$$\int_0^{M(fx,ffx,t)} \psi(s)ds = \int_0^{M(\frac{x}{2},\frac{x}{4},t)} e^s ds$$

$$
\begin{aligned}
&= \int_0^{\frac{t}{t+\frac{x}{2}-\frac{x}{4}}} e^s ds \\
&= e^{\frac{t}{t+\frac{x}{2}-\frac{x}{4}}} - 1 \\
&> e^{\frac{t}{t+10x-\frac{x}{4}}} - 1 \\
&= \int_0^{M(10x,\frac{x}{4},t)} e^s ds \\
&= \int_0^{M(gx,ffx,t)} \psi(s) ds \\
&\geq \int_0^{\min\{M(gx,gfx,t),M(fx,gx,t),M(ffx,gfx,t),M(fx,gfx,t),M(gx,ffx,t)\}} \psi(s) ds \\
&= \int_0^{\theta(M(gx,gfx,t),M(fx,gx,t),M(ffx,gfx,t),M(fx,gfx,t),M(gx,ffx,t))} \psi(s) ds.
\end{aligned}
$$

Therefore, we get condition (iii) of Theorem 3.1.1 holds.

Now, all the required hypotheses of Theorem 3.1.1 are satisfied. Thus we deduce the existence of a common fixed point of f and g. Here, a point 0 is a common fixed point of f and g.

Example 3.1.10. Let $X = [0, 1]$ with the usual metric d and $a * b = ab$. Let M be a usual fuzzy metric space on $(X, M, *)$ which is define by $M(x, y, t) = \dfrac{t}{t + d(x, y)}$ for all $x, y \in X$ and $t > 0$. Let $f, g : X \to X$ define by

$$
f(x) = \begin{cases} 0 & ; x \in [0, \frac{1}{4}] \cup [\frac{1}{2}, 1]; \\ \frac{x}{4} & ; x \in (\frac{1}{4}, \frac{1}{2}) \end{cases}
$$

and

$$
g(x) = \begin{cases} 0 & ; x \in [0, \frac{1}{4}] \cup [\frac{1}{2}, 1]; \\ \frac{x}{2} & ; x \in (\frac{1}{4}, \frac{1}{2}). \end{cases}
$$

Let Lebesgue integrable $\psi : \mathbb{R}_+ \to \mathbb{R}$ define by $\psi(s) = e^s$. Now, we show that all hypothesis of Theorem 3.1.1 holds.

- Let us prove that f and g satisfy the (CLRg) property.

  Consider the sequence $\{x_n\}$ in X which is define by

  $$x_n = \frac{n}{n+1} \; ; n = 1, 2, 3 \ldots$$

  Since

  $$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0 = g1.$$

  Thus f and g satisfy the (CLRg) property with this sequence.

- From Example 2.4.18, we conclude that (f, g) is R-weakly commuting of type (P) for all $R > 0$.

- We prove that
$$\int_0^{M(fx,fy,t)} \psi(s)ds \geq \int_0^{M(gx,gy,t)} \psi(s)ds$$
for $x, y \in X$. We distinguish the following cases.

  Case 1: $x, y \in [0, \frac{1}{4}] \cup [\frac{1}{2}, 1]$.

  In this case, we have
$$\int_0^{M(fx,fy,t)} \psi(s)ds = \int_0^{M(gx,gy,t)} \psi(s)ds.$$

  Case 2: $x \in [0, \frac{1}{4}] \cup [\frac{1}{2}, 1]$ and $y \in (\frac{1}{4}, \frac{1}{2})$.

  In this case, we have
$$\begin{aligned}
\int_0^{M(fx,fy,t)} \psi(s)ds &= \int_0^{M(0,\frac{y}{4},t)} e^s ds \\
&= \int_0^{\frac{t}{t+\frac{y}{4}}} e^s ds \\
&= e^{\frac{t}{t+\frac{y}{4}}} - 1 \\
&> e^{\frac{t}{t+\frac{y}{2}}} - 1 \\
&= \int_0^{\frac{t}{t+\frac{y}{2}}} e^s ds \\
&= \int_0^{M(0,\frac{y}{2},t)} e^s ds \\
&= \int_0^{M(gx,gy,t)} \psi(s)ds.
\end{aligned}$$

  Case 3: $x \in (\frac{1}{4}, \frac{1}{2})$ and $y \in [0, \frac{1}{4}] \cup [\frac{1}{2}, 1]$.

  In this case, we have
$$\begin{aligned}
\int_0^{M(fx,fy,t)} \psi(s)ds &= \int_0^{M(\frac{x}{4},0,t)} e^s ds \\
&= \int_0^{\frac{t}{t+\frac{x}{4}}} e^s ds \\
&= e^{\frac{t}{t+\frac{x}{4}}} - 1 \\
&> e^{\frac{t}{t+\frac{x}{2}}} - 1
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\frac{t}{t+\frac{x}{2}}} e^s ds \\
&= \int_0^{M(\frac{x}{2},0,t)} e^s ds \\
&= \int_0^{M(gx,gy,t)} \psi(s) ds.
\end{aligned}
$$

Case 4: $x, y \in (\frac{1}{4}, \frac{1}{2})$.

In this case, we have

$$
\begin{aligned}
\int_0^{M(fx,fy,t)} \psi(s) ds &= \int_0^{M(\frac{x}{4},\frac{y}{4},t)} e^s ds \\
&= \int_0^{\frac{t}{t+|\frac{x}{4}-\frac{y}{4}|}} e^s ds \\
&= e^{\frac{t}{t+|\frac{x}{4}-\frac{y}{4}|}} - 1 \\
&= e^{\frac{t}{t+\frac{|x-y|}{4}}} - 1 \\
&\geq e^{\frac{t}{t+\frac{|x-y|}{2}}} - 1 \\
&= e^{\frac{t}{t+|\frac{x}{2}-\frac{y}{2}|}} - 1 \\
&= \int_0^{M(\frac{x}{2},\frac{y}{2},t)} e^s ds \\
&= \int_0^{M(gx,gy,t)} \psi(s) ds.
\end{aligned}
$$

Therefore, we can conclude that

$$
\int_0^{M(fx,fy,t)} \psi(s) ds \geq \int_0^{M(gx,gy,t)} \psi(s) ds
$$

for $x, y \in X$. So the condition (ii) in Theorem 3.1.1 holds.

- We show that

$$
\int_0^{M(fx,ffx,t)} \psi(s) ds > \int_0^{\eta(x)} \psi(s) ds
$$

for $fx \neq ffx$ and

$$
\eta(x) = \theta(M(gx,gfx,t), M(fx,gx,t), M(ffx,gfx,t), M(fx,gfx,t), M(gx,ffx,t))
$$

for some $\theta \in \Theta$.

Now, let $\theta : (0,1]^5 \to [0,1]$ define by

$$
\theta(x_1, x_2, x_3, x_4, x_5) = \min\{x_1, x_2, x_3, x_4, x_5\}.
$$

If $fx \neq ffx$, then we get $x \in (\frac{1}{4}, \frac{1}{2})$. Thus we only show for this case.
We obtain that

$$\int_0^{M(fx,ffx,t)} \psi(s)ds$$

$$\begin{aligned}
&= \int_0^{M(\frac{x}{4},0,t)} e^s ds \\
&= \int_0^{\frac{t}{t+\frac{x}{4}}} e^s ds \\
&= e^{\frac{t}{t+\frac{x}{4}}} - 1 \\
&> e^{\frac{t}{t+\frac{x}{2}}} - 1 \\
&= \int_0^{M(\frac{x}{2},0,t)} e^s ds \\
&= \int_0^{M(gx,ffx,t)} \psi(s)ds \\
&\geq \int_0^{\min\{M(gx,gfx,t),M(fx,gx,t),M(ffx,gfx,t),M(fx,gfx,t),M(gx,ffx,t)\}} \psi(s)ds \\
&= \int_0^{\theta(M(gx,gfx,t),M(fx,gx,t),M(ffx,gfx,t),M(fx,gfx,t),M(gx,ffx,t))} \psi(s)ds.
\end{aligned}$$

Therefore, we get condition (iii) of Theorem 3.1.1 holds.

Now, all the required hypotheses of Theorem 3.1.1 are satisfied. Thus we deduce the existence of a common fixed point of f and g. Here, a point 0 is a common fixed point of f and g.

For example of (f, g) being R-weakly commuting of type (A_g), see [117]

3.2 Common Tripled Fixed Point Theorems in Abstract Metric Spaces

In this section, we extend and unify common tripled fixed point results in [67] and study condition which guarantee the uniqueness of common tripled fixed point. We also provide illustrative example in support of our results. Now, we introduce the following concepts.

Definition 3.2.1. Let (X, d) be a K-metric space with a cone P having non-empty interior (normal or non-normal). Mappings $F : X^3 \to X$ and $g : X \to X$ are said to satisfy $E.A$ property if there exist sequences $\{x_n\}, \{y_n\}, \{z_n\} \in X$ such that

$$\begin{aligned} \lim_{n\to\infty} F(x_n, y_n, z_n) &= \lim_{n\to\infty} g(x_n) = x, \\ \lim_{n\to\infty} F(y_n, z_n, x_n) &= \lim_{n\to\infty} g(y_n) = y, \\ \lim_{n\to\infty} F(z_n, x_n, y_n) &= \lim_{n\to\infty} g(z_n) = z \end{aligned}$$

for some $x, y, z \in X$.

Definition 3.2.2. Let (X, d) be a K-metric space with a cone P having non-empty interior (normal or non-normal). Mappings $F : X^3 \to X$ and $g : X \to X$ are said to satisfy CLR_g property if there exist sequences $\{x_n\}, \{y_n\}, \{z_n\} \in X$ such that

$$\begin{aligned} \lim_{n\to\infty} F(x_n, y_n, z_n) &= \lim_{n\to\infty} g(x_n) = gx, \\ \lim_{n\to\infty} F(y_n, z_n, x_n) &= \lim_{n\to\infty} g(y_n) = gy, \\ \lim_{n\to\infty} F(z_n, x_n, y_n) &= \lim_{n\to\infty} g(z_n) = gz \end{aligned}$$

for some $x, y, z \in X$.

3.2.1 Existence Results

Theorem 3.2.3. *Let (X, d) be a K-metric space with a cone P having non-empty interior (normal or non-normal) and $F : X^3 \longrightarrow X$ and $g : X \to X$ be mappings satisfy CLR_g property. Suppose that for any $x, y, z, u, v, w \in X$, following condition*

$$\begin{aligned} d(F(x,y,z), F(u,v,w)) \preceq\ & a_1 d(F(x,y,z), gx) + a_2 d(F(y,z,x), gy) \\ & + a_3 d(F(z,x,y), gz) + a_4 d(F(u,v,w), gu) \\ & + a_5 d(F(v,w,u), gv) + a_6 d(F(w,u,v), gw) \\ & + a_7 d(F(u,v,w), gx) + a_8 d(F(v,w,u), gy) \\ & + a_9 d(F(w,u,v), gz) + a_{10} d(F(x,y,z), gu) \\ & + a_{11} d(F(y,z,x), gv) + a_{12} d(F(z,x,y), gw) \\ & + a_{13} d(gx, gu) + a_{14} d(gy, gv) + a_{15} d(gz, gw), \end{aligned}$$

holds, where a_i, $i = 1, \cdots, 15$ are nonnegative real numbers such that $\sum_{i=1}^{15} a_i < 1$. Then F and g have a tripled coincidence point.

Proof. Since F and g satisfy CLR_g property, there exist sequences $\{x_n\}, \{y_n\}, \{z_n\} \in X$ such that

$$\begin{aligned}
\lim_{n\to\infty} F(x_n, y_n, z_n) &= \lim_{n\to\infty} g(x_n) = gx, && (3.2.1)\\
\lim_{n\to\infty} F(y_n, z_n, x_n) &= \lim_{n\to\infty} g(y_n) = gy, && (3.2.2)\\
\lim_{n\to\infty} F(z_n, x_n, y_n) &= \lim_{n\to\infty} g(z_n) = gz && (3.2.3)
\end{aligned}$$

for some $x, y, z \in X$.

Now, we prove that $F(x, y, z) = gx$, $F(y, z, x) = gy$ and $F(z, x, y) = gz$. Note that for each $n \in \mathbb{N}$, we have

$$d(F(x,y,z), gx) \preceq d(F(x,y,z), F(x_n,y_n,z_n)) + d(F(x_n,y_n,z_n), gx). \qquad (3.2.4)$$

On the other hand, applying given contractive condition and using triangular inequality, we obtain that

$$\begin{aligned}
d(F(x,y,z), F(x_n,y_n,z_n)) &\preceq a_1 d(F(x,y,z), gx) + a_2 d(F(y,z,x), gy)\\
&\quad + a_3 d(F(z,x,y), gz) + a_4 d(F(x_n,y_n,z_n), gx_n)\\
&\quad + a_5 d(F(y_n,z_n,x_n), gy_n) + a_6 d(F(z_n,x_n,y_n), gz_n)\\
&\quad + a_7 d(F(x_n,y_n,z_n), gx) + a_8 d(F(y_n,z_n,x_n), gy)\\
&\quad + a_9 d(F(z_n,x_n,y_n), gz) + a_{10} d(F(x,y,z), gx_n)\\
&\quad + a_{11} d(F(y,z,x), gy_n) + a_{12} d(F(z,x,y), gz_n)\\
&\quad + a_{13} d(gx, gx_n) + a_{14} d(gy, gy_n) + a_{15} d(gz, gz_n)\\
&\preceq a_1 d(F(x,y,z), gx) + a_2 d(F(y,z,x), gy)\\
&\quad + a_3 d(F(z,x,y), gz)\\
&\quad + a_4 [d(F(x_n,y_n,z_n), gx) + d(gx, gx_n)]\\
&\quad + a_5 [d(F(y_n,z_n,x_n), gy) + d(gy, gy_n)]\\
&\quad + a_6 [d(F(z_n,x_n,y_n), gz) + d(gz, gz_n)]\\
&\quad + a_7 d(F(x_n,y_n,z_n), gx) + a_8 d(F(y_n,z_n,x_n), gy)\\
&\quad + a_9 d(F(z_n,x_n,y_n), gz)\\
&\quad + a_{10} [d(F(x,y,z), gx) + d(gx, gx_n)]
\end{aligned}$$

$$+a_{11}[d(F(y,z,x),gy)+d(gy,gy_n)]$$
$$+a_{12}[d(F(z,x,y),gz)+d(gz,gz_n)]$$
$$+a_{13}d(gx,gx_n)+a_{14}d(gy,gy_n)+a_{15}d(gz,gz_n),$$

for all $n \in \mathbb{N}$. Combining above inequality with (3.2.4), we have

$$\begin{aligned}
d(F(x,y,z),gx) \preceq\ & a_1 d(F(x,y,z),gx)+a_2 d(F(y,z,x),gy) \\
& +a_3 d(F(z,x,y),gz) \\
& +a_4[d(F(x_n,y_n,z_n),gx)+d(gx,gx_n)] \\
& +a_5[d(F(y_n,z_n,x_n),gy)+d(gy,gy_n)] \\
& +a_6[d(F(z_n,x_n,y_n),gz)+d(gz,gz_n)] \\
& +a_7 d(F(x_n,y_n,z_n),gx)+a_8 d(F(y_n,z_n,x_n),gy) \\
& +a_9 d(F(z_n,x_n,y_n),gz) \\
& +a_{10}[d(F(x,y,z),gx)+d(gx,gx_n)] \\
& +a_{11}[d(F(y,z,x),gy)+d(gy,gy_n)] \\
& +a_{12}[d(F(z,x,y),gz)+d(gz,gz_n)] \\
& +a_{13}d(gx,gx_n)+a_{14}d(gy,gy_n) \\
& +a_{15}d(gz,gz_n)+d(F(x_n,y_n,z_n),gx)
\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}
& (1-a_1-a_{10})d(F(x,y,z),gx) \\
& \quad -(a_2+a_{11})d(F(y,z,x),gy) \\
& \quad -(a_3+a_{12})d(F(z,x,y),gz) \preceq (1+a_4+a_7)d(F(x_n,y_n,z_n),gx) \\
& \qquad +(a_5+a_8)d(F(y_n,z_n,x_n),gy) \\
& \qquad +(a_6+a_9)d(F(z_n,x_n,y_n),gz) \\
& \qquad +(a_4+a_{10}+a_{13})d(gx,gx_n) \\
& \qquad +(a_5+a_{11}+a_{14})d(gy,gy_n) \\
& \qquad +(a_6+a_{12}+a_{15})d(gz,gz_n) \qquad (3.2.5)
\end{aligned}$$

for all $n \in \mathbb{N}$. Similarly, we obtain

$$\begin{aligned}
(1-a_1-a_{10})d(F(y,z,x),gy) \\
-(a_2+a_{11})d(F(z,x,y),gz) \\
-(a_3+a_{12})d(F(x,y,z),gx) \quad &\preceq \quad (1+a_4+a_7)d(F(y_n,z_n,x_n),gy) \\
&+(a_5+a_8)d(F(z_n,x_n,y_n),gz) \\
&+(a_6+a_9)d(F(x_n,y_n,z_n),gx) \\
&+(a_4+a_{10}+a_{13})d(gy,gy_n) \\
&+(a_5+a_{11}+a_{14})d(gz,gz_n) \\
&+(a_6+a_{12}+a_{15})d(gx,gx_n), \qquad (3.2.6)
\end{aligned}$$

and

$$\begin{aligned}
(1-a_1-a_{10})d(F(z,x,y),gz) \\
-(a_2+a_{11})d(F(x,y,z),gx) \\
-(a_3+a_{12})d(F(y,z,x),gy) \quad &\preceq \quad (1+a_4+a_7)d(F(z_n,x_n,y_n),gz) \\
&+(a_5+a_8)d(F(x_n,y_n,z_n),gx) \\
&+(a_6+a_9)d(F(y_n,z_n,x_n),gy) \\
&+(a_4+a_{10}+a_{13})d(gz,gz_n) \\
&+(a_5+a_{11}+a_{14})d(gx,gx_n) \\
&+(a_6+a_{12}+a_{15})d(gy,gy_n) \qquad (3.2.7)
\end{aligned}$$

for all $n \in \mathbb{N}$. Adding (3.2.5), (3.2.6) and (3.2.7), we get
$(1-a_1-a_2-a_3-a_{10}-a_{11}-a_{12})$
$\times\,[d(F(x,y,z),gx)+d(F(y,z,x),gy)+d(F(z,x,y),gz)]$

$$\begin{aligned}
\preceq \quad &(1+a_4+a_5+a_6+a_7+a_8+a_9)d(F(x_n,y_n,z_n),gx) \\
&+(1+a_4+a_5+a_6+a_7+a_8+a_9)d(F(y_n,z_n,x_n),gy) \\
&+(1+a_4+a_5+a_6+a_7+a_8+a_9)d(F(z_n,x_n,y_n),gz_n) \\
&+(a_4+a_5+a_6+a_{10}+a_{11}+a_{12}a_{13}+a_{14}+a_{15})d(gx,gx_n) \\
&+(a_4+a_5+a_6+a_{10}+a_{11}+a_{12}a_{13}+a_{14}+a_{15})d(gy,gy_n) \\
&+(a_4+a_5+a_6+a_{10}+a_{11}+a_{12}a_{13}+a_{14}+a_{15})d(gz,gz_n)
\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
&d(F(x,y,z),gx)+d(F(y,z,x),gy)+d(F(z,x,y),gz)\\
&\quad\preceq\ \alpha d(F(x_n,y_n,z_n),gx)+\alpha d(F(y_n,z_n,x_n),gy)+\alpha d(F(z_n,x_n,y_n),gz_n)\\
&\qquad+\beta d(gx,gx_n)+\beta d(gy,gy_n)+\beta d(gz,gz_n),
\end{aligned}$$

where

$$\alpha=\frac{2}{1-a_1-a_2-a_3-a_{10}-a_{11}-a_{12}}$$
$$\beta=\frac{1}{1-a_1-a_2-a_3-a_{10}-a_{11}-a_{12}}.$$

From (3.2.1), (3.2.2) and (3.2.3), for any $c \in E$ with $0_E \ll c$, there exists $N \in \mathbb{N}$ such that

$$\begin{aligned}
d(F(x_n,y_n,z_n),gx) &\preceq \frac{c}{6\max\{\alpha,\beta\}},\\
d(F(y_n,z_n,x_n),gy) &\preceq \frac{c}{6\max\{\alpha,\beta\}},\\
d(F(z_n,x_n,y_n),gz) &\preceq \frac{c}{6\max\{\alpha,\beta\}},\\
d(gx_n,gx) &\preceq \frac{c}{6\max\{\alpha,\beta\}},\\
d(gy_n,gy) &\preceq \frac{c}{6\max\{\alpha,\beta\}},\\
d(gz_n,gz) &\preceq \frac{c}{6\max\{\alpha,\beta\}},
\end{aligned}$$

for all $n \geq N$. Thus, for all $n \geq N$, we have

$$d(F(x,y,z),gx)+d(F(y,z,x),gy)+d(F(z,x,y),gz)\preceq \frac{c}{6}+\frac{c}{6}+\frac{c}{6}+\frac{c}{6}+\frac{c}{6}+\frac{c}{6}=c.$$

It follows that $d(F(x,y,z),gx)=d(F(y,z,x),gy)=d(F(z,x,y),gz)=0_E$, that is $F(x,y,z)=gx$, $F(y,z,x)=gy$ and $F(z,x,y)=gz$. □

Corollary 3.2.4. *Let (X,d) be a K-metric space with a cone P having non-empty interior (normal or non-normal) and $F : X^3 \longrightarrow X$ and $g : X \to X$ be mappings satisfy E.A. property and $g(X)$ be closed subspace of X. Suppose that for any $x,y,z,u,v,w \in X$, following condition*

$$\begin{aligned} d(F(x,y,z),F(u,v,w)) &\preceq a_1 d(F(x,y,z),gx) + a_2 d(F(y,z,x),gy) \\ &+a_3 d(F(z,x,y),gz) + a_4 d(F(u,v,w),gu) \\ &+a_5 d(F(v,w,u),gv) + a_6 d(F(w,u,v),gw) \\ &+a_7 d(F(u,v,w),gx) + a_8 d(F(v,w,u),gy) \\ &+a_9 d(F(w,u,v),gz) + a_{10} d(F(x,y,z),gu) \\ &+a_{11} d(F(y,z,x),gv) + a_{12} d(F(z,x,y),gw) \\ &+a_{13} d(gx,gu) + a_{14} d(gy,gv) + a_{15} d(gz,gw), \end{aligned}$$

holds, where a_i, $i = 1, \cdots, 15$ are nonnegative real numbers such that $\sum_{i=1}^{15} a_i < 1$. Then F and g have a tripled coincidence point.

Proof. Since F and g satisfy $E.A$ property, there exist sequences $\{x_n\}, \{y_n\}, \{z_n\} \in X$ such that

$$\begin{aligned} \lim_{n\to\infty} F(x_n,y_n,z_n) &= \lim_{n\to\infty} g(x_n) = p, \\ \lim_{n\to\infty} F(y_n,z_n,x_n) &= \lim_{n\to\infty} g(y_n) = q, \\ \lim_{n\to\infty} F(z_n,x_n,y_n) &= \lim_{n\to\infty} g(z_n) = r, \end{aligned}$$

for some $p, q, r \in X$. It follows from $g(X)$ is a closed subspace of X that $p = gx$, $q = gy$ and $r = gz$ for some $x, y, z \in X$ and then F and g satisfy the CLR_g property. By Theorem 3.2.3, we get F and g have a tripled coincidence point. □

Corollary 3.2.5. *[67]] Let (X, d) be a K-metric space with a cone P having non-empty interior (normal or non-normal) and $F : X^3 \longrightarrow X$ and $g : X \to X$ be mappings such that $F(X^3) \subseteq g(X)$. Suppose that for any $x, y, z, u, v, w \in X$, following condition*

$$\begin{aligned} d(F(x,y,z),F(u,v,w)) &\preceq a_1 d(F(x,y,z),gx) + a_2 d(F(y,z,x),gy) \\ &+a_3 d(F(z,x,y),gz) + a_4 d(F(u,v,w),gu) \\ &+a_5 d(F(v,w,u),gv) + a_6 d(F(w,u,v),gw) \\ &+a_7 d(F(u,v,w),gx) + a_8 d(F(v,w,u),gy) \end{aligned}$$

$$+a_9d(F(w,u,v),gz)+a_{10}d(F(x,y,z),gu)$$
$$+a_{11}d(F(y,z,x),gv)+a_{12}d(F(z,x,y),gw)$$
$$+a_{13}d(gx,gu)+a_{14}d(gy,gv)+a_{15}d(gz,gw),$$

holds, where a_i, $i=1,\cdots,15$ *are nonnegative real numbers such that* $\sum_{i=1}^{15} a_i < 1$. *Then* F *and* g *have a tripled coincidence point provided that* $g(X)$ *is a complete subspace of* X.

Corollary 3.2.6. *Let* (X,d) *be a* K*-metric space with a cone* P *having non-empty interior (normal or non-normal) and* $F: X^3 \longrightarrow X$ *and* $g: X \to X$ *be mappings satisfy* CLR_g *property. Suppose that for any* $x,y,z,u,v,w \in X$, *following*

$$\begin{aligned}
d(F(x,y,z),F(u,v,w)) \preceq\ & \alpha_1d(F(x,y,z),gx)+\alpha_1d(F(y,z,x),gy)\\
&+\alpha_1d(F(z,x,y),gz)+\alpha_2d(F(u,v,w),gu)\\
&+\alpha_2d(F(v,w,u),gv)+\alpha_2d(F(w,u,v),gw)\\
&+\alpha_3d(F(u,v,w),gx)+\alpha_3d(F(v,w,u),gy)\\
&+\alpha_3d(F(w,u,v),gz)+\alpha_4d(F(x,y,z),gu)\\
&+\alpha_4d(F(y,z,x),gv)+\alpha_4d(F(z,x,y),gw)\\
&+\alpha_5d(gx,gu)+\alpha_5d(gy,gv)+\alpha_5d(gz,gw),
\end{aligned}$$

holds where α_i, $i=1,\cdots,5$ *are nonnegative real numbers such that* $\sum_{i=1}^{5}\alpha_i < 1/3$. *Then* F *and* g *have a tripled coincidence point.*

Proof. It suffices to take $a_1=a_2=a_3=\alpha_1$, $a_4=a_5=a_6=\alpha_2$, $a_7=a_8=a_9=\alpha_3$, $a_{10}=a_{11}=a_{12}=\alpha_4$ and $a_{13}=a_{14}=a_{15}=\alpha_5$ in Theorem 3.2.3 with $\sum_{i=1}^{5}\alpha_i < 1/3$. □

Corollary 3.2.7. *Let* (X,d) *be a* K*-metric space with a cone* P *having non-empty interior (normal or non-normal) and* $F: X^3 \longrightarrow X$ *and* $g: X \to X$ *be mappings satisfy E.A. property and* $g(X)$ *be closed subspace of* X. *Suppose that for any*

$x, y, z, u, v, w \in X$, following

$$\begin{aligned} d(F(x,y,z),F(u,v,w)) \preceq\ & \alpha_1 d(F(x,y,z),gx) + \alpha_1 d(F(y,z,x),gy) \\ & + \alpha_1 d(F(z,x,y),gz) + \alpha_2 d(F(u,v,w),gu) \\ & + \alpha_2 d(F(v,w,u),gv) + \alpha_2 d(F(w,u,v),gw) \\ & + \alpha_3 d(F(u,v,w),gx) + \alpha_3 d(F(v,w,u),gy) \\ & + \alpha_3 d(F(w,u,v),gz) + \alpha_4 d(F(x,y,z),gu) \\ & + \alpha_4 d(F(y,z,x),gv) + \alpha_4 d(F(z,x,y),gw) \\ & + \alpha_5 d(gx,gu) + \alpha_5 d(gy,gv) + \alpha_5 d(gz,gw), \end{aligned}$$

holds where α_i, $i = 1, \cdots, 5$ are nonnegative real numbers such that $\sum_{i=1}^{5} \alpha_i < 1/3$. Then F and g have a tripled coincidence point.

Proof. It follows immediately from Corollary 3.2.6. □

Corollary 3.2.8. *[67] Let (X, d) be a K-metric space with a cone P having non-empty interior (normal or non-normal) and $F : X^3 \longrightarrow X$ and $g : X \to X$ be mappings such that $F(X^3) \subseteq g(X)$ and for any $x, y, z, u, v, w \in X$, following*

$$\begin{aligned} d(F(x,y,z),F(u,v,w)) \preceq\ & \alpha_1 d(F(x,y,z),gx) + \alpha_1 d(F(y,z,x),gy) \\ & + \alpha_1 d(F(z,x,y),gz) + \alpha_2 d(F(u,v,w),gu) \\ & + \alpha_2 d(F(v,w,u),gv) + \alpha_2 d(F(w,u,v),gw) \\ & + \alpha_3 d(F(u,v,w),gx) + \alpha_3 d(F(v,w,u),gy) \\ & + \alpha_3 d(F(w,u,v),gz) + \alpha_4 d(F(x,y,z),gu) \\ & + \alpha_4 d(F(y,z,x),gv) + \alpha_4 d(F(z,x,y),gw) \\ & + \alpha_5 d(gx,gu) + \alpha_5 d(gy,gv) + \alpha_5 d(gz,gw), \end{aligned}$$

holds where α_i, $i = 1, \cdots, 5$ are nonnegative real numbers such that $\sum_{i=1}^{5} \alpha_i < 1/3$. Then F and g have a tripled coincidence point provided that $g(X)$ is a complete subspace of X.

Next, we prove the existence of common tripled fixed point theorem for W-compatible mapping.

Theorem 3.2.9. *Let $F : X^3 \longrightarrow X$ and $g : X \to X$ be two mappings which satisfy all the conditions of Theorem 3.2.3. If F and g are W-compatible, then F and g have a unique common tripled fixed point. Moreover, common tripled fixed point of F and g is of the form (u, u, u) for some $u \in X$.*

Proof. First, we will show that the tripled point of coincidence is unique. Suppose that (x, y, z) and $(x^*, y^*, z^*) \in X^3$ with

$$\begin{cases} gx = F(x, y, z) \\ gy = F(y, z, x) \\ gz = F(z, x, y), \end{cases} \quad \text{and} \quad \begin{cases} gx^* = F(x^*, y^*, z^*) \\ gy^* = F(y^*, z^*, x^*) \\ gz^* = F(z^*, x^*, y^*). \end{cases}$$

Using contractive condition in Theorem 3.2.3, we obtain

$$\begin{aligned} d(gx, gx^*) &= d(F(x, y, z), F(x^*, y^*, z^*)) \\ &\preceq a_1 d(F(x, y, z), gx) + a_2 d(F(y, z, x), gy) + a_3 d(F(z, x, y), gz) \\ &\quad + a_4 d(F(x^*, y^*, z^*), gx^*) + a_5 d(F(y^*, z^*, x^*), gy^*) \\ &\quad + a_6 d(F(z^*, x^*, y^*), gz^*) + a_7 d(F(x^*, y^*, z^*), gx) \\ &\quad + a_8 d(F(y^*, z^*, x^*), gy) + a_9 d(F(z^*, x^*, y^*), gz) \\ &\quad + a_{10} d(F(x, y, z), gx^*) + a_{11} d(F(y, z, x), gy^*) + a_{12} d(F(z, x, y), gz^*) \\ &\quad + a_{13} d(gx, gx^*) + a_{14} d(gy, gy^*) + a_{15} d(gz, gz^*) \\ &= (a_7 + a_{10} + a_{13}) d(gx^*, gx) + (a_8 + a_{11} + a_{14}) d(gy^*, gy) \\ &\quad + (a_9 + a_{12} + a_{15}) d(gz^*, gz). \end{aligned}$$

Similarly, we have

$$\begin{aligned} d(gy, gy^*) &= d(F(y, z, x), F(y^*, z^*, x^*)) \\ &\preceq (a_7 + a_{10} + a_{13}) d(gy^*, gy) + (a_8 + a_{11} + a_{14}) d(gz^*, gz) \\ &\quad + (a_9 + a_{12} + a_{15}) d(gx^*, gx) \end{aligned}$$

and

$$\begin{aligned} d(gz, gz^*) &= d(F(z, x, y), F(z^*, x^*, y^*)) \\ &\preceq (a_7 + a_{10} + a_{13}) d(gz^*, gz) + (a_8 + a_{11} + a_{14}) d(gx^*, gx) \\ &\quad + (a_9 + a_{12} + a_{15}) d(gy^*, gy). \end{aligned}$$

Adding above three inequalities, we get

$$d(gx, gx^*) + d(gy, gy^*) + d(gz, gz^*) \preceq (\sum_{i=7}^{15} a_i)[d(gx, gx^*) + d(gy, gy^*) + d(gz, gz^*)].$$

Since $\sum_{i=7}^{15} a_i < 1$, we obtain that

$$d(gx, gx^*) + d(gy, gy^*) + d(gz, gz^*) = 0_E,$$

which implies that

$$gx = gx^*, \quad gy = gy^* \quad \text{and} \quad gz = gz^*. \tag{3.2.8}$$

This show the uniqueness of the tripled point of coincidence of F and g, that is, (gx, gy, gz).

From the contractive condition in Theorem 3.2.3, we have

$$\begin{aligned} d(gx, gy^*) &= d(F(x, y, z), F(y^*, z^*, x^*)) \\ &\preceq a_1 d(F(x, y, z), gx) + a_2 d(F(y, z, x), gy) + a_3 d(F(z, x, y), gz) \\ &\quad + a_4 d(F(y^*, z^*, x^*), gy^*) + a_5 d(F(z^*, x^*, y^*), gz^*) \\ &\quad + a_6 d(F(x^*, y^*, z^*), gx^*) + a_7 d(F(y^*, z^*, x^*), gx) \\ &\quad + a_8 d(F(z^*, x^*, y^*), gy) + a_9 d(F(x^*, y^*, z^*), gz) \\ &\quad + a_{10} d(F(x, y, z), gy^*) + a_{11} d(F(y, z, x), gz^*) + a_{12} d(F(z, x, y), gx^*) \\ &\quad + a_{13} d(gx, gy^*) + a_{14} d(gy, gz^*) + a_{15} d(gz, gx^*) \\ &= (a_7 + a_{10} + a_{13}) d(gy^*, gx) + (a_8 + a_{11} + a_{14}) d(gz^*, gy) \\ &\quad + (a_9 + a_{12} + a_{15}) d(gx^*, gz). \end{aligned}$$

Similarly, we get

$$\begin{aligned} d(gy, gz^*) &\preceq (a_7 + a_{10} + a_{13}) d(gz^*, gy) + (a_8 + a_{11} + a_{14}) d(gx^*, gz) \\ &\quad + (a_9 + a_{12} + a_{15}) d(gy^*, gx), \end{aligned}$$

and

$$\begin{aligned} d(gz, gx^*) &\preceq (a_7 + a_{10} + a_{13}) d(gx^*, gz) + (a_8 + a_{11} + a_{14}) d(gy^*, gx) \\ &\quad + (a_9 + a_{12} + a_{15}) d(gz^*, gy). \end{aligned}$$

Adding above inequalities, we obtain

$$d(gx, gy^*) + d(gy, gz^*) + d(gz, gx^*) \preceq (\sum_{i=7}^{15} a_i)(d(gx, gy^*) + d(gy, gz^*) + d(gz, gx^*)).$$

It follows from $\sum_{i=7}^{15} a_i < 1$ that

$$gx = gy^*, \quad gy = gz^* \quad \text{and} \quad gz = gx^*. \tag{3.2.9}$$

From (3.2.8) and (3.2.9), we can conclude that

$$gx = gy = gz. \tag{3.2.10}$$

This implies that (gx, gx, gx) is the unique tripled point of coincidence of F and g.

Now, let $u = gx$, then we have $u = gx = F(x, y, z) = gy = F(y, z, x) = gz = F(z, x, y)$. Since F and g are W-compatible, we have

$$F(gx, gy, gz) = g(F(x, y, z)),$$

which due to (3.2.10) gives that

$$F(u, u, u) = gu.$$

Consequently, (u, u, u) is a tripled coincidence point of F and g, and so (gu, gu, gu) is a tripled point of coincidence of F and g, and by its uniqueness, we get $gu = gx$. Thus, we obtain

$$u = gx = gu = F(u, u, u).$$

Hence, (u, u, u) is the unique common tripled fixed point of F and g. This completes the proof. □

Corollary 3.2.10. *Let $F : X^3 \longrightarrow X$ and $g : X \to X$ be two mappings which satisfy all the conditions of Corollary 3.2.4. If F and g are W-compatible, then F and g have a unique common tripled fixed point. Moreover, common tripled fixed point of F and g is of the form (u, u, u) for some $u \in X$.*

Proof. It is similar to the proof of Theorem 3.2.9. □

Corollary 3.2.11. *[67] Let $F : X^3 \longrightarrow X$ and $g : X \to X$ be two mappings which satisfy all the conditions of Corollary 3.2.5. If F and g are W-compatible, then F and g have a unique common tripled fixed point. Moreover, common tripled fixed point of F and g is of the form (u, u, u) for some $u \in X$.*

Here, we give some illustrative examples which demonstrate the validity of the hypotheses and degree of utility of our results. These examples can not conclude the existence of tripled coincidence point and common tripled fixed point by using main results of Aydi et al. [67].

3.2.2 Examples

Example 3.2.12. Let $X = [0, \frac{1}{2}]$ and $E = \mathbb{R}^2$ with the usual norm. Define the cone $P = \{(x, y) \in E : x, y \geq 0\}$ (*this cone is normal*) and $d : X^2 \to E$ by $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. It is easy to see that (X, d) is a K-metric space over a normal solid cone P.

Consider the mappings $F : X^3 \to X$ and $g : X \to X$ are defined as

$$F(x,y,z) = \begin{cases} \frac{1}{20} & ; (x,y,z) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \\ \frac{x^2+y^2+z^2}{60} & ; (x,y,z) \neq (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \end{cases} \quad \text{and} \quad gx = \begin{cases} \frac{1}{2} & ; x = \frac{1}{2} \\ \frac{x}{10} & ; x \neq \frac{1}{2}. \end{cases}$$

Since $F(X^3) = [0, \frac{1}{80}) \cup \{\frac{1}{20}\} \not\subseteq g(X) = [0, \frac{1}{20}) \cup \{\frac{1}{2}\}$, the main results of Aydi et al. [67] can not applied in this case.

Next, we show that our results can be used for this case.

- Let us prove that f and g satisfy the (CLRg) property.

  Consider the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X which is define by

  $$x_n = \frac{1}{3n}, \ y_n = \frac{1}{4n}, \ \text{and } z_n = \frac{1}{5n} \ ; n = 1, 2, 3 \ldots$$

  Since

  $$\begin{aligned} \lim_{n\to\infty} F(x_n, y_n, z_n) &= \lim_{n\to\infty} g(x_n) = g0, \\ \lim_{n\to\infty} F(y_n, z_n, x_n) &= \lim_{n\to\infty} g(y_n) = g0, \\ \lim_{n\to\infty} F(z_n, x_n, y_n) &= \lim_{n\to\infty} g(z_n) = g0. \end{aligned}$$

  Thus F and g satisfy the CLR_g property with these sequences.

- Next, we will show that F and g are W-compatible.

  It easy to see that $F(x, y, z) = gx$, $F(y, z, x) = gy$ and $F(z, x, y) = gz$ if and only if $x = y = z = 0$. Since

  $$F(g0, g0, g0) = g(F(0, 0, 0)),$$

mappings F and g are W-compatible.

- Finally, we prove that, for $x, y, z, u, v, w \in X$,

$$\begin{aligned} d(F(x,y,z),F(u,v,w)) \preceq\ & a_1 d(F(x,y,z),gx) + a_2 d(F(y,z,x),gy) \\ & + a_3 d(F(z,x,y),gz) + a_4 d(F(u,v,w),gu) \\ & + a_5 d(F(v,w,u),gv) + a_6 d(F(w,u,v),gw) \\ & + a_7 d(F(u,v,w),gx) + a_8 d(F(v,w,u),gy) \\ & + a_9 d(F(w,u,v),gz) + a_{10} d(F(x,y,z),gu) \\ & + a_{11} d(F(y,z,x),gv) + a_{12} d(F(z,x,y),gw) \\ & + a_{13} d(gx,gu) + a_{14} d(gy,gv) + a_{15} d(gz,gw), \end{aligned}$$

where $a_1 = a_4 = \frac{2}{9}$, $a_2 = a_3 = a_5 = a_6 = a_7 = a_8 = a_9 = a_{10} = a_{11} = a_{12} = 0$ and $a_{13} = a_{14} = a_{15} = \frac{1}{6}$ such that $\sum_{i=1}^{15} a_i < 1$.

For $x, y, z, u, v, w \in X$, we distinguish the following cases.

Case 1:$(x,y,z) \neq (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(u,v,w) \neq (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. In this case, we have

$$\begin{aligned} d(F(x,y,z),F(u,v,w)) &= \left(\left| \frac{x^2+y^2+z^2}{60} - \frac{u^2+v^2+w^2}{60} \right|, \right. \\ & \qquad \left. \alpha \left| \frac{x^2+y^2+z^2}{60} - \frac{u^2+v^2+w^2}{60} \right| \right) \\ &\preceq \left(\frac{|x^2-u^2|}{60} + \frac{|y^2-v^2|}{60} + \frac{|z^2-w^2|}{60}, \right. \\ & \qquad \left. \alpha \frac{|x^2-u^2|}{60} + \alpha \frac{|y^2-v^2|}{60} + \alpha \frac{|z^2-w^2|}{60} \right) \\ &\preceq \left(\frac{|x-u|}{60} + \frac{|y-v|}{60} + \frac{|z-w|}{60}, \right. \\ & \qquad \left. \alpha \frac{|x-u|}{60} + \alpha \frac{|y-v|}{60} + \alpha \frac{|z-w|}{60} \right) \\ &= \frac{1}{6} \left(\frac{|x-u|}{10}, \alpha \frac{|x-u|}{10} \right) + \frac{1}{6} \left(\frac{|y-v|}{10}, \alpha \frac{|y-v|}{10} \right) \\ & \quad + \frac{1}{6} \left(\frac{|z-w|}{10}, \alpha \frac{|z-w|}{10} \right) \\ &= a_{13} d(gx,gu) + a_{14} d(gy,gv) + a_{15} d(gz,gw) \\ &\preceq a_1 d(F(x,y,z),gx) + a_4 d(F(u,v,w),gu) \\ & \quad + a_{13} d(gx,gu) + a_{14} d(gy,gv) + a_{15} d(gz,gw). \end{aligned}$$

Case 2:$(x,y,z) \neq (\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and $(u,v,w) = (\frac{1}{2},\frac{1}{2},\frac{1}{2})$. In this case, we have

$$\begin{aligned}
d(F(x,y,z),F(u,v,w)) &= \left(\left|\frac{x^2+y^2+z^2}{60} - \frac{1}{20}\right|, \alpha\left|\frac{x^2+y^2+z^2}{60} - \frac{1}{20}\right|\right) \\
&\preceq (\frac{1}{20}, \frac{\alpha}{20}) \\
&\preceq \frac{2}{9}(\frac{9}{20}, \frac{9\alpha}{20}) \\
&= a_4 d(F(u,v,w),gu) \\
&\preceq a_1 d(F(x,y,z),gx) + a_4 d(F(u,v,w),gu) \\
&\quad + a_{13} d(gx,gu) + a_{14} d(gy,gv) + a_{15} d(gz,gw).
\end{aligned}$$

Case 3:$(x,y,z) = (\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and $(u,v,w) \neq (\frac{1}{2},\frac{1}{2},\frac{1}{2})$. In this case, we have

$$\begin{aligned}
d(F(x,y,z),F(u,v,w)) &= \left(\left|\frac{1}{20} - \frac{u^2+v^2+w^2}{60}\right|, \alpha\left|\frac{1}{20} - \frac{u^2+v^2+w^2}{60}\right|\right) \\
&\preceq (\frac{1}{20}, \frac{\alpha}{20}) \\
&\preceq \frac{2}{9}(\frac{9}{20}, \frac{9\alpha}{20}) \\
&= a_1 d(F(x,y,z),gx) \\
&\preceq a_1 d(F(x,y,z),gx) + a_4 d(F(u,v,w),gu) \\
&\quad + a_{13} d(gx,gu) + a_{14} d(gy,gv) + a_{15} d(gz,gw).
\end{aligned}$$

Case 4:$(x,y,z) = (\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and $(u,v,w) = (\frac{1}{2},\frac{1}{2},\frac{1}{2})$. Clearly,

$$\begin{aligned}
d(F(x,y,z),F(u,v,w)) &\preceq a_1 d(F(x,y,z),gx) + a_2 d(F(y,z,x),gy) \\
&\quad + a_3 d(F(z,x,y),gz) + a_4 d(F(u,v,w),gu) \\
&\quad + a_5 d(F(v,w,u),gv) + a_6 d(F(w,u,v),gw) \\
&\quad + a_7 d(F(u,v,w),gx) + a_8 d(F(v,w,u),gy) \\
&\quad + a_9 d(F(w,u,v),gz) + a_{10} d(F(x,y,z),gu) \\
&\quad + a_{11} d(F(y,z,x),gv) + a_{12} d(F(z,x,y),gw) \\
&\quad + a_{13} d(gx,gu) + a_{14} d(gy,gv) + a_{15} d(gz,gw).
\end{aligned}$$

Hence, all hypotheses of Theorem 3.2.3 and Theorem 3.2.9 are hold. Clearly $(0,0,0)$ is the unique common tripled fixed point of F and g.

Example 3.2.13. Let $X = [0,1]$ and $E = C^1_{\mathbb{R}}[0,1]$ with the norm $\|f\| = \|f\|_\infty + \|f^{'}\|_\infty$ for all $f \in E$. Define the cone $P = \{f \in E : f(t) \geq 0 \text{ for } t \in [0,1]\}$ (*this* *cone is not normal*) and $d : X^2 \to E$ by $d(x, y) = |x - y|\varphi$ for a fixed $\varphi \in P$ (e.g., $\varphi(t) = e^t$ for $t \in [0, 1]$). It is easy to see that (X, d) is a K-metric space over a non-normal solid cone.

Consider the mappings $F : X^3 \to X$ and $g : X \to X$ are defined as

$$F(x,y,z) = \begin{cases} \frac{1}{9} & ; (x,y,z) = (1,1,1) \\ \frac{x+y+z}{90} & ; (x,y,z) \neq (1,1,1) \end{cases} \quad \text{and} \quad gx = \begin{cases} 1 & ; x = 1 \\ \frac{x}{9} & ; x \neq 1. \end{cases}$$

Since $F(X^3) = [0, \frac{1}{30}) \cup \{\frac{1}{9}\} \not\subseteq g(X) = [0, \frac{1}{9}) \cup \{1\}$, the main results of Aydi et al. [67] can not applied in this case.

Next, we show that our results can be used for this case.

- Let us prove that f and g satisfy the (CLRg) property.

  Consider the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X which is defined by

  $$x_n = \frac{1}{2n}, \ y_n = \frac{1}{3n}, \text{ and } z_n = \frac{1}{4n} \ ; n = 1, 2, 3 \ldots$$

  Since

  $$\begin{aligned} \lim_{n\to\infty} F(x_n, y_n, z_n) &= \lim_{n\to\infty} g(x_n) = g0, \\ \lim_{n\to\infty} F(y_n, z_n, x_n) &= \lim_{n\to\infty} g(y_n) = g0, \\ \lim_{n\to\infty} F(z_n, x_n, y_n) &= \lim_{n\to\infty} g(z_n) = g0. \end{aligned}$$

  Thus F and g satisfy the CLR_g property with these sequences.

- Next, we will show that F and g are W-compatible.

  It obtain that $F(x, y, z) = gx$, $F(y, z, x) = gy$ and $F(z, x, y) = gz$ if and only if $x = y = z = 0$. Since

  $$F(g0, g0, g0) = g(F(0, 0, 0)),$$

  mappings F and g are W-compatible.

- Finally, we prove that, for $x, y, z, u, v, w \in X$,

$$
\begin{aligned}
d(F(x,y,z),F(u,v,w)) \preceq\ & a_1 d(F(x,y,z),gx) + a_2 d(F(y,z,x),gy) \\
& + a_3 d(F(z,x,y),gz) + a_4 d(F(u,v,w),gu) \\
& + a_5 d(F(v,w,u),gv) + a_6 d(F(w,u,v),gw) \\
& + a_7 d(F(u,v,w),gx) + a_8 d(F(v,w,u),gy) \\
& + a_9 d(F(w,u,v),gz) + a_{10} d(F(x,y,z),gu) \\
& + a_{11} d(F(y,z,x),gv) + a_{12} d(F(z,x,y),gw) \\
& + a_{13} d(gx,gu) + a_{14} d(gy,gv) + a_{15} d(gz,gw),
\end{aligned}
$$

where $a_1 = a_4 = \frac{1}{4}$, $a_2 = a_3 = a_5 = a_6 = a_7 = a_8 = a_9 = a_{10} = a_{11} = a_{12} = 0$ and $a_{13} = a_{14} = a_{15} = \frac{1}{10}$ such that $\sum_{i=1}^{15} a_i < 1$.

For $x, y, z, u, v, w \in X$, we distinguish the following cases.

Case 1:$(x,y,z) \neq (1,1,1)$ and $(u,v,w) \neq (1,1,1)$. In this case, we have

$$
\begin{aligned}
d(F(x,y,z),F(u,v,w)) &= \left| \frac{x+y+z}{90} - \frac{u+v+w}{90} \right| \varphi \\
&\preceq \frac{1}{10}\frac{|x-u|}{9}\varphi + \frac{1}{10}\frac{|y-v|}{9}\varphi + \frac{1}{10}\frac{|z-w|}{9}\varphi \\
&= a_{13} d(gx,gu) + a_{14} d(gy,gv) + a_{15} d(gz,gw) \\
&\preceq a_1 d(F(x,y,z),gx) + a_4 d(F(u,v,w),gu) \\
&\quad + a_{13} d(gx,gu) + a_{14} d(gy,gv) + a_{15} d(gz,gw).
\end{aligned}
$$

Case 2:$(x,y,z) \neq (1,1,1)$ and $(u,v,w) = (1,1,1)$. In this case, we have

$$
\begin{aligned}
d(F(x,y,z),F(u,v,w)) &= \left| \frac{x+y+z}{90} - \frac{1}{9} \right| \varphi \\
&\preceq \frac{1}{9}\varphi \\
&\preceq (\frac{1}{4})(\frac{8}{9})\varphi \\
&= a_4 d(F(u,v,w),gu) \\
&\preceq a_1 d(F(x,y,z),gx) + a_4 d(F(u,v,w),gu) \\
&\quad + a_{13} d(gx,gu) + a_{14} d(gy,gv) + a_{15} d(gz,gw).
\end{aligned}
$$

Case 3:$(x,y,z) = (1,1,1)$ and $(u,v,w) \neq (1,1,1)$. In this case, we have

$$\begin{aligned}
d(F(x,y,z), F(u,v,w)) &= \left|\frac{1}{9} - \frac{u+v+w}{90}\right|\varphi \\
&\preceq \frac{1}{9}\varphi \\
&\preceq (\frac{1}{4})(\frac{8}{9})\varphi \\
&= a_1 d(F(x,y,z), gx) \\
&\preceq a_1 d(F(x,y,z), gx) + a_4 d(F(u,v,w), gu) \\
&\quad + a_{13} d(gx, gu) + a_{14} d(gy, gv) + a_{15} d(gz, gw).
\end{aligned}$$

Case 4:$(x,y,z) = (1,1,1)$ and $(u,v,w) = (1,1,1)$. Clearly,

$$\begin{aligned}
d(F(x,y,z), F(u,v,w)) &\preceq a_1 d(F(x,y,z), gx) + a_2 d(F(y,z,x), gy) \\
&\quad + a_3 d(F(z,x,y), gz) + a_4 d(F(u,v,w), gu) \\
&\quad + a_5 d(F(v,w,u), gv) + a_6 d(F(w,u,v), gw) \\
&\quad + a_7 d(F(u,v,w), gx) + a_8 d(F(v,w,u), gy) \\
&\quad + a_9 d(F(w,u,v), gz) + a_{10} d(F(x,y,z), gu) \\
&\quad + a_{11} d(F(y,z,x), gv) + a_{12} d(F(z,x,y), gw) \\
&\quad + a_{13} d(gx, gu) + a_{14} d(gy, gv) + a_{15} d(gz, gw).
\end{aligned}$$

Therefore, all hypotheses of Theorem 3.2.3 and Theorem 3.2.9 are hold. It is easy to see that a point $(0,0,0)$ is the unique common tripled fixed point of F and g.

CHAPTER 4 SYSTEM OF VARIATIONAL INEQUALITY PROBLEMS

4.1 Systems of Hierarchical Variational Inequality Problems

In this section,

Throughout this section we always assume that the following conditions are satisfied:

(C1) $A_i : H \to H$ is an α_i-inverse-strongly monotone mapping and $VI(C, A_i)$ is the set of solutions to variational inequality problem with $A = A_i$, for all $i = 1, 2, 3$;

(C2) K_i and $K_{i,\beta}, \beta \in (0,1), i = 1, 2, 3$, are the mappings defined by

$$\begin{cases} K_i := P_{C_i}(I - \lambda A_i), \quad \lambda \in (0, 2\alpha_i], \\ K_{i,\beta} = (1-\beta)I + \beta K_i, \quad \beta \in (0,1), \end{cases} \tag{4.1.1}$$

respectively.

We have the following result.

4.1.1 Existence Result

Theorem 4.1.1. *Let C be a bounded closed convex subset of a real Hilbert space H. Let A_i and $VI(C, A_i)$ satisfy the condition (C1) and let $f_i : H \to H$ be contractions with a contractive constant $h_i \in (0,1)$, for all $i = 1, 2, 3$. Then there exists a unique element $(x^*, y^*, z^*) \in VI(C, A_1) \times VI(C, A_2) \times VI(C, A_3)$ such that the following three inequalities are satisfied*

$$\begin{cases} \langle x^* - f_1(y^*), x - x^* \rangle \geq 0, \quad \forall x \in VI(C, A_1), \\ \langle y^* - f_2(z^*), y - y^* \rangle \geq 0, \quad \forall y \in VI(C, A_2), \\ \langle z^* - f_3(x^*), z - z^* \rangle \geq 0, \quad \forall z \in VI(C, A_3). \end{cases} \tag{4.1.2}$$

Proof. The proof is a consequence of the Banach's contraction principle but it is given here for the sake of completeness. By Proposition 2.7.14, Lemma 2.7.18 (3) and Lemma 2.7.13 (4), $VI(C, A_1), VI(C, A_2)$ and $VI(C, A_3)$ are nonempty closed and convex. Therefore the metric projection $P_{VI(C,A_i)}$ is well defined for each $i = 1, 2, 3$.

Since f_i is a contractions mapping for each $i = 1, 2, 3$ then we have

$$P_{VI(C,A_1)} f_1 \circ P_{VI(C,A_2)} f_2 \circ P_{VI(C,A_3)} f_3$$

is a contractions. Hence there exists a unique element $x^* \in H$ such that

$$x^* = (P_{VI(C,A_1)} f_1 \circ P_{VI(C,A_2)} f_2 \circ P_{VI(C,A_3)} f_3) x^*.$$

Putting $z^* = P_{VI(C,A_3)} f_3(x^*)$ and $y^* = P_{VI(C,A_2)} f_2(z^*)$, then $z^* \in VI(C, A_3)$, $y^* \in VI(C, A_2)$ and $x^* = P_{VI(C,A_1)} f_1(y^*)$.

Suppose that there is an element $(\hat{x}, \hat{y}, \hat{z}) \in VI(C, A_1) \times VI(C, A_2) \times VI(C, A_3)$ such that the following three inequalities are satisfied

$$\begin{aligned}
&\langle \hat{x} - f_1(\hat{y}), x - \hat{x} \rangle \geq 0, \quad \forall x \in VI(C, A_1), \\
&\langle \hat{y} - f_2(\hat{z}), y - \hat{y} \rangle \geq 0, \quad \forall y \in VI(C, A_2), \\
&\langle \hat{z} - f_3(\hat{x}), z - \hat{z} \rangle \geq 0, \quad \forall z \in VI(C, A_3).
\end{aligned}$$

Then

$$\begin{aligned}
\hat{x} &= P_{VI(C,A_1)} f_1(\hat{y}), \\
\hat{y} &= P_{VI(C,A_2)} f_2(\hat{z}), \\
\hat{z} &= P_{VI(C,A_3)} f_3(\hat{x}).
\end{aligned}$$

Therefore

$$\hat{x} = (P_{VI(C,A_1)} f_1 \circ P_{VI(C,A_2)} f_2 \circ P_{VI(C,A_3)} f_3) \hat{x}$$

This implies that $\hat{x} = x^*$, $\hat{y} = y^*$ and $\hat{z} = z^*$. This completes the proof. □

4.1.2 Approximation Result

Theorem 4.1.2. *Let C be a closed convex subset of a real Hilbert space H and $VI(C, A_i) \neq \emptyset$. Let $A_i, VI(C, A_i), K_i$ and $K_{i,\beta}$ satisfy the conditions (C1) and (C2), and let $f_i : H \to H$ be contractions with a contractive constant $h_i \in (0, 1)$, for all $i = 1, 2, 3$. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be three sequences defined by*

$$\begin{cases}
x_0, y_0, z_0 \in H, \\
x_{n+1} = (1 - \alpha_n) K_{1,\beta} x_n + \alpha_n f_1(K_{2,\beta} y_n), \\
y_{n+1} = (1 - \alpha_n) K_{2,\beta} y_n + \alpha_n f_2(K_{3,\beta} z_n), \\
z_{n+1} = (1 - \alpha_n) K_{3,\beta} z_n + \alpha_n f_3(K_{1,\beta} x_n), \qquad n = 0, 1, 2, \dots,
\end{cases} \tag{4.1.3}$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *satisfying* $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$. *Then the sequences* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *generated be (4.1.3) converge to* x^*, y^* *and* z^* *respectively, where* (x^*, y^*, z^*) *is the unique element in* $VI(C, A_1) \times VI(C, A_2) \times VI(C, A_3)$ *verifying (4.1.2).*

Proof. (i) First we prove that $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are bounded.

From Lemma 2.7.18, it follow that $K_{i,\beta}$ is strongly quasi-nonexpansive and $F(K_{i,\beta}) = F(K_i) = VI(C, A_i)$ for each $i = 1, 2, 3$. Since f_i is contraction with the coefficient h_i for each $i = 1, 2, 3$ and $x^* \in F(K_{1,\beta}), y^* \in F(K_{2,\beta}), z^* \in F(K_{3,\beta})$, we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq (1 - \alpha_n)\|K_{1,\beta}x_n - x^*\| + \alpha_n\|f_1(K_{2,\beta}y_n) - x^*\| \\
&\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\|f_1(K_{2,\beta}y_n) - f_1(y^*)\| + \alpha_n\|f_1(y^*) - x^*\| \\
&\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n h_1\|K_{2,\beta}y_n - y^*\| + \alpha_n\|f_1(y^*) - x^*\| \\
&\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n h_1\|y_n - y^*\| + \alpha_n\|f_1(y^*) - x^*\| \\
&\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n h\|y_n - y^*\| + \alpha_n\|f_1(y^*) - x^*\|,
\end{aligned}$$

where $h = \max\{h_1, h_2, h_3\}$. Similarly, we can also prove that

$$\|y_{n+1} - y^*\| \leq (1 - \alpha_n)\|y_n - y^*\| + \alpha_n h\|z_n - z^*\| + \alpha_n\|f_2(z^*) - y^*\|$$

and

$$\|z_{n+1} - z^*\| \leq (1 - \alpha_n)\|z_n - z^*\| + \alpha_n h\|x_n - x^*\| + \alpha_n\|f_3(x^*) - z^*\|.$$

This implies that

$$\begin{aligned}
&\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| + \|z_{n+1} - z^*\| \\
&\quad\leq (1 - \alpha_n(1 - h))\Big[\|x_n - x^*\| + \|y_n - y^*\| + \|z_n - z^*\|\Big] \\
&\qquad + \alpha_n(1 - h)\frac{\|f_1(y^*) - x^*\| + \|f_2(z^*) - y^*\| + \|f_3(x^*) - z^*\|}{1 - h} \\
&\quad\leq \max\Big\{\|x_n - x^*\| + \|y_n - y^*\| + \|z_n - z^*\|, \\
&\qquad \frac{\|f_1(y^*) - x^*\| + \|f_2(z^*) - y^*\| + \|f_3(x^*) - z^*\|}{1 - h}\Big\}.
\end{aligned}$$

By induction, we have

$$
\begin{aligned}
&\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| + \|z_{n+1} - z^*\| \\
&\qquad \leq \max\Big\{ \|x_0 - x^*\| + \|y_0 - y^*\| + \|z_0 - z^*\|, \\
&\qquad\qquad \frac{\|f_1(y^*) - x^*\| + \|f_2(z^*) - y^*\| + \|f_3(x^*) - z^*\|}{1-h} \Big\},
\end{aligned}
$$

for all $n \geq 1$.

Hence $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are bounded. Consequently, $\{K_{1,\beta}x_n\}, \{K_{2,\beta}y_n\}$ and $\{K_{3,\beta}z_n\}$ are bounded.

(ii) Next we prove that for each $n \geq 1$ the following inequality holds.

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 + \|z_{n+1} - z^*\|^2 \\
&\quad \leq (1-\alpha_n)^2(\|x_n - x^*\|^2 + \|y_n - y^*\|^2 + \|z_n - z^*\|^2) \\
&\qquad +2\alpha_n h\big(\|x_{n+1} - x^*\|\|y_n - y^*\| \qquad (4.1.4) \\
&\qquad +\|y_{n+1} - y^*\|\|z_n - z^*\| + \|z_{n+1} - z^*\|\|x_n - x^*\|\big) \\
&\qquad +2\alpha_n(\langle f_1(y^*) - x^*, x_{n+1} - x^*\rangle + \langle f_2(z^*) - y^*, y_{n+1} - y^*\rangle \qquad (4.1.5) \\
&\qquad +\langle f_3(x^*) - z^*, z_{n+1} - z^*\rangle). \qquad (4.1.6)
\end{aligned}
$$

From (4.1.3) and Lemma 2.7.15, we have

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)(K_{1,\beta}x_n - x^*) + \alpha_n(f_1(K_{2,\beta}y_n) - x^*)\|^2 \\
&\leq \|(1-\alpha_n)(K_{1,\beta}x_n - x^*)\|^2 + 2\alpha_n\langle f_1(K_{2,\beta}y_n) - x^*, x_{n+1} - x^*\rangle \\
&= (1-\alpha_n)^2\|K_{1,\beta}x_n - x^*\|^2 + 2\alpha_n\langle f_1(K_{2,\beta}y_n) - f_1(y^*), x_{n+1} - x^*\rangle \\
&\quad +2\alpha_n\langle f_1(y^*) - x^*, x_{n+1} - x^*\rangle \\
&\leq (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\|f_1(K_{2,\beta}y_n) - f_1(y^*)\|\|x_{n+1} - x^*\| \\
&\quad +2\alpha_n\langle f_1(y^*) - x^*, x_{n+1} - x^*\rangle \\
&\leq (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n h_1\|K_{2,\beta}y_n - y^*\|\|x_{n+1} - x^*\| \\
&\quad +2\alpha_n\langle f_1(y^*) - x^*, x_{n+1} - x^*\rangle \\
&\leq (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n h\|y_n - y^*\|\|x_{n+1} - x^*\| \\
&\quad +2\alpha_n\langle f_1(y^*) - x^*, x_{n+1} - x^*\rangle. \qquad (4.1.7)
\end{aligned}
$$

Similarly, we can also prove that

$$
\begin{aligned}
\|y_{n+1} - y^*\|^2 &\leq (1-\alpha_n)^2\|y_n - y^*\|^2 + 2\alpha_n h\|z_n - z^*\|\|y_{n+1} - y^*\| \\
&\quad +2\alpha_n\langle f_2(z^*) - y^*, y_{n+1} - y^*\rangle \qquad (4.1.8)
\end{aligned}
$$

and

$$\begin{aligned} \|z_{n+1} - z^*\|^2 &\leq (1-\alpha_n)^2 \|z_n - z^*\|^2 + 2\alpha_n h \|x_n - x^*\| \|z_{n+1} - z^*\| \\ &\quad + 2\alpha_n \langle f_3(x^*) - z^*, z_{n+1} - z^* \rangle. \end{aligned} \tag{4.1.9}$$

Adding up the inequality (4.1.7), (4.1.8) and (4.1.9), the inequality (4.1.6) is proved.

(iii) Next we prove that if there exists a subsequence $\{n_k\} \subset \{n\}$ such that

$$\begin{aligned} 0 &\leq \liminf_{k\to\infty} \big\{ (\|x_{n_k+1} - x^*\|^2 + \|y_{n_k+1} - y^*\|^2 + \|z_{n_k+1} - z^*\|^2) \\ &\quad - (\|x_{n_k} - x^*\|^2 + \|y_{n_k} - y^*\|^2 + \|z_{n_k} - z^*\|^2) \big\}, \end{aligned}$$

then

$$\begin{aligned} 0 &\geq \limsup_{k\to\infty} \big\{ \langle f_1(y^*) - x^*, x_{n_k+1} - x^* \rangle + \langle f_2(z^*) - y^*, y_{n_k+1} - y^* \rangle \\ &\quad + \langle f_3(x^*) - z^*, z_{n_k+1} - z^* \rangle \big\}. \end{aligned}$$

Since the norm $\|\cdot\|^2$ is convex and $\lim_{n\to\infty} \alpha_n = 0$, By (4.1.3), we have

$$\begin{aligned} 0 &\leq \liminf_{k\to\infty} \big\{ (\|x_{n_k+1} - x^*\|^2 + \|y_{n_k+1} - y^*\|^2 + \|z_{n_k+1} - z^*\|^2) \\ &\quad - (\|x_{n_k} - x^*\|^2 + \|y_{n_k} - y^*\|^2 + \|z_{n_k} - z^*\|^2) \big\} \\ &\leq \liminf_{k\to\infty} \big\{ (1-\alpha_{n_k}) \|K_{1,\beta} x_{n_k} - x^*\|^2 + \alpha_{n_k} \|f_1(K_{2,\beta} y_{n_k}) - x^*\|^2 \\ &\quad + (1-\alpha_{n_k}) \|K_{2,\beta} y_{n_k} - y^*\|^2 + \alpha_{n_k} \|f_2(K_{3,\beta} z_{n_k}) - y^*\|^2 \\ &\quad + (1-\alpha_{n_k}) \|K_{3,\beta} z_{n_k} - z^*\|^2 + \alpha_{n_k} \|f_3(K_{1,\beta} x_{n_k}) - z^*\|^2 \\ &\quad - (\|x_{n_k} - x^*\|^2 + \|y_{n_k} - y^*\|^2 + \|z_{n_k} - z^*\|^2) \big\} \\ &= \liminf_{k\to\infty} \big\{ (\|K_{1,\beta} x_{n_k} - x^*\|^2 - \|x_{n_k} - x^*\|^2) + (\|K_{2,\beta} y_{n_k} - y^*\|^2 - \|y_{n_k} - y^*\|^2) \\ &\quad + (\|K_{3,\beta} z_{n_k} - z^*\|^2 - \|z_{n_k} - z^*\|^2) \big\} \\ &\leq \limsup_{k\to\infty} \big\{ (\|K_{1,\beta} x_{n_k} - x^*\|^2 - \|x_{n_k} - x^*\|^2) + (\|K_{2,\beta} y_{n_k} - y^*\|^2 - \|y_{n_k} - y^*\|^2) \\ &\quad + (\|K_{3,\beta} z_{n_k} - z^*\|^2 - \|z_{n_k} - z^*\|^2) \big\} \\ &\leq 0. \end{aligned}$$

This implies that

$$\begin{aligned} \lim_{k\to\infty} (\|K_{1,\beta} x_{n_k} - x^*\|^2 - \|x_{n_k} - x^*\|^2) &= \lim_{k\to\infty} (\|K_{2,\beta} y_{n_k} - y^*\|^2 - \|y_{n_k} - y^*\|^2) \\ &= \lim_{k\to\infty} (\|K_{3,\beta} z_{n_k} - z^*\|^2 - \|z_{n_k} - z^*\|^2) = 0. \end{aligned}$$

Since the sequence $\{\|K_{1,\beta}x_{n_k}-x^*\|+\|x_{n_k}-x^*\|\}$, $\{\|K_{2,\beta}y_{n_k}-y^*\|+\|y_{n_k}-y^*\|\}$ and $\{\|K_{3,\beta}z_{n_k}-z^*\|+\|z_{n_k}-z^*\|\}$ are bounded, we have

$$\begin{aligned}\lim_{k\to\infty}(\|K_{1,\beta}x_{n_k}-x^*\|-\|x_{n_k}-x^*\|) &= \lim_{k\to\infty}(\|K_{2,\beta}y_{n_k}-y^*\|-\|y_{n_k}-y^*\|)\\ &= \lim_{k\to\infty}(\|K_{3,\beta}z_{n_k}-z^*\|-\|z_{n_k}-z^*\|)=0\end{aligned}$$

By Lemma 2.7.18, $K_{1,\beta}, K_{2,\beta}$ and $K_{3,\beta}$ are strongly quasi-nonexpansive. We have

$$K_{1,\beta}x_{n_k}-x_{n_k}\to 0,\quad K_{2,\beta}y_{n_k}-y_{n_k}\to 0\quad\text{and}\quad K_{3,\beta}z_{n_k}-z_{n_k}\to 0$$

Consequence, we obtain that

$$x_{n_k}-x_{n_k+1}\to 0\quad y_{n_k}-y_{n_k+1}\to 0\quad\text{and}\quad z_{n_k}-z_{n_k+1}\to 0$$

It follows from the boundedness of $\{x_{n_k}\}$ that there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_l}}\rightharpoonup p$ and

$$\begin{aligned}\lim_{l\to\infty}\langle f_1(y^*)-x^*, x_{n_{k_l}}-x^*\rangle &= \limsup_{k\to\infty}\langle f_1(y^*)-x^*, x_{n_k}-x^*\rangle\\ &= \limsup_{k\to\infty}\langle f_1(y^*)-x^*, x_{n_k+1}-x^*\rangle\end{aligned}$$

By Lemma 2.7.18, $I-K_{1,\beta}$ is demiclosed at zero, and so $p\in F(K_{1,\beta})=VI(C,A_1)$. Hence from (4.1.2) we have

$$\lim_{l\to\infty}\langle f_1(y^*)-x^*, x_{n_{k_l}}-x^*\rangle=\langle f_1(y^*)-x^*, p-x^*\rangle\leq 0.$$

Therefore

$$\limsup_{k\to\infty}\langle f_1(y^*)-x^*, x_{n_k+1}-x^*\rangle=\lim_{l\to\infty}\langle f_1(y^*)-x^*, x_{n_{k_l}}-x^*\rangle\leq 0.$$

Similarly, we can also prove that

$$\limsup_{k\to\infty}\langle f_2(z^*)-y^*, y_{n_k+1}-y^*\rangle\leq 0$$

and

$$\limsup_{k\to\infty}\langle f_3(x^*)-z^*, z_{n_k+1}-z^*\rangle\leq 0.$$

Hence, we have the desired inequality.

(iv) Finally we prove that the sequence $\{x_n\},\{y_n\}$ and $\{z_n\}$ generated be (4.1.3) converge to x^*, y^* and z^* respectively.

It is clearly that

$$\|x_{n+1}-x^*\|\|y_n-y^*\|+\|y_{n+1}-y^*\|\|z_n-z^*\|+\|z_{n+1}-z^*\|\|x_n-x^*\|$$
$$\begin{aligned} &\leq \left(\|x_n-x^*\|^2+\|y_n-y^*\|^2+\|z_n-z^*\|^2\right)^{\frac{1}{2}} \\ &\quad \times\left(\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2+\|z_{n+1}-z^*\|^2\right)^{\frac{1}{2}} \end{aligned} \quad (4.1.10)$$

Substituting (4.1.10) into (4.1.6), we have

$$\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2+\|z_{n+1}-z^*\|^2$$
$$\begin{aligned} &\leq (1-\alpha_n)^2(\|x_n-x^*\|^2+\|y_n-y^*\|^2+\|z_n-z^*\|^2) \\ &\quad +2\alpha_n h\big\{\left(\|x_n-x^*\|^2+\|y_n-y^*\|^2+\|z_n-z^*\|^2\right)^{\frac{1}{2}} \\ &\quad \times\left(\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2+\|z_{n+1}-z^*\|^2\right)^{\frac{1}{2}}\big\} \\ &\quad +2\alpha_n(\langle f_1(y^*)-x^*, x_{n+1}-x^*\rangle+\langle f_2(z^*)-y^*, y_{n+1}-y^*\rangle \\ &\quad +\langle f_3(x^*)-z^*, z_{n+1}-z^*\rangle). \end{aligned} \quad (4.1.11)$$

Set

$$\begin{aligned} a_n &:= \|x_n-x^*\|^2+\|y_n-y^*\|^2+\|z_n-z^*\|^2, \\ b_n &:= 2(\langle f_1(y^*)-x^*, x_{n+1}-x^*\rangle+\langle f_2(z^*)-y^*, y_{n+1}-y^*\rangle \\ &\quad +\langle f_3(x^*)-z^*, z_{n+1}-z^*\rangle). \end{aligned}$$

Then we have the following statement:

- From (i), $\{a_n\}$ is bounded sequence;
- From (4.1.11), $a_{n+1} \leq (1-\alpha_n)^2 a_n \hat{\alpha}\sqrt{a_n}\sqrt{a_{n+1}}+\alpha_n b_n, \forall n \geq 1$;
- From (iii), whenever $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ satisfying

  $$\liminf_{k\to\infty}(a_{n_k+1}-a_{n_k}) \geq 0,$$

  it follows that $\limsup_{k\to\infty} b_{n_k} \leq 0$;

By Lemma 2.7.17, we have

$$\lim_{n\to\infty}(\|x_n-x^*\|^2+\|y_n-y^*\|^2+\|z_n-z^*\|^2)=0.$$

Hence we obtain that

$$\lim_{n\to\infty}\|x_n-x^*\|=\lim_{n\to\infty}\|y_n-y^*\|=\lim_{n\to\infty}\|z_n-z^*\|=0.$$

This completes the proof. □

4.1.3 Consequence Results

Using Theorem 4.1.2, we can prove the following results.

Theorem 4.1.3. *Let C be a closed convex subset of a real Hilbert space H and $VI(C, A_i) \neq \emptyset$. Let $A_i, VI(C, A_i), K_i$ and $K_{i,\beta}$ satisfy the conditions (C1) and (C2) for each $i = 1, 2, 3$, and let $F : H \to H$ be a μ-Lipschitzian and r-strongly monotone mapping. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be three sequences defined by*

$$\begin{cases} x_0, y_0, z_0 \in H, \\ x_{n+1} = (1-\alpha_n)K_{1,\beta}x_n + \alpha_n f_1(K_{2,\beta}y_n), \\ y_{n+1} = (1-\alpha_n)K_{2,\beta}y_n + \alpha_n f_2(K_{3,\beta}z_n), \\ z_{n+1} = (1-\alpha_n)K_{3,\beta}z_n + \alpha_n f_3(K_{1,\beta}x_n), \qquad n = 0, 1, 2, \ldots, \end{cases}$$

where $f_1 := I - \rho F, f_2 := I - \eta F, f_3 := I - \xi F$ with $\rho, \eta, \xi \in (0, \frac{2r}{\mu^2})$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge to x^*, y^* and z^* respectively, where (x^*, y^*, z^*) is the unique element in $VI(C, A_1) \times VI(C, A_2) \times VI(C, A_3)$ such that the following three inequalities are satisfied

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in VI(C, A_1), \\ \langle \eta F(z^*) + y^* - z^*, y - y^* \rangle \geq 0, \quad \forall y \in VI(C, A_2), \\ \langle \xi F(x^*) + z^* - x^*, z - z^* \rangle \geq 0, \quad \forall z \in VI(C, A_3). \end{cases} \tag{4.1.12}$$

Proof. It is easy to see that f_1, f_2, f_3 are contraction mappings and all the condition in Theorem 4.1.2 are satisfied. By Theorem 4.1.2, we have the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge to $(x^*, y^*, z^*) \in VI(C, A_1) \times VI(C, A_2) \times VI(C, A_3)$ such that the following three inequalities are satisfied

$$\begin{cases} \langle x^* - f_1(y^*), x - x^* \rangle \geq 0, \quad \forall x \in VI(C, A_1), \\ \langle y^* - f_2(z^*), y - y^* \rangle \geq 0, \quad \forall y \in VI(C, A_2), \\ \langle z^* - f_3(x^*), z - z^* \rangle \geq 0, \quad \forall z \in VI(C, A_3). \end{cases} \tag{4.1.13}$$

Substituting $f_1 := I - \rho F, f_2 := I - \eta F, f_3 := I - \xi F$ into (4.1.13), we obtain that the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge to $(x^*, y^*, z^*) \in VI(C, A_1) \times VI(C, A_2) \times VI(C, A_3)$ such that the following three inequalities are satisfied

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0 \quad \forall x \in VI(C, A_1), \\ \langle \eta F(z^*) + y^* - z^*, y - y^* \rangle \geq 0 \quad \forall y \in VI(C, A_2), \\ \langle \xi F(x^*) + z^* - x^*, z - z^* \rangle \geq 0 \quad \forall z \in VI(C, A_3). \end{cases}$$

This completes the proof □

In Theorem 4.1.2 and Theorem 4.1.3, if $A_i = I - T_i$ where $T_i : H \to H$ is nonexpansive mapping. Then A_i is $\frac{1}{2}$ inverse strongly-monotone and $VI(C, A_i) = F(T_i)$, for each $i = 1, 2, 3$. We obtain the following corollary.

Corollary 4.1.4. *Let C be a closed convex subset of a real Hilbert space H and $VI(C, A_i) \neq \emptyset$. Let $T_i : H \to H$ be nonexpansive mapping and $A_i = I - T_i, VI(C, A_i), K_i$ and $K_{i,\beta}$ satisfy the conditions (C1) and (C2). Let $f_i : H \to H$ be contractions with a contractive constant $h_i \in (0, 1)$, for all $i = 1, 2, 3$. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be three sequences defined by*

$$\begin{cases} x_0, y_0, z_0 \in H, \\ x_{n+1} = (1 - \alpha_n)K_{1,\beta}x_n + \alpha_n f_1(K_{2,\beta}y_n), \\ y_{n+1} = (1 - \alpha_n)K_{2,\beta}y_n + \alpha_n f_2(K_{3,\beta}z_n), \\ z_{n+1} = (1 - \alpha_n)K_{3,\beta}z_n + \alpha_n f_3(K_{1,\beta}x_n), \qquad n = 0, 1, 2, \ldots, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}^*$ converge to x^*, y^* and z^* respectively, where (x^*, y^*, z^*) is the unique element in $F(T_1) \times F(T_2) \times F(T_3)$ such that the following three inequalities are satisfied

$$\begin{cases} \langle x^* - f_1(y^*), x - x^* \rangle \geq 0, \quad \forall x \in F(T_1), \\ \langle y^* - f_2(z^*), y - y^* \rangle \geq 0, \quad \forall y \in F(T_2), \\ \langle z^* - f_3(x^*), z - z^* \rangle \geq 0, \quad \forall z \in F(T_3). \end{cases}$$

Corollary 4.1.5. *Let C be a closed convex subset of a real Hilbert space H and $VI(C, A_i) \neq \emptyset$. Let $T_i : H \to H$ be nonexpansive mapping and $A_i = I - T_i, VI(C, A_i), K_i$ and $K_{i,\beta}$ satisfy the conditions (C1) and (C2) for each $i = 1, 2, 3$. Let $F : H \to H$ be a μ-Lipschitzian and r-strongly monotone mapping. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be three sequences defined by*

$$\begin{cases} x_0, y_0, z_0 \in H, \\ x_{n+1} = (1 - \alpha_n)K_{1,\beta}x_n + \alpha_n f_1(K_{2,\beta}y_n), \\ y_{n+1} = (1 - \alpha_n)K_{2,\beta}y_n + \alpha_n f_2(K_{3,\beta}z_n), \\ z_{n+1} = (1 - \alpha_n)K_{3,\beta}z_n + \alpha_n f_3(K_{1,\beta}x_n), \qquad n = 0, 1, 2, \ldots, \end{cases}$$

where $f_1 := I - \rho F, f_2 := I - \eta F, f_3 := I - \xi F$ with $\rho, \eta, \xi \in (0, \frac{2r}{\mu^2})$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequences

$\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge to x^*, y^* and z^* respectively, where (x^*, y^*, z^*) is the unique element in $F(T_1) \times F(T_2) \times F(T_3)$ such that the following three inequalities are satisfied

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in F(T_1), \\ \langle \eta F(z^*) + y^* - z^*, y - y^* \rangle \geq 0, & \forall y \in F(T_2), \\ \langle \xi F(x^*) + z^* - x^*, z - z^* \rangle \geq 0 & \forall z \in F(T_3). \end{cases}$$

Corollary 4.1.6. *Let C be a closed convex subset of a real Hilbert space H and $VI(C, A_i) \neq \emptyset$. Let $A_i = I - P_{C_i}, VI(C, A_i), K_i$ and $K_{i,\beta}$ satisfy the conditions (C1) and (C2). Let $f_i : H \to H$ be contractions with a contractive constant $h_i \in (0, 1)$, for all $i = 1, 2, 3$. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be three sequences defined by*

$$\begin{cases} x_0, y_0, z_0 \in H, \\ x_{n+1} = (1 - \alpha_n)K_{1,\beta}x_n + \alpha_n f_1(K_{2,\beta}y_n), \\ y_{n+1} = (1 - \alpha_n)K_{2,\beta}y_n + \alpha_n f_2(K_{3,\beta}z_n), \\ z_{n+1} = (1 - \alpha_n)K_{3,\beta}z_n + \alpha_n f_3(K_{1,\beta}x_n), \qquad n = 0, 1, 2, \ldots, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}^*$ converge to x^*, y^* and z^* respectively, where (x^*, y^*, z^*) is the unique element in $C_1 \times C_2 \times C_3$ such that the following three inequalities are satisfied

$$\begin{cases} \langle x^* - f_1(y^*), x - x^* \rangle \geq 0, & \forall x \in C_1, \\ \langle y^* - f_2(z^*), y - y^* \rangle \geq 0, & \forall y \in C_2, \\ \langle z^* - f_3(x^*), z - z^* \rangle \geq 0, & \forall z \in C_3. \end{cases}$$

Corollary 4.1.7. *Let C be a closed convex subset of a real Hilbert space H and $VI(C, A_i) \neq \emptyset$. Let $A_i = I - P_{C_i}, VI(C, A_i), K_i$ and $K_{i,\beta}$ satisfy the conditions (C1) and (C2) for each $i = 1, 2, 3$. Let $F : H \to H$ be a μ-Lipschitzian and r-strongly monotone mapping. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be three sequences defined by*

$$\begin{cases} x_0, y_0, z_0 \in H, \\ x_{n+1} = (1 - \alpha_n)K_{1,\beta}x_n + \alpha_n f_1(K_{2,\beta}y_n), \\ y_{n+1} = (1 - \alpha_n)K_{2,\beta}y_n + \alpha_n f_2(K_{3,\beta}z_n), \\ z_{n+1} = (1 - \alpha_n)K_{3,\beta}z_n + \alpha_n f_3(K_{1,\beta}x_n), \qquad n = 0, 1, 2, \ldots, \end{cases}$$

where $f_1 := I - \rho F, f_2 := I - \eta F, f_3 := I - \xi F$ with $\rho, \eta, \xi \in (0, \frac{2r}{\mu^2})$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequences

$\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge to x^*, y^* and z^* respectively, where (x^*, y^*, z^*) is the unique element in $C_1 \times C_2 \times C_3$ such that the following three inequalities are satisfied

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C_1, \\ \langle \eta F(z^*) + y^* - z^*, y - y^* \rangle \geq 0, & \forall y \in C_2, \\ \langle \xi F(x^*) + z^* - x^*, z - z^* \rangle \geq 0 & \forall z \in C_3. \end{cases}$$

Setting $A_1 = A_2 = A_3$ in Theorem 4.1.2, we obtain the following corollary.

Corollary 4.1.8. *Let C be a closed convex subset of a real Hilbert space H and $VI(C, A_1) \neq \emptyset$. Let $A_1, VI(C, A_1), K_1$ and $K_{1,\beta}$ satisfy the conditions (C1) and (C2), and let $f_i : H \to H$ be contractions with a contractive constant $h_i \in (0,1)$, for all $i = 1, 2, 3$. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be three sequences defined by*

$$\begin{cases} x_0, y_0, z_0 \in H, \\ x_{n+1} = (1 - \alpha_n) K_{1,\beta} x_n + \alpha_n f_1(K_{1,\beta} y_n), \\ y_{n+1} = (1 - \alpha_n) K_{1,\beta} y_n + \alpha_n f_2(K_{1,\beta} z_n), \\ z_{n+1} = (1 - \alpha_n) K_{1,\beta} z_n + \alpha_n f_3(K_{1,\beta} x_n), & n = 0, 1, 2, \ldots, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge to x^*, y^* and z^* respectively, where (x^*, y^*, z^*) is the unique element in $VI(C, A_1) \times VI(C, A_1) \times VI(C, A_1)$ such that the following three inequalities are satisfied

$$\begin{cases} \langle x^* - f_1(y^*), x - x^* \rangle \geq 0 & \forall x \in VI(C, A_1), \\ \langle y^* - f_2(z^*), x - y^* \rangle \geq 0, & \forall x \in VI(C, A_1), \\ \langle z^* - f_3(x^*), x - z^* \rangle \geq 0, & \forall x \in VI(C, A_1). \end{cases}$$

Corollary 4.1.9. *Let C be a closed convex subset of a real Hilbert space H and $VI(C, A_1) \neq \emptyset$. Let $A_1, VI(C, A_1)$ and $K_{1,\beta}$ satisfy the conditions (C1) and (C2), and let $F : H \to H$ be a μ-Lipschitzian and r-strongly monotone mapping. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be three sequences defined by*

$$\begin{cases} x_0, y_0, z_0 \in H, \\ x_{n+1} = (1 - \alpha_n) K_{1,\beta} x_n + \alpha_n f_1(K_{1,\beta} y_n), \\ y_{n+1} = (1 - \alpha_n) K_{1,\beta} y_n + \alpha_n f_2(K_{1,\beta} z_n), \\ z_{n+1} = (1 - \alpha_n) K_{1,\beta} z_n + \alpha_n f_3(K_{1,\beta} x_n), & n = 0, 1, 2, \ldots, \end{cases}$$

where $f_1 := I - \rho F, f_2 := I - \eta F, f_3 := I - \xi F$ *with* $\rho, \eta, \xi \in (0, \frac{2r}{\mu^2})$ *and* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *satisfying* $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$. *Then the sequences* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *converge to* x^*, y^* *and* z^* *respectively, where* (x^*, y^*, z^*) *is the unique element in* $VI(C, A_1) \times VI(C, A_1) \times VI(C, A_1)$ *such that the following three inequalities are satisfied*

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0 \quad \forall x \in VI(C, A_1), \\ \langle \eta F(z^*) + y^* - z^*, x - y^* \rangle \geq 0 \quad \forall x \in VI(C, A_1), \\ \langle \xi F(x^*) + z^* - x^*, x - z^* \rangle \geq 0 \quad \forall x \in VI(C, A_1). \end{cases}$$

Corollary 4.1.10. *Let* C *be a closed convex subset of a real Hilbert space* H *and* $VI(C, A_1) \neq \emptyset$. *Let* $T : H \to H$ *be nonexpansive mapping and* $A_1 = I - T, VI(C, A_1), K_1$ *and* $K_{1,\beta}$ *satisfy the conditions (C1) and (C2). Let* $f_i : H \to H$ *be contractions with a contractive constant* $h_i \in (0,1)$, *for all* $i = 1, 2, 3$. *Let* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *be three sequences defined by*

$$\begin{cases} x_0, y_0, z_0 \in H, \\ x_{n+1} = (1 - \alpha_n) K_{1,\beta} x_n + \alpha_n f_1(K_{1,\beta} y_n), \\ y_{n+1} = (1 - \alpha_n) K_{1,\beta} y_n + \alpha_n f_2(K_{1,\beta} z_n), \\ z_{n+1} = (1 - \alpha_n) K_{1,\beta} z_n + \alpha_n f_3(K_{1,\beta} x_n), \qquad n = 0, 1, 2, \ldots, \end{cases}$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *satisfying* $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$. *Then the sequences* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *converge to* x^*, y^* *and* z^* *respectively, where* (x^*, y^*, z^*) *is the unique element in* $F(T) \times F(T) \times F(T)$ *such that the following three inequalities are satisfied*

$$\begin{cases} \langle x^* - f_1(y^*), x - x^* \rangle \geq 0 \quad \forall x \in F(T), \\ \langle y^* - f_2(z^*), x - y^* \rangle \geq 0, \quad \forall x \in F(T), \\ \langle z^* - f_3(x^*), x - z^* \rangle \geq 0, \quad \forall x \in F(T). \end{cases}$$

Corollary 4.1.11. *Let* C *be a closed convex subset of a real Hilbert space* H *and* $VI(C, A_1) \neq \emptyset$. *Let* $T : H \to H$ *be nonexpansive mapping and* $A_1 = I - T, VI(C, A_1), K_1$ *and* $K_{1,\beta}$ *satisfy the conditions (C1) and (C2). Let* $F : H \to H$ *be a* μ*-Lipschitzian and* r*-strongly monotone mapping. Let* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *be three sequences de-*

fined by

$$\begin{cases} x_0, y_0, z_0 \in H, \\ x_{n+1} = (1-\alpha_n)K_{1,\beta}x_n + \alpha_n f_1(K_{1,\beta}y_n), \\ y_{n+1} = (1-\alpha_n)K_{1,\beta}y_n + \alpha_n f_2(K_{1,\beta}z_n), \\ z_{n+1} = (1-\alpha_n)K_{1,\beta}z_n + \alpha_n f_3(K_{1,\beta}x_n), \qquad n = 0,1,2,\dots, \end{cases}$$

where $f_1 := I - \rho F, f_2 := I - \eta F, f_3 := I - \xi F$ *with* $\rho, \eta, \xi \in (0, \frac{2r}{\mu^2})$ *and* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *satisfying* $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$. *Then the sequences* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *converge to* x^*, y^* *and* z^* *respectively, where* (x^*, y^*, z^*) *is the unique element in* $F(T) \times F(T) \times F(T)$ *such that the following three inequalities are satisfied*

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0 \quad \forall x \in F(T), \\ \langle \eta F(z^*) + y^* - z^*, x - y^* \rangle \geq 0 \quad \forall x \in F(T), \\ \langle \xi F(x^*) + z^* - x^*, x - z^* \rangle \geq 0 \quad \forall x \in F(T). \end{cases}$$

Corollary 4.1.12. *Let* C *be a closed convex subset of a real Hilbert space* H *and* $VI(C, A_1) \neq \emptyset$. *Let* $A_1 = I - P_C, VI(C, A_1), K_1$ *and* $K_{1,\beta}$ *satisfy the conditions (C1) and (C2). Let* $f_i : H \to H$ *be contractions with a contractive constant* $h_i \in (0,1)$, *for all* $i = 1,2,3$. *Let* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *be three sequences defined by*

$$\begin{cases} x_0, y_0, z_0 \in H, \\ x_{n+1} = (1-\alpha_n)K_{1,\beta}x_n + \alpha_n f_1(K_{1,\beta}y_n), \\ y_{n+1} = (1-\alpha_n)K_{1,\beta}y_n + \alpha_n f_2(K_{1,\beta}z_n), \\ z_{n+1} = (1-\alpha_n)K_{1,\beta}z_n + \alpha_n f_3(K_{1,\beta}x_n), \qquad n = 0,1,2,\dots, \end{cases}$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *satisfying* $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$. *Then the sequences* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *converge to* x^*, y^* *and* z^* *respectively, where* (x^*, y^*, z^*) *is the unique element in* $C \times C \times C$ *such that the following three inequalities are satisfied*

$$\begin{cases} \langle x^* - f_1(y^*), x - x^* \rangle \geq 0, \quad \forall x \in C, \\ \langle y^* - f_2(z^*), x - y^* \rangle \geq 0, \quad \forall x \in C, \\ \langle z^* - f_3(x^*), x - z^* \rangle \geq 0, \quad \forall x \in C. \end{cases}$$

Corollary 4.1.13. *Let* C *be a closed convex subset of a real Hilbert space* H *and* $VI(C, A_1) \neq \emptyset$. *Let* $A_1 = I - P_C, VI(C, A_1), K_1$ *and* $K_{1,\beta}$ *satisfy the conditions (C1) and (C2). Let* $F : H \to H$ *be a* μ*-Lipschitzian and* r*-strongly monotone mapping.*

Let $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *be three sequences defined by*

$$\begin{cases} x_0, y_0, z_0 \in H, \\ x_{n+1} = (1-\alpha_n)K_{1,\beta}x_n + \alpha_n f_1(K_{1,\beta}y_n), \\ y_{n+1} = (1-\alpha_n)K_{1,\beta}y_n + \alpha_n f_2(K_{1,\beta}z_n), \\ z_{n+1} = (1-\alpha_n)K_{1,\beta}z_n + \alpha_n f_3(K_{1,\beta}x_n), \qquad n = 0, 1, 2, \ldots, \end{cases}$$

where $f_1 := I - \rho F, f_2 := I - \eta F, f_3 := I - \xi F$ *with* $\rho, \eta, \xi \in (0, \frac{2r}{\mu^2})$ *and* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *satisfying* $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*. Then the sequences* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *converge to* x^*, y^* *and* z^* *respectively, where* (x^*, y^*, z^*) *is the unique element in* $C \times C \times C$ *such that the following three inequalities are satisfied*

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0 \quad \forall x \in C, \\ \langle \eta F(z^*) + y^* - z^*, x - y^* \rangle \geq 0 \quad \forall x \in C), \\ \langle \xi F(x^*) + z^* - x^*, x - z^* \rangle \geq 0 \quad \forall x \in C. \end{cases}$$

Setting $A_1 = A_2 = A_3, f_1 = f_2 = f_3$ and $x_0 = y_0 = z_0$ in Theorem 4.1.2, we obtain the following corollary.

Corollary 4.1.14. *Let* C *be a closed convex subset of a real Hilbert space* H *and* $VI(C, A_1) \neq \emptyset$*. Let* $A_1, VI(C, A_1), K_1$ *and* $K_{1,\beta}$ *satisfy the conditions (C1) and (C2), and let* $f : H \to H$ *be contractions with a contractive constant* $h \in (0,1)$*. Let* $\{x_n\}$ *be the sequences defined by*

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1-\alpha_n)K_{1,\beta}x_n + \alpha_n f(K_{1,\beta}x_n), \qquad n = 0, 1, 2, \ldots, \end{cases}$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *satisfying* $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*. Then the sequences* $\{x_n\}$ *converge to* $x^* \in VI(C, A_1)$ *such that the following three inequalities are satisfied*

$$\langle x^* - f_1(x^*), x - x^* \rangle \geq 0, \qquad \forall x \in VI(C, A_1).$$

Corollary 4.1.15. *Let* C *be a closed convex subset of a real Hilbert space* H *and* $VI(C, A_1) \neq \emptyset$*. Let* $A_1, VI(C, A_1), K_1$ *and* $K_{1,\beta}$ *satisfy the conditions (C1) and (C2), and let* $F : H \to H$ *be a* μ*-Lipschitzian and* r*-strongly monotone mapping. Let* $\{x_n\}$ *be the sequences defined by*

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1-\alpha_n)K_{1,\beta}x_n + \alpha_n (I - \rho F)(K_{1,\beta}x_n), \qquad n = 0, 1, 2, \ldots, \end{cases}$$

where $\rho \in (0, \frac{2r}{\mu^2})$ *and* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *satisfying* $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*. Then the sequences* $\{x_n\}$ *converge to* $x^* \in VI(C, A_1)$ *such that the following three inequalities are satisfied*

$$\langle F(x^*), x - x^* \rangle \geq 0, \qquad \forall x \in VI(C, A_1).$$

Corollary 4.1.16. *Let* C *be a closed convex subset of a real Hilbert space* H *and* $VI(C, A_1) \neq \emptyset$*. Let* $T : H \to H$ *be nonexpansive mapping and* $A_1 = I - T, VI(C, A_1), K_1$ *and* $K_{1,\beta}$ *satisfy the conditions (C1) and (C2). Let* $f : H \to H$ *be contractions with a contractive constant* $h \in (0,1)$*. Let* $\{x_n\}$ *be the sequences defined by*

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1 - \alpha_n) K_{1,\beta} x_n + \alpha_n f(K_{1,\beta} x_n), \qquad n = 0, 1, 2, \ldots, \end{cases}$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *satisfying* $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*. Then the sequences* $\{x_n\}$ *converge to* $x^* \in F(T)$ *such that the following three inequalities are satisfied*

$$\langle x^* - f_1(x^*), x - x^* \rangle \geq 0, \qquad \forall x \in F(T).$$

Corollary 4.1.17. *Let* C *be a closed convex subset of a real Hilbert space* H *and* $VI(C, A_1) \neq \emptyset$*. Let* $T : H \to H$ *be nonexpansive mapping and* $A_1 = I - T, VI(C, A_1), K_1$ *and* $K_{1,\beta}$ *satisfy the conditions (C1) and (C2). Let* $F : H \to H$ *be a* μ*-Lipschitzian and* r*-strongly monotone mapping. Let* $\{x_n\}$ *be the sequences defined by*

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1 - \alpha_n) K_{1,\beta} x_n + \alpha_n (I - \rho F)(K_{1,\beta} x_n), \qquad n = 0, 1, 2, \ldots, \end{cases}$$

where $\rho \in (0, \frac{2r}{\mu^2})$ *and* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *satisfying* $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*. Then the sequences* $\{x_n\}$ *converge to* $x^* \in F(T)$ *such that the following three inequalities are satisfied*

$$\langle F(x^*), x - x^* \rangle \geq 0, \qquad \forall x \in F(T).$$

Corollary 4.1.18. *Let* C *be a closed convex subset of a real Hilbert space* H *and* $VI(C, A_1) \neq \emptyset$*. Let* $A_1 = I - P_C, VI(C, A_1), K_1$ *and* $K_{1,\beta}$ *satisfy the conditions (C1) and (C2). Let* $f : H \to H$ *be contractions with a contractive constant* $h \in (0,1)$*. Let* $\{x_n\}$ *be the sequences defined by*

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1 - \alpha_n) K_{1,\beta} x_n + \alpha_n f(K_{1,\beta} x_n), \qquad n = 0, 1, 2, \ldots, \end{cases}$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *satisfying* $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$. *Then the sequences* $\{x_n\}$ *converge to* $x^* \in C$ *such that the following three inequalities are satisfied*

$$\langle x^* - f_1(x^*), x - x^*\rangle \geq 0, \qquad \forall x \in C.$$

Corollary 4.1.19. *Let* C *be a closed convex subset of a real Hilbert space* H *and* $VI(C, A_1) \neq \emptyset$. *Let* $A_1 = I - P_C, VI(C, A_1), K_1$ *and* $K_{1,\beta}$ *satisfy the conditions (C1) and (C2). Let* $F : H \to H$ *be a* μ*-Lipschitzian and* r*-strongly monotone mapping. Let* $\{x_n\}$ *be the sequences defined by*

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1-\alpha_n)K_{1,\beta}x_n + \alpha_n(I - \rho F)(K_{1,\beta}x_n), \qquad n = 0, 1, 2, \ldots, \end{cases}$$

where $\rho \in (0, \frac{2r}{\mu^2})$ *and* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *satisfying* $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$. *Then the sequences* $\{x_n\}$ *converge to* $x^* \in C$ *such that the following three inequalities are satisfied*

$$\langle F(x^*), x - x^*\rangle \geq 0, \qquad \forall x \in C.$$

4.2 Systems of Hierarchical Variational Inclusion Problems

Throughout this section, we always assume that the following conditions are satisfied:

(C1) $M_i : H \to 2^H$ is a multi-valued maximal monotone mapping, $A_i : H \to H$ is an α_i-inverse-strongly monotone mapping and Ω_i is the set of solutions to variational inclusion problem with $A = A_i, M = M_i$ and $\Omega_i \neq \emptyset$, for all $i = 1, 2, 3$;

(C2) K_i and $K_{i,\beta}, \beta \in (0,1), i = 1, 2, 3$, are the mappings defined by

$$\begin{cases} K_i := J_{M_i,\lambda}(I - \lambda A_i), \quad \lambda \in (0, 2\alpha_i], \\ K_{i,\beta} = (1-\beta)I + \beta K_i, \quad \beta \in (0,1), \end{cases} \tag{4.2.1}$$

respectively.

Next, there are our main results.

4.2.1 Existence Result

Theorem 4.2.1. *Let A_i, M_i, Ω_i, K_i and $K_{i,\beta}$ satisfy the conditions (C1) and (C2), and let $f_i : H \to H$ be contractions with a contractive constant $h_i \in (0,1)$, for all $i = 1,2,3$. Then there exists a unique element $(x^*, y^*, z^*) \in \Omega_1 \times \Omega_2 \times \Omega_3$ such that the following three inequalities are satisfied*

$$\begin{cases} \langle x^* - f_1(y^*), x - x^* \rangle \geq 0, \quad \forall x \in \Omega_1, \\ \langle y^* - f_2(z^*), y - y^* \rangle \geq 0, \quad \forall y \in \Omega_2, \\ \langle z^* - f_3(x^*), z - z^* \rangle \geq 0, \quad \forall z \in \Omega_3. \end{cases} \tag{4.2.2}$$

Proof. The proof is a consequence of the Banach's contraction principle but it is given here for the sake of completeness. By Proposition 2.7.9 (4) and Lemma 2.7.10 (1), Ω_1, Ω_2 and Ω_3 are nonempty closed and convex. Therefore the metric projection P_{Ω_i} is well defined for each $i = 1,2,3$.

Since f_i is a contraction mapping for each $i = 1,2,3$, then, we have $P_{\Omega_i} f_i$ is a contraction and also have

$$P_{\Omega_1} f_1 \circ P_{\Omega_2} f_2 \circ P_{\Omega_3} f_3$$

is a contraction. Hence there exists a unique element $x^* \in H$ such that

$$x^* = (P_{\Omega_1} f_1 \circ P_{\Omega_2} f_2 \circ P_{\Omega_3} f_3) x^*.$$

Putting $z^* = P_{\Omega_3} f_3(x^*)$ and $y^* = P_{\Omega_2} f_2(z^*)$, then $z^* \in \Omega_3$, $y^* \in \Omega_2$ and $x^* = P_{\Omega_1} f_1(y^*)$.

Suppose that there is an element $(\hat{x}, \hat{y}, \hat{z}) \in \Omega_1 \times \Omega_2 \times \Omega_3$ such that the following three inequalities are satisfied

$$\begin{aligned} \langle \hat{x} - f_1(\hat{y}), x - \hat{x} \rangle \geq 0, \quad \forall x \in \Omega_1, \\ \langle \hat{y} - f_2(\hat{z}), y - \hat{y} \rangle \geq 0, \quad \forall y \in \Omega_2, \\ \langle \hat{z} - f_3(\hat{x}), z - \hat{z} \rangle \geq 0, \quad \forall z \in \Omega_3. \end{aligned}$$

Then

$$\begin{aligned} \hat{x} &= P_{\Omega_1} f_1(\hat{y}), \\ \hat{y} &= P_{\Omega_2} f_2(\hat{z}), \\ \hat{z} &= P_{\Omega_3} f_3(\hat{x}). \end{aligned}$$

Therefore

$$\hat{x} = (P_{\Omega_1} f_1 \circ P_{\Omega_2} f_2 \circ P_{\Omega_3} f_3) \hat{x}$$

This implies that $\hat{x} = x^*$, $\hat{y} = y^*$ and $\hat{z} = z^*$. This completes the proof. $\square$

4.2.2 Approximation Result

Theorem 4.2.2. *Let* A_i, M_i, Ω_i, K_i *and* $K_{i,\beta}$ *satisfy the conditions (C1) and (C2), and let* $f_i : H \to H$ *be contractions with a contractive constant* $h_i \in (0,1)$*, for all* $i = 1,2,3$*. Let* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *be three sequences defined by*

$$\begin{cases} x_0, y_0, z_0 \in H, \\ x_{n+1} = (1-\alpha_n)K_{1,\beta}x_n + \alpha_n f_1(K_{2,\beta}y_n), \\ y_{n+1} = (1-\alpha_n)K_{2,\beta}y_n + \alpha_n f_2(K_{3,\beta}z_n), \\ z_{n+1} = (1-\alpha_n)K_{3,\beta}z_n + \alpha_n f_3(K_{1,\beta}x_n), \qquad n = 0,1,2,\ldots, \end{cases} \tag{4.2.3}$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *satisfying* $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*. Then the sequences* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *generated be (4.2.3) converge to* x^*, y^* *and* z^* *respectively, where* (x^*, y^*, z^*) *is the unique element in* $\Omega_1 \times \Omega_2 \times \Omega_3$ *verifying (4.2.2).*

Proof. (i) First we prove that sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are bounded.

From Lemma 2.7.10, it follow that $K_{i,\beta}$ is strongly quasi-nonexpansive and $F(K_{i,\beta}) = F(K_i) = \Omega_i$ for each $i = 1,2,3$. Since f_i is contraction with the coefficient h_i for each $i = 1,2,3$ and $x^* \in F(K_{1,\beta}), y^* \in F(K_{2,\beta})$ and $z^* \in F(K_{3,\beta})$, it follows that

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq (1-\alpha_n)\|K_{1,\beta}x_n - x^*\| + \alpha_n\|f_1(K_{2,\beta}y_n) - x^*\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|f_1(K_{2,\beta}y_n) - f_1(y^*)\| + \alpha_n\|f_1(y^*) - x^*\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n h_1\|K_{2,\beta}y_n - y^*\| + \alpha_n\|f_1(y^*) - x^*\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n h_1\|y_n - y^*\| + \alpha_n\|f_1(y^*) - x^*\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n h\|y_n - y^*\| + \alpha_n\|f_1(y^*) - x^*\|, \end{aligned}$$

where $h = \max\{h_1, h_2, h_3\}$. Similarly, we can also compute that

$$\|y_{n+1} - y^*\| \leq (1-\alpha_n)\|y_n - y^*\| + \alpha_n h\|z_n - z^*\| + \alpha_n\|f_2(z^*) - y^*\|$$

and

$$\|z_{n+1} - z^*\| \leq (1-\alpha_n)\|z_n - z^*\| + \alpha_n h\|x_n - x^*\| + \alpha_n\|f_3(x^*) - z^*\|.$$

This implies that

$$
\begin{aligned}
&\|x_{n+1}-x^*\|+\|y_{n+1}-y^*\|+\|z_{n+1}-z^*\| \\
&\quad\le (1-\alpha_n(1-h))\Big[\|x_n-x^*\|+\|y_n-y^*\|+\|z_n-z^*\|\Big] \\
&\qquad +\alpha_n(1-h)\frac{\|f_1(y^*)-x^*\|+\|f_2(z^*)-y^*\|+\|f_3(x^*)-z^*\|}{1-h} \\
&\quad\le \max\Big\{\|x_n-x^*\|+\|y_n-y^*\|+\|z_n-z^*\|, \\
&\qquad \frac{\|f_1(y^*)-x^*\|+\|f_2(z^*)-y^*\|+\|f_3(x^*)-z^*\|}{1-h}\Big\}.
\end{aligned}
$$

By induction, we have

$$
\begin{aligned}
&\|x_{n+1}-x^*\|+\|y_{n+1}-y^*\|+\|z_{n+1}-z^*\| \\
&\quad\le \max\Big\{\|x_0-x^*\|+\|y_0-y^*\|+\|z_0-z^*\|, \\
&\qquad \frac{\|f_1(y^*)-x^*\|+\|f_2(z^*)-y^*\|+\|f_3(x^*)-z^*\|}{1-h}\Big\},
\end{aligned}
$$

for all $n \ge 1$.

Hence $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are bounded. Consequently, $\{K_{1,\beta}x_n\}, \{K_{2,\beta}y_n\}$ and $\{K_{3,\beta}z_n\}$ are bounded.

(ii) Next we prove that for each $n \ge 1$ the following inequality holds.

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2+\|z_{n+1}-z^*\|^2 \\
&\le (1-\alpha_n)^2(\|x_n-x^*\|^2+\|y_n-y^*\|^2+\|z_n-z^*\|^2) \\
&\quad +2\alpha_n h(\|x_{n+1}-x^*\|\|y_n-y^*\|+\|y_{n+1}-y^*\|\|z_n-z^*\|+\|z_{n+1}-z^*\|\|x_n-x^*\|) \\
&\quad +2\alpha_n(\langle f_1(y^*)-x^*, x_{n+1}-x^*\rangle+\langle f_2(z^*)-y^*, y_{n+1}-y^*\rangle \\
&\quad +\langle f_3(x^*)-z^*, z_{n+1}-z^*\rangle). \qquad (4.2.4)
\end{aligned}
$$

From (4.2.3) and Lemma 2.7.15, we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|(1-\alpha_n)(K_{1,\beta}x_n-x^*)+\alpha_n(f_1(K_{2,\beta}y_n)-x^*)\|^2\\
&\leq \|(1-\alpha_n)(K_{1,\beta}x_n-x^*)\|^2+2\alpha_n\langle f_1(K_{2,\beta}y_n)-x^*,x_{n+1}-x^*\rangle\\
&= (1-\alpha_n)^2\|K_{1,\beta}x_n-x^*\|^2+2\alpha_n\langle f_1(K_{2,\beta}y_n)-f_1(y^*),x_{n+1}-x^*\rangle\\
&\quad +2\alpha_n\langle f_1(y^*)-x^*,x_{n+1}-x^*\rangle\\
&\leq (1-\alpha_n)^2\|x_n-x^*\|^2+2\alpha_n\|f_1(K_{2,\beta}y_n)-f_1(y^*)\|\|x_{n+1}-x^*\|\\
&\quad +2\alpha_n\langle f_1(y^*)-x^*,x_{n+1}-x^*\rangle\\
&\leq (1-\alpha_n)^2\|x_n-x^*\|^2+2\alpha_n h_1\|K_{2,\beta}y_n-y^*\|\|x_{n+1}-x^*\|\\
&\quad +2\alpha_n\langle f_1(y^*)-x^*,x_{n+1}-x^*\rangle\\
&\leq (1-\alpha_n)^2\|x_n-x^*\|^2+2\alpha_n h\|y_n-y^*\|\|x_{n+1}-x^*\|\\
&\quad +2\alpha_n\langle f_1(y^*)-x^*,x_{n+1}-x^*\rangle. \qquad (4.2.5)
\end{aligned}$$

Similarly, we can also prove that

$$\begin{aligned}
\|y_{n+1}-y^*\|^2 &\leq (1-\alpha_n)^2\|y_n-y^*\|^2+2\alpha_n h\|z_n-z^*\|\|y_{n+1}-y^*\|\\
&\quad +2\alpha_n\langle f_2(z^*)-y^*,y_{n+1}-y^*\rangle \qquad (4.2.6)
\end{aligned}$$

and

$$\begin{aligned}
\|z_{n+1}-z^*\|^2 &\leq (1-\alpha_n)^2\|z_n-z^*\|^2+2\alpha_n h\|x_n-x^*\|\|z_{n+1}-z^*\|\\
&\quad +2\alpha_n\langle f_3(x^*)-z^*,z_{n+1}-z^*\rangle. \qquad (4.2.7)
\end{aligned}$$

Adding up the inequality (4.2.5), (4.2.6) and (4.2.7), the inequality (4.2.4) is proved.

(iii) Next, we prove that if there exists a subsequence $\{n_k\}\subset\{n\}$ such that

$$\begin{aligned}
0 \leq \liminf_{k\to\infty}\big\{&(\|x_{n_k+1}-x^*\|^2+\|y_{n_k+1}-y^*\|^2+\|z_{n_k+1}-z^*\|^2)\\
&-(\|x_{n_k}-x^*\|^2+\|y_{n_k}-y^*\|^2+\|z_{n_k}-z^*\|^2)\big\},
\end{aligned}$$

then

$$\begin{aligned}
0 \geq \limsup_{k\to\infty}\big\{&\langle f_1(y^*)-x^*,x_{n_k+1}-x^*\rangle+\langle f_2(z^*)-y^*,y_{n_k+1}-y^*\rangle\\
&+\langle f_3(x^*)-z^*,z_{n_k+1}-z^*\rangle\big\}.
\end{aligned}$$

Since the norm $\|\cdot\|^2$ is convex and $\lim_{n\to\infty}\alpha_n = 0$, By (4.2.3), we have

$$\begin{aligned}
0 &\leq \liminf_{k\to\infty}\big\{(\|x_{n_k+1}-x^*\|^2+\|y_{n_k+1}-y^*\|^2+\|z_{n_k+1}-z^*\|^2)\\
&\quad -(\|x_{n_k}-x^*\|^2+\|y_{n_k}-y^*\|^2+\|z_{n_k}-z^*\|^2)\big\}\\
&\leq \liminf_{k\to\infty}\big\{(1-\alpha_{n_k})\|K_{1,\beta}x_{n_k}-x^*\|^2+\alpha_{n_k}\|f_1(K_{2,\beta}y_{n_k})-x^*\|^2\\
&\quad +(1-\alpha_{n_k})\|K_{2,\beta}y_{n_k}-y^*\|^2+\alpha_{n_k}\|f_2(K_{3,\beta}z_{n_k})-y^*\|^2\\
&\quad +(1-\alpha_{n_k})\|K_{3,\beta}z_{n_k}-z^*\|^2+\alpha_{n_k}\|f_3(K_{1,\beta}x_{n_k})-z^*\|^2\\
&\quad -(\|x_{n_k}-x^*\|^2+\|y_{n_k}-y^*\|^2+\|z_{n_k}-z^*\|^2)\big\}\\
&= \liminf_{k\to\infty}\big\{(\|K_{1,\beta}x_{n_k}-x^*\|^2-\|x_{n_k}-x^*\|^2)+(\|K_{2,\beta}y_{n_k}-y^*\|^2-\|y_{n_k}-y^*\|^2)\\
&\quad +(\|K_{3,\beta}z_{n_k}-z^*\|^2-\|z_{n_k}-z^*\|^2)\big\}\\
&\leq \limsup_{k\to\infty}\big\{(\|K_{1,\beta}x_{n_k}-x^*\|^2-\|x_{n_k}-x^*\|^2)+(\|K_{2,\beta}y_{n_k}-y^*\|^2-\|y_{n_k}-y^*\|^2)\\
&\quad +(\|K_{3,\beta}z_{n_k}-z^*\|^2-\|z_{n_k}-z^*\|^2)\big\}\\
&\leq 0.
\end{aligned}$$

This implies that

$$\begin{aligned}
\lim_{k\to\infty}(\|K_{1,\beta}x_{n_k}-x^*\|^2-\|x_{n_k}-x^*\|^2) &= \lim_{k\to\infty}(\|K_{2,\beta}y_{n_k}-y^*\|^2-\|y_{n_k}-y^*\|^2)\\
&= \lim_{k\to\infty}(\|K_{3,\beta}z_{n_k}-z^*\|^2-\|z_{n_k}-z^*\|^2)=0.
\end{aligned}$$

Since the sequence $\{\|K_{1,\beta}x_{n_k}-x^*\|+\|x_{n_k}-x^*\|\}$, $\{\|K_{2,\beta}y_{n_k}-y^*\|+\|y_{n_k}-y^*\|\}$ and $\{\|K_{3,\beta}z_{n_k}-z^*\|+\|z_{n_k}-z^*\|\}$ are bounded, we have

$$\begin{aligned}
\lim_{k\to\infty}(\|K_{1,\beta}x_{n_k}-x^*\|-\|x_{n_k}-x^*\|) &= \lim_{k\to\infty}(\|K_{2,\beta}y_{n_k}-y^*\|-\|y_{n_k}-y^*\|)\\
&= \lim_{k\to\infty}(\|K_{3,\beta}z_{n_k}-z^*\|-\|z_{n_k}-z^*\|)=0
\end{aligned}$$

By Lemma 2.7.10, $K_{1,\beta}, K_{2,\beta}$ and $K_{3,\beta}$ are strongly quasi-nonexpansive. We have

$$K_{1,\beta}x_{n_k}-x_{n_k}\to 0,\quad K_{2,\beta}y_{n_k}-y_{n_k}\to 0\quad\text{and}\quad K_{3,\beta}z_{n_k}-z_{n_k}\to 0.$$

Consequence, we obtain that

$$x_{n_k}-x_{n_k+1}\to 0\quad y_{n_k}-y_{n_k+1}\to 0\quad\text{and}\quad z_{n_k}-z_{n_k+1}\to 0.$$

It follows from the boundedness of $\{x_{n_k}\}$ and H is reflexive that there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_l}}\rightharpoonup p$ and

$$\begin{aligned}
\lim_{l\to\infty}\langle f_1(y^*)-x^*, x_{n_{k_l}}-x^*\rangle &= \limsup_{k\to\infty}\langle f_1(y^*)-x^*, x_{n_k}-x^*\rangle\\
&= \limsup_{k\to\infty}\langle f_1(y^*)-x^*, x_{n_k+1}-x^*\rangle.
\end{aligned}$$

By Lemma 2.7.10, $I - K_{1,\beta}$ is demiclosed at zero, and so $p \in F(K_{1,\beta}) = \Omega_1$. Hence from (4.2.2) we have

$$\lim_{l\to\infty}\langle f_1(y^*) - x^*, x_{n_{k_l}} - x^*\rangle = \langle f_1(y^*) - x^*, p - x^*\rangle \leq 0.$$

Therefore

$$\limsup_{k\to\infty}\langle f_1(y^*) - x^*, x_{n_k+1} - x^*\rangle = \lim_{l\to\infty}\langle f_1(y^*) - x^*, x_{n_{k_l}} - x^*\rangle \leq 0.$$

Similarly, we can also prove that

$$\limsup_{k\to\infty}\langle f_2(z^*) - y^*, y_{n_k+1} - y^*\rangle \leq 0$$

and

$$\limsup_{k\to\infty}\langle f_3(x^*) - z^*, z_{n_k+1} - z^*\rangle \leq 0.$$

Hence, we have the desired inequality.

(iv) Finally, we prove that the sequence $\{x_n\}, \{y_n\}$ and $\{z_n\}$ generated be (4.2.3) converge to x^*, y^* and z^* respectively.

It is clearly that

$$\begin{aligned}\|x_{n+1} - x^*\|\|y_n - y^*\| + \|y_{n+1} - y^*\|\|z_n - z^*\| + \|z_{n+1} - z^*\|\|x_n - x^*\| \\ \leq \big(\|x_n - x^*\|^2 + \|y_n - y^*\|^2 + \|z_n - z^*\|^2\big)^{\frac{1}{2}} \\ \times\big(\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 + \|z_{n+1} - z^*\|^2\big)^{\frac{1}{2}}. \end{aligned} \tag{4.2.8}$$

Substituting (4.2.8) into (4.2.4), we have

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 + \|z_{n+1} - z^*\|^2 \\ &\leq (1-\alpha_n)^2(\|x_n - x^*\|^2 + \|y_n - y^*\|^2 + \|z_n - z^*\|^2) \\ &\quad +2\alpha_n h\{\big(\|x_n - x^*\|^2 + \|y_n - y^*\|^2 + \|z_n - z^*\|^2\big)^{\frac{1}{2}} \\ &\quad \times\big(\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 + \|z_{n+1} - z^*\|^2\big)^{\frac{1}{2}}\} \\ &\quad +2\alpha_n(\langle f_1(y^*) - x^*, x_{n+1} - x^*\rangle + \langle f_2(z^*) - y^*, y_{n+1} - y^*\rangle \\ &\quad +\langle f_3(x^*) - z^*, z_{n+1} - z^*\rangle). \end{aligned} \tag{4.2.9}$$

Set

$$\begin{aligned} a_n &:= \|x_n - x^*\|^2 + \|y_n - y^*\|^2 + \|z_n - z^*\|^2, \\ b_n &:= 2(\langle f_1(y^*) - x^*, x_{n+1} - x^*\rangle + \langle f_2(z^*) - y^*, y_{n+1} - y^*\rangle \\ &\quad +\langle f_3(x^*) - z^*, z_{n+1} - z^*\rangle). \end{aligned}$$

Then, we have the following statement:

- From (i), $\{a_n\}$ is bounded sequence;
- From (4.2.9), $a_{n+1} \leq (1-\alpha_n)^2 a_n \hat{\alpha}\sqrt{a_n}\sqrt{a_{n+1}} + \alpha_n b_n, \forall n \geq 1$;
- From (iii), whenever $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ satisfying

$$\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0,$$

  it follows that $\limsup_{k\to\infty} b_{n_k} \leq 0$;

By Lemma 2.7.17, we have

$$\lim_{n\to\infty}(\|x_n - x^*\|^2 + \|y_n - y^*\|^2 + \|z_n - z^*\|^2) = 0.$$

Hence, we obtain that

$$\lim_{n\to\infty}\|x_n - x^*\| = \lim_{n\to\infty}\|y_n - y^*\| = \lim_{n\to\infty}\|z_n - z^*\| = 0.$$

This completes the proof. □

4.2.3 Consequence Results

Using Theorem 4.2.2, we can prove the following results.

Theorem 4.2.3. *Let A_i, M_i, Ω_i, K_i and $K_{i,\beta}$ satisfy the conditions (C1) and (C2), and let $F : H \to H$ be a μ-Lipschitzian and r-strongly monotone mapping. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be three sequences defined by*

$$\begin{cases} x_0, y_0, z_0 \in H, \\ x_{n+1} = (1-\alpha_n)K_{1,\beta}x_n + \alpha_n f_1(K_{2,\beta}y_n), \\ y_{n+1} = (1-\alpha_n)K_{2,\beta}y_n + \alpha_n f_2(K_{3,\beta}z_n), \\ z_{n+1} = (1-\alpha_n)K_{3,\beta}z_n + \alpha_n f_3(K_{1,\beta}x_n), \qquad n = 0,1,2,\ldots, \end{cases}$$

where $f_1 := I - \rho F, f_2 := I - \eta F, f_3 := I - \xi F$ with $\rho, \eta, \xi \in (0, \frac{2r}{\mu^2})$ and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\alpha_n \to 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge to x^*, y^* and z^* respectively, where (x^*, y^*, z^*) is the unique element in $\Omega_1 \times \Omega_2 \times \Omega_3$ such that the following three inequalities are satisfied

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega_1, \\ \langle \eta F(z^*) + y^* - z^*, y - y^* \rangle \geq 0, \quad \forall y \in \Omega_2, \\ \langle \xi F(x^*) + z^* - x^*, z - z^* \rangle \geq 0, \quad \forall z \in \Omega_3. \end{cases} \tag{4.2.10}$$

Proof. It is easy to see that f_1, f_2, f_3 are contraction mappings and all the condition in Theorem 4.2.2 are satisfied. By Theorem 4.2.2, we have the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge to $(x^*, y^*, z^*) \in \Omega_1 \times \Omega_2 \times \Omega_3$ such that the following three inequalities are satisfied

$$\begin{cases} \langle x^* - f_1(y^*), x - x^* \rangle \geq 0, & \forall x \in \Omega_1, \\ \langle y^* - f_2(z^*), y - y^* \rangle \geq 0, & \forall y \in \Omega_2, \\ \langle z^* - f_3(x^*), z - z^* \rangle \geq 0, & \forall z \in \Omega_3. \end{cases} \tag{4.2.11}$$

Substituting $f_1 := I - \rho F, f_2 := I - \eta F, f_3 := I - \xi F$ into (4.2.11), we obtain that the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge to $(x^*, y^*, z^*) \in \Omega_1 \times \Omega_2 \times \Omega_3$ such that the following three inequalities are satisfied

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in \Omega_1, \\ \langle \eta F(z^*) + y^* - z^*, y - y^* \rangle \geq 0, & \forall y \in \Omega_2, \\ \langle \xi F(x^*) + z^* - x^*, z - z^* \rangle \geq 0, & \forall z \in \Omega_3. \end{cases}$$

This completes the proof □

Setting $A_1 = A_2 = A_3$ in Theorem 4.2.2, we obtain the following corollary.

Corollary 4.2.4. *Let A_1, M_1, Ω_1, K_1 and $K_{1,\beta}$ satisfy the conditions (C1) and (C2), and let $f_i : H \to H$ be contractions with a contractive constant $h_i \in (0,1)$, for all $i = 1, 2, 3$. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be three sequences defined by*

$$\begin{cases} x_0, y_0, z_0 \in H, \\ x_{n+1} = (1 - \alpha_n) K_{1,\beta} x_n + \alpha_n f_1(K_{1,\beta} y_n), \\ y_{n+1} = (1 - \alpha_n) K_{1,\beta} y_n + \alpha_n f_2(K_{1,\beta} z_n), \\ z_{n+1} = (1 - \alpha_n) K_{1,\beta} z_n + \alpha_n f_3(K_{1,\beta} x_n), & n = 0, 1, 2, \ldots, \end{cases} \tag{4.2.12}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ generated be (4.2.3) converge to x^*, y^* and z^* respectively, where (x^*, y^*, z^*) is the unique element in $\Omega_1 \times \Omega_1 \times \Omega_1$ such that the following three inequalities are satisfied

$$\begin{cases} \langle x^* - f_1(y^*), x - x^* \rangle \geq 0, & \forall x \in \Omega_1, \\ \langle y^* - f_2(z^*), x - y^* \rangle \geq 0, & \forall x \in \Omega_1, \\ \langle z^* - f_3(x^*), x - z^* \rangle \geq 0, & \forall x \in \Omega_1. \end{cases} \tag{4.2.13}$$

Corollary 4.2.5. *Let A_1, M_1, Ω, K_1 and $K_{1,\beta}$ satisfy the conditions (C1) and (C2), and let $F : H \to H$ be a μ-Lipschitzian and r-strongly monotone mapping. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be three sequences defined by*

$$\begin{cases} x_0, y_0, z_0 \in H, \\ x_{n+1} = (1-\alpha_n)K_{1,\beta}x_n + \alpha_n f_1(K_{1,\beta}y_n), \\ y_{n+1} = (1-\alpha_n)K_{1,\beta}y_n + \alpha_n f_2(K_{1,\beta}z_n), \\ z_{n+1} = (1-\alpha_n)K_{1,\beta}z_n + \alpha_n f_3(K_{1,\beta}x_n), \qquad n = 0, 1, 2, \dots, \end{cases}$$

where $f_1 := I - \rho F, f_2 := I - \eta F, f_3 := I - \xi F$ with $\rho, \eta, \xi \in (0, \frac{2r}{\mu^2})$ and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge to x^*, y^* and z^* respectively, where (x^*, y^*, z^*) is the unique element in $\Omega_1 \times \Omega_1 \times \Omega_1$ such that the following three inequalities are satisfied

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega_1, \\ \langle \eta F(z^*) + y^* - z^*, x - y^* \rangle \geq 0, \quad \forall x \in \Omega_1, \\ \langle \xi F(x^*) + z^* - x^*, x - z^* \rangle \geq 0, \quad \forall x \in \Omega_1. \end{cases} \tag{4.2.14}$$

Setting $A_1 = A_2 = A_3, f_1 = f_2 = f_3$ and $x_0 = y_0 = z_0$ in Theorem 4.2.2, we obtain the following corollary.

Corollary 4.2.6. *Let A_1, M_1, Ω_1, K_1 and $K_{1,\beta}$ satisfy the conditions (C1) and (C2), and let $f : H \to H$ be contractions with a contractive constant $h \in (0,1)$. Let $\{x_n\}$ be the sequences defined by*

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1-\alpha_n)K_{1,\beta}x_n + \alpha_n f(K_{1,\beta}x_n), \qquad n = 0, 1, 2, \dots, \end{cases} \tag{4.2.15}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequences $\{x_n\}$ converge to $x^* \in \Omega_1$ such that the following three inequalities are satisfied

$$\langle x^* - f_1(x^*), x - x^* \rangle \geq 0, \qquad \forall x \in \Omega_1,$$

Corollary 4.2.7. *Let A_1, M_1, Ω_1, K_1 and $K_{1,\beta}$ satisfy the conditions (C1) and (C2), and let $F : H \to H$ be a μ-Lipschitzian and r-strongly monotone mapping. Let $\{x_n\}$ be the sequences defined by*

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1-\alpha_n)K_{1,\beta}x_n + \alpha_n (I - \rho F)(K_{1,\beta}x_n), \qquad n = 0, 1, 2, \dots, \end{cases} \tag{4.2.16}$$

where $\rho \in (0, \frac{2r}{\mu^2})$ *and* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *satisfying* $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*. Then the sequences* $\{x_n\}$ *converge to* $x^* \in \Omega_1$ *such that the following three inequalities are satisfied*

$$\langle F(x^*), x - x^* \rangle \geq 0, \qquad \forall x \in \Omega_1, \tag{4.2.17}$$

CHAPTER 5 ITERATION ALGORITHMS FOR FIXED POINT AND OPTIMIZATION PROBLEMS

5.1 Iterative Algorithms for Solving Hierarchical Fixed Point Problem of Nonexpansive Mapping

In This section deals with a method for approximating a solution of the fixed point problem: find $\tilde{x} \in F(T)$, where H is a Hilbert space, C is a closed convex subset of H, f is a ρ-contraction from C into H, $0 < \rho < 1$, A is a strongly positive linear bounded operator with coeffient $\bar{\gamma} > 0$, $0 < \gamma < \bar{\gamma}/\rho$, T is a nonexpansive mapping on C and $P_{F(T)}$ denotes the metric projection on the set of fixed point of T. We prove a strong convergence theorem by using the projection method which solves the variational inequality $\langle (A - \gamma f)\tilde{x} + \tau(I - S)\tilde{x}, x - \tilde{x}\rangle \geq 0$ for $x \in F(T)$, where $\tau \in [0, \infty)$.

Theorem 5.1.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to H$ be a ρ-contraction with $\rho \in (0, 1)$. Let $S, T : C \to C$ be two nonexpansive mappings with $F(T) \neq \emptyset$. Let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$. $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$ and $0 < \gamma < \bar{\gamma}/\rho$. Starting with an arbitrary initial guess $x_0 \in C$ and $\{x_n\}$ is a sequence generated by*

$$\begin{aligned} y_n &= \beta_n S x_n + (1 - \beta_n) x_n, \\ x_{n+1} &= P_C[\alpha_n \gamma f(x_n) + (I - \alpha_n A) T y_n], \quad \forall n \geq 1. \end{aligned} \tag{5.1.1}$$

Suppose that the following conditions are satisfied:

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

$(C2)$ $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = \tau = 0$*;*

$(C3)$ $\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$ *and* $\lim_{n\to\infty} \frac{|\beta_n - \beta_{n-1}|}{\beta_n} = 0$ *or*

$(C4)$ $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *and* $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$.

Then the sequence $\{x_n\}$ *converges strongly to a point* $\tilde{x} \in H$*, which is the unique solution of the variational inequality:*

$$\tilde{x} \in F(T), \qquad \langle (A - \gamma f)\tilde{x}, x - \tilde{x}\rangle \geq 0, \quad \forall x \in F(T). \tag{5.1.2}$$

Equivalently, we have $P_{F(T)}(I - A + \gamma f)\tilde{x} = \tilde{x}$*.*

Proof. We first show the uniqueness of a solution of the variational inequality (5.1.2), which is indeed a consequence of the strong monotonicity of $A - \gamma f$. Suppose $\bar{x} \in F(T)$ and $\tilde{x} \in F(T)$ both are solutions to 5.1.2, then $\langle (A - \gamma f)\bar{x}, \bar{x} - \tilde{x}\rangle \leq 0$ and $\langle (A - \gamma f)\tilde{x}, \tilde{x} - \bar{x}\rangle \leq 0$. It follows that

$$\begin{aligned}
\langle (A - \gamma f)\bar{x}, \bar{x} - \tilde{x}\rangle + \langle (A - \gamma f)\tilde{x}, \tilde{x} - \bar{x}\rangle &= \langle (A - \gamma f)\bar{x}, \bar{x} - \tilde{x}\rangle \\
&\quad -\langle (A - \gamma f)\tilde{x}, \bar{x} - \tilde{x}\rangle \\
&= \langle (A - \gamma f)\bar{x} - (A - \gamma f)\tilde{x}, \bar{x} - \tilde{x}\rangle \\
&\leq 0.
\end{aligned}$$

The strongly monotonicity of $A - \gamma f$ (Lemma 2.7.11) implies that $\bar{x} = \tilde{x}$ and the uniqueness is proved.

Next, we prove the sequence $\{x_n\}$ is bounded. Since $\alpha_n \to 0$ and $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = 0$ by condition $(C1)$ and $(C2)$, respectively , we can assume, without loss of generality, that $\alpha_n < \|A\|^{-1}$ and $\beta_n < \alpha_n$ for all $n \geq 1$. Take $u \in F(T)$ and from 5.1.1, we have

$$\begin{aligned}
\|x_{n+1} - u\| &= \|P_C[\alpha_n \gamma f(x_n) + (I - \alpha_n A)Ty_n] - P_C[u]\| \\
&\leq \|\alpha_n \gamma f(x_n) + (I - \alpha_n A)Ty_n - u\| \\
&\leq \alpha_n \gamma \|f(x_n) - f(u)\| + \alpha_n \|\gamma f(u) - Au\| + \|(I - \alpha_n A)(Ty_n - u)\|.
\end{aligned}$$

Since $\|I - \alpha_n A\| \leq 1 - \alpha_n \bar{\gamma}$ and by Lemma 2.7.2, we note that

$$\begin{aligned}
\|x_{n+1} - u\| &\leq \alpha_n \gamma \|f(x_n) - f(u)\| + \alpha_n \|\gamma f(u) - Au\| + (1 - \alpha_n \bar{\gamma})\|Ty_n - u\| \\
&\leq \alpha_n \gamma \rho \|x_n - u\| + \alpha_n \|\gamma f(u) - Au\| + (1 - \alpha_n \bar{\gamma})\|Ty_n - Tu\| \\
&\leq \alpha_n \gamma \rho \|x_n - u\| + \alpha_n \|\gamma f(u) - Au\| + (1 - \alpha_n \bar{\gamma})\|y_n - u\| \\
&\leq \alpha_n \gamma \rho \|x_n - u\| + \alpha_n \|\gamma f(u) - Au\| \\
&\quad + (1 - \alpha_n \bar{\gamma})\Big[\beta_n \|Sx_n - Su\| + \beta_n \|Su - u\| + (1 - \beta_n)\|x_n - u\|\Big]
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n\gamma\rho\|x_n-u\|+\alpha_n\|\gamma f(u)-Au\| \\
&\quad +(1-\alpha_n\bar{\gamma})\Big[\beta_n\|x_n-u\|+\beta_n\|Su-u\|+(1-\beta_n)\|x_n-u\|\Big] \\
&= \Big(1-\alpha_n(\bar{\gamma}-\gamma\rho)\Big)\|x_n-u\|+\alpha_n\|\gamma f(u)-Au\| \\
&\quad +(1-\alpha_n\bar{\gamma})\beta_n\|Su-u\| \\
&\leq \Big(1-\alpha_n(\bar{\gamma}-\gamma\rho)\Big)\|x_n-u\|+\alpha_n\|\gamma f(u)-Au\|+\beta_n\|Su-u\| \\
&\leq \Big(1-\alpha_n(\bar{\gamma}-\gamma\rho)\Big)\|x_n-u\|+\alpha_n\|\gamma f(u)-Au\|+\alpha_n\|Su-u\| \\
&= \Big(1-\alpha_n(\bar{\gamma}-\gamma\rho)\Big)\|x_n-u\|+\alpha_n\Big[\|\gamma f(u)-Au\|+\|Su-u\|\Big] \\
&= \Big(1-\alpha_n(\bar{\gamma}-\gamma\rho)\Big)\|x_n-u\| \\
&\quad +\alpha_n(\bar{\gamma}-\gamma\rho)\frac{\|\gamma f(u)-Au\|+\|Su-u\|}{(\bar{\gamma}-\gamma\rho)}.
\end{aligned}
$$

By induction, we can obtain

$\|x_{n+1}-u\| \leq Max\Big\{\|x_0-u\|, \frac{\|\gamma f(u)-Au\|+\|Su-u\|}{(\bar{\gamma}-\gamma\rho)}\Big\}$,

which implies that the sequence $\{x_n\}$ is bounded and so are the sequence $\{f(x_n)\}$, $\{Sx_n\}$, and $\{ATy_n\}$.

Set $w_n := \alpha_n\gamma f(x_n)+(I-\alpha_nA)Ty_n, \quad n\geq 1$. We get

$$
\begin{aligned}
\|x_{n+1}-x_n\| &= \|P_C[w_{n+1}]-P_C[w_n]\| \\
&\leq \|w_{n+1}-w_n\|. \qquad (5.1.3)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|x_{n+1}-x_n\| &\leq \|(\alpha_n\gamma f(x_n)+(I-\alpha_nA)Ty_n) \\
&\quad -(\alpha_{n-1}\gamma f(x_{n-1})+(I-\alpha_{n-1}A)Ty_{n-1})\| \\
&\leq \alpha_n\gamma\|f(x_n)-f(x_{n-1})\|+|\alpha_n-\alpha_{n-1}|\|\gamma f(x_{n-1})-ATy_{n-1}\| \\
&\quad +(1-\alpha_n\bar{\gamma})\|Ty_n-Ty_{n-1}\| \\
&\leq \alpha_n\gamma\rho\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|\gamma f(x_{n-1})-ATy_{n-1}\| \\
&\quad +(1-\alpha_n\bar{\gamma})\|y_n-y_{n-1}\|. \qquad (5.1.4)
\end{aligned}
$$

By (5.1.3) and (5.1.4), we get

$$
\begin{aligned}
\|x_{n+1}-x_n\| &\leq \alpha_n\gamma\rho\|w_n-w_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|\gamma f(x_{n-1})-ATy_{n-1}\| \\
&\quad +(1-\alpha_n\bar{\gamma})\|y_n-y_{n-1}\|. \qquad (5.1.5)
\end{aligned}
$$

From (5.1.1), we obtain

$$\begin{aligned}
\|y_n - y_{n-1}\| &= \|(\beta_n Sx_n + (1-\beta_n)x_n) - (\beta_{n-1}Sx_{n-1} + (1-\beta_{n-1})x_{n-1})\| \\
&= \|\beta_n(Sx_n - Sx_{n-1}) + (\beta_n - \beta_{n-1})(Sx_{n-1} - x_{n-1}) \\
&\quad +(1-\beta_n)(x_n - x_{n-1})\| \\
&\leq \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|Sx_{n-1} - x_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|M, \qquad (5.1.6)
\end{aligned}$$

where M is a constant such that

$$\sup_{n\in\mathbb{N}}\Big\{\|\gamma f(x_{n-1}) - ATy_{n-1}\| + \|Sx_{n-1} - x_{n-1}\|\Big\} \leq M.$$

Substituting (5.1.6) into (5.1.4) to obtain

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \alpha_n\gamma\rho\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|\gamma f(x_{n-1}) - ATy_{n-1}\| \\
&\quad +(1-\alpha_n\bar{\gamma})\Big[\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|M\Big] \\
&\leq \alpha_n\gamma\rho\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M \\
&\quad +(1-\alpha_n\bar{\gamma})\Big[\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|M\Big] \\
&= \Big(1 - \alpha_n(\bar{\gamma} - \gamma\rho)\Big)\|x_n - x_{n-1}\| \\
&\quad +M\Big[|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|\Big] \qquad (5.1.7) \\
&\leq \Big(1 - \alpha_n(\bar{\gamma} - \gamma\rho)\Big)\|w_n - w_{n-1}\| \\
&\quad +M\Big[|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|\Big].
\end{aligned}$$

At the same time , we can write (5.1.7) as

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \Big(1 - \alpha_n(\bar{\gamma} - \gamma\rho)\Big)\|w_n - w_{n-1}\| \\
&\quad +M\alpha_n\Big[\frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} + \frac{|\beta_n - \beta_{n-1}|}{\alpha_n}\Big] \\
&\leq \Big(1 - \alpha_n(\bar{\gamma} - \gamma\rho)\Big)\|w_n - w_{n-1}\| \\
&\quad +M\alpha_n\Big[\frac{|\alpha_n - \alpha_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n}\Big]. \qquad (5.1.8)
\end{aligned}$$

From (5.1.7), $(C4)$ and Lemma 2.7.2 or (from (5.1.8), $(C3)$ and Lemma 2.7.2), we can deduce that $\|x_{n+1} - x_n\| \to 0$, respectively.

From (5.1.1), we have

$$
\begin{aligned}
\|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \\
&= \|x_n - x_{n+1}\| + \|P_C[w_n] - P_C[Tx_n]\| \\
&\leq \|x_n - x_{n+1}\| + \|w_n - Tx_n\| \\
&= \|x_n - x_{n+1}\| + \|\alpha_n \gamma f(x_n) + (I - \alpha_n A)Ty_n - Tx_n\| \\
&\leq \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - ATx_n\| + (1 - \alpha_n\bar{\gamma})\|Ty_n - Tx_n\| \\
&\leq \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - ATx_n\| + (1 - \alpha_n\bar{\gamma})\|y_n - x_n\| \\
&= \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - ATx_n\| + (1 - \alpha_n\bar{\gamma})\beta_n\|Sx_n - x_n\|.
\end{aligned}
$$

Notice that $\alpha_n \to 0$, $\beta_n \to 0$, and $\|x_{n+1} - x_n\| \to 0$, so we obtain

$$\|x_n - Tx_n\| \to 0. \tag{5.1.9}$$

Next, we prove

$$\limsup_{n\to\infty}\langle \gamma f(z) - Az, x_n - z\rangle \leq 0. \tag{5.1.10}$$

where $z = P_{F(T)}(I - A + \gamma f)z$. Since the sequence $\{x_n\}$ is bounded we can take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(z) - Az, x_n - z\rangle = \lim_{k\to\infty}\langle \gamma f(z) - Az, x_{n_k} - z\rangle$$

and $x_{n_k} \rightharpoonup \tilde{x}$. From (5.1.8) and by Lemma 2.7.1, it follows that $\tilde{x} \in F(T)$. Hence, by Lemma 2.7.13 that

$$
\begin{aligned}
\limsup_{n\to\infty}\langle \gamma f(z) - Az, x_n - z\rangle &= \lim_{k\to\infty}\langle \gamma f(z) - Az, x_{n_k} - z\rangle \\
&= \langle \gamma f(z) - Az, \tilde{x} - z\rangle \\
&= \langle (I - A + \gamma f)z - z, \tilde{x} - z\rangle \\
&\leq 0
\end{aligned}
$$

Now, by Lemma 2.7.13, we observe that

$$\langle P_C[w_n] - w_n, P_C[w_n] - z\rangle \leq 0,$$

and so

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &= \langle P_C[w_n] - z, P_C[w_n] - z\rangle \\
&= \langle P_C[w_n] - w_n, P_C[w_n] - z\rangle + \langle w_n - z, P_C[w_n] - z\rangle \\
&\leq \langle w_n - z, P_C[w_n] - z\rangle \\
&= \langle \alpha_n \gamma f(x_n) + (I - \alpha_n A)Ty_n - z, x_{n+1} - z\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\leq \alpha_n\gamma\|f(x_n)-f(z)\|\|x_{n+1}-z\|+\alpha_n\langle\gamma f(z)-Az,x_{n+1}-z\rangle\\
&\quad+(1-\alpha_n\bar{\gamma})\|Ty_n-z\|\|x_{n+1}-z\|\\
&\leq \alpha_n\gamma\rho\|x_n-z\|\|x_{n+1}-z\|+\alpha_n\langle\gamma f(z)-Az,x_{n+1}-z\rangle\\
&\quad+(1-\alpha_n\bar{\gamma})\|y_n-z\|\|x_{n+1}-z\|\\
&= \alpha_n\gamma\rho\|x_n-z\|\|x_{n+1}-z\|+\alpha_n\langle\gamma f(z)-Az,x_{n+1}-z\rangle\\
&\quad+(1-\alpha_n\bar{\gamma})\|\beta_nSx_n+(1-\beta_n)x_n-z\|\|x_{n+1}-z\|\\
&\leq \alpha_n\gamma\rho\|x_n-z\|\|x_{n+1}-z\|+\alpha_n\langle\gamma f(z)-Az,x_{n+1}-z\rangle\\
&\quad+(1-\alpha_n\bar{\gamma})\Big[\beta_n\|Sx_n-Sz\|+\beta_n\|Sz-z\|\\
&\quad+(1-\beta_n)\|x_n-z\|\Big]\|x_{n+1}-z\|\\
&\leq \alpha_n\gamma\rho\|x_n-z\|\|x_{n+1}-z\|+\alpha_n\langle\gamma f(z)-Az,x_{n+1}-z\rangle\\
&\quad+(1-\alpha_n\bar{\gamma})\Big[\beta_n\|x_n-z\|+\beta_n\|Sz-z\|\\
&\quad+(1-\beta_n)\|x_n-z\|\Big]\|x_{n+1}-z\|\\
&= \Big(1-\alpha_n(\bar{\gamma}-\gamma\rho)\Big)\|x_n-z\|\|x_{n+1}-z\|\\
&\quad+\alpha_n\langle\gamma f(z)-Az,x_{n+1}-z\rangle\\
&\quad+(1-\alpha_n\bar{\gamma})\beta_n\|Sz-z\|\|x_{n+1}-z\|\\
&\leq \frac{[1-\alpha_n(\bar{\gamma}-\gamma\rho)]}{2}\Big[\|x_n-z\|^2+\|x_{n+1}-z\|^2\Big]\\
&\quad+\alpha_n\langle\gamma f(z)-Az,x_{n+1}-z\rangle\\
&\quad+(1-\alpha_n\bar{\gamma})\beta_n\|Sz-z\|\|x_{n+1}-z\|.
\end{aligned}
$$

Hence, it follows that

$$
\begin{aligned}
\|x_{n+1}-z\|^2 &\leq \frac{1-\alpha_n(\bar{\gamma}-\gamma\rho)}{1+\alpha_n(\bar{\gamma}-\gamma\rho)}\|x_n-z\|^2\\
&\quad+\frac{2\alpha_n}{1+\alpha_n(\bar{\gamma}-\gamma\rho)}\langle\gamma f(z)-Az,x_{n+1}-z\rangle\\
&\quad+\frac{2(1-\alpha_n\bar{\gamma})\beta_n}{1+\alpha_n(\bar{\gamma}-\gamma\rho)}\|Sz-z\|\|x_{n+1}-z\|\\
&= \left[\frac{2\alpha_n(\bar{\gamma}-\gamma\rho)}{1+\alpha_n(\bar{\gamma}-\gamma\rho)}\right]\left[\frac{1}{\alpha_n(\bar{\gamma}-\gamma\rho)}\langle\gamma f(z)-Az,x_{n+1}-z\rangle\right.\\
&\quad\left.+\frac{\beta_n(1-\alpha_n\bar{\gamma})}{\alpha_n(\bar{\gamma}-\gamma\rho)}\|Sz-z\|\|x_{n+1}-z\|\right]\\
&\quad+\left[1-\frac{2\alpha_n(\bar{\gamma}-\gamma\rho)}{1+\alpha_n(\bar{\gamma}-\gamma\rho)}\right]\|x_n-z\|^2.
\end{aligned}
$$

We observe that

$$\limsup_{n\to\infty}\left[\frac{1}{\alpha_n(\bar{\gamma}-\gamma\rho)}\langle\gamma f(z)-Az,x_{n+1}-z\rangle+\frac{\beta_n(1-\alpha_n\bar{\gamma})}{\alpha_n(\bar{\gamma}-\gamma\rho)}\|Sz-z\|\|x_{n+1}-z\|\right]\leq 0.$$

Thus, by Lemma 2.7.6, $x_n \to z$ as $n \to \infty$. This is completes. □

Under different conditions on data we obtain the following result.

Theorem 5.1.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to H$ be a ρ-contraction (possibly nonself) with $\rho \in (0,1)$. Let $S, T : C \to C$ be two nonexpansive mappings with $F(T) \neq \emptyset$. Let A be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \bar{\gamma}/\rho$. $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$. Starting with an arbitrary initial guess $x_0 \in C$ nd $\{x_n\}$ is a sequence generated by*

$$\begin{aligned} y_n &= \beta_n S x_n + (1-\beta_n)x_n, \\ x_{n+1} &= P_C[\alpha_n\gamma f(x_n) + (I-\alpha_n A)Ty_n], \quad \forall n \geq 1. \end{aligned}$$

Suppose that the following conditions are satisfied:

$(C1)$ $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$*;*

$(C2)$ $\lim_{n\to\infty}\frac{\beta_n}{\alpha_n} = \tau \in (0,\infty)$*;*

$(C5)$ $\lim_{n\to\infty}\frac{|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|}{\alpha_n\beta_n} = 0$*;*

$(C6)$ *there exists a constant $K > 0$ such that $\frac{1}{\alpha_n}|\frac{1}{\beta_n} - \frac{1}{\beta_{ni1}}| \leq K$.*

Then the sequence $\{x_n\}$ converges strongly to a point $\tilde{x} \in H$, which is the unique solution of the variational inequality:

$$\tilde{x} \in F(T), \quad \langle\frac{1}{\tau}(A-\gamma f)\tilde{x} + (I-S)\tilde{x}, x-\tilde{x}\rangle \geq 0, \quad \forall x \in F(T). \tag{5.1.11}$$

Proof. First of all, we show that (5.1.11) has the unique solution. Indeed, let $\bar{x}$ and $\tilde{x}$ be two solutions. Then

$$\langle(A-\gamma f)\tilde{x}, \tilde{x}-\bar{x}\rangle \leq \tau\langle(I-S)\tilde{x}, \bar{x}-\tilde{x}\rangle. \tag{5.1.12}$$

Analogously, we have

$$\langle(A-\gamma f)\bar{x}, \bar{x}-\tilde{x}\rangle \leq \tau\langle(I-S)\bar{x}, \tilde{x}-\bar{x}\rangle. \tag{5.1.13}$$

Adding (5.1.12) and (5.1.13), by Lemma 2.7.11, we obtain

$$\begin{aligned}(\bar{\gamma}-\gamma\rho)\|\tilde{x}-\bar{x}\|^2 &\leq \langle (A-\gamma f)\tilde{x}-(A-\gamma f)\bar{x}, \tilde{x}-\bar{x}\rangle \\ &\leq -\tau\langle (I-S)\tilde{x}-(I-S)\bar{x}, \tilde{x}-\bar{x}\rangle \\ &\leq 0,\end{aligned}$$

and so $\tilde{x}=\bar{x}$. From $(C2)$, we can assume, without loss of generality, that $\beta_n \leq (\tau+1)\alpha_n$ for all $n \geq 1$. By a similar argument in Theorem 5.1.1, we have

$$\begin{aligned}\|x_{n+1}-u\| &\leq \alpha_n\gamma\rho\|x_n-u\|+\alpha_n\|\gamma f(u)-Au\| \\ &\quad +(1-\alpha_n\bar{\gamma})\Big[\ \|x_n-u\|+\beta_n\|Su-u\|+(1-\beta_n)\|x_n-u\|\Big] \\ &= \Big(1-\alpha_n(\bar{\gamma}-\gamma\rho)\Big)\|x_n-u\|+\alpha_n\|\gamma f(u)-Au\| \\ &\quad +(1-\alpha_n\bar{\gamma})\beta_n\|Su-u\| \\ &\leq \Big(1-\alpha_n(\bar{\gamma}-\gamma\rho)\Big)\|x_n-u\|+\alpha_n\|\gamma f(u)-Au\|+\beta_n\|Su-u\| \\ &\leq \Big(1-\alpha_n(\bar{\gamma}-\gamma\rho)\Big)\|x_n-u\|+\alpha_n\|\gamma f(u)-Au\| \\ &\quad +(\tau+1)\alpha_n\|Su-u\| \\ &= \Big(1-\alpha_n(\bar{\gamma}-\gamma\rho)\Big)\|x_n-u\|+\alpha_n\Big[\|\gamma f(u)-Au\| \\ &\quad +(\tau+1)\|Su-u\|\Big] \\ &= \Big(1-\alpha_n(\bar{\gamma}-\gamma\rho)\Big)\|x_n-u\| \\ &\quad +\alpha_n(\bar{\gamma}-\gamma\rho)\frac{\|\gamma f(u)-Au\|+(\tau+1)\|Su-u\|}{(\bar{\gamma}-\gamma\rho)}.\end{aligned}$$

By induction, we obtain

$$\|x_n-u\| \leq max\Bigg\{\|x_0-u\|, \frac{1}{\bar{\gamma}-\gamma\rho}\Big[\|\gamma f(u)-Au\|+(\tau+1)\|Su-u\|\Big]\Bigg\}$$

which implies that the sequence $\{x_n\}$ is bounded. Since $(C5)$ implies $(C4)$ then, from Theorem 5.1.1, we can deduce $\|x_{n+1}-x_n\| \to 0$.

From (5.1.1), we note that

$$\begin{aligned}x_{n+1} &= P_C[w_n]-w_n+w_n+y_n-y_n \\ &= P_C[w_n]-w_n+\alpha_n\gamma f(x_n)+(Ty_n-y_n)+(y_n-\alpha_n ATy_n).\end{aligned}$$

Hence, it follows that

$$\begin{aligned} x_n - x_{n+1} &= (w_n - P_C[w_n]) + \alpha_n(Ax_n - \gamma f(x_n) + (y_n - Ty_n) \\ &\quad + (x_n - y_n) + \alpha_n(ATy_n - Ax_n) \\ &= (w_n - P_C[w_n]) + \alpha_n(A - \gamma f)x_n + (I - T)y_n \\ &\quad + \beta_n(I - S)x_n) + \alpha_n A(Ty_n - x_n) \end{aligned}$$

and so

$$\begin{aligned} \frac{x_n - x_{n+1}}{(1-\alpha_n)\beta_n} &= \frac{1}{(1-\alpha_n)\beta_n}(w_n - P_C[w_n]) \\ &\quad + \frac{\alpha_n}{(1-\alpha_n)\beta_n}(A - \gamma f)x_n + \frac{1}{(1-\alpha_n)\beta_n}(I - T)y_n \\ &\quad + \frac{1}{(1-\alpha_n)}(I - S)x_n) + \frac{\alpha_n}{(1-\alpha_n)\beta_n}A(Ty_n - x_n). \end{aligned}$$

Set $v_n := \frac{x_n - x_{n+1}}{(1-\alpha_n)\beta_n}$. Then, we have

$$\begin{aligned} v_n &= \frac{1}{(1-\alpha_n)\beta_n}(w_n - P_C[w_n]) \\ &\quad + \frac{\alpha_n}{(1-\alpha_n)\beta_n}(A - \gamma f)x_n + \frac{1}{(1-\alpha_n)\beta_n}(I - T)y_n \\ &\quad + \frac{1}{(1-\alpha_n)}(I - S)x_n) + \frac{\alpha_n}{(1-\alpha_n)\beta_n}A(Ty_n - x_n). \end{aligned} \tag{5.1.14}$$

From (5.1.7) in Theorem5.1.1 and ($C6$), we obtain

$$\begin{aligned} \frac{\|x_{n+1} - x_n\|}{\beta_n} &\leq \Big(1 - \alpha_n(\bar{\gamma} - \gamma\rho)\Big)\frac{\|x_n - x_{n-1}\|}{\beta_n} + M\Big[\frac{|\alpha_n - \alpha_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n}\Big] \\ &= \Big(1 - \alpha_n(\bar{\gamma} - \gamma\rho)\Big)\frac{\|x_n - x_{n-1}\|}{\beta_n} + \Big(1 - \alpha_n(\bar{\gamma} - \gamma\rho)\Big)\frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} \\ &\quad - \Big(1 - \alpha_n(\bar{\gamma} - \gamma\rho)\Big)\frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} + M\Big[\frac{|\alpha_n - \alpha_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n}\Big] \\ &= \Big(1 - \alpha_n(\bar{\gamma} - \gamma\rho)\Big)\frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} \\ &\quad + \Big(1 - \alpha_n(\bar{\gamma} - \gamma\rho)\Big)\|x_n - x_{n-1}\|\Big[\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}\Big] \\ &\quad + M\Big[\frac{|\alpha_n - \alpha_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n}\Big] \\ &\leq \Big(1 - \alpha_n(\bar{\gamma} - \gamma\rho)\Big)\frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} + \|x_n - x_{n-1}\|\Big|\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}\Big| \\ &\quad + M\Big[\frac{|\alpha_n - \alpha_{n-1}|}{\beta_n} + \frac{|\beta_n - \beta_{n-1}|}{\beta_n}\Big] \end{aligned}$$

$$
\begin{aligned}
&\le \Big(1-\alpha_n(\bar{\gamma}-\gamma\rho)\Big)\frac{\|x_n-x_{n-1}\|}{\beta_{n-1}}+\alpha_n K\|x_n-x_{n-1}\| \\
&\quad +M\Big[\frac{|\alpha_n-\alpha_{n-1}|}{\beta_n}+\frac{|\beta_n-\beta_{n-1}|}{\beta_n}\Big] \\
&\le \Big(1-\alpha_n(\bar{\gamma}-\gamma\rho)\Big)\frac{\|w_n-w_{n-1}\|}{\beta_{n-1}}+\alpha_n K\|x_n-x_{n-1}\| \\
&\quad +M\Big[\frac{|\alpha_n-\alpha_{n-1}|}{\beta_n}+\frac{|\beta_n-\beta_{n-1}|}{\beta_n}\Big].
\end{aligned}
$$

This together with Lemma 2.7.6 and $(C2)$ imply that

$$
\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\beta_n}=\lim_{n\to\infty}\frac{\|w_{n+1}-w_n\|}{\beta_n}=\lim_{n\to\infty}\frac{\|w_{n+1}-w_n\|}{\alpha_n}=0.
$$

From (5.1.14), for $z\in F(T)$, we have

$$
\begin{aligned}
\langle v_n, x_n-z\rangle &= \frac{1}{(1-\alpha_n)\beta_n}\langle w_n-P_C[w_n], P_C[w_{n-1}]-z\rangle \\
&\quad +\frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (A-\gamma f)x_n, x_n-z\rangle \\
&\quad +\frac{1}{(1-\alpha_n)\beta_n}\langle (I-T)y_n, x_n-z\rangle+\frac{1}{(1-\alpha_n)}\langle (I-S)x_n, x_n-z\rangle \\
&\quad +\frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (A(Ty_n-x_n), x_n-z\rangle \\
&= \frac{1}{(1-\alpha_n)\beta_n}\langle w_n-P_C[w_n], P_C[w_n]-z\rangle \\
&\quad +\frac{1}{(1-\alpha_n)\beta_n}\langle w_n-P_C[w_n], P_C[w_{n-1}]-P_C[w_n]\rangle \\
&\quad +\frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (A-\gamma f)x_n-(A-\gamma f)z, x_n-z\rangle \\
&\quad +\frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (A-\gamma f)z, x_n-z\rangle \\
&\quad +\frac{1}{(1-\alpha_n)}\langle (I-S)x_n-(I-S)z, x_n-z\rangle \\
&\quad +\frac{1}{(1-\alpha_n)}\langle (I-S)z, x_n-z\rangle \\
&\quad +\frac{1}{(1-\alpha_n)\beta_n}\langle (I-T)y_n, x_n-z\rangle \\
&\quad +\frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (A(Ty_n-x_n), x_n-z\rangle. \qquad (5.1.15)
\end{aligned}
$$

By Lemma 2.7.13 and Lemma 2.7.11, we obtain

$$
\begin{aligned}
\langle v_n, x_n-z\rangle &\ge \frac{1}{(1-\alpha_n)\beta_n}\langle w_n-P_C[w_n], P_C[w_{n-1}]-P_C[w_n]\rangle \\
&\quad +\frac{(\bar{\gamma}-\gamma\rho)\alpha_n}{(1-\alpha_n)\beta_n}\|x_n-z\|^2+\frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (A-\gamma f)z, x_n-z\rangle
\end{aligned}
$$

$$+\frac{1}{(1-\alpha_n)}\langle (I-S)z, x_n - z\rangle + \frac{1}{(1-\alpha_n)\beta_n}\langle (I-T)y_n, x_n - z\rangle$$
$$+\frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (A(Ty_n - x_n), x_n - z\rangle.$$

Now, we observe that

$$\begin{aligned}
\|x_n - z\|^2 \leq\ & \frac{(1-\alpha_n)\beta_n}{(\bar\gamma-\gamma\rho)\alpha_n}\langle v, x_n - z\rangle - \frac{\beta_n}{(\bar\gamma-\gamma\rho)\alpha_n}\langle (I-S)z, x_n - z\rangle \\
& -\frac{1}{(\bar\gamma-\gamma\rho)}\langle (A-\gamma f)z, x_n - z\rangle - \frac{1}{(\bar\gamma-\gamma\rho)\alpha_n}\langle (I-T)y_n, x_n - z\rangle \\
& -\frac{1}{(\bar\gamma-\gamma\rho)}\langle (A(Ty_n - x_n), x_n - z\rangle \\
& -\frac{1}{(\bar\gamma-\gamma\rho)\alpha_n}\langle w_n - P_C[w_n], P_C[w_{n-1}] - P_C[w_n]\rangle \\
\leq\ & \frac{(1-\alpha_n)\beta_n}{(\bar\gamma-\gamma\rho)\alpha_n}\langle v, x_n - z\rangle - \frac{\beta_n}{(\bar\gamma-\gamma\rho)\alpha_n}\langle (I-S)z, x_n - z\rangle \\
& -\frac{1}{(\bar\gamma-\gamma\rho)}\langle (A-\gamma f)z, x_n - z\rangle - \frac{1}{(\bar\gamma-\gamma\rho)\alpha_n}\langle (I-T)y_n, x_n - z\rangle \\
& -\frac{1}{(\bar\gamma-\gamma\rho)}\langle (A(Ty_n - x_n), x_n - z\rangle + \frac{\|w_n - w_{n-1}\|}{(\bar\gamma-\gamma\rho)}\|w_n - P_C[w_n]\|.
\end{aligned}$$

From $(C1)$ and $(C2)$, we have $\beta_n \to 0$. Hence, from 5.1.1, we deduce $\|y_n - x_n\| \to 0$ and $\|x_{n+1} - Ty_n\| \to 0$. Therefore

$$\|y_n - Ty_n\| \leq \|y_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - Ty_n\| \to 0.$$

Since $v_n \to 0$, $(I-T)y_n \to 0$, $A(Ty)_n - x_n) \to 0$ and $\frac{\|w_n - w_{n-1}\|}{(\bar\gamma-\gamma\rho)} \to 0$, every weak cluster point of $\{x_n\}$ is also a strong cluster point. Note that the sequence $\{x_n\}$ is bounded, thus there exist a subsequence $\{x_{n_k}\}$ converging to a point $\tilde{x} \in H$. For all $z \in F(T)$, it follows from (5.1.15) that

$$\begin{aligned}
\langle (A-\gamma f)x_{n_k}, x_{n_k} - z\rangle =\ & \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle v_{n_k}, x_{n_k} - z\rangle - \frac{1}{\alpha_{n_k}}\langle (I-T)y_{n_k}, x_{n_k} - z\rangle \\
& -\frac{\beta_{n_k}}{\alpha_{n_k}}\langle (I-S)x_{n_k}, x_{n_k} - z\rangle - \langle A(Ty_{n_k} - x_{n_k}), x_{n_k} - z)\rangle \\
& -\frac{1}{\alpha_{n_k}}\langle w_{n_k} - P_C[w_{n_k}], P_C[w_{n_{k-1}}] - z\rangle \\
\leq\ & \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle v_{n_k}, x_{n_k} - z\rangle - \frac{1}{\alpha_{n_k}}\langle (I-T)y_{n_k}, x_{n_k} - z\rangle \\
& -\frac{\beta_{n_k}}{\alpha_{n_k}}\langle (I-S)x_{n_k}, x_{n_k} - z\rangle - \langle A(Ty_{n_k} - x_{n_k}), x_{n_k} - z)\rangle \\
& -\frac{1}{\alpha_{n_k}}\langle w_{n_k} - P_C[w_{n_k}], P_C[w_{n_{k-1}}] - P_C[w_{n_k}]\rangle \\
& -\langle A(Ty_{n_k} - x_{n_k}), x_{n_k} - z)\rangle \\
& -\frac{1}{\alpha_{n_k}}\langle w_{n_k} - P_C[w_{n_k}], P_C[w_{n_{k-1}}] - z\rangle
\end{aligned}$$

$$\leq \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle v_{n_k}, x_{n_k}-z\rangle - \frac{1}{\alpha_{n_k}}\langle (I-T)y_{n_k}, x_{n_k}-z\rangle$$
$$-\frac{\beta_{n_k}}{\alpha_{n_k}}\langle (I-S)x_{n_k}, x_{n_k}-z\rangle - \langle A(Ty_{n_k}-x_{n_k}), x_{n_k}-z)\rangle$$
$$+\frac{\|w_{n_k}-w_{n_{k-1}}\|}{\alpha_{n_k}}\|w_{n_k}-P_C[w_{n_k}]\|.$$

Letting $k\to\infty$, we obtain

$$\langle (A-\gamma f)\tilde{x}, \tilde{x}-z\rangle \leq -\tau\langle (I-S)\tilde{x}, \tilde{x}-z\rangle, \quad \forall z\in F(T).$$

By Lemma 2.2.13, (5.1.11) has the unique solution, it follows that $\omega_w(x_n)=\{\tilde{x}\}$. Therefore, $x_n\to\tilde{x}$ as $n\to\infty$. This is completes the proof. □

From Theorem 5.1.2, we can deduce the following interesting corollary.

Corollary 5.1.3. *[8] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f: C\to H$ be a ρ-contraction (possibly nonself) with $\rho\in(0,1)$. Let $S,T: C\to C$ be two nonexpansive mappings with $F(T)\neq\emptyset$. $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ Starting with an arbitrary initial guess $x_0\in C$ and $\{x_n\}$ is a sequence generated by*

$$\begin{aligned} y_n &= \beta_n S x_n + (1-\beta_n)x_n, \\ x_{n+1} &= P_C[\alpha_n f(x_n) + (1-\alpha_n)Ty_n], \quad \forall n\geq 1. \end{aligned}$$

Suppose that the following conditions are satisfied:

$(C1)$ $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$*;*

$(C2)$ $\lim_{n\to\infty}\frac{\beta_n}{\alpha_n}=\tau\in(0,\infty)$*;*

$(C5)$ $\lim_{n\to\infty}\frac{|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|}{\alpha_n\beta_n}=0$*;*

$(C6)$ *there exists a constant $K>0$ such that $\frac{1}{\alpha_n}|\frac{1}{\beta_n}-\frac{1}{\beta_{ni1}}|\leq K$.*

Then the sequence $\{x_n\}$ converges strongly to a point $\tilde{x}\in H$, which is the unique solution of the variational inequality:

$$\tilde{x}\in F(T), \quad \langle\frac{1}{\tau}(I-f)\tilde{x}+(I-S)\tilde{x}, x-\tilde{x}\rangle\geq 0, \quad \forall x\in F(T). \tag{5.1.16}$$

Proof. As a matter of fact, if we take $A = I$ and $\gamma = 1$ in Theorem 5.1.2. This complete the proof. □

Corollary 5.1.4. *[8] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S, T : C \to C$ be two nonexpansive mappings with $F(T) \neq \emptyset$. $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$. Starting with an arbitrary initial guess $x_0 \in C$ and suppose $\{x_n\}$ is a sequence generated by*

$$\begin{aligned} y_n &= \beta_n S x_n + (1 - \beta_n) x_n, \\ x_{n+1} &= P_C[(1 - \alpha_n) T y_n], \quad \forall n \geq 1. \end{aligned} \tag{5.1.17}$$

Suppose that the following conditions are satisfied:

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

$(C2)$ $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = 1$;

$(C5)$ $\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|}{\alpha_n \beta_n} = 0$;

$(C6)$ *there exists a constant* $K > 0$ *such that* $\frac{1}{\alpha_n} |\frac{1}{\beta_n} - \frac{1}{\beta_{ni1}}| \leq K$.

Then the sequence $\{x_n\}$ converges strongly to a point $\tilde{x} \in H$, which is the unique solution of the variational inequality:

$$\tilde{x} \in F(T), \qquad \langle (I - \frac{S}{2})\tilde{x}, x - \tilde{x} \rangle \geq 0, \quad \forall x \in F(T). \tag{5.1.18}$$

Proof. As a matter of fact, if we take $A = I$, $f = 0$ and $\gamma = 1$ in Theorem 5.1.2 . This is completes the proof. □

Remark 5.1.5. Prototypes for the iterative parameters are, for example, $\alpha_n = n^{-\theta}$ and $\beta_n = n^{-\omega}$ (with $\theta, \omega > 0$). Since $|\alpha_n - \alpha_{n-1}| \approx n^{-\theta}$ and $|\beta_n - \beta_{n-1}| \approx n^{-\omega}$, it is not difficult to prove that $(C5)$ is satisfied for $0 < \theta, \omega < 1$and $(C6)$ is satisfied if $\theta + \omega \leq 1$.

Remark 5.1.6. Our results improve and extend the results of Yao et al. [8] by we take $A = I$ and $\gamma = 1$ in Theorems 5.1.1 and 5.1.2.

Example 5.1.7. Let $H = \mathbb{R}$, $C = [-\frac{1}{4}, \frac{1}{4}]$, $T = I$, $S = -I$, $A = I$, $f(x) = x^2$, $P_C = I$, $\beta_n = \frac{1}{\sqrt{n}}$, $\alpha_n = \frac{1}{\sqrt{n}}$ for every $n \in \mathbb{N}$, we have $\tau = 1$ and choose $\bar{\gamma} = \frac{1}{2}$, $\rho = \frac{1}{3}$ and $\gamma = 1$. Then $\{x_n\}$ is the sequence

$$x_{n+1} = \frac{x_n^2}{\sqrt{n}} + (1 - \frac{1}{\sqrt{n}})(1 - \frac{2}{\sqrt{n}})x_n, \tag{5.1.19}$$

and $x_n \to \tilde{x} = 0$ as $n \to \infty$, where $\tilde{x} = 0$ is the unique solution of the variational inequality

$$\tilde{x} \in F(T) = [-\frac{1}{4}, \frac{1}{4}], \quad \langle (3\tilde{x} - \tilde{x}^2), x - \tilde{x} \rangle \geq 0, \quad \forall x \in F(T) = [-\frac{1}{4}, \frac{1}{4}]. \tag{5.1.20}$$

5.2 Iteration Algorithm for Solving Hierarchical Fixed Point Problem of Strictly Pseudo-Contractive Mapping

In this section, we introduce a new iterative scheme that converges strongly to a common fixed point of a countable family of strictly pseudo-contractive mappings in a real Hilbert space which is also a solution of variational inequality problem related to quadratic minimization problems.

Let us consider the net iterative scheme as follows:

$$\begin{cases} y_n = P_C[\beta_n S x_n + (1 - \beta_n) x_n], \\ x_{n+1} = P_C[\alpha_n f(x_n) + \sum\limits_{i=1}^{n} (\alpha_{i-1} - \alpha_i) V_i y_n], \quad \forall n \geq 1, \end{cases} \tag{5.2.1}$$

where $V_i = k_i I + (1 - k_i) T_i$, $f : C \to H$ is a ρ-contraction mapping, $S : C \to H$ is a nonexpansive mapping, $\{T_i\}_{i=1}^{\infty} : C \to C$ is a countable family of k_i-strict pseudo-contraction mappings and $\bigcap\limits_{i=1}^{\infty} F(T_i) \neq \emptyset$. Set $\alpha_0 = 1$, $\{\alpha_n\} \subset (0, 1)$ is a strictly decreasing sequence and $\{\beta_n\} \subset (0, 1)$. As we will see the convergence of the scheme depends on the choice of the parameters $\{\alpha_n\}$ and $\{\beta_n\}$. We list some possible hypotheses on them:

(H1) there exists $\gamma > 0$ such that $\beta_n \leq \gamma \alpha_n$;

(H2) $\lim\limits_{n \to \infty} \beta_n / \alpha_n = \tau \in [0, \infty)$;

(H3) $\lim\limits_{n \to \infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(H4) $\sum\limits_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$;

(H5) $\sum_{n=1}^{\infty}|\beta_n - \beta_{n-1}| < \infty$;

(H6) $\lim_{n\to\infty}|\alpha_n - \alpha_{n-1}|/\alpha_n = 0$;

(H7) $\lim_{n\to\infty}|\beta_n - \beta_{n-1}|/\beta_n = 0$;

(H8) $\lim_{n\to\infty}[|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|]/\alpha_n\beta_n = 0$;

(H9) there exists a constant $K > 0$ such that $\frac{1}{\alpha_n}|\frac{1}{\beta_n} - \frac{1}{\beta_{ni1}}| \leq K$.

Proposition 5.2.1. *Assume that (H1) holds. Then $\{x_n\}$ and $\{y_n\}$ are bounded.*

Proof. Let $z \in \bigcap_{i=1}^{\infty} F(T_i) = \bigcap_{i=1}^{\infty} F(V_i)$

$$\begin{aligned}
\|x_{n+1} - z\| &= \left\| P_C[\alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)V_i y_n] - P_C[z] \right\| \\
&\leq \left\| \alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)V_i y_n - z \right\| \\
&= \left\| \alpha_n(f(x_n) - z) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)(V_i y_n - z) \right\| \\
&\leq \alpha_n\|f(x_n) - f(z)\| + \alpha_n\|f(z) - z\| + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|V_i y_n - z\| \\
&\leq \alpha_n\rho\|x_n - z\| + \alpha_n\|f(z) - z\| + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|y_n - z\| \\
&\leq \alpha_n\rho\|x_n - z\| + \alpha_n\|f(z) - z\| \\
&\quad + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|\beta_n S x_n + (1 - \beta_n)x_n - z\| \\
&\leq \alpha_n\rho\|x_n - z\| + \alpha_n\|f(z) - z\| \\
&\quad + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)(\beta_n\|Sx_n - Sz\| + \beta_n\|Sz - z\| \\
&\quad + (1 - \beta_n)\|x_n - z\|) \\
&\leq \alpha_n\rho\|x_n - z\| + \alpha_n\|f(z) - z\| \\
&\quad + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)(\beta_n\|x_n - z\| + \beta_n\|Sz - z\| \\
&\quad + (1 - \beta_n)\|x_n - z\|) \\
&= \alpha_n\rho\|x_n - z\| + \alpha_n\|f(z) - z\| \\
&\quad + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)(\|x_n - z\| + \beta_n\|Sz - z\|)
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_n\rho\|x_n - z\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)(\|x_n - z\| + \beta_n\|Sz - z\|) \\
&= (1-\alpha_n(1-\rho))\|x_n - z\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)\beta_n\|Sz - z\| \\
&\leq (1-\alpha_n(1-\rho))\|x_n - z\| + \alpha_n\|f(z) - z\| + \beta_n\|Sz - z\| \\
&\leq (1-\alpha_n(1-\rho))\|x_n - z\| + \alpha_n[\|f(z) - z\| + \gamma\|Sz - z\|]. \qquad (5.2.2)
\end{aligned}
$$

So, by induction, one can obtain that

$$
\|x_n - z\| \leq \max\left\{\|x_0 - z\|, \frac{1}{1-\rho}[\|f(z) - z\| + \gamma\|Sz - z\|\right\}. \qquad (5.2.3)
$$

Hence $\{x_n\}$ is bounded. Of course $\{y_n\}$ is bounded too. □

Proposition 5.2.2. *Suppose that (H1) and (H3) hold. Also, assume that either (H4) and (H5) hold, or (H6) and (H7) hold. Then*

(1) $\{x_n\}$ is asymptotically regular, that is,

$$
\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0, \qquad (5.2.4)
$$

(2) the weak cluster points set $\omega_w(x_n) \subset \bigcap_{i=1}^{\infty} F(T_i)$.

Proof. Set $u_n = \alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)V_i y_n$. From (5.2.1) and since P_C is a nonexpansive mapping, we have

$$
\begin{aligned}
\|x_{n+1} - x_n\| &= \|P_C[u_n] - P_C[u_{n-1}]\| \\
&\leq \|u_n - u_{n-1}\| \qquad (5.2.5) \\
&= \Bigg\|\alpha_n(f(x_n) - f(x_{n-1})) + (\alpha_n - \alpha_{n-1})f(x_{n-1}) \\
&\quad + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)(V_i y_n - V_i y_{n-1}) + (\alpha_{n-1} - \alpha_n)V_n y_{n-1}\Bigg\| \\
&\leq \alpha_n\|f(x_n) - f(x_{n-1})\| + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|y_n - y_{n-1}\| \\
&\quad |\alpha_n - \alpha_{n-1}|(\|f(x_{n-1})\| + \|V_n y_{n-1}\|) \\
&\leq \alpha_n\rho\|x_n - x_{n-1}\| + (1-\alpha_n)\|y_n - y_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|(\|f(x_{n-1})\| + \|V_n y_{n-1}\|). \qquad (5.2.6)
\end{aligned}
$$

By definition of y_n one obtain that

$$
\begin{aligned}
\|y_n - y_{n-1}\| &= \|P_C[\beta_n Sx_n + (1-\beta_n)x_n] - P_C[\beta_{n-1}Sx_{n-1} + (1-\beta_{n-1})x_{n-1}]\| \\
&\leq \|(\beta_n Sx_n + (1-\beta_n)x_n) - (\beta_{n-1}Sx_{n-1} + (1-\beta_{n-1})x_{n-1})\|
\end{aligned}
$$

$$\begin{aligned} &= \|\beta_n(Sx_n - Sx_{n-1}) + (\beta_n - \beta_{n-1})Sx_{n-1} \\ &\quad +(1-\beta_{n-1})(x_n - x_{n-1}) + (\beta_{n-1} - \beta_n)x_{n-1}\| \\ &\leq \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|Sx_{n-1}\| + \|x_{n-1}\|). \end{aligned} \tag{5.2.7}$$

So, substituting (5.2.7) in (5.2.6), we obtain

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \alpha_n\rho\|x_n - x_{n-1}\| + (1-\alpha_n)[\|x_n - x_{n-1}\| \\ &\quad +|\beta_n - \beta_{n-1}|(\|Sx_{n-1}\| + \|x_{n-1}\|)] \\ &\quad +|\alpha_n - \alpha_{n-1}|(\|f(x_{n-1})\| + \|V_n y_{n-1}\|) \\ &\leq (1-(1-\rho)\alpha_n)\|x_n - x_{n-1}\| \\ &\quad +|\beta_n - \beta_{n-1}|(\|Sx_{n-1}\| + \|x_{n-1}\|) \\ &\quad +|\alpha_n - \alpha_{n-1}|(\|f(x_{n-1})\| + \|V_n y_{n-1}\|). \end{aligned} \tag{5.2.8}$$

By Proposition 5.2.1, we say

$$M := \max\left\{\sup_{n\geq 1}\{\|Sx_{n-1}\| + \|x_{n-1}\|\}, \sup_{n\geq 1}\{\|f(x_{n-1})\| + \|V_n y_{n-1}\|\}\right\}.$$

So, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq (1-(1-\rho)\alpha_n)\|x_n - x_{n-1}\| \\ &\quad +M[|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|]. \end{aligned} \tag{5.2.9}$$

So, if (H4) and (H5) hold, we obtain the asymptotic regularity by Lemma 2.7.6, if instead, (H6) and (H7) hold, from (H1), we can write

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq (1-(1-\rho)\alpha_n)\|x_n - x_{n-1}\| + \\ &\quad M\alpha_n\left[\frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} + \frac{|\beta_n - \beta_{n-1}|}{\alpha_n}\right] \\ &\leq (1-(1-\rho)\alpha_n)\|x_n - x_{n-1}\| + \\ &\quad M\alpha_n\left[\frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} + \gamma\frac{|\beta_n - \beta_{n-1}|}{\beta_n}\right]. \end{aligned} \tag{5.2.10}$$

By Lemma 2.7.6, we obtain the asymptotic regularity.

In order to prove (2), since $V_i x_n \in C$ for each $i \geq 1$ and $\sum_{n=1}^{\infty}(\alpha_{n-1} - \alpha_n) + \alpha_n = 1$, we have

$$\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)V_i x_n + \alpha_n p \in C, \quad \forall p \in C. \tag{5.2.11}$$

Now, fixing a $p \in \bigcap_{i=1}^{\infty} F(Vi)$, from (5.2.1), we have

$$\begin{aligned}
\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(x_n - V_i x_n) &= P_C[u_n] + (1-\alpha_n)x_n + \alpha_n p - x_{n+1} \\
&\quad -\Big(\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i x_n + \alpha_n p\Big) \\
&= P_C[u_n] - P_C\Big[\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i x_n + \alpha_n p\Big] \\
&\quad +(1-\alpha_n)(x_n - x_{n+1}) + \alpha_n(p - x_{n+1}).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\textstyle\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle x_n - V_i x_n, x_n - z\rangle \\
&= \Big\langle P_C[u_n] - P_C\Big[\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i x_n + \alpha_n p\Big], x_n - z\Big\rangle \\
&\quad +(1-\alpha_n)\langle x_n - x_{n+1}, x_n - z\rangle + \alpha_n\langle p - x_{n+1}, x_n - z\rangle \\
&\le \Big\|u_n - \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i x_n + \alpha_n p\Big\|\|x_n - z\| \\
&\quad +(1-\alpha_n)\|x_n - x_{n+1}\|\|x_n - z\| + \alpha_n\|p - x_{n+1}\|\|x_n - z\| \\
&= \Big\|\alpha_n(f(x_n) - p) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(V_i y_n - V_i x_n)\Big\|\|x_n - z\| \\
&\quad +(1-\alpha_n)\|x_n - x_{n+1}\|\|x_n - z\| + \alpha_n\|p - x_{n+1}\|\|x_n - z\| \\
&\le \alpha_n\|f(x_n) - p\|\|x_n - z\| + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|y_n - x_n\|\|x_n - z\| \\
&\quad +(1-\alpha_n)\|x_n - x_{n+1}\|\|x_n - z\| + \alpha_n\|p - x_{n+1}\|\|x_n - z\| \\
&\le \alpha_n\|f(x_n) - p\|\|x_n - z\| + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\beta_n\|Sx_n - x_n\|\|x_n - z\| \\
&\quad +(1-\alpha_n)\|x_n - x_{n+1}\|\|x_n - z\| + \alpha_n\|p - x_{n+1}\|\|x_n - z\| \\
&= \alpha_n\|f(x_n) - p\|\|x_n - z\| + (1-\alpha_n)\beta_n\|Sx_n - x_n\|\|x_n - z\| \\
&\quad +(1-\alpha_n)\|x_n - x_{n+1}\|\|x_n - z\| + \alpha_n\|p - x_{n+1}\|\|x_n - z\|. \qquad (5.2.12)
\end{aligned}$$

Now, from Lemma 2.7.5 and (3.12), we get

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|x_n - V_i x_n\|^2 &\le \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle x_n - V_i x_n, x_n - z\rangle \\
&\le \alpha_n\|f(x_n) - p\|\|x_n - z\| \\
&\quad +(1-\alpha_n)\beta_n\|Sx_n - x_n\|\|x_n - z\| \\
&\quad +(1-\alpha_n)\|x_n - x_{n+1}\|\|x_n - z\| \\
&\quad +\alpha_n\|p - x_{n+1}\|\|x_n - z\|.
\end{aligned}$$

By (H1) and (H3), it follows that $\beta_n \to 0$, as $n \to \infty$, so that

$$\lim_{n\to\infty} \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i)\|x_n - V_i x_n\|^2 = 0. \tag{5.2.13}$$

Since $(\alpha_{i-1} - \alpha_i)\|x_n - V_i x_n\|^2 \leq \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i)\|x_n - V_i x_n\|^2$ for each $i \geq 1$ and $\{\alpha_n\}$ is strictly decreasing, one has

$$\lim_{n\to\infty} \|x_n - V_i x_n\| = 0, \quad \forall i \geq 1. \tag{5.2.14}$$

Hence, we obtain

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = \lim_{n\to\infty} \frac{\|x_n - V_i x_n\|}{(1 - k_i)} = 0, \quad \forall i \geq 1.$$

Since $\{x_n\}$ is asymptotically regular and demiclosedness principle, we obtain the proposition. □

Corollary 5.2.3. *Suppose that the hypotheses of Proposition 5.2.2 hold. Then*

(i) $\lim_{n\to\infty} \|x_n - y_n\| = 0$*;*

(ii) $\lim_{n\to\infty} \|x_n - V_i y_n\| = 0, \quad \forall i \geq 1$*;*

(iii) $\lim_{n\to\infty} \|y_n - V_i y_n\| = 0, \quad \forall i \geq 1.$

Proof. To prove (i), we can observe that

$$\|x_n - y_n\| \leq \beta_n \|x_n - S x_n\|.$$

Since $\beta_n \to 0$ as $n \to \infty$, we obtain (i).
To prove (ii), we observe that

$$\|y_n - V_i x_n\| \leq \|y_n - x_n\| + \|x_n - V_i x_n\|, \ \ \forall i \geq 1$$

and

$$\|x_n - V_i y_n\| \leq \|x_n - y_n\| + \|y_n - V_i x_n\|, \ \ \forall i \geq 1.$$

Since $\|y_n - x_n\| \to 0$ and $\|x_n - V_i x_n\| \to 0$ as $n \to \infty$, $\forall i \geq 1$, then $\|y_n - V_i x_n\| \to 0$, that is, we obtain (ii). To prove (iii), we can observe that

$$\|y_n - V_i y_n\| \leq \|x_n - y_n\| + \|x_n - V_i y_n\|, \ \ \forall i \geq 1.$$

By (i) and (ii), we obtain (iii). □

Theorem 5.2.4. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $f : C \to H$ be a ρ-contraction mapping, $S : C \to H$ be a nonexpansive mapping and $\{T_i\}_{i=1}^{\infty} : C \to C$ be a countable family of k_i-strict pseudo-contraction mappings and $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\alpha_0 = 1$, and $x_1 \in C$ and define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = P_C[\beta_n S x_n + (1-\beta_n)x_n], \\ x_{n+1} = P_C[\alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)V_i y_n], \quad \forall n \geq 1, \end{cases} \tag{5.2.15}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\alpha_n\}$ is a strictly decreasing sequence, $V_i = k_i I + (1-k_i)T_i$, $\{\beta_n\} \subset (0,1)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions (H2) with $\tau = 0$, (H3), either (H4) and (H5) , or (H6) and (H7). Then the sequence $\{x_n\}$ converges strongly to a point $z \in \mathcal{F}$, which is the unique solution of the variational inequality:

$$\langle (I-f)z, x-z\rangle \geq 0, \quad \forall x \in \mathcal{F}. \tag{5.2.16}$$

Proof. First of all, since $P_{\mathcal{F}}f$ is a contraction. By Banach contraction principle, so there exists a unique $z \in \mathcal{F}$ such that $z = P_{\mathcal{F}}f(z)$, Moreover, from Lemma 2.7.13, we have

$$\langle f(z) - z, y - z\rangle \leq 0, \quad \forall y \in \mathcal{F}.$$

Since (H2) implies (H1), thus $\{x_n\}$ is bounded. Moreover, since either (H4) and (H5), or (H6) and (H7), then $\{x_n\}$ is asymptotically regular. Similarly, by Proposition 5.2.2, the weak cluster points set of x_n, that is, $\omega_w(x_n)$, is a subset of $\mathcal{F}$.
Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle = \lim_{k\to\infty}\langle f(z) - z, x_{n_k} - z\rangle,$$

and $x_{n_k} \to x'$. By Proposition 5.2.2 it follows that $x' \in \mathcal{F}$. Then

$$\lim_{k\to\infty}\langle f(z) - z, x_{n_k} - z\rangle = \langle f(z) - z, x' - z\rangle \leq 0.$$

Set $u_n = \alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)V_i y_n$, we obtain

$$\|x_{n+1} - z\|^2 = \langle P_C[u_n] - u_n, P_C[u_n] - z\rangle + \langle u_n - z, x_{n+1} - z\rangle. \tag{5.2.17}$$

By Lemma 2.7.13, we have

$$\langle P_C[u_n] - u_n, P_C[u_n] - z\rangle \leq 0. \tag{5.2.18}$$

From (5.2.17) and (5.2.18), it follows that

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\leq \langle u_n - z, x_{n+1}-z\rangle \\
&= \alpha_n\langle f(x_n)-f(z), x_{n+1}-z\rangle + \alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle V_i y_n - z, x_{n+1}-z\rangle \\
&\leq \alpha_n\rho\|x_n-z\|\|x_{n+1}-z\| + \alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad (1-\alpha_n)\|y_n-z\|\|x_{n+1}-z\| \\
&\leq \alpha_n\rho\|x_n-z\|\|x_{n+1}-z\| + \alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad (1-\alpha_n)\|\beta_n Sx_n + (1-\beta_n)x_n - z\|\|x_{n+1}-z\| \\
&\leq \alpha_n\rho\|x_n-z\|\|x_{n+1}-z\| + \alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad (1-\alpha_n)\|x_n-z\|\|x_{n+1}-z\| + (1-\alpha_n)\beta_n\|Sz-z\|\|x_{n+1}-z\| \\
&= [1-\alpha_n(1-\rho)]\|x_n-z\|\|x_{n+1}-z\| + \alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad +(1-\alpha_n)\beta_n\|Sz-z\|\|x_{n+1}-z\| \\
&\leq \left[\frac{1-\alpha_n(1-\rho)}{2}\right]\left[\|x_n-z\|^2 + \|x_{n+1}-z\|^2\right] \\
&\quad +\alpha_n\langle f(z)-z, x_{n+1}-z\rangle + (1-\alpha_n)\beta_n\|Sz-z\|\|x_{n+1}-z\| \\
&\leq \left[1-\frac{2(1-\rho)\alpha_n}{1+(1-\rho)\alpha_n}\right]\|x_n-z\|^2 \\
&\quad +\left[\frac{2\alpha_n}{1+(1-\rho)\alpha_n}\right]\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad +\left[\frac{2(1-\alpha_n)\beta_n}{1+(1-\rho)\alpha_n}\right]\|Sz-z\|\|x_{n+1}-z\| \\
&= \left[1-\frac{2(1-\rho)\alpha_n}{1+(1-\rho)\alpha_n}\right]\|x_n-z\|^2 \\
&\quad +\left[\frac{2(1-\rho)\alpha_n}{1+(1-\rho)\alpha_n}\right]\left\{\frac{1}{1-\rho}\langle f(z)-z, x_{n+1}-z\rangle\right. \\
&\quad \left.+\frac{(1-\alpha_n)\beta_n}{(1-\rho)\alpha_n}\|Sz-z\|\|x_{n+1}-z\|\right\}.
\end{aligned}$$

Let

$$\gamma_n = \frac{2(1-\rho)\alpha_n}{1+(1-\rho)\alpha_n}$$

and

$$\delta_n = \frac{2(1-\rho)\alpha_n}{1+(1-\rho)\alpha_n}\left\{\frac{1}{1-\rho}\langle f(z)-z, x_{n+1}-z\rangle + \frac{(1-\alpha_n)\beta_n}{(1-\rho)\alpha_n}\|Sz-z\|\|x_{n+1}-z\|\right\},$$

for all $n \geq 1$. Since

$$\limsup_{n\to\infty}\left\{\frac{1}{1-\rho}\langle f(z)-z, x_{n+1}-z\rangle + \frac{(1-\alpha_n)\beta_n}{(1-\rho)\alpha_n}\|Sz-z\|\|x_{n+1}-z\|\right\} \leq 0,$$

$\sum\limits_{i=1}^{\infty}\alpha_n = \infty$ and $\frac{2(1-\rho)\alpha_n}{1+(1-\rho)\alpha_n} \geq (1-\rho)\alpha_n$, we have

$$\sum_{n=1}^{\infty}\gamma_n = \infty \text{ and } \limsup_{n\to\infty}\frac{\delta_n}{\gamma_n} \leq 0.$$

Hence, by Lemma 2.7.6, we conclude that $x_n \to z$ as $n \to \infty$. This completes the proof. □

Remark 5.2.5. In the iterative scheme (5.2.15), if we set $f \equiv 0$, then we get $x_n \to z = P_{\mathcal{F}}0$. In this case, from (5.2.16), it follows that

$$\langle z, z - x\rangle \leq 0, \quad \forall x \in \mathcal{F}.$$

That is

$$\|z\|^2 \leq \langle z, x\rangle \leq \|z\|\|x\|, \quad \forall x \in \mathcal{F}.$$

Therefore, the point z is the unique solution to the following quadratic minimization problem:

$$z = arg\min_{x\in\mathcal{F}}\|x\|^2.$$

By changing the restrictions on parameters in Theorem 5.2.4, we obtain the following results.

Theorem 5.2.6. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $f : C \to H$ be a ρ-contraction mapping, $S : C \to C$ be a nonexpansive mapping and $\{T_i\}_{i=1}^{\infty} : C \to C$ be a countable family of k_i-strict pseudo-contraction mappings and $\mathcal{F} = \bigcap\limits_{i=1}^{\infty}F(T_i) \neq \emptyset$. Let $\alpha_0 = 1$, and $x_1 \in C$ and define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = P_C[\beta_n Sx_n + (1-\beta_n)x_n] = \beta_n Sx_n + (1-\beta_n)x_n \\ x_{n+1} = P_C[\alpha_n f(x_n) + \sum\limits_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i y_n], \quad \forall n \geq 1 \end{cases} \tag{5.2.19}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\alpha_n\}$ is a strictly decreasing sequence, $V_i = k_iI + (1-k_i)T_i$, $\{\beta_n\} \subset (0,1)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions (H2) with $\tau \in (0,\infty)$, (H3), (H8) and (H9). Then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \mathcal{F}$, which is the unique solution of the variational inequality:

$$\langle\frac{1}{\tau}(I-f)x^* + (I-S)x^*, x - x^*\rangle \geq 0, \quad \forall x \in \mathcal{F}. \tag{5.2.20}$$

Proof. First, we shows that (5.2.20) has the unique solution. Let x' and x^* be two solutions. Then, since x' is solution, for $y = x^*$ one has

$$\langle (I-f)x', x'-x^*\rangle \leq \tau\langle (I-S)x', x^*-x'\rangle \tag{5.2.21}$$

and

$$\langle (I-f)x^*, x^*-x'\rangle \leq \tau\langle (I-S)x^*, x'-x^*\rangle. \tag{5.2.22}$$

Adding (5.2.21) and (5.2.22), we obtain

$$\begin{aligned}(1-\rho)\|x'-x^*\|^2 &\leq \langle (I-f)x'-(I-f)x^*, x'-x^*\rangle \\ &\leq -\rho\langle (I-S)x'-(I-S)x^*, x'-x^*\rangle \leq 0\end{aligned}$$

so $x' = x^*$. Also now the condition (H2) with $0 < \tau < \infty$ implies (H1) so the sequence $\{x_n\}$ is bounded. Moreover, since (H8) implies (H6) and (H7), then $\{x_n\}$ is asymptotically regular.

Similarly, by Proposition 5.2.2, the weak cluster points set of x_n, i.e., $\omega_w(x_n)$, is a subset of $\mathcal{F}$.

From (5.2.5)-(5.2.9), we observe that

$$\begin{aligned}\frac{\|x_{n+1}-x_n\|}{\beta_n} &\leq \frac{\|u_n-u_{n-1}\|}{\beta_n} \\ &\leq [1-(1-\rho))\alpha_n]\frac{\|x_n-x_{n-1}\|}{\beta_n} + M\left[\frac{|\alpha_n-\alpha_{n-1}|}{\beta_n} + \frac{|\beta_n-\beta_{n-1}|}{\beta_n}\right] \\ &= [1-(1-\rho))\alpha_n]\frac{\|x_n-x_{n-1}\|}{\beta_{n-1}} \\ &\quad +[1-(1-\rho))\alpha_n]\|x_n-x_{n-1}\|\left[\frac{1}{\beta_n}-\frac{1}{\beta_{n-1}}\right] \\ &\quad +M\left[\frac{|\alpha_n-\alpha_{n-1}|}{\beta_n} + \frac{|\beta_n-\beta_{n-1}|}{\beta_n}\right] \\ &\leq [1-(1-\rho))\alpha_n]\frac{\|x_n-x_{n-1}\|}{\beta_{n-1}} + \|x_n-x_{n-1}\|\left[\frac{1}{\beta_n}-\frac{1}{\beta_{n-1}}\right] \\ &\quad +M\left[\frac{|\alpha_n-\alpha_{n-1}|}{\beta_n} + \frac{|\beta_n-\beta_{n-1}|}{\beta_n}\right] \\ &\leq [1-(1-\rho))\alpha_n]\frac{\|x_n-x_{n-1}\|}{\beta_{n-1}} + \alpha_n K\|x_n-x_{n-1}\| \\ &\quad +M\left[\frac{|\alpha_n-\alpha_{n-1}|}{\beta_n} + \frac{|\beta_n-\beta_{n-1}|}{\beta_n}\right] \\ &\leq [1-(1-\rho))\alpha_n]\frac{\|u_n-u_{n-1}\|}{\beta_{n-1}} + \alpha_n K\|x_n-x_{n-1}\| \\ &\quad +M\left[\frac{|\alpha_n-\alpha_{n-1}|}{\beta_n} + \frac{|\beta_n-\beta_{n-1}|}{\beta_n}\right].\end{aligned}$$

Let $\gamma_n = (1-\rho)\alpha_n$ and $\delta_n = \alpha_n K\|x_n - x_{n-1}\| + M\Big[\frac{|\alpha_n-\alpha_{n-1}|}{\beta_n} + \frac{|\beta_n-\beta_{n-1}|}{\beta_n}\Big]$. From condition (H3) and (H8), we have

$$\sum_{i=1}^{\infty}\gamma_n = \infty \quad and \quad \lim_{n\to\infty}\frac{\delta_n}{\gamma_n} = 0.$$

By Lemma 2.7.6, we obtain

$$\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\beta_n} = 0, \quad \lim_{n\to\infty}\frac{\|u_{n+1}-u_n\|}{\beta_n} = \lim_{n\to\infty}\frac{\|u_{n+1}-u_n\|}{\alpha_n} = 0.$$

From (5.2.19), we have

$$\begin{aligned} x_n - x_{n-1} &= (1-\alpha_n)x_n - \Big[P_C[u_n] - u_n + \alpha_n f(x_n) \\ &\quad + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(V_i y_n - y_n) + (1-\alpha_n)y_n\Big] \\ &= (1-\alpha_n)\beta_n(x_n - Sx_n) + (u_n - P_C[u_n]) \\ &\quad + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(y_n - V_i y_n) + \alpha_n(x_n - f(x_n)). \end{aligned}$$

It follows that

$$\begin{aligned} \frac{x_n - x_{n-1}}{(1-\alpha_n)\beta_n} &= (x_n - Sx_n) + \frac{1}{(1-\alpha_n)\beta_n}(u_n - P_C[u_n]) \\ &\quad + \frac{1}{(1-\alpha_n)\beta_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(y_n - V_i y_n) + \frac{\alpha_n}{(1-\alpha_n)\beta_n}(x_n - f(x_n)). \end{aligned}$$

Let $v_n = \frac{x_n - x_{n-1}}{(1-\alpha_n)\beta_n}$. For all $z \in \mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) = \bigcap_{i=1}^{\infty} F(V_i)$, we get

$$\begin{aligned} \langle v_n, x_n - z\rangle &= \frac{1}{(1-\alpha_n)\beta_n}\langle u_n - P_C[u_n], P_C[u_{n-1}] - z\rangle \\ &\quad + \frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)x_n, x_n - z\rangle + \langle x_n - Sx_n, x_n - z\rangle \\ &\quad + \frac{1}{(1-\alpha_n)\beta_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle y_n - V_i y_n, x_n - z\rangle. \end{aligned} \tag{5.2.23}$$

By Lemma 2.7.11, we have

$$\begin{aligned} \langle x_n - Sx_n, x_n - z\rangle &= \langle (I-S)x_n - (I-S)z, x_n - z\rangle + \langle (I-S)z, x_n - z\rangle \\ &\geq \langle (I-S)z, x_n - z\rangle, \end{aligned} \tag{5.2.24}$$

$$\begin{aligned} \langle (I-f)x_n, x_n - z\rangle &= \langle (I-f)x_n - (I-f)z, x_n - z\rangle + \langle (I-f)z, x_n - z\rangle \\ &\geq (1-\rho)\|x_n - z\|^2 + \langle (I-f)z, x_n - z\rangle \end{aligned} \tag{5.2.25}$$

and

$$\begin{aligned}\langle y_n - V_i y_n, x_n - z\rangle &= \langle (I-V_i)y_n - (I-V_i)z, x_n - y_n\rangle \\ &\quad + \langle (I-V_i)y_n - (I-V_i)z, y_n - z\rangle \\ &\geq \langle (I-V_i)y_n - (I-V_i)z, x_n - y_n\rangle \\ &= \beta_n \langle (I-V_i)y_n, x_n - Sx_n\rangle, \quad \forall i \geq 1. \end{aligned} \tag{5.2.26}$$

By Lemma 2.7.13, we obtain

$$\begin{aligned}\langle u_n - P_C[u_n], P_C[u_{n-1}] - z\rangle &= \langle u_n - P_C[u_n], P_C[u_{n-1}] - P_C[u_n]\rangle \\ &\quad + \langle u_n - P_C[u_n], P_C[u_n] - z\rangle \\ &\geq \langle u_n - P_C[u_n], P_C[u_{n-1}] - P_C[u_n]\rangle. \end{aligned} \tag{5.2.27}$$

Now, from(5.2.23)-(5.2.27), it follows that

$$\begin{aligned}\langle v_n, x_n - z\rangle &\geq \frac{1}{(1-\alpha_n)\beta_n}\langle u_n - P_C[u_n], P_C[u_{n-1}] - P_C[u_n]\rangle \\ &\quad + \frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)z, x_n - z\rangle + \langle (I-S)z, x_n - z\rangle \\ &\quad + \frac{1}{(1-\alpha_n)}\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\langle (I-V_i)y_n, x_n - Sx_n\rangle \\ &\quad + \frac{(1-\rho)\alpha_n}{(1-\alpha_n)\beta_n}\|x_n - z\|^2. \end{aligned} \tag{5.2.28}$$

We observe from (5.2.28) that

$$\begin{aligned}\|x_n - z\|^2 &\leq \frac{(1-\alpha_n)\beta_n}{(1-\rho)\alpha_n}\Big[\langle v_n, x_n - z\rangle - \langle (I-S)z, x_n - z\rangle\Big] \\ &\quad + \frac{\|u_{n-1} - u_n\|}{(1-\rho)\alpha_n}\|u_n - P_C[u_n]\| - \frac{1}{1-\rho}\langle (I-f)z, x_n - z\rangle \\ &\quad - \frac{\beta_n}{(1-\rho)\alpha_n}\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\langle (I-V_i)y_n, x_n - Sx_n\rangle, \end{aligned} \tag{5.2.29}$$

since $v_n \to 0$ and $(I-V_i)y_n \to 0$, as $n \to \infty$, then every weak cluster point of $\{x_n\}$ is also a strong cluster point. By Proposition 5.2.2, $\{x_n\}$ is bounded, thus there exists a subsequence $\{x_{n_k}\}$ converging to x^*.

For all $z \in \mathcal{F}$ by (5.2.23), we compute

$$\begin{aligned}
\langle (I-f)x_{n_k}, x_{n_k}-z\rangle &= \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle v_{n_k}, x_{n_k}-z\rangle \\
&\quad -\frac{1}{\alpha_{n_k}}\langle u_{n_k}-P_C[u_{n_k}], P_C[u_{n_k-1}]-z\rangle \\
&\quad -\frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle x_{n_k}-Sx_{n_k}, x_{n_k}-z\rangle \\
&\quad -\frac{1}{\alpha_{n_k}}\sum_{i=1}^{n_k}(\alpha_{i-1}-\alpha_i)\langle y_{n_k}-V_iy_{n_k}, x_{n_k}-z\rangle \\
&\le \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle v_{n_k}, x_{n_k}-z\rangle \\
&\quad -\frac{\beta_{n_k}}{(\alpha_{n_k}}\sum_{i=1}^{n_k}(\alpha_{i-1}-\alpha_i\langle (I-V_i)y_{n_k}, x_{n_k}-Sx_{n_k}\rangle \\
&\quad -\frac{1}{\alpha_{n_k}}\|u_{n_k-1}-u_{n_k}\|\|u_{n_k}-P_C[u_{n_k}]\| \\
&\quad -\frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle (I-S)z, x_{n_k}-z\rangle. \qquad (5.2.30)
\end{aligned}$$

Since $v_n \to 0, (I-V_i)y_n \to 0$ for all $i \ge 1$, and $\|u_n - u_{n-1}\|/\alpha_n \to 0$, letting $k \to \infty$ in (5.2.30), we obtain

$$\langle (I-f)x^*, x^*-z\rangle \le -\tau\langle (I-S)z, x^*-z\rangle, \quad \forall z \in \mathcal{F}.$$

Since (5.2.20) has the unique solution, it follows that $\omega_w(x_n) = \{x^*\}$. Since every weak cluster point of $\{x_n\}$ is also a strong cluster point, we conclude that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

If we take $T_i = T$, for all $i \ge 1$, where $T : C \to C$ is a k-strict pseudo-contraction mapping in Theorem 5.2.4, then we get the following result:

Corollary 5.2.7. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $f : C \to H$ be a ρ-contraction mapping, $S : C \to H$ be a nonexpansive mapping and $T : C \to C$ be a k-strict pseudo-contraction mapping such that $F(T) \neq \emptyset$. Let $x_1 \in C$ and define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = P_C[\beta_n Sx_n + (1-\beta_n)x_n], \\ x_{n+1} = P_C[\alpha_n f(x_n) + (1-\alpha_n)Vy_n], \quad \forall n \ge 1, \end{cases} \qquad (5.2.31)$$

where $V = kI + (1-k)T, \{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ are sequences satisfying the conditions (H2) with $\tau = 0$, (H3), either (H4) and (H5) , or (H6) and (H7).

Then the sequence $\{x_n\}$ *converges strongly to a point* $z \in F(T)$*, which is the unique solution of the variational inequality:*

$$\langle (I-f)z, x-z\rangle \geq 0, \quad \forall x \in F(T).$$

Taking $k_i = 0$, for all $i \geq 1$ in Theorem 5.2.4, then we get the following result:

Corollary 5.2.8. *Let* C *be a nonempty closed and convex subset of a real Hilbert space* H*. Let* $f : C \to H$ *be a* ρ*-contraction mapping,* $S : C \to H$ *be a nonexpansive mapping and* $\{T_i\}_{i=1}^{\infty} : C \to C$ *be a countable family of nonexpansive mappings and* $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$*. Let* $\alpha_0 = 1$*,* $x_1 \in C$ *and define the sequence* $\{x_n\}$ *by*

$$\begin{cases} y_n = P_C[\beta_n S x_n + (1-\beta_n)x_n], \\ x_{n+1} = P_C[\alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)T_i y_n], \quad \forall n \geq 1, \end{cases} \tag{5.2.32}$$

where $\{\alpha_n\} \subset (0,1)$ *and* $\{\alpha_n\}$ *is a strictly decreasing sequence,* $\{\beta_n\} \subset (0,1)$ *and* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences satisfying the conditions (H2) with* $\tau = 0$*, (H3), either (H4) and (H5) , or (H6) and (H7). Then the sequence* $\{x_n\}$ *converges strongly to a point* $z \in \mathcal{F}$*, which is the unique solution of the variational inequality:*

$$\langle (I-f)z, x-z\rangle \geq 0, \quad \forall x \in \mathcal{F}.$$

If we take $k = 0$ in Corollary 5.2.7, then we get the following result:

Corollary 5.2.9. *Let* C *be a nonempty closed and convex subset of a real Hilbert space* H*. Let* $f : C \to H$ *be a* ρ*-contraction mapping,* $S : C \to H$ *be a nonexpansive mapping and* $T : C \to C$ *be a nonexpansive mapping such that* $F(T) \neq \emptyset$*. Let* $x_1 \in C$ *and define the sequence* $\{x_n\}$ *by*

$$\begin{cases} y_n = P_C[\beta_n S x_n + (1-\beta_n)x_n], \\ x_{n+1} = P_C[\alpha_n f(x_n) + (1-\alpha_n)T y_n], \quad \forall n \geq 1, \end{cases} \tag{5.2.33}$$

where $\{\alpha_n\} \subset (0,1), \{\beta_n\} \subset (0,1)$ *and* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences satisfying the conditions (H2) with* $\tau = 0$*, (H3), either (H4) and (H5) , or (H6) and (H7). Then the sequence* $\{x_n\}$ *converges strongly to a point* $z \in F(T)$*, which is the unique solution of the variational inequality:*

$$\langle (I-f)z, x-z\rangle \geq 0, \quad \forall x \in F(T).$$

If we take $T_i = T$, for all $i \geq 1$, where $T : C \to C$ is a k-strict pseudo-contraction mapping in Theorem 5.2.6, then we obtain the following result:

Corollary 5.2.10. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $f : C \to H$ be a ρ-contraction mapping, $S : C \to C$ be a nonexpansive mapping and $T : C \to C$ be a k-strict pseudo-contraction mapping and $\mathcal{F} = F(T) \neq \emptyset$. Let $x_1 \in C$ and define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = \beta_n S x_n + (1-\beta_n)x_n, \\ x_{n+1} = P_C[\alpha_n f(x_n) + (1-\alpha_n)V y_n], \quad \forall n \geq 1, \end{cases} \tag{5.2.34}$$

where $V = kI + (1-k)T$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions (H2) with $\tau \in (0,\infty)$, (H3), (H8) and (H9). Then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \mathcal{F}$, which is the unique solution of the variational inequality:

$$\langle \frac{1}{\tau}(I-f)x^* + (I-S)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{F}. \tag{5.2.35}$$

If we take $k_i = 0$, for all $i \geq 1$ in Theorem 5.2.6, then we get the following result:

Corollary 5.2.11. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $f : C \to H$ be a ρ-contraction mapping, $S : C \to C$ be a nonexpansive mapping and $\{T_i\}_{i=1}^{\infty} : C \to C$ be a countable family of nonexpansive mappings and $\mathcal{F} = \bigcap\limits_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\alpha_0 = 1$, $x_1 \in C$ and define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = \beta_n S x_n + (1-\beta_n)x_n, \\ x_{n+1} = P_C[\alpha_n f(x_n) + \sum\limits_{i=1}^{n}(\alpha_{i-1} - \alpha_i)T_i y_n], \quad \forall n \geq 1, \end{cases} \tag{5.2.36}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\alpha_n\}$ is a strictly decreasing sequence, $\{\beta_n\} \subset (0,1)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions (H2) with $\tau \in (0,\infty)$, (H3), (H8) and (H9). Then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \mathcal{F}$, which is the unique solution of the variational inequality:

$$\langle \frac{1}{\tau}(I-f)x^* + (I-S)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{F}. \tag{5.2.37}$$

If $k = 0$ in Corollary 5.2.10, then we get the following Corollary:

Corollary 5.2.12. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $f : C \to H$ be a ρ-contraction mapping, $S, T : C \to C$ be nonexpansive mappings and $\mathcal{F} = F(T) \neq \emptyset$. Let $x_1 \in C$ and define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = \beta_n S x_n + (1-\beta_n)x_n, \\ x_{n+1} = P_C[\alpha_n f(x_n) + (1-\alpha_n)T y_n], \quad \forall n \geq 1, \end{cases} \tag{5.2.38}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions (H2) with $\tau \in (0,\infty)$, (H3), (H8) and (H9). Then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \mathcal{F}$, which is the unique solution of the variational inequality:

$$\langle \frac{1}{\tau}(I-f)x^* + (I-S)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{F}. \tag{5.2.39}$$

Remark 5.2.13. Theorem 5.2.4 and Theorem 5.2.6 extend and improve the result of Gu et al. [54] from the countable family of nonexpansive mappings to more general the countable family of strictly pseudo contraction mappings.

5.3 Iterative Algorithm for Solving Triple Hierarchical Fixed Point Problem

In this section, we introduce an iterative algorithm for solving the monotone variational inequality over triple hierarchical fixed point problem. Always in this section, we may assume that the set $\Theta := VI(\Upsilon, I-\phi)$ is nonempty where $\Omega := VI(F(T), A-\gamma f)$ and $\Upsilon := VI(\Omega, B)$.

Theorem 5.3.1. *Let H be a real Hilbert space, C be a closed convex subset of H. Let $A : C \to H$ be a strongly positive linear bounded operator, $f : C \to H$ be a ρ-contraction, γ be a positive real number such that $\frac{\bar{\gamma}-1}{\rho} < \gamma < \frac{\bar{\gamma}}{\rho}$. Let $T : C \to C$ be a nonexpansive mapping, $B : C \to C$ be a β-strongly monotone and L-Lipschitz continuous. Let $\phi : C \to C$ be a k-contraction mapping with $k \in [0,1)$. Suppose $\{x_n\}$ is a sequence generated by the following algorithm $x_0 \in C$ arbitrarily*

$$\begin{cases} z_n = TP_C[I - \delta_n(A-\gamma f)]x_n, \\ y_n = (I - \mu\beta_n B)z_n, \\ x_{n+1} = \alpha_n \phi(x_n) + (1-\alpha_n)y_n, \ \forall n \geq 0, \end{cases} \tag{5.3.1}$$

where $\{\alpha_n\}, \{\delta_n\} \subset [0,1]$. If $\mu \in (0, \frac{2\beta}{L^2})$ is used and if $\{\beta_n\} \subset (0,1]$ satisfy the following conditions:

(C1): $\Sigma_{n=1}^{\infty}|\delta_{n+1}-\delta_n|<\infty$, $\Sigma_{n=1}^{\infty}\delta_n=\infty$;

(C2): $\Sigma_{n=1}^{\infty}|\beta_{n+1}-\beta_n|<\infty$;

(C3): $\Sigma_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$, $\lim_{n\to\infty}\alpha_n=0$;

(C4): $\delta_n\le\beta_n$ *and* $\beta_n\le\alpha_n$.

Then $\{x_n\}$ *converges strongly to* $x^*\in\Upsilon$*, which is the unique solution of the variational inequality:*

$$\text{Find } x^*\in\Upsilon \text{ such that } \langle(I-\phi)x^*, x-x^*\rangle\ge 0,\ \forall x\in\Upsilon, \tag{5.3.2}$$

where $\Upsilon:=VI(\Omega,B):=VI\Big(VI(F(T),A-\gamma f),B\Big)$.

Proof. We will divide the proof into four steps.

Step 1. We will show $\{x_n\}$ is bounded. For any $q\in\Theta$, we have

$$\begin{aligned}
\|z_n-q\| &= \|TP_C[I-\delta_n(A-\gamma f)]x_n-TP_Cq\|\\
&\le \|[I-\delta_n(A-\gamma f)]x_n-q\|\\
&\le \delta_n\|\gamma f(x_n)-\gamma f(q)\|+\delta_n\|\gamma f(q)-Aq\|+\|I-\delta_nA\|\|x_n-q\|\\
&\le \delta_n\gamma\rho\|x_n-q\|+\delta_n\|\gamma f(q)-Aq\|+(1-\delta_n\bar{\gamma})\|x_n-q\|\\
&= [1-(\bar{\gamma}-\gamma\rho)\delta_n]\|x_n-q\|+\delta_n\|\gamma f(q)-Aq\|. \qquad (5.3.3)
\end{aligned}$$

By Lemma 2.7.7, it is found that

$$\begin{aligned}
\|y_n-q\| &= \|(I-\mu\beta_nB)z_n-(I-\mu\beta_nB)q\|\\
&\le (1-\beta_n\tau)\|z_n-q\|\\
&\le (1-\beta_n\tau)\Big\{[1-(\bar{\gamma}-\gamma\rho)\delta_n]\|x_n-q\|\\
&\quad +\delta_n\|\gamma f(q)-Aq\|\Big\}. \qquad (5.3.4)
\end{aligned}$$

From (5.3.1), we get

$$\begin{aligned}
\|x_{n+1}-q\| &\le \alpha_n\|\phi(x_n)-\phi(q)\|+\alpha_n\|\phi(q)-q\|+(1-\alpha_n)\|y_n-q\|\\
&\le \alpha_nk\|x_n-q\|+\alpha_n\|\phi(q)-\phi(q)\|+(1-\alpha_n)\|y_n-q\|\\
&\le \alpha_nk\|x_n-q\|+(1-\alpha_n)(1-\beta_n\tau)\Big\{[1-(\bar{\gamma}-\gamma\rho)\delta_n]\|x_n-q\|\\
&\quad +\delta_n\|\gamma f(q)-Aq\|\Big\}
\end{aligned}$$

$$
\begin{aligned}
&\le \alpha_n\|x_n-q\|+(1-\alpha_n)(1-\beta_n\tau)[1-(\bar{\gamma}-\gamma\rho)\delta_n]\|x_n-q\| \\
&\quad +(1-\alpha_n)(1-\beta_n\tau)\delta_n\|\gamma f(q)-Aq\| \\
&= \alpha_n\|x_n-q\|+(1-\alpha_n)[1-(\bar{\gamma}-\gamma\rho)\delta_n-\beta_n\tau \\
&\quad +\beta_n\tau(\bar{\gamma}-\gamma\rho)\delta_n]\|x_n-q\| \\
&\quad +(1-\alpha_n)(1-\beta_n\tau)\delta_n\|\gamma f(q)-Aq\| \\
&= \alpha_n\|x_n-q\|+(1-\alpha_n)\Big[1-\{(\bar{\gamma}-\gamma\rho)\delta_n+\beta_n\tau \\
&\quad -\beta_n\tau(\bar{\gamma}-\gamma\rho)\delta_n\}\Big]\|x_n-q\| \\
&\quad +(1-\alpha_n)(1-\beta_n\tau)\delta_n\|\gamma f(q)-Aq\| \\
&= \alpha_n\|x_n-q\|+\Big[1-\alpha_n-\{(\bar{\gamma}-\gamma\rho)\delta_n+\beta_n\tau \\
&\quad -\beta_n\tau(\bar{\gamma}-\gamma\rho)\delta_n\}(1-\alpha_n)\Big]\|x_n-q\| \\
&\quad +(1-\alpha_n)(1-\beta_n\tau)\delta_n\|\gamma f(q)-Aq\| \\
&= \Big[1-(1-\alpha_n)\{(\bar{\gamma}-\gamma\rho)\delta_n+\beta_n\tau-\beta_n\tau(\bar{\gamma}-\gamma\rho)\delta_n\}\Big]\|x_n-q\| \\
&\quad +(1-\alpha_n)(1-\beta_n\tau)\delta_n\|\gamma f(q)-Aq\| \\
&= \Big[1-(1-\alpha_n)\{(\bar{\gamma}-\gamma\rho)\delta_n(1-\beta_n\tau)+\beta_n\tau\}\Big]\|x_n-q\| \\
&\quad +(1-\alpha_n)(1-\beta_n\tau)\delta_n\|\gamma f(q)-Aq\| \\
&= [1-(1-\alpha_n)(\bar{\gamma}-\gamma\rho)\delta_n(1-\beta_n\tau)-(1-\alpha_n)\beta_n\tau]\|x_n-q\| \\
&\quad +(1-\alpha_n)(1-\beta_n\tau)\delta_n\|\gamma f(q)-Aq\| \\
&= [1-(1-\alpha_n)(\bar{\gamma}-\gamma\rho)\delta_n(1-\beta_n\tau)]\|x_n-q\|-(1-\alpha_n)\beta_n\tau\|x_n-q\| \\
&\quad +(1-\alpha_n)(1-\beta_n\tau)\delta_n\|\gamma f(q)-Aq\| \\
&\le [1-(\bar{\gamma}-\gamma\rho)(1-\alpha_n)(1-\beta_n\tau)\delta_n]\|x_n-q\| \\
&\quad +(\bar{\gamma}-\gamma\rho)(1-\alpha_n)(1-\beta_n\tau)\delta_n\frac{\|\gamma f(q)-Aq\|}{\bar{\gamma}-\gamma\rho} \\
&= (1-\sigma_n)\|x_n-q\|+\sigma_n\frac{\|\gamma f(q)-Aq\|}{\bar{\gamma}-\gamma\rho},
\end{aligned}
$$

where $\sigma_n := (\bar{\gamma}-\gamma\rho)(1-\alpha_n)(1-\beta_n\tau)\delta_n$. Then, by mathematical induction implies that

$$
\|x_n-q\| \;\le\; \max\Big\{\|x_0-q\|,\frac{\|\gamma f(q)-Aq\|}{\bar{\gamma}-\gamma\rho}\Big\},\ \forall n\ge 0.
$$

Therefore $\{x_n\}$ is bounded and so are $\{y_n\}$, $\{z_n\}$, $\{Ax_n\}$, $\{Bx_n\}$, $\{\phi(x_n)\}$ and $\{f(x_n)\}$.

Step 2. We claim that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ and $\lim_{n\to\infty}\|x_n-Tx_n\|=0$.

From (5.3.1), we have

$$\begin{aligned}
\|z_{n+1}-z_n\| &= \|TP_C[I-\delta_{n+1}(A-\gamma f)]x_{n+1}-TP_C[I-\delta_n(A-\gamma f)]x_n\| \\
&\leq \|P_C[I-\delta_{n+1}(A-\gamma f)]x_{n+1}-P_C[I-\delta_n(A-\gamma f)]x_n\| \\
&\leq \|[I-\delta_{n+1}(A-\gamma f)]x_{n+1}-[I-\delta_n(A-\gamma f)]x_n\| \\
&= \|\delta_{n+1}(\gamma f(x_{n+1})-\gamma f(x_n))+(\delta_{n+1}-\delta_n)\gamma f(x_n) \qquad (5.3.5) \\
&\quad +(I-\delta_{n+1}A)(x_{n+1}-x_n)+(\delta_n-\delta_{n+1})Ax_n\| \\
&\leq \delta_{n+1}\gamma\|f(x_{n+1})-f(x_n)\|+(1-\delta_{n+1}\bar{\gamma})\|x_{n+1}-x_n\| \\
&\quad +|\delta_{n+1}-\delta_n|(\|\gamma f(x_n)\|+\|Ax_n\|) \\
&\leq \delta_{n+1}\gamma\rho\|x_{n+1}-x_n\|+(1-\delta_{n+1}\bar{\gamma})\|x_{n+1}-x_n\| \\
&\quad +|\delta_{n+1}-\delta_n|(\|\gamma f(x_n)\|+\|Ax_n\|) \\
&= [1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}]\|x_{n+1}-x_n\| \\
&\quad +|\delta_{n+1}-\delta_n|(\|\gamma f(x_n)\|+\|Ax_n\|) \qquad (5.3.6)
\end{aligned}$$

and

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \|(I-\mu\beta_{n+1}B)z_{n+1}-(I-\mu\beta_n B)z_n\| \\
&\leq \|(I-\mu\beta_{n+1}B)z_{n+1}-(I-\mu\beta_{n+1}B)z_n\| \\
&\quad +\|(I-\mu\beta_{n+1}B)z_n-(I-\mu\beta_n B)z_n\| \\
&\leq (1-\beta_n\tau)\|z_{n+1}-z_n\|+\mu|\beta_{n+1}-\beta_n|\|Bz_n\|. \qquad (5.3.7)
\end{aligned}$$

Using (5.3.5) and (5.3.8), we get

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &= \|\alpha_{n+1}\phi(x_{n+1})+(1-\alpha_{n+1})y_{n+1}-\alpha_n\phi(x_n)-(1-\alpha_n)y_n\| \\
&\leq \alpha_{n+1}\|\phi(x_{n+1})-\phi(x_n)\|+|\alpha_{n+1}-\alpha_n|\|\phi(x_{n+1})\| \\
&\quad +(1-\alpha_{n+1})\|y_{n+1}-y_n\|+|\alpha_{n+1}-\alpha_n|\|y_n\| \\
&\leq \alpha_{n+1}k\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|(\|\phi(x_{n+1})\|+\|y_n\|) \\
&\quad +(1-\alpha_{n+1})\|y_{n+1}-y_n\| \\
&\leq \alpha_{n+1}k\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|(\|\phi(x_{n+1})\|+\|y_n\|) \\
&\quad +(1-\alpha_{n+1})\Big\{(1-\beta_n\tau)\|z_{n+1}-z_n\|+\mu|\beta_{n+1}-\beta_n|\|Bz_n\|\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_{n+1}k\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|(\|\phi(x_{n+1})\| + \|y_n\|) \\
&\quad +(1-\alpha_{n+1})(1-\beta_n\tau)\|z_{n+1}-z_n\| \\
&\quad +(1-\alpha_{n+1})\mu|\beta_{n+1}-\beta_n|\|Bz_n\| \\
&\leq \alpha_{n+1}\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|(\|\phi(x_{n+1})\| + \|y_n\|) \\
&\quad +(1-\alpha_{n+1})\mu|\beta_{n+1}-\beta_n|\|Bz_n\| \\
&\quad +(1-\alpha_{n+1})\Big\{[1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}]\|x_{n+1}-x_n\| \\
&\quad +|\delta_{n+1}-\delta_n|(\|\gamma f(x_n)\| + \|Ax_n\|)\Big\} \\
&\leq \alpha_{n+1}\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|(\|\phi(x_{n+1})\| + \|y_n\|) \\
&\quad +(1-\alpha_{n+1})\mu|\beta_{n+1}-\beta_n|\|Bz_n\| \\
&\quad +(1-\alpha_{n+1})[1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}]\|x_{n+1}-x_n\| \\
&\quad +(1-\alpha_{n+1})|\delta_{n+1}-\delta_n|(\|\gamma f(x_n)\| + \|Ax_n\|) \\
&\leq [1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}(1-\alpha_{n+1})]\|x_{n+1}-x_n\| \\
&\quad +|\alpha_{n+1}-\alpha_n|(\|\phi(x_{n+1})\| + \|y_n\|) \\
&\quad +\mu|\beta_{n+1}-\beta_n|\|Bz_n\| + |\delta_{n+1}-\delta_n|(\|\gamma f(x_n)\| + \|Ax_n\|) \\
&\leq [1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}(1-\alpha_{n+1})]\|x_{n+1}-x_n\| \\
&\quad +\{|\alpha_{n+1}-\alpha_n| + |\beta_{n+1}-\beta_n| + |\delta_{n+1}-\delta_n|\}M, \qquad (5.3.8)
\end{aligned}
$$

where M is some constant such that

$$\sup_{n\geq 0}\Big\{\|\phi(x_n)\| + \|y_n\|, \mu\|Bz_n\|, \|\gamma f(x_n)\| + \|Ax_n\|\Big\} \leq M.$$

From (C1)-(C3) and the boundedness of $\{x_n\}$, $\{y_n\}$, $\{Ax_n\}$, $\{Bz_n\}$, $\{\phi(x_n)\}$ and $\{f(x_n)\}$. By Lemma 2.7.6, then we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \qquad (5.3.9)$$

On the other hand, we note that

$$
\begin{aligned}
\|z_n - Tx_n\| &= \|TP_C[I-\delta_n(A-\gamma f)]x_n - Tx_n\| \\
&= \|TP_C[I-\delta_n(A-\gamma f)]x_n - TP_Cx_n\| \\
&\leq \|[I-\delta_n(A-\gamma f)]x_n - x_n\| \\
&\leq \delta_n\|(A-\gamma f)]x_n\|,
\end{aligned}
$$

by (C3)-(C4) and it follows that

$$\lim_{n\to\infty}\|z_n - Tx_n\| = 0. \qquad (5.3.10)$$

From (5.3.1), we compute

$$\begin{aligned}\|x_{n+1}-z_n\| &= \|\alpha_n\phi(x_n)+(1-\alpha_n)y_n-z_n\| \\ &= \|\alpha_n\phi(x_n)+(1-\alpha_n)(I-\mu\beta_nB)z_n-z_n\| \\ &\leq \alpha_n\|\phi(x_n)-z_n\|+(1-\alpha_n)\|(I-\mu\beta_nB)z_n-z_n\| \\ &\leq \alpha_nk\|x_n-z_n\|+\alpha_n\|\phi(z_n)-z_n\|+(1-\alpha_n)\mu\beta_n\|Bz_n\|.\end{aligned}$$

By (C3) and (C4), it follows that

$$\lim_{n\to\infty}\|x_{n+1}-z_n\|=0. \tag{5.3.11}$$

Since

$$\|x_n-Tx_n\| \leq \|x_n-x_{n+1}\|+\|x_{n+1}-z_n\|+\|z_n-Tx_n\|.$$

By (5.3.9), (5.3.10) and (5.3.11), we obtain

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{5.3.12}$$

From (5.3.1), we compute

$$\begin{aligned}\|x_{n+1}-y_n\| &= \|\alpha_n\phi(x_n)+(1-\alpha_n)y_n-y_n\| \\ &= \|\alpha_n\phi(x_n)+y_n-\alpha_ny_n-y_n\| \\ &\leq \alpha_n\|\phi(x_n)-y_n\|. \end{aligned} \tag{5.3.13}$$

By (C3), it follows that

$$\lim_{n\to\infty}\|x_{n+1}-y_n\|=0. \tag{5.3.14}$$

Since

$$\|x_n-y_n\| \leq \|x_n-x_{n+1}\|+\|x_{n+1}-y_n\|.$$

From (5.3.9) and (5.3.14), we obtain

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{5.3.15}$$

Step 3. First, $\limsup_{n\to\infty}\langle u_n-x^*,\gamma f(x^*)-Ax^*\rangle \leq 0$ is proven. Choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x_n-x^*,\gamma f(x^*)-Ax^*\rangle=\lim_{i\to\infty}\langle x_{n_i}-x^*,\gamma f(x^*)-Ax^*\rangle.$$

The boundedness of $\{x_{n_i}\}$ implies the existences of a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ and a point $\hat{x} \in H$ such that $\{x_{n_{i_j}}\}$ converges weakly to $\hat{x}$. We may assume without loss of generality that $\lim_{i\to\infty}\langle x_{n_i}, w\rangle = \langle \hat{x}, w\rangle,\ w \in H$. Assume $\hat{x} \neq T(\hat{x})$. By $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ with $F(T) \neq \emptyset$ guarantee that

$$\begin{aligned}
\liminf_{i\to\infty}\|x_{n_i} - \hat{x}\| &< \liminf_{i\to\infty}\|x_{n_i} - T(\hat{x})\| \\
&= \liminf_{i\to\infty}\|x_{n_i} - T(x_{n_i}) + T(x_{n_i}) - T(\hat{x})\| \\
&= \liminf_{i\to\infty}\|T(x_{n_i}) - T(\hat{x})\| \\
&\leq \liminf_{i\to\infty}\|x_{n_i} - \hat{x}\|.
\end{aligned}$$

which has a contradiction. Therefore $\hat{x} \in F(T)$. From $x^* \in VI(F(T), A - \gamma f)$, we find

$$\begin{aligned}
\limsup_{n\to\infty}\langle x_n - x^*, \gamma f(x^*) - Ax^*\rangle &= \lim_{i\to\infty}\langle x_{n_i} - x^*, \gamma f(x^*) - Ax^*\rangle \\
&= \langle \hat{x} - x^*, \gamma f(x^*) - Ax^*\rangle \\
&\leq 0.
\end{aligned}$$

Setting $u_n = [I - \delta_n(A - \gamma f)]x_n$ and by (C3)-(C4), we notice that

$$\|u_n - x_n\| \leq \delta_n\|(A - \gamma f)\| \to 0.$$

Hence,we get

$$\limsup_{n\to\infty}\langle u_n - x^*, \gamma f(x^*) - Ax^*\rangle \leq 0. \tag{5.3.16}$$

Second, $\limsup_{n\to\infty}\langle x^* - x_{n+1}, Bx^*\rangle \leq 0$ is proven. From $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ guarantees the existences of a subsequence $\{x_{n_k+1}\}$ of $\{x_{n_k}\}$ and a point $\bar{x} \in H$ such that $\limsup_{n\to\infty}\langle x^* - x_{n+1}, Bx^*\rangle = \lim_{k\to\infty}\langle x^* - x_{n_k+1}, Bx^*\rangle$ and $\lim_{k\to\infty}\langle x_{n_k}, w\rangle = \lim_{k\to\infty}\langle x_{n_k+1}, w\rangle = \langle \bar{x}, w\rangle,\ w \in H$. By the same discussion as in the proof of $\hat{x} \in F(T)$, we have $\bar{x} \in F(T)$. Let $y \in F(T)$ be fixed arbitrarily. Then, it follows from $T : C \to C$ is a nonexpansive mappings with $F(T) \neq \emptyset$, $A : C \to H$ be a strongly positive linear bounded operator and $f : C \to H$ be a contraction that, for all $n \in \mathbb{N}$. From (5.3.1)

$$\begin{aligned}
\|z_n - y\| &= \|TP_C u_n - TP_C y\| \\
&\leq \|u_n - y\|.
\end{aligned} \tag{5.3.17}$$

By (C3)-(C4), we observe that

$$\begin{aligned}\|u_n - y\| &= \|[I - \delta_n(A - \gamma f)]x_n - y\| \\ &\leq \|x_n - y\| + \delta_n\|(A - \gamma f)x_n\| \\ &\leq \|x_n - y\|. \end{aligned} \tag{5.3.18}$$

Using (5.3.17) and (5.3.18)

$$\begin{aligned}\|u_n - y\|^2 &= \|[I - \delta_n(A - \gamma f)]x_n - y\|^2 \\ &= \Big\|\delta_n(\gamma f(x_n) - Ay) + (I - \delta_n A)(x_n - y)\Big\|^2 \\ &\leq (1 - \delta_n\bar{\gamma})^2\|x_n - y\|^2 + 2\delta_n\langle\gamma f(x_n) - Ay, u_n - y\rangle \\ &\leq (1 - 2\delta_n\bar{\gamma} + \delta_n^2\bar{\gamma}^2)\|x_n - y\|^2 + 2\delta_n\gamma\rho\|x_n - y\|\|u_n - y\| \\ &\quad +2\delta_n\langle\gamma f(y) - Ay, u_n - y\rangle \\ &\leq (1 - 2\delta_n\bar{\gamma} + \delta_n^2\bar{\gamma}^2)\|x_n - y\|^2 + 2\delta_n\gamma\rho\|x_n - y\|^2 \\ &\quad +2\delta_n\langle\gamma f(y) - Ay, u_n - y\rangle \\ &= [1 - 2\delta_n(\bar{\gamma} - \gamma\rho)]\|x_n - y\|^2 + \delta_n^2\bar{\gamma}^2\|x_n - y\|^2 + \\ &\quad 2\delta_n\langle\gamma f(y) - Ay, u_n - y\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}0 &\leq \Big(\|x_n - y\|^2 - \|u_n - y\|^2\Big) - 2\delta_n(\bar{\gamma} - \gamma\rho)\|x_n - y\|^2 + \delta_n^2\bar{\gamma}^2\|x_n - y\|^2 \\ &\quad +2\delta_n\langle\gamma f(y) - Ay, u_n - y\rangle \\ &= (\|x_n - y\| + \|u_n - y\|)(\|x_n - y\| - \|u_n - y\|) - 2\delta_n(\bar{\gamma} - \gamma\rho)\|x_n - y\|^2 \\ &\quad +\delta_n^2\bar{\gamma}^2\|x_n - y\|^2 + 2\delta_n\langle\gamma f(y) - Ay, u_n - y\rangle \\ &\leq M_2\|x_n - u_n\| - 2\delta_n(\bar{\gamma} - \gamma\rho)\|x_n - y\|^2 + \delta_n^2\bar{\gamma}^2\|x_n - y\|^2 \\ &\quad +2\delta_n\langle\gamma f(y) - Ay, u_n - y\rangle,\end{aligned}$$

where $M_2 := \sup\{\|x_n - y\| + \|u_n - y\| : n \in \mathbb{N}\} < \infty$, for every $n \in \mathbb{N}$. By the weak convergence of $\{u_{n_i}\}$ to $\bar{x} \in F(T)$, $\lim_{n\to\infty}\|u_n - x_n\| = 0$ and (C3)-(C4), we get $\langle(\gamma f - A)y, \bar{x} - y\rangle \leq 0$ for all $y \in F(T)$. A mapping A be a strongly positive linear bounded operator and f be a contraction ensures $\langle(\gamma f - A)y, \bar{x} - y\rangle \leq 0$ for all $y \in F(T)$, that is, $\bar{x} \in VI(F(T), A - \gamma f)$. Thus $x^* \in VI(VI(F(T), A - \gamma f), B)$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle x^*-x_n, Bx^*\rangle &= \limsup_{i\to\infty}\langle x^*-x_{n_i}, Bx^*\rangle\\ &= \langle x^*-\bar{x}, Bx^*\rangle\\ &\le 0.\end{aligned}$$

From (5.3.15), we notice that

$$\limsup_{n\to\infty}\langle x^*-y_n, Bx^*\rangle \le 0. \tag{5.3.19}$$

Third, $\limsup_{n\to\infty}\langle x_n-x^*, \phi(x^*)-x^*\rangle \le 0$ is proven. Choose a subsequence $\{x_{n_g}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x_n-x^*, \phi(x^*)-x^*\rangle = \lim_{g\to\infty}\langle x_{n_g}-x^*, \phi(x^*)-x^*\rangle.$$

The boundedness of $\{x_{n_g}\}$ implies the existences of a subsequence $\{x_{n_{g_h}}\}$ of $\{x_{n_g}\}$ and a point $\tilde{x}\in H$ such that $\{x_{n_{g_h}}\}$ converges weakly to $\tilde{x}$. By $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, we have $\lim_{h\to\infty}\langle x_{n_{g_h}+1}, w\rangle = \langle\tilde{x}, w\rangle,\ w\in H$. We may assume without loss of generality that $\lim_{i\to\infty}\langle x_{n_g}, w\rangle = \langle\tilde{x}, w\rangle,\ w\in H$. Assume $\tilde{x}\neq T(\tilde{x})$. By $\lim_{n\to\infty}\|x_n-Tx_n\|=0$ with $F(T)\neq\emptyset$ guarantee that

$$\begin{aligned}\liminf_{g\to\infty}\|x_{n_g}-\tilde{x}\| &< \liminf_{g\to\infty}\|x_{n_g}-T(\tilde{x})\|\\ &= \liminf_{g\to\infty}\|x_{n_g}-T(x_{n_g})+T(x_{n_g})-T(\tilde{x})\|\\ &= \liminf_{g\to\infty}\|T(x_{n_g})-T(\tilde{x})\|\\ &\le \liminf_{g\to\infty}\|x_{n_g}-\tilde{x}\|.\end{aligned}$$

This is a contradiction, that is, $\tilde{x}\in F(T)$. From $x^*\in VI\Big(VI(VI(F(T),A-\gamma f),B),I-\phi\Big)$, we find

$$\begin{aligned}\limsup_{n\to\infty}\langle x_n-x^*, \phi(x^*)-x^*\rangle &= \lim_{g\to\infty}\langle x_{n_g}-x^*, \phi(x^*)-x^*\rangle\\ &= \langle\tilde{x}-x^*, \phi(x^*)-x^*\rangle\\ &\le 0. \end{aligned}\tag{5.3.20}$$

Step 4. Finally, we prove $\lim_{n\to\infty}\|x_n-x^*\|=0$. By Lemma 2.7.7, we compute

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\alpha_n\phi(x_n)+(1-\alpha_n)y_n-x^*\|^2\\ &= \Big\|\alpha_n(\phi(x_n)-\phi(x^*))+\alpha_n(\phi(x^*)-x^*)+(1-\alpha_n)(y_n-x^*)\Big\|^2\end{aligned}$$

$$
\begin{aligned}
&\le \alpha_n\|\phi(x_n)-\phi(x^*)\|^2+(1-\alpha_n)\|y_n-x^*\|^2\\
&\quad +2\alpha_n\langle \phi(x^*)-x^*,x_{n+1}-x^*\rangle\\
&\le \alpha_n k^2\|x_n-x^*\|^2+(1-\alpha_n)\|(I-\mu\beta_n B)z_n-x^*\|^2\\
&\quad +2\alpha_n\langle \phi(x^*)-x^*,x_{n+1}-x^*\rangle\\
&= \alpha_n k^2\|x_n-x^*\|^2+(1-\alpha_n)\|(z_n-\mu\beta_n Bz_n)\\
&\quad -(x^*-\mu\beta_n Bx^*)-\mu\beta_n Bx^*\|^2+2\alpha_n\langle \phi(x^*)-x^*,x_{n+1}-x^*\rangle\\
&\le \alpha_n k^2\|x_n-x^*\|^2+(1-\alpha_n)\Big\{\|(z_n-\mu\beta_n Bz_n)-(x^*-\mu\beta_n Bx^*)\|^2\\
&\quad +2\mu\beta_n\langle x^*-y_n,Bx^*\rangle\Big\}+2\alpha_n\langle \phi(x^*)-x^*,x_{n+1}-x^*\rangle\\
&\le \alpha_n k^2\|x_n-x^*\|^2+(1-\alpha_n)(1-\tau\beta_n)^2\|z_n-x^*\|^2\\
&\quad +2\mu\beta_n\langle x^*-y_n,Bx^*\rangle+2\alpha_n\langle \phi(x^*)-x^*,x_{n+1}-x^*\rangle\\
&\le \alpha_n k^2\|x_n-x^*\|^2+(1-\alpha_n)(1-\tau\beta_n)\|u_n-x^*\|^2\\
&\quad +2\mu\beta_n\langle x^*-y_n,Bx^*\rangle+2\alpha_n\langle \phi(x^*)-x^*,x_{n+1}-x^*\rangle\\
&= \alpha_n k^2\|x_n-x^*\|^2+(1-\alpha_n)(1-\tau\beta_n)\|[I-\delta_n(A-\gamma f)]x_n-x^*\|^2\\
&\quad +2\mu\beta_n\langle x^*-y_n,Bx^*\rangle+2\alpha_n\langle \phi(x^*)-x^*,x_{n+1}-x^*\rangle\\
&= \alpha_n k^2\|x_n-x^*\|^2+(1-\alpha_n)(1-\tau\beta_n)\|(I-\delta_n A)(x_n-x^*)\\
&\quad +\delta_n(\gamma f(x_n)-Ax^*)\|^2+2\mu\beta_n\langle x^*-y_n,Bx^*\rangle\\
&\quad +2\alpha_n\langle \phi(x^*)-x^*,x_{n+1}-x^*\rangle\\
&\le \alpha_n k^2\|x_n-x^*\|^2+(1-\alpha_n)(1-\tau\beta_n)\Big\{(1-\delta_n\bar{\gamma})^2\|x_n-x^*\|^2\\
&\quad +2\delta_n\langle \gamma f(x_n)-Ax^*,u_n-x^*\rangle\Big\}+2\mu\beta_n\langle x^*-y_n,Bx^*\rangle\\
&\quad +2\alpha_n\langle \phi(x^*)-x^*,x_{n+1}-x^*\rangle\\
&\le \alpha_n k\|x_n-x^*\|^2+(1-\alpha_n)(1-\tau\beta_n)\Big\{(1-2\delta_n\bar{\gamma}+\delta_n^2\bar{\gamma}^2)\|x_n-x^*\|^2\\
&\quad +2\delta_n\langle \gamma f(x_n)-\gamma f(x^*),u_n-x^*\rangle+2\delta_n\langle \gamma f(x^*)-Ax^*,u_n-x^*\rangle\Big\}\\
&\quad +2\mu\beta_n\langle x^*-y_n,Bx^*\rangle+2\alpha_n\langle \phi(x^*)-x^*,x_{n+1}-x^*\rangle\\
&\le \alpha_n k\|x_n-x^*\|^2+(1-\alpha_n)(1-\tau\beta_n)\Big\{(1-2\delta_n\bar{\gamma})\|x_n-x^*\|^2\\
&\quad +\delta_n^2\bar{\gamma}^2\|x_n-x^*\|^2+2\delta_n\gamma\rho\|x_n-x^*\|\|u_n-x^*\|+2\delta_n\langle \gamma f(x^*)-Ax^*,u_n-x^*\rangle\Big\}\\
&\quad +2\mu\beta_n\langle x^*-y_n,Bx^*\rangle+2\alpha_n\langle \phi(x^*)-x^*,x_{n+1}-x^*\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)(1-\tau\beta_n)[1-2\delta_n(\bar{\gamma}-\gamma\rho)]\|x_n - x^*\|^2 \\
&\quad +(1-\alpha_n)(1-\tau\beta_n)\delta_n^2\bar{\gamma}^2\|x_n - x^*\|^2 + 2\delta_n\langle \gamma f(x^*) - Ax^*, u_n - x^*\rangle \\
&\quad +2\mu\beta_n\langle x^* - y_n, Bx^*\rangle + 2\alpha_n\langle \phi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&= \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)[1-2\delta_n(\bar{\gamma}-\gamma\rho) - \tau\beta_n + \tau\beta_n 2\delta_n(\bar{\gamma}-\gamma\rho)]\|x_n - x^*\|^2 \\
&\quad +(1-\alpha_n)(1-\tau\beta_n)\delta_n^2\bar{\gamma}^2\|x_n - x^*\|^2 + 2\delta_n\langle \gamma f(x^*) - Ax^*, u_n - x^*\rangle \\
&\quad +2\mu\beta_n\langle x^* - y_n, Bx^*\rangle + 2\alpha_n\langle \phi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&= \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)[1-\{2\delta_n(\bar{\gamma}-\gamma\rho) + \tau\beta_n - \tau\beta_n 2\delta_n(\bar{\gamma}-\gamma\rho)\}]\|x_n - x^*\|^2 \\
&\quad +(1-\alpha_n)(1-\tau\beta_n)\delta_n^2\bar{\gamma}^2\|x_n - x^*\|^2 + 2\delta_n\langle \gamma f(x^*) - Ax^*, u_n - x^*\rangle \\
&\quad +2\mu\beta_n\langle x^* - y_n, Bx^*\rangle + 2\alpha_n\langle \phi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&= [1-(1-\alpha_n)\{2\delta_n(\bar{\gamma}-\gamma\rho) + \tau\beta_n - \tau\beta_n 2\delta_n(\bar{\gamma}-\gamma\rho)\}]\|x_n - x^*\|^2 \\
&\quad +(1-\alpha_n)(1-\tau\beta_n)\delta_n^2\bar{\gamma}^2\|x_n - x^*\|^2 + 2\delta_n\langle \gamma f(x^*) - Ax^*, u_n - x^*\rangle \\
&\quad +2\mu\beta_n\langle x^* - y_n, Bx^*\rangle + 2\alpha_n\langle \phi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&= [1-(1-\alpha_n)\{2\delta_n(\bar{\gamma}-\gamma\rho)(1-\tau\beta_n) + \tau\beta_n\}]\|x_n - x^*\|^2 \\
&\quad +(1-\alpha_n)(1-\tau\beta_n)\delta_n^2\bar{\gamma}^2\|x_n - x^*\|^2 + 2\delta_n\langle \gamma f(x^*) - Ax^*, u_n - x^*\rangle \\
&\quad +2\mu\beta_n\langle x^* - y_n, Bx^*\rangle + 2\alpha_n\langle \phi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&= [1-(1-\alpha_n)2\delta_n(\bar{\gamma}-\gamma\rho)(1-\tau\beta_n)]\|x_n - x^*\|^2 - (1-\alpha_n)\tau\beta_n\|x_n - x^*\|^2 \\
&\quad +(1-\alpha_n)(1-\tau\beta_n)\delta_n^2\bar{\gamma}^2\|x_n - x^*\|^2 + 2\delta_n\langle \gamma f(x^*) - Ax^*, u_n - x^*\rangle \\
&\quad +2\mu\beta_n\langle x^* - y_n, Bx^*\rangle + 2\alpha_n\langle \phi(x^*) - x^*, x_{n+1} - x^*\rangle \\
&\le [1-2(\bar{\gamma}-\gamma\rho)(1-\alpha_n)(1-\tau\beta_n)\delta_n]\|x_n - x^*\|^2 \\
&\quad +(1-\alpha_n)(1-\tau\beta_n)\delta_n^2\bar{\gamma}^2\|x_n - x^*\|^2 + 2\delta_n\langle \gamma f(x^*) - Ax^*, u_n - x^*\rangle \\
&\quad +2\mu\beta_n\langle x^* - y_n, Bx^*\rangle + 2\alpha_n\langle \phi(x^*) - x^*, x_{n+1} - x^*\rangle. \qquad (5.3.21)
\end{aligned}
$$

Since $\{x_n\},\ \{Ax_n\},\ \{Bx_n\},\ \{\phi(x_n)\}$ and $\{f(x_n)\}$ are all bounded, we can choose a constant $M_1 > 0$ such that

$$
\sup_n \frac{1}{\bar{\gamma}-\gamma\rho}\Big\{\frac{\delta_n\bar{\gamma}^2}{2}\|x_n - x^*\|^2\Big\} \le M_1.
$$

It follows that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le [1-2(\bar{\gamma}-\gamma\rho)(1-\alpha_n)(1-\tau\beta_n)\delta_n]\|x_n - x^*\|^2 \\
&\quad +2(\bar{\gamma}-\gamma\rho)(1-\alpha_n)(1-\tau\beta_n)\delta_n\varsigma_n, \qquad (5.3.22)
\end{aligned}
$$

where

$$
\begin{aligned}
\varsigma_n &= \delta_n M_1 + \frac{1}{(\bar{\gamma}-\gamma\rho)(1-\alpha_n)(1-\tau\beta_n)}\langle \gamma f(x^*) - Ax^*, u_n - x^*\rangle \\
&\quad + \frac{\mu\beta_n}{(\bar{\gamma}-\gamma\rho)(1-\alpha_n)(1-\tau\beta_n)\delta_n}\langle x^* - y_n, Bx^*\rangle \\
&\quad + \frac{\alpha_n}{(\bar{\gamma}-\gamma\rho)(1-\alpha_n)(1-\tau\beta_n)\delta_n}\langle \phi(x^*) - x^*, x_{n+1} - x^*\rangle.
\end{aligned}
$$

By (5.3.16), (5.3.19), (5.3.20) and (C3)-(C4) then we get $\limsup_{n\to\infty}\varsigma_n \le 0$. Applying Lemma 2.7.6, we can conclude that $x_n \to x^*$. This completes the proof. □

Next, the following example shows that all conditions of Theorem 5.3.1 are satisfied.

Example 5.3.2. For instance, let $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{1}{2n}$ and $\delta_n = \frac{1}{3n}$. We will show that the condition (C1) is achieves. Then, clearly, the sequences $\{\delta_n\}$

$$\Sigma_{n=1}^{\infty}\delta_n = \Sigma_{n=1}^{\infty}\frac{1}{3n} = \infty$$

and

$$
\begin{aligned}
\Sigma_{n=1}^{\infty}|\delta_{n+1}-\delta_n| &= \Sigma_{n=1}^{\infty}|\tfrac{1}{3(n+1)} - \tfrac{1}{3n}| \\
&\le |\tfrac{1}{3\cdot 1} - \tfrac{1}{3\cdot 2}| + |\tfrac{1}{3\cdot 2} - \tfrac{1}{3\cdot 3}| + |\tfrac{1}{3\cdot 3} - \tfrac{1}{3\cdot 4}| + \dots \\
&= \tfrac{1}{3}.
\end{aligned}
$$

The sequence $\{\delta_n\}$ satisfy the condition (C1).

Next, we will show that the condition (C2) is achieves. We compute

$$
\begin{aligned}
\Sigma_{n=1}^{\infty}|\beta_{n+1}-\beta_n| &= \Sigma_{n=1}^{\infty}|\tfrac{1}{2(n+1)} - \tfrac{1}{2n}| \\
&\le |\tfrac{1}{2\cdot 1} - \tfrac{1}{2\cdot 2}| + |\tfrac{1}{2\cdot 2} - \tfrac{1}{2\cdot 3}| + |\tfrac{1}{2\cdot 3} - \tfrac{1}{2\cdot 4}| + \dots \\
&= \tfrac{1}{2}.
\end{aligned}
$$

The sequence $\{\beta_n\}$ satisfy the condition (C2).

Next, we will show that the condition (C3) is achieves. We compute

$$
\begin{aligned}
\Sigma_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| &= \Sigma_{n=1}^{\infty}|\tfrac{1}{n+1} - \tfrac{1}{n}| \\
&\le |\tfrac{1}{1} - \tfrac{1}{2}| + |\tfrac{1}{2} - \tfrac{1}{3}| + |\tfrac{1}{3} - \tfrac{1}{4}| + \dots \\
&= 1
\end{aligned}
$$

and

$$\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\frac{1}{n} = 0,$$

The sequence $\{\alpha_n\}$ satisfy the condition (C3).

Finally, we will show that the condition (C4) is achieves.

$$\frac{1}{3n} < \frac{1}{2n} \quad \text{and} \quad \frac{1}{2n} < \frac{1}{n}.$$

Corollary 5.3.3. *Let H be a real Hilbert space, C be a closed convex subset of H. Let $A : C \to H$ be an inverse-strongly monotone. Let $T : C \to C$ be a nonexpansive mapping. Let $B : C \to C$ be a β-strongly monotone and L-Lipschitz continuous. Suppose $\{x_n\}$ is a sequence generated by the following algorithm $x_0 \in C$ arbitrarily*

$$\begin{cases} z_n = T(I - \delta_n A)x_n, \\ y_n = (I - \mu\beta_n B)z_n, \\ x_{n+1} = (1 - \alpha_n)y_n, \ \forall n \geq 0, \end{cases} \tag{5.3.23}$$

$\{\alpha_n\}, \{\delta_n\} \subset [0, 1]$. *If $\mu \in (0, \frac{2\beta}{L^2})$ is used and if $\{\beta_n\} \subset (0, 1]$ satisfy the following conditions:*

(C1): $\Sigma_{n=1}^{\infty}|\delta_{n+1} - \delta_n| < \infty$, $\Sigma_{n=1}^{\infty}\delta_n = \infty$;

(C2): $\Sigma_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$;

(C3): $\Sigma_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, $\lim_{n\to\infty} \alpha_n = 0$;

(C4): $\delta_n \leq \beta_n$ and $\beta_n \leq \alpha_n$.

Then $\{x_n\}$ converges strongly to $x^* \in VI(F(T), A)$, which is the unique solution of the variational inequality:

$$\text{Find } x^* \in VI(F(T), A) \text{ such that } \langle Bx^*, x - x^*\rangle \geq 0, \ \forall x \in VI(F(T), A). \tag{5.3.24}$$

Proof. Putting P_C is the identity and $f, \phi \equiv 0$ in Theorem 5.3.1, we can obtain desired conclusion immediately. □

Remark 5.3.4. Corollary 5.3.3 generalizes and improves the results of Iiduka [7].

□

Corollary 5.3.5. *Let H be a real Hilbert space, C be a closed convex subset of H. Let $A : C \to H$ be a strongly positive linear bounded operator, $f : C \to H$ be a ρ-contraction, γ be a positive real number such that $\frac{\bar{\gamma}-1}{\rho} < \gamma < \frac{\bar{\gamma}}{\rho}$. Let $T : C \to C$ be a nonexpansive mapping. Suppose $\{x_n\}$ is a sequence generated by the following* *algorithm* $x_0 \in C$ *arbitrarily*

$$\begin{cases} z_n = TP_C[I - \delta_n(A - \gamma f)]x_n, \\ y_n = (I - \mu\beta_n B)z_n, \\ x_{n+1} = \alpha_n(x_n) + (1 - \alpha_n)y_n, \ \forall n \geq 0, \end{cases} \tag{5.3.25}$$

where $\{\alpha_n\}, \{\delta_n\} \subset [0,1]$. *If* $\mu \in (0, \frac{2\beta}{L^2})$ *is used and if* $\{\beta_n\} \subset (0,1]$ *satisfy the following conditions:*

(C1): $\Sigma_{n=1}^{\infty}|\delta_{n+1} - \delta_n| < \infty$, $\Sigma_{n=1}^{\infty}\delta_n = \infty$;

(C2): $\Sigma_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$;

(C3): $\Sigma_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, $\lim_{n\to\infty} \alpha_n = 0$;

(C4): $\delta_n \leq \beta_n$ *and* $\beta_n \leq \alpha_n$.

Then $\{x_n\}$ *converges strongly to* $x^* \in \Omega$, *which is the unique solution of the variational inequality:*

$$\text{Find } x^* \in \Omega \text{ such that } \langle Bx^*, x - x^*\rangle \geq 0, \ \forall x \in \Omega. \tag{5.3.26}$$

Proof. Putting ϕ is the identity in Theorem 5.3.1, we can obtain desired conclusion immediately. □

Remark 5.3.6. Corollary 5.3.5 generalizes and improves the results of Marino and Xu [22].

5.4 Iteration Algorithm for Solvig Hierarchical Generalized Variational Inequality Problem

In this section, we consider and study the convex feasibility problem (CFP) in the case that each C_m is a solution set of generalized variational inequality $GVI(C, B_m, A_m)$ and we introduce an iterative algorithm for solve the following the HGVIP: find $\tilde{x} \in \cap_{m=1}^{r} GVI(C, B_m, A_m)$ such that

$$\langle(\gamma f - \mu G)\tilde{x}, x - \tilde{x}\rangle \leq 0, \quad \forall x \in \cap_{m=1}^{r} GVI(C, B_m, A_m). \tag{5.4.1}$$

Theorem 5.4.1. *Let* C *be a nonempty closed and convex subset of a real Hilbert space* H *such that* $C \pm C \subset C$. *Let* $f : C \to C$ *be a contraction with coefficient* $k \in (0,1)$. *Let* $G : C \to C$ *be a* ξ*-strongly monotone and* L*-Lipschitz continuous* *mapping. let* $A_m : C \to H$ *be a relaxed* (η_m, ρ_m)*-cocoercive and* ν_m*-Lipschitz continuous mapping and* $B_m : C \to H$ *be a relaxed* $(\hat{\eta}_m, \hat{\rho}_m)$*-cocoercive and* $\hat{\nu}_m$*-Lipschitz continuous mapping for each* $1 \leq m \leq r$*. Let* $p_m = \sqrt{1 - 2\lambda_m\rho_m + \lambda_m^2\nu_m^2 + 2\lambda_m\eta_m\nu_m^2}$ *and* $q_m = \sqrt{1 - 2\hat{\lambda}_m\hat{\rho}_m + \hat{\lambda}_m^2\hat{\nu}_m^2 + 2\hat{\lambda}_m\hat{\eta}_m\hat{\nu}_m^2}$*, where* $\{\lambda_m\}$ *and* $\{\hat{\lambda}_m\}$ *are two positive sequences for each* $1 \leq m \leq r$*. Assume that* $\cap_{m=1}^r GVI(C, B_m, A_m) \neq \emptyset$*,* $\xi > 0, L > 0,\ 0 < \mu < 2\xi/L^2,\ 0 < \gamma < \mu(\xi - \mu L^2/2)/k = \pi/k$ *and* $p_m, q_m \in [0, \frac{1}{2})$*, for each* $1 \leq m \leq r$*. Given the initial guess* $x_1 \in C$ *and* $\{x_n\}$ *is a sequence generated by*

$$x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n\mu G)\Sigma_{m=1}^r \beta_{(m,n)} P_C(\hat{\lambda}_m B_m x_n - \lambda_m A_m x_n),\ \ \forall n \geq 1, \quad (5.4.2)$$

where $\{\alpha_n\}, \{\beta_{(1,n)}\}, \{\beta_{(2,n)}\}, ..., \{\beta_{(r,n)}\}$ *are sequences in* $(0,1)$*, satisfying the following conditions:*

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0, \Sigma_{n=1}^{\infty}\alpha_n = \infty$ *and* $\Sigma_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$;

$(C2)$ $\Sigma_{m=1}^r \beta_{(m,n)} = 1, \forall n \geq 1\ , \Sigma_{n=1}^{\infty}|\beta_{(m,n+1)} - \beta_{(m,n)}| < \infty$
and $\lim_{n\to\infty} \beta_{(m,n)} = \beta_m \in (0,1), \forall 1 \leq m \leq r.$

Then $\{x_n\}$ *converges strongly to a common element* $\tilde{x} \in \cap_{m=1}^r GVI(C, B_m, A_m)$*, which is the unique solution of the following problem:*

$$\langle(\gamma f - \mu G)\tilde{x}, x - \tilde{x}\rangle \leq 0, \quad \forall x \in \cap_{m=1}^r GVI(C, B_m, A_m). \quad (5.4.3)$$

Proof. Put $T_m = P_C(\hat{\lambda}_m B_m - \lambda_m A_m), \forall 1 \leq m \leq r$. For each $x, y \in C$ and for each $m \geq 1$, we have

$$\begin{aligned} \|T_m x - T_m y\| &= \|P_C(\hat{\lambda}_m B_m - \lambda_m A_m)x - P_C(\hat{\lambda}_m B_m - \lambda_m A_m)y\| \\ &\leq \|(\hat{\lambda}_m B_m - \lambda_m A_m)x - (\hat{\lambda}_m B_m - \lambda_m A_m)y\| \\ &\leq \|(x-y) - \lambda_m(A_m x - A_m y)\| \\ &\quad + \|(x-y) - \hat{\lambda}_m(B_m x - B_m y)\|. \end{aligned} \quad (5.4.4)$$

It follows from the assumption that each A_m is relaxed (η_m, ρ_m)-cocoercive and ν_m-Lipschitz continuous that

$$\begin{aligned} \|(x-y) - \lambda_m(A_m x - A_m y)\|^2 &= \|x-y\|^2 - 2\lambda_m\langle A_m x - A_m y, x - y\rangle \\ &\quad + \lambda_m^2\|A_m x - A_m y\|^2 \end{aligned}$$

$$\begin{aligned}
&\leq \|x-y\|^2 - 2\lambda_m\big[(-\eta_m)\|A_mx - A_my\|^2 \\
&\quad +\rho_m\|x-y\|^2\big] + \lambda_m^2\nu_m^2\|x-y\|^2 \\
&\leq (1 - 2\lambda_m\rho_m + \lambda_m^2\nu_m^2)\|x-y\|^2 \\
&\quad +2\lambda_m\eta_m\nu_m^2\|x-y\|^2 \\
&= p_m^2\|x-y\|^2.
\end{aligned}$$

This shows that

$$\|(x-y) - \lambda_m(A_mx - A_my)\| \leq p_m\|x-y\|. \tag{5.4.5}$$

In a similar way, we can obtain that

$$\|(x-y) - \hat{\lambda}_m(B_mx - B_my)\| \leq q_m\|x-y\|. \tag{5.4.6}$$

Substituting (3.4) and (3.5) into (3.3), we have

$$\begin{aligned}
\|T_mx - T_my\| &\leq (p_m + q_m)\|x-y\| \\
&\leq \|x-y\|.
\end{aligned}$$

Hence T_m is a nonexpansive mapping and $F(T_m) = F(P_C(\hat{\lambda}_mB_m - \lambda_mA_m)) = GVI(C, B_m, A_m)$ for each $1 \leq m \leq r$.

Put $S_n = \Sigma_{m=1}^r\beta_{(m,n)}T_m$. By Lemma 2.7.19, we conclude that S_n is a nonexpansive mapping and $F(S_n) = \cap_{m=1}^rGVI(C, B_m, A_m), \forall n \geq 1$. We can rewrite the algorithm (5.4.2) as

$$x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n\mu G)S_nx_n. \tag{5.4.7}$$

Step 1: We will show that $\{x_n\}$ is bounded.

Take $v \in F(S_n) = \cap_{m=1}^rGVI(C, B_m, A_m)$, from (5.4.7) and lemma 2.7.3, we have

$$\begin{aligned}
\|x_{n+1} - v\| &= \|\alpha_n\gamma f(x_n) + (I - \alpha_n\mu G)S_nx_n - v\| \\
&= \|\alpha_n(\gamma f(x_n) - \mu Gv) + (I - \alpha_n\mu G)S_nx_n - (I - \alpha_n\mu G)v\| \\
&\leq \alpha_n\|\gamma(f(x_n) - f(v)) + \gamma f(v) - \mu Gv\| + (1 - \alpha_n\pi)\|x_n - v\|
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n\gamma k\|x_n - v\| + \alpha_n\|\gamma f(v) - \mu Gv\| + (1 - \alpha_n\pi)\|x_n - v\| \\
&= (1 - \alpha_n(\pi - \gamma k))\|x_n - v\| + \alpha_n\|\gamma f(v) - \mu Gv\| \\
&\leq max\bigg\{\|x_n - v\|, \frac{\|\gamma f(v) - \mu Gv\|}{\pi - \gamma k}\bigg\}.
\end{aligned}$$

By induction, we btain

$$\|x_n - v\| \leq \max\left\{\|x_1 - v\|, \frac{\|\gamma f(v) - \mu Gv\|}{\pi - \gamma k}\right\}.$$

Hence $\{x_n\}$ is bounded.

Since S_n is nonexpansive mappings for $n \geq 1$, we see that

$$\begin{aligned}\|S_n x_n - v\| &= \|S_n x_n - S_n v\| \\ &\leq \|x_n - v\| \\ &\leq \max\left\{\|x_1 - v\|, \frac{\|\gamma f(v) - \mu Gv\|}{\pi - \gamma k}\right\}.\end{aligned}$$

Therefore, $\{S_n x_n\}$ is bounded. Since G is a L-Lipschitz continuous mapping, we have

$$\begin{aligned}\|GS_n x_n - Gv\| &= \|GS_n x_n - GS_n v\| \\ &\leq L\|S_n x_n - S_n v\| \\ &\leq L\|x_n - v\| \\ &\leq \max\left\{L\|x_1 - v\|, L\frac{\|\gamma f(v) - \mu Gv\|}{\pi - \gamma k}\right\}.\end{aligned}$$

Hence $\{GS_n x_n\}$ is bounded. Since f is contraction, so $f(x_n)$ is bounded.

Step 2: We will show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

From (5.4.7), we consider

$$\begin{aligned}x_{n+1} - x_n &= [\alpha_n \gamma f(x_n) + (I - \alpha_n \mu G)S_n x_n] \\ &\quad -[\alpha_{n-1}\gamma f(x_{n-1}) + (I - \alpha_{n-1}\mu G)S_{n-1}x_{n-1}] \\ &= \alpha_n \gamma(f(x_n) - f(x_{n-1})) + [(I - \alpha_n \mu G)S_n x_n - (I - \alpha_n \mu G)S_{n-1}x_{n-1}] \\ &\quad +(\alpha_n - \alpha_{n-1})\gamma f(x_{n-1}) + (\alpha_{n-1} - \alpha_n)\mu GS_{n-1}x_{n-1},\end{aligned}$$

it follows that

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \alpha_n \gamma k\|x_n - x_{n-1}\| + (1 - \alpha_n \pi)\|S_n x_n - S_{n-1}x_{n-1}\| \\ &\quad +|\alpha_n - \alpha_{n-1}|(\gamma\|f(x_{n-1})\| + \mu\|GS_{n-1}x_{n-1}\|) \\ &\leq \alpha_n \gamma k\|x_n - x_{n-1}\| + (1 - \alpha_n \pi)\|S_n x_n - S_{n-1}x_{n-1}\| \\ &\quad +|\alpha_n - \alpha_{n-1}|M_1, \qquad (5.4.8)\end{aligned}$$

where $M_1 = \sup_{n\geq 1}\{\gamma\|f(x_n)\| + \mu\|GS_nx_n\|\}$. On the other hand, we note that

$$\begin{aligned}
\|S_nx_n - S_{n-1}x_{n-1}\| &\leq \|S_nx_n - S_nx_{n-1}\| + \|S_nx_{n-1} - S_{n-1}x_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + \|\Sigma_{m=1}^r\beta_{(m,n)}T_mx_{n-1} - \Sigma_{m=1}^r\beta_{(m,n-1)}T_mx_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + M_2\Sigma_{m=1}^r|\beta_{(m,n)} - \beta_{(m,n-1)}|, \qquad (5.4.9)
\end{aligned}$$

where $M_2 = \max\{\sup_{n\geq 1}\|T_mx_n\|, \forall 1 \leq m \leq r\}$.
Substituting (5.4.9) into (5.4.8) yields

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \alpha_n\gamma k\|x_n - x_{n-1}\| + (1-\alpha_n\pi)\|x_n - x_{n-1}\| + M_1|\alpha_n - \alpha_{n-1}| \\
&\quad + M_2\Sigma_{m=1}^r|\beta_{(m,n)} - \beta_{(m,n-1)}| \\
&\leq \alpha_n\gamma k\|x_n - x_{n-1}\| + (1-\alpha_n\pi)\|x_n - x_{n-1}\| \\
&\quad + M_3(|\alpha_n - \alpha_{n-1}| + \Sigma_{m=1}^r|\beta_{(m,n)} - \beta_{(m,n-1)}|),
\end{aligned}$$

where M_3 is an appropriate constant such that $M_3 \geq \max\{M_1, M_2\}$.
By conditions $(C1)$ and $(C2)$ and Lemma 2.7.6, we obtain that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \qquad (5.4.10)$$

Step 3: We will show that $\lim_{n\to\infty}\|Sx_n - x_n\| = 0$.
Define a mapping $S : C \to C$ by

$$Sx = \Sigma_{m=1}^r\beta_mT_mx, \forall x \in C,$$

where $\beta_m = \lim_{n\to\infty}\beta_{(m,n)}$. From Lemma 2.7.19, we see that S is a nonexpansive mapping and

$$F(S) = \cap_{m=1}^rF(T_m) = \cap_{m=1}^rGVI(C, B_m, A_m), \forall n \geq 1.$$

From (5.4.7), we observe that

$$\begin{aligned}
\|x_{n+1} - S_nx_n\| &= \alpha_n\|\gamma f(x_n) + \mu GS_nx_n\| \\
&\leq \alpha_n(\gamma\|f(x_n) - f(v)\| + \|\gamma f(v) + \mu GS_nv\| + \mu\|GS_nx_n - GS_nv\|).
\end{aligned}$$

It follows from the condition $(C1)$ and the boundedness of $\{f(x_n)\}$ and $\{GS_nx_n\}$, we obtain that

$$\lim_{n\to\infty}\|x_{n+1} - S_nx_n\| = 0. \qquad (5.4.11)$$

We observe that

$$\begin{aligned}\|x_n - S_n x_n\| &= \|x_n - x_{n+1} + x_{n+1} - S_n x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - S_n x_n\|.\end{aligned}$$

From (5.4.10) and (5.4.11), we obtain

$$\lim_{n\to\infty} \|x_n - S_n x_n\| = 0. \tag{5.4.12}$$

Now, we show that $Sx_n - x_n \to 0$ as $n \to \infty$. Note that

$$\begin{aligned}\|Sx_n - x_n\| &= \|Sx_n - S_n x_n + S_n x_n - x_n\| \\ &\leq \|\Sigma_{m=1}^r \beta_m T_m x_n - \Sigma_{m=1}^r \beta_{(m,n)} T_m x_n\| + \|S_n x_n - x_n\| \\ &\leq M_2(\Sigma_{m=1}^r |\beta_m - \beta_{(m,n)}|) + \|S_n x_n - x_n\|.\end{aligned}$$

By the condition $(C2)$ and (5.4.12), we have

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{5.4.13}$$

From the boundedness of x_n, we deduced that x_n converges weakly in $F(S)$, say $x_n \rightharpoonup p$, by Lemma 2.7.1 and (5.4.13), we obtain $p = Sp$. So, we have

$$\omega_w(x_n) \subset F(S). \tag{5.4.14}$$

By Lemma 2.7.4, $\mu G - \gamma f$ is strongly monotone, so the variational inequality (5.4.3) has a unique solution $\tilde{x} \in F(S) = \cap_{m=1}^r GVI(C, B_m, A_m)$.

Step 4: We show that $\limsup_{n\to\infty} \langle (\gamma f - \mu G)\tilde{x}, x_n - \tilde{x}\rangle \leq 0$.

Indeed, since $\{x_n\}$ is bounded, then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that

$$\limsup_{n\to\infty} \langle (\gamma f - \mu G)\tilde{x}, x_n - \tilde{x}\rangle = \lim_{i\to\infty} \langle (\gamma f - \mu G)\tilde{x}, x_{n_i} - \tilde{x}\rangle.$$

Without loss of generality, we may further assume that $x_{n_i} \rightharpoonup p$. It follows from (5.4.14) that $p \in F(S)$. Since $\tilde{x}$ is the unique solution of (5.4.3), we obtain

$$\begin{aligned}\limsup_{n\to\infty} \langle (\gamma f - \mu G)\tilde{x}, x_n - \tilde{x}\rangle &= \lim_{i\to\infty} \langle (\gamma f - \mu G)\tilde{x}, x_{n_i} - \tilde{x}\rangle \\ &= \langle (\gamma f - \mu G)\tilde{x}, p - \tilde{x}\rangle \leq 0. \end{aligned} \tag{5.4.15}$$

Step 5: Finally, we will show that $x_n \to \tilde{x}$ as $n \to \infty$.

From Lemma 2.7.15, we have

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &= \|\alpha_n(\gamma f(x_n) - \mu G\tilde{x}) + (I - \alpha_n \mu G)S_n x_n - \mu G(I - \alpha_n \mu G)\tilde{x}\|^2 \\
&\leq (1 - \alpha_n \pi)^2 \|x_n - \tilde{x}\|^2 + 2\alpha_n \langle \gamma f(x_n) - \mu G\tilde{x}, x_{n+1} - \tilde{x}\rangle \\
&\leq (1 - \alpha_n \pi)^2 \|x_n - \tilde{x}\|^2 + 2\alpha_n \gamma \langle f(x_n) - f(\tilde{x}), x_{n+1} - \tilde{x}\rangle \\
&\quad +2\alpha_n \langle \gamma f(\tilde{x}) - \mu G\tilde{x}, x_{n+1} - \tilde{x}\rangle \\
&\leq (1 - \alpha_n \pi)^2 \|x_n - \tilde{x}\|^2 + 2\alpha_n \gamma k \|x_n - \tilde{x}\| \|x_{n+1} - \tilde{x}\| \\
&\quad +2\alpha_n \langle \gamma f(\tilde{x}) - \mu G\tilde{x}, x_{n+1} - \tilde{x}\rangle \\
&\leq (1 - \alpha_n \pi)^2 \|x_n - \tilde{x}\|^2 + \alpha_n \gamma k (\|x_n - \tilde{x}\|^2 + \|x_{n+1} - \tilde{x}\|^2) \\
&\quad +2\alpha_n \langle \gamma f(\tilde{x}) - \mu G\tilde{x}, x_{n+1} - \tilde{x}\rangle \\
&\leq \frac{1 - 2\alpha_n \pi + (\alpha_n \pi)^2 + \alpha_n \gamma k}{1 - \alpha_n \gamma k} \|x_n - \tilde{x}\|^2 \\
&\quad + \frac{2\alpha_n}{1 - \alpha_n \gamma k} \langle \gamma f(\tilde{x}) - \mu G\tilde{x}, x_{n+1} - \tilde{x}\rangle \\
&= [1 - \frac{2\alpha_n(\pi - \gamma k)}{1 - \alpha_n \gamma k}] \|x_n - \tilde{x}\|^2 + \frac{(\alpha_n \pi)^2}{1 - \alpha_n \gamma k} \|x_n - \tilde{x}\|^2 \\
&\quad + \frac{2\alpha_n}{1 - \alpha_n \gamma k} \langle \gamma f(\tilde{x}) - \mu G\tilde{x}, x_{n+1} - \tilde{x}\rangle \\
&= (1 - \theta_n)\|x_n - \tilde{x}\|^2 + \delta_n,
\end{aligned}$$

where $\theta_n := \frac{2\alpha_n(\pi - \gamma k)}{1 - \alpha_n \gamma k}$ and $\delta_n := \frac{\alpha_n}{1 - \alpha_n \gamma k}[\alpha_n \pi^2 \|x_n - \tilde{x}\|^2 + 2\langle \gamma f(\tilde{x}) - \mu G\tilde{x}, x_{n+1} - \tilde{x}\rangle]$.

Note that,

$$\theta_n := \frac{2\alpha_n(\pi - \gamma k)}{1 - \alpha_n \gamma k} \leq \frac{2(\pi - \gamma k)}{1 - \gamma k}\alpha_n.$$

By the condition $(C1)$, we obtain that

$$\lim_{n \to \infty} \theta_n = 0. \tag{5.4.16}$$

On the other hand, we have

$$\theta_n := \frac{2\alpha_n(\pi - \gamma k)}{1 - \alpha_n \gamma k} \geq 2\alpha_n(\pi - \gamma k).$$

From the condition $(C1)$, we have

$$\sum_{n=1}^{\infty} \theta_n = \infty. \tag{5.4.17}$$

Put $M = \sup_{n \in \mathbb{N}}\{\|x_n - \tilde{x}\|\}$, we have

$$\frac{\delta_n}{\theta_n} = \frac{1}{2(\pi - \gamma k)}[\alpha_n \pi^2 M + 2\langle \gamma f(\tilde{x}) - \mu G\tilde{x}, x_{n+1} - \tilde{x}\rangle].$$

From the condition $(C1)$ and (3.14), we have

$$\limsup_{n\to\infty} \frac{\delta_n}{\theta_n} \leq 0. \tag{5.4.18}$$

Hence, by Lemma 2.7.6, (5.4.16), (5.4.17) and (5.4.18), we conclude that

$$\lim_{n\to\infty} \|x_n - \tilde{x}\| = 0.$$

This completes the proof. □

If $B_m = I$, the identity mapping and $\hat{\lambda_m} = 1$, then Theorem 5.4.1 is reduced to the following result on the classical variational inequality (2.6.1).

Corollary 5.4.2. *Let C be a nonempty closed and convex subset of a real Hilbert space H such that $C \pm C \subset C$. Let $f : C \to C$ be a contraction with coefficient $k \in (0,1)$. Let $G : C \to C$ be a ξ-strongly monotone and L-Lipschitz continuous mapping. Let $A_m : C \to H$ be a relaxed (η_m, ρ_m)-cocoercive and ν_m-Lipschitz continuous mapping, for each $1 \leq m \leq r$. Let $p_m = \sqrt{1 - 2\lambda_m\rho_m + \lambda_m^2\nu_m^2 + 2\lambda_m\eta_m\nu_m^2}$, where $\{\lambda_m\}$ is a positive sequence, for each $1 \leq m \leq r$. Assume that $\cap_{m=1}^{r} VI(C, A_m) \neq \emptyset$, $\xi > 0, L > 0$, $0 < \mu < 2\xi/L^2$, $0 < \gamma < \mu(\xi - \mu L^2/2)/k = \pi/k$ and $p_m \in [0,1)$, for each $1 \leq m \leq r$. Given the initial guess $x_1 \in C$ and $\{x_n\}$ is a sequence generated by*

$$x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n\mu G)\Sigma_{m=1}^{r}\beta_{(m,n)}P_C(x_n - \lambda_m A_m x_n), \;\; \forall n \geq 1,$$

where $\{\alpha_n\}, \{\beta_{(1,n)}\}, \{\beta_{(2,n)}\}, ..., \{\beta_{(r,n)}\}$ are sequences in $(0,1)$, satisfying the following conditions:

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0, \Sigma_{n=1}^{\infty}\alpha_n = \infty$ *and* $\Sigma_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$;

$(C2)$ $\Sigma_{m=1}^{r}\beta_{(m,n)} = 1, \forall n \geq 1$ $,\Sigma_{n=1}^{\infty}|\beta_{(m,n+1)} - \beta_{(m,n)}| < \infty$ *and* $\lim_{n\to\infty} \beta_{(m,n)} = \beta_m \in (0,1), \forall 1 \leq m \leq r$,

Then the sequence $\{x_n\}$ converges strongly to a common element $\tilde{x} \in \cap_{m=1}^{r} VI(C, A_m)$, which is the unique solution of the following problem:

$$\langle(\gamma f - \mu G)\tilde{x}, x - \tilde{x}\rangle \leq 0, \;\; \forall x \in \cap_{m=1}^{r} VI(C, A_m).$$

If $r = 1$, then Theorem 5.4.1 is reduced to the following Corollary.

Corollary 5.4.3. *Let C be a nonempty closed and convex subset of a real Hilbert space H such that $C \pm C \subset C$. Let $f : C \to C$ be a contraction with coefficient $k \in (0,1)$. Let $G : C \to C$ be a ξ-strongly monotone and L-Lipschitz continuous mapping. Let $A : C \to H$ be a relaxed (η, ρ)-cocoercive and ν-Lipschitz continuous mapping. Let $B : C \to H$ be a relaxed $(\hat{\eta}, \hat{\rho})$-cocoercive and $\hat{\nu}$-Lipschitz continuous mapping. Let $p = \sqrt{1 - 2\lambda\rho + \lambda^2\nu^2 + 2\lambda\eta\nu^2}$ and $q = \sqrt{1 - 2\hat{\lambda}\hat{\rho} + \hat{\lambda}^2\hat{\nu}^2 + 2\hat{\lambda}\hat{\eta}\hat{\nu}^2}$, where λ and $\hat{\lambda}$ are two positive real numbers. Assume that $GVI(C, B, A) \neq \emptyset$, $\xi > 0, L > 0$, $0 < \mu < 2\xi/L^2$, $0 < \gamma < \mu(\xi - \mu L^2/2)/k = \pi/k$ and $p, q \in [0, \frac{1}{2})$. Given the initial guess $x_1 \in C$ and $\{x_n\}$ is a sequence generated by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G) P_C(\hat{\lambda} B x_n - \lambda A x_n), \;\; \forall n \geq 1,$$

where $\{\alpha_n\}$ is a sequences in $(0,1)$, satisfying the following conditions:

$$\lim_{n\to\infty} \alpha_n = 0, \;\; \Sigma_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \Sigma_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Then the sequence $\{x_n\}$ converges strongly to a common element $\tilde{x} \in GVI(C, B, A)$, which is the unique solution of the HGVIP (2.6.3):

$$\langle (\gamma f - \mu G)\tilde{x}, x - \tilde{x} \rangle \leq 0, \;\; \forall x \in GVI(C, B, A).$$

For the variational inequality (2.6.1), we can obtain from Corollary 5.4.3 the following immediately.

Corollary 5.4.4. *Let C be a nonempty closed and convex subset of a real Hilbert space H such that $C \pm C \subset C$. Let $f : C \to C$ be a contraction with coefficient $k \in (0,1)$. Let $G : C \to C$ be a ξ-strongly monotone and L-Lipschitz continuous mapping. Let $A : C \to H$ be a relaxed (η, ρ)-cocoercive and ν-Lipschitz continuous mapping. Let $p = \sqrt{1 - 2\lambda\rho + \lambda^2\nu^2 + 2\lambda\eta\nu^2}$, where λ is a positive real number. Assume that $VI(C, A) \neq \emptyset$, $\xi > 0, L > 0$, $0 < \mu < 2\xi/L^2$, $0 < \gamma < \mu(\xi - \mu L^2/2)/k = \pi/k$ and $p \in [0, 1)$. Given the initial guess $x_1 \in C$ and $\{x_n\}$ is a sequence generated by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G) P_C(x_n - \lambda A x_n), \;\; \forall n \geq 1,$$

where $\{\alpha_n\}$ is a sequences in $(0,1)$, satisfying the following conditions:

$$\lim_{n\to\infty} \alpha_n = 0, \;\; \Sigma_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \Sigma_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Then the sequence $\{x_n\}$ converges strongly to a common element $\tilde{x} \in VI(C, A)$, which is the unique solution of the HVIP (2.6.2):

$$\langle (\gamma f - \mu G)\tilde{x}, x - \tilde{x} \rangle \leq 0, \quad \forall x \in VI(C, A).$$

Remark 5.4.5. (1) If we take $G = A$ and $\mu = 1$, where A is a strongly positive linear bounded operator on C in Theorem 5.4.1, then our iterative algorithm define by (5.4.2) converges strongly to $\tilde{x} \in \cap_{m=1}^{r} GVI(C, B_m, A_m)$, such that $\langle (\gamma f - A)\tilde{x}, x - \tilde{x} \rangle \leq 0, \quad \forall x \in \cap_{m=1}^{r} GVI(C, B_m, A_m)$, Equivalently, $\tilde{x}$ is the unique solution to the minimization problem:

$$\min_{x \in \cap_{m=1}^{r} GVI(C, B_m, A_m)} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

(2) If we taking $G = I$ and $\gamma = \mu = 1$, where I is a identity mapping in Theorem 5.4.1, then our iterative algorithm define by (5.4.2) converges strongly to a common element $\tilde{x} \in \cap_{m=1}^{r} GVI(C, B_m, A_m)$, such that $\langle (f - I)\tilde{x}, x - \tilde{x} \rangle \leq 0, \quad \forall x \in \cap_{m=1}^{r} GVI(C, B_m, A_m)$.

In case, $f = 0$, our iterative algorithm define by (5.4.2) converges strongly to $\tilde{x}$ which is the unique solution to the quadratic minimiztion problem:

$$z = \operatorname*{arg\,min}_{x \in \cap_{m=1}^{r} GVI(C, B_m, A_m)} \|x\|^2. \tag{5.4.19}$$

In case, $f = u$, where u is fixed element in C, our iterative algorithm define by (5.4.2) converges strongly to a common element $\tilde{x} \in \cap_{m=1}^{r} GVI(C, B_m, A_m)$, such that $\langle u - \tilde{x}, x - \tilde{x} \rangle \leq 0, \quad \forall x \in \cap_{m=1}^{r} GVI(C, B_m, A_m)$.

(3) Note that, our iterative algorithm define by (5.4.2) are more flexible in solving the HGVIP than the one introduced by Yu and Liang [51].

5.5 Iteration Algorithm for Solving Hierarchical Equilibrium and Generalized Variational Inequality Problem

In this section, we introduce the convex feasibility problem (CFP) in the case that each is a solution set of the generalized variational inequality and the equilibrium problem and show a new approach method to find a common element in the intersection of the set of the solutions of a finite family of equilibrium problems and the intersection of the set of the solutions of a finite family of generalized variational inequality problems in a real Hilbert space which is is a unique solution of the hierarchical equilibrium and generalized variational inequality problems(HEGVIP).

Let $I = \{1, 2, ..., l\}$ be a finite index set. For each $i \in I$, let F_i be a bi-function from $C \times C$ into $\Re$ satisfying (A1)-(A4). Denote $T^i_{r_n} : H \to C$ by

$$T^i_{r_n}(x) = \left\{ z \in C : F_i(z, y) + \frac{1}{r_n}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}.$$

Theorem 5.5.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H such that $C \pm C \subset C$. For each $i \in I$, let F_i be a bi-function from $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $f : C \to C$ be a contraction with coefficient $k \in (0, 1)$. Let $G : C \to C$ be a ξ-strongly monotone and L-Lipschitz continuous mapping. let $A_m : C \to H$ be a relaxed (η_m, ρ_m)-cocoercive and ν_m-Lipschitz continuous mapping and $B_m : C \to H$ be a relaxed $(\hat{\eta}_m, \hat{\rho}_m)$-cocoercive and $\hat{\nu}_m$-Lipschitz continuous mapping for each $1 \leq m \leq r$. Let $p_m = \sqrt{1 - 2\lambda_m\rho_m + \lambda_m^2\nu_m^2 + 2\lambda_m\eta_m\nu_m^2}$ and $q_m = \sqrt{1 - 2\hat{\lambda}_m\hat{\rho}_m + \hat{\lambda}_m^2\hat{\nu}_m^2 + 2\hat{\lambda}_m\hat{\eta}_m\hat{\nu}_m^2}$, where $\{\lambda_m\}$ and $\{\hat{\lambda}_m\}$ are two positive sequences for each $1 \leq m \leq r$. Assume that $\Omega = (\cap_{i=1}^l EP(F_i)) \cap (\cap_{m=1}^r GVI(C, B_m, A_m))$ $\neq \emptyset$, $\xi > 0, L > 0$, $0 < \mu < 2\xi/L^2$, $0 < \gamma < \mu(\xi - \mu L^2/2)/k = \pi/k$ and $p_m, q_m \in [0, \frac{1}{2})$, for each $1 \leq m \leq r$. Given $\{x_n\}$ is a sequence generated by*

$$\begin{cases} x_1 \in C, \\ u_n^i = T^i_{r_n} x_n, \qquad \forall i \in I, \\ v_n = \frac{u_n^1 + u_n^2 + \cdots + u_n^l}{l}, \\ y_n^m = P_C(\hat{\lambda}_m B_m v_n - \lambda_m A_m v_n), \qquad \forall m = 1, 2, \ldots, r, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G)\sum_{m=1}^r \beta_{(m,n)} y_n^m, \ \ \forall n \geq 1, \end{cases} \tag{5.5.1}$$

where $\{\alpha_n\}, \{\beta_{(m,n)}\} \subset (0, 1), \forall 1 \leq m \leq r$ and $\{r_n\} \subset (0, +\infty)$ satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^\infty \alpha_n = \infty$ *and* $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;

(C2) $\sum_{m=1}^r \beta_{(m,n)} = 1, \forall n \geq 1$,$\sum_{n=1}^\infty |\beta_{(m,n+1)} - \beta_{(m,n)}| < \infty$ *and* $\lim_{n\to\infty} \beta_{(m,n)} = \beta_m \in (0, 1), \forall 1 \leq m \leq r.$

(C3) $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^\infty |r_{n+1} - r_n| < \infty$;

Then the sequence $\{x_n\}$ converges strongly to a common element $c \in \Omega$, which is the unique solution of the HEGVIP:

$$\langle(\gamma f - \mu G)c, x - c\rangle \leq 0, \quad \forall x \in \Omega. \tag{5.5.2}$$

Proof. We will proceed with the following steps.

Step 1: We prove that $P_C(\hat{\lambda}_m B_m - \lambda_m A_m)$ is a nonexpansive mapping. Put $T_m = P_C(\hat{\lambda}_m B_m - \lambda_m A_m), \forall 1 \leq m \leq r$. $\forall x, y \in C$ and $1 \leq m \leq r$, we have

$$\begin{aligned} \|T_m x - T_m y\| &\leq \|(x-y) - \lambda_m(A_m x - A_m y)\| \\ &\quad + \|(x-y) - \hat{\lambda}_m(B_m x - B_m y)\|. \end{aligned} \tag{5.5.3}$$

It follows from the assumption that each A_m is relaxed (η_m, ρ_m)-cocoercive and ν_m-Lipschitz continuous that $\|(x-y) - \lambda_m(A_m x - A_m y)\|^2 \leq p_m^2\|x-y\|^2$. This shows that $\|(x-y) - \lambda_m(A_m x - A_m y)\| \leq p_m\|x-y\|$. In a similar way, we can obtain that $\|(x-y) - \hat{\lambda}_m(B_m x - B_m y)\| \leq q_m\|x-y\|$. So, we have

$$\|T_m x - T_m y\| \leq (p_m + q_m)\|x - y\| \leq \|x - y\|. \tag{5.5.4}$$

Hence T_m is a nonexpansive mapping and

$$F(T_m) = F(P_C(\hat{\lambda}_m B_m - \lambda_m A_m)) = GVI(C, B_m, A_m), \forall 1 \leq m \leq r$$

. Put $S_n = \sum_{m=1}^{r} \beta_{(m,n)} T_m$. By Lemma 2.7.19, we conclude that S_n is a nonexpansive mapping and $F(S_n) = \cap_{m=1}^{r} GVI(C, B_m, A_m), \forall n \geq 1$. We can rewrite the algorithm (5.5.1) as

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G)S_n v_n. \tag{5.5.5}$$

Step 2: We prove that the sequence $\{x_n\}, \{y_n^m\}, \{v_n\}$ and $\{u_n^i\}$ are bounded. Take $c \in \Omega$. For each $i \in I$, we have

$$\|u_n^i - c\| = \|T_{r_n}^i x_n - T_{r_n}^i c\| \leq \|x_n - c\|, \quad \forall n \geq 1. \tag{5.5.6}$$

From (5.5.1) and (5.5.6) we have

$$\|v_n - c\| \leq \|x_n - c\|, \quad \forall n \geq 1 \tag{5.5.7}$$

For each $1 \leq m \leq r$, we have

$$\|y_n^m - c\| \leq (p_m + q_n)\|v_n - c\| \leq \|x_n - c\| \tag{5.5.8}$$

From (5.5.1), (5.5.8) and Lemma 2.7.3 , we have

$$\begin{aligned}\|x_{n+1}-c\| &\leq \alpha_n\|\gamma(f(x_n)-f(c))+\gamma f(c)-\mu Gc\| \\ &\quad +(1-\alpha_n\pi)\|\sum_{m=1}^{r}\beta_{(m,n)}y_n^m-c\| \\ &\leq max\left\{\|x_n-c\|,\frac{\|\gamma f(c)-\mu Gc\|}{\pi-\gamma k}\right\},\forall n\geq 1.\end{aligned}$$

By induction, we obtain $\|x_n-c\|\leq max\left\{\|x_1-c\|,\frac{\|\gamma f(c)-\mu Gc\|}{\pi-\gamma k}\right\},\forall n\geq 1$. Hence $\{x_n\}$ is bounded. Also, we know that $\{y_n^m\},\{v_n\}$ and $\{u_n^i\}$ are all $\forall 1\leq m\leq r,\forall 1\leq i\leq l$. Since S_n is nonexpansive mappings for $n\geq 1$, we see that

$$\|S_nx_n-c\| \leq \|x_n-c\|\leq \max\left\{\|x_1-c\|,\frac{\|\gamma f(c)-\mu Gc\|}{\pi-\gamma k}\right\}.$$

Therefore, $\{S_nx_n\}$ is bounded. Since G is a L-Lipschitz continuous mapping, we have $\|GS_nx_n-Gc\|\leq L\|x_n-c\|\leq \max\left\{L\|x_1-c\|,L\frac{\|\gamma f(c)-\mu Gc\|}{\pi-\gamma k}\right\}$. Hence $\{GS_nx_n\}$ is bounded. Since f is contraction and $\{x_n\}$ is bounded, so $\{f(x_n)\}$ is bounded.

Step 3: We prove that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

From (5.5.1), we consider

$$\begin{aligned}x_{n+1}-x_n &= [(I-\alpha_n\mu G)\sum_{m=1}^{r}\beta_{(m,n)}y_n^m-(I-\alpha_n\mu G)\sum_{m=1}^{r}\beta_{(m,n-1)}y_{n-1}^m] \\ &\quad +(\alpha_n-\alpha_{n-1})\gamma f(x_{n-1})+(\alpha_{n-1}-\alpha_n)\mu G\sum_{m=1}^{r}\beta_{(m,n-1)}y_{n-1}^m \\ &\quad +\alpha_n\gamma(f(x_n)-f(x_{n-1})),\end{aligned}$$

it follows that

$$\begin{aligned}\|x_{n+1}-x_n\| &\leq (1-\alpha_n\pi)\|\sum_{m=1}^{r}\beta_{(m,n)}y_n^m-\sum_{m=1}^{r}\beta_{(m,n-1)}y_{n-1}^m\| \\ &\quad +\alpha_n\gamma k\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|M_1,\end{aligned} \tag{5.5.9}$$

where $M_1=\sup_{n\geq 1}\{\gamma\|f(x_n)\|+\mu\|G\sum_{m=1}^{r}\beta_{(m,n)}y_n^m\|\}$, and

$$\|\sum_{m=1}^{r}\beta_{(m,n)}y_n^m-\sum_{m=1}^{r}\beta_{(m,n-1)}y_{n-1}^m\|\leq M_2\sum_{m=1}^{r}|\beta_{(m,n)}-\beta_{(m,n-1)}|+\|v_n-v_{n-1}\|, \tag{5.5.10}$$

where $M_2=\max\{\sup_{n\geq 1}\|y_n^m\|,\forall 1\leq m\leq r\}$. $\forall\in I$, since $u_{n-1}^i,u_n^i\in C$, we have

$$F_i(u_n^i,u_{n-1}^i)+\frac{1}{r_n}\langle u_{n-1}^i-u_n^i,u_n^i-x_n\rangle\geq 0, \tag{5.5.11}$$

and

$$F_i(u_{n-1}^i, u_n^i) + \frac{1}{r_{n-1}}\langle u_n^i - u_{n-1}^i, u_{n-1}^i - x_{n-1}\rangle \geq 0. \quad (5.5.12)$$

From (5.5.11), (5.5.12) and (A2), we see that

$$\begin{aligned} 0 &\leq r_n[F_i(u_n^i, u_{n-1}^i) + F_i(u_{n-1}^i, u_n^i)] + \langle u_{n-1}^i - u_n^i, u_n^i - x_n - \frac{r_n}{r_{n-1}}(u_{n-1}^i - x_{n-1})\rangle \\ &\leq \langle u_{n-1}^i - u_n^i, u_n^i - x_n - \frac{r_n}{r_{n-1}}(u_{n-1}^i - x_{n-1})\rangle \end{aligned}$$

which implies

$$\langle u_{n-1}^i - u_n^i, u_{n-1}^i - u_n^i + x_n - x_{n-1} + x_{n-1} - u_{n-1}^i + \frac{r_n}{r_{n-1}}(u_{n-1}^i - x_{n-1})\rangle \leq 0. \quad (5.5.13)$$

It follows from (5.5.13) that

$$\|u_n^i - u_{n-1}^i\| \leq \|x_n - x_{n-1}\| + |\frac{r_n - r_{n-1}}{r_{n-1}}|\|x_{n-1} - u_{n-1}^i\|, \quad \forall n \geq 1. \quad (5.5.14)$$

Without loss of generality, let us assume that there exists a real number d such that $r_n > d > 0$ for all $n \geq 1$. Since $v_n = \frac{1}{l}(u_n^1 + u_n^2 + \ldots + u_n^l)$, by (5.5.14), we have

$$\|v_n - v_{n-1}\| \leq \frac{1}{l}\sum_{i=1}^{l}\|u_n^i - u_{n-1}^i\| \leq \|x_n - x_{n-1}\| + \frac{|r_n - r_{n-1}|}{d}M_3, \quad \forall n \geq 1, \quad (5.5.15)$$

where $M_3 = \max\{\sup_{n\geq 1}\frac{1}{l}\sum_{i=1}^{l}\|x_{n-1} - u_{n-1}^i\|, \forall 1 \leq i \leq l\}$. From (5.5.9), (5.5.10) and (5.5.17), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq (1 - \alpha_n[\pi - \gamma k])\|x_n - x_{n-1}\| \\ &\quad + M\Bigg[|\alpha_n - \alpha_{n-1}| + \sum_{m=1}^{r}|\beta_{(m,n)} - \beta_{(m,n-1)}| \\ &\quad + \frac{|r_n - r_{n-1}|}{d}\Bigg], \end{aligned} \quad (5.5.16)$$

where M is appropriate constant such that $M \geq \max\{M_1, M_2, M_3\}$. By conditions $(C1), (C2)$ and $(C3)$ and Lemma 2.7.6 , we obtain that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \quad (5.5.17)$$

Define a mapping $S : C \to C$ by

$$Sx = \sum_{m=1}^{r}\beta_m P_C(\hat{\lambda}_m B_m x - \lambda_m A_m x), \forall x \in C, \quad (5.5.18)$$

where $\beta_m = \lim_{n\to\infty}\beta_{(m,n)}$. From Lemma 2.7.19, we see that S is a nonexpansive mapping and $F(S) = \cap_{m=1}^{r}F(T_m) = \cap_{m=1}^{r}GVI(C, B_m, A_m), \forall n \geq 1$.

Step 4: We will show that $\lim_{n\to\infty}\|Sx_n - x_n\| = 0$.
By Lemma 2.7.21, we see that

$$\begin{aligned}\|u_n^i - c\|^2 &\leq \langle T_{r_n}^i x_n - T_{r_n}^i c, x_n - c\rangle \\ &= \|x_n - c\|^2 - \|u_n^i - x_n\|^2. \qquad (5.5.19)\end{aligned}$$

From (5.5.19) and Lemma 2.7.22,

$$\|v_n - c\|^2 \leq \frac{1}{l}\sum_{i=1}^{l}\|u_n^i - c\|^2 \leq \|x_n - c\|^2 - \frac{1}{l}\sum_{i=1}^{l}\|u_n^i - x_n\|^2. \qquad (5.5.20)$$

From (5.5.20) and Lemma 2.7.15, we have

$$\begin{aligned}\|x_{n+1} - c\|^2 &\leq (1-\alpha_n\pi)^2\|v_n - x\|^2 + 2\alpha_n\langle\gamma f(x_n) - \mu Gc, x_{n+1} - c\rangle \\ &\leq \|v_n - x\|^2 + 2\alpha_n\|\gamma f(x_n) - \mu Gc\|\|x_{n+1} - c\| \\ &\leq \|x_n - c\|^2 - \frac{1}{l}\sum_{i=1}^{l}\|u_n^i - x_n\|^2 + 2\alpha_n\|\gamma f(x_n) - \mu Gc\|\|x_{n+1} - c\|.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{1}{l}\sum_{i=1}^{l}\|u_n^i - x_n\|^2 &\leq \left[\|x_n - c\| - \|x_{n+1} - c\|\right]\|x_{n+1} - x_n\| \\ &\quad +2\alpha_n\|\gamma f(x_n) - \mu Gc\|\|x_{n+1} - c\|. \qquad (5.5.21)\end{aligned}$$

Letting $n \to \infty$ in the equality (5.5.21), we obtain

$$\lim_{n\to\infty}\|u_n^i - x_n\| = 0, \quad \forall i \in I. \qquad (5.5.22)$$

By Lemma 2.7.22, we get

$$\|v_n - x_n\|^2 = \left\|\sum_{i=1}^{l}\frac{1}{l}[u_n^i - x_n]\right\|^2 \leq \frac{1}{l}\sum_{i=1}^{l}\|u_n^i - x_n\|^2. \qquad (5.5.23)$$

Hence

$$\lim_{n\to\infty}\|v_n - x_n\| = 0. \qquad (5.5.24)$$

Furthermore, it is easy to prove that

$$\lim_{n\to\infty}\|v_n - u_n^i\| = 0, \quad \forall i \in I. \qquad (5.5.25)$$

From (5.5.5), we observe that

$$\|x_{n+1} - S_n v_n\| = \alpha_n\|\gamma f(x_n) + \mu G S_n v_n\|. \qquad (5.5.26)$$

Hence, $\lim_{n\to\infty}\|x_{n+1}-S_nv_n\| = 0$. Since $\|x_n-S_nv_n\| \le \|x_{n+1}-S_nv_n\|+\|x_n-x_{n+1}\|$, it follows that

$$\lim_{n\to\infty}\|x_n - S_nv_n\| = 0. \tag{5.5.27}$$

From (5.5.22), (5.5.24), (5.5.25), (5.5.27) and S_n is nonexapnsive, we have

$$\lim_{n\to\infty}\|x_n - S_nx_n\| = \lim_{n\to\infty}\|v_n - S_nv_n\| = \lim_{n\to\infty}\|u_n^i - S_nu_n^i\| = 0, \quad \forall i \in I. \tag{5.5.28}$$

Now, we show that $Sx_n - x_n \to 0$ as $n \to \infty$. Note that

$$\begin{aligned}\|Sx_n - x_n\| &\le \|\sum_{m=1}^{r}\beta_mT_mx_n - \sum_{m=1}^{r}\beta_{(m,n)}T_mx_n\| + \|S_nx_n - x_n\| \\ &\le M_2(\sum_{m=1}^{r}|\beta_m - \beta_{(m,n)}|) + \|S_nx_n - x_n\|.\end{aligned}$$

By the condition $(C2)$ and (5.5.28), we have

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0. \tag{5.5.29}$$

From the boundedness of x_n, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \rightharpoonup z$ as $j \to \infty$, by Lemma 2.7.1 and (5.5.29), we obtain $z = Sz$. So, we have

$$z \in F(S) = \cap_{m=1}^{r}GVI(C, B_m, A_m), \forall n \ge 1. \tag{5.5.30}$$

From (5.5.22), we also have $u_{n_j}^i \rightharpoonup z$ as $j \to \infty, \forall i \in I$, since $F_i(u_{n_j}^i, y) + \frac{1}{r_{n_j}}\langle y - u_{n_j}^i, u_{n_j}^i - x_{n_j}\rangle \ge 0, \forall y \in C$, it follows from (A2) that

$$\begin{aligned}\frac{1}{r_{n_j}}\langle y - u_{n_j}^i, u_{n_j}^i - x_{n_j}\rangle &\ge F_i(y, u_{n_j}^i) + F_i(u_{n_j}^i, y) + \frac{1}{r_{n_j}}\langle y - u_{n_j}^i, u_{n_j}^i - x_{n_j}\rangle \\ \langle y - u_{n_j}^i, \frac{u_{n_j}^i - x_{n_j}}{r_{n_j}}\rangle &\ge F_i(y, u_{n_j}^i), \quad \forall y \in C.\end{aligned} \tag{5.5.31}$$

For (5.5.22) and (A4), we have

$$F_i(y, z) \le 0, \quad \forall y \in C. \tag{5.5.32}$$

Put $y_t = ty + (1-t)z, t \in (0,1)$. Then $y_t \in C$ and $F_i(y_t, z) \le 0$ for all $i \in I$. By (A1) and (A4), we obtain $0 = F_i(y_t, y_t) \le tF_i(y_t, y) + (1-t)F_i(y_t, z) \le tF_i(y_t, y), \forall i \in I$. By (A3), we get

$$F_i(z, y) \ge \lim_{t\downarrow 0} F_i(ty + (1-t)z, y) = \lim_{t\downarrow 0} F_i(y_t, y) \ge 0, \forall i \in I.$$

It follows that $z \in \bigcap_{i=i}^{l} EP(F_i)$. Hence, $z \in \Omega$. So, we have $\omega_w(x_n) \subset \Omega$. By Lemma 2.7.4, $\mu G - \gamma f$ is strongly monotone, so the variational inequality (5.5.2) has a unique solution $c \in \Omega$.

Step 5: We will show that $\limsup_{n\to\infty}\langle(\gamma f - \mu G)c, x_n - c\rangle \leq 0$.
Indeed, since $\{x_n\}$ is bounded, then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f - \mu G)c, x_n - c\rangle = \lim_{i\to\infty}\langle(\gamma f - \mu G)c, x_{n_i} - c\rangle. \tag{5.5.33}$$

Without loss of generality, we may further assume that $x_{n_i} \rightharpoonup z$. It follows from (5.5.13) that $z \in \Omega$. Since z is the unique solution of (5.5.2), we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle(\gamma f - \mu G)c, x_n - c\rangle &= \lim_{i\to\infty}\langle(\gamma f - \mu G)c, x_{n_i} - c\rangle \\ &= \langle(\gamma f - \mu G)c, z - c\rangle \leq 0. \end{aligned}\tag{5.5.34}$$

Step 6: Finally, we will show that $x_n \to c$ as $n \to \infty$.
From Lemma 2.7.3, Lemma 2.7.15 and (5.5.1) we have

$$\begin{aligned}\|x_{n+1} - c\|^2 &\leq (1 - \alpha_n\pi)^2\|\frac{1}{l}\sum_{i=1}^{l}(u_n^i - c)\|^2 + 2\alpha_n\gamma\langle f(x_n) - f(c), x_{n+1} - c\rangle \\ &\quad +2\alpha_n\langle\gamma f(c) - \mu Gc, x_{n+1} - c\rangle \\ &\leq [1 - \frac{2\alpha_n(\pi - \gamma k)}{1 - \alpha_n\gamma k}]\|x_n - c\|^2 + \frac{(\alpha_n\pi)^2}{1 - \alpha_n\gamma k}\|x_n - c\|^2 \\ &\quad +\frac{2\alpha_n}{1 - \alpha_n\gamma k}\langle\gamma f(c) - \mu Gc, x_{n+1} - c\rangle \\ &= (1 - \theta_n)\|x_n - c\|^2 + \delta_n,\end{aligned}$$

where $\theta_n := \frac{2\alpha_n(\pi - \gamma k)}{1 - \alpha_n\gamma k}$ and $\delta_n := \frac{\alpha_n}{1 - \alpha_n\gamma k}[\alpha_n\pi^2\|x_n - c\|^2 + 2\langle\gamma f(c) - \mu Gc, x_{n+1} - c\rangle]$.
Note that,

$$\theta_n := \frac{2\alpha_n(\pi - \gamma k)}{1 - \alpha_n\gamma k} \leq \frac{2(\pi - \gamma k)}{1 - \gamma k}\alpha_n. \tag{5.5.35}$$

By (C1), we obtain that $\lim_{n\to\infty}\theta_n = 0$. On the other hand, we have

$$\theta_n := \frac{2\alpha_n(\pi - \gamma k)}{1 - \alpha_n\gamma k} \geq 2\alpha_n(\pi - \gamma k). \tag{5.5.36}$$

From (C1), we have $\sum_{n=1}^{\infty}\theta_n = \infty$. Put $M = \sup_{n\in N}\{\|x_n - c\|\}$, we have

$$\frac{\delta_n}{\theta_n} = \frac{1}{2(\pi - \gamma k)}[\alpha_n\pi^2 M + 2\langle\gamma f(c) - \mu Gc, x_{n+1} - c\rangle]. \tag{5.5.37}$$

It follows that $\limsup_{n\to\infty} \frac{\delta_n}{\theta_n} \leq 0$. Hence, by Lemma 2.7.6, we conclude that

$$\lim_{n\to\infty} \|x_n - c\| = 0. \tag{5.5.38}$$

Therefore $x_n \to c$ as $n \to \infty$. This completes the proof. □ □

As direct consequences of Theorem 5.5.1, we obtain corollaries.

Corollary 5.5.2. *Let C be a nonempty closed and convex subset of a real Hilbert space H such that $C \pm C \subset C$. Let F be a bi-function from $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $f : C \to C$ be a contraction with coefficient $k \in (0,1)$. Let $G : C \to C$ be a ξ-strongly monotone and L-Lipschitz continuous mapping. let $A_m : C \to H$ be a relaxed (η_m, ρ_m)-cocoercive and ν_m-Lipschitz continuous mapping and $B_m : C \to H$ be a relaxed $(\hat{\eta}_m, \hat{\rho}_m)$-cocoercive and $\hat{\nu}_m$-Lipschitz continuous mapping for each $1 \leq m \leq r$. Let $p_m = \sqrt{1 - 2\lambda_m\rho_m + \lambda_m^2\nu_m^2 + 2\lambda_m\eta_m\nu_m^2}$ and $q_m = \sqrt{1 - 2\hat{\lambda}_m\hat{\rho}_m + \hat{\lambda}_m^2\hat{\nu}_m^2 + 2\hat{\lambda}_m\hat{\eta}_m\hat{\nu}_m^2}$, where $\{\lambda_m\}$ and $\{\hat{\lambda}_m\}$ are two positive sequences for each $1 \leq m \leq r$. Assume that $\Delta = EP(F) \cap (\cap_{m=1}^r GVI(C, B_m, A_m)) \neq \emptyset$, $\xi > 0, L > 0$, $0 < \mu < 2\xi/L^2$, $0 < \gamma < \mu(\xi - \mu L^2/2)/k = \pi/k$ and $p_m, q_m \in [0, \frac{1}{2})$, for each $1 \leq m \leq r$. Given $\{x_n\}$ is a sequence generated by*

$$\begin{cases} x_1 \in C, \\ u_n = T_{r_n} x_n, \\ y_n^m = P_C(\hat{\lambda}_m B_m u_n - \lambda_m A_m u_n), \qquad \forall m = 1, 2, \ldots, r, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G) \sum_{m=1}^r \beta_{(m,n)} y_n^m, \ \ \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_{(m,n)}\} \subset (0,1), \forall 1 \leq m \leq r$ and $\{r_n\} \subset (0, +\infty)$ satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^\infty \alpha_n = \infty$ *and* $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;

(C2) $\sum_{m=1}^r \beta_{(m,n)} = 1, \forall n \geq 1$,$\sum_{n=1}^\infty |\beta_{(m,n+1)} - \beta_{(m,n)}| < \infty$ *and* $\lim_{n\to\infty} \beta_{(m,n)} = \beta_m \in (0,1), \forall 1 \leq m \leq r$.

(C3) $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^\infty |r_{n+1} - r_n| < \infty$;

Then the sequence $\{x_n\}$ converges strongly to a common element $c \in \Delta$, which is the unique solution of the HEGVIP: $\langle (\gamma f - \mu G)c, x - c \rangle \leq 0, \ \ \forall x \in \Delta$.

Corollary 5.5.3. *Let C be a nonempty closed and convex subset of a real Hilbert space H such that $C \pm C \subset C$. Let F be a bi-function from $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $f : C \to C$ be a contraction with coefficient $k \in (0,1)$. Let $G : C \to C$ be a ξ-strongly monotone and L-Lipschitz continuous mapping. Let $A : C \to H$ be a relaxed (η, ρ)-cocoercive and ν-Lipschitz continuous mapping. Let $B : C \to H$ be a relaxed $(\hat{\eta}, \hat{\rho})$-cocoercive and $\hat{\nu}$-Lipschitz continuous mapping. Let $p = \sqrt{1 - 2\lambda\rho + \lambda^2\nu^2 + 2\lambda\eta\nu^2}$ and $q = \sqrt{1 - 2\hat{\lambda}\hat{\rho} + \hat{\lambda}^2\hat{\nu}^2 + 2\hat{\lambda}\hat{\eta}\hat{\nu}^2}$, where λ and $\hat{\lambda}$ are two positive real numbers. Assume that $\Lambda = EP(F) \cap GVI(C, B, A) \neq \emptyset$, $\xi > 0, L > 0$, $0 < \mu < 2\xi/L^2$, $0 < \gamma < \mu(\xi - \mu L^2/2)/k = \pi/k$ and $p, q \in [0, \frac{1}{2})$. Given the initial guess $x_1 \in C$ and $\{x_n\}$ is a sequence generated by*

$$\begin{cases} x_1 \in C, \\ u_n = T_{r_n} x_n, \\ y_n = P_C(\hat{\lambda} B u_n - \lambda A u_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G) y_n, \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0, +\infty)$, satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(C2) $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

Then the sequence $\{x_n\}$ converges strongly to a common element $c \in \Lambda$, which is the unique solution of the HEGVIP: $\langle (\gamma f - \mu G)c, x - c \rangle \leq 0, \quad \forall x \in \Lambda$.

If $F_i(x, y) \equiv 0, \forall (x, y) \in C \times C$ in Theorem 5.5.1, for all $i \in I$. Then, from the algorithm (5.5.1), we have $u_n^i \equiv P_C x_n$, for all $i \in I$. So we have the following result.

Corollary 5.5.4. *Let C be a nonempty closed and convex subset of a real Hilbert space H such that $C \pm C \subset C$. Let $f : C \to C$ be a contraction with coefficient $k \in (0,1)$. Let $G : C \to C$ be a ξ-strongly monotone and L-Lipschitz continuous mapping. let $A_m : C \to H$ be a relaxed (η_m, ρ_m)-cocoercive and ν_m-Lipschitz continuous mapping and $B_m : C \to H$ be a relaxed $(\hat{\eta}_m, \hat{\rho}_m)$-cocoercive and $\hat{\nu}_m$-Lipschitz continuous mapping for each $1 \leq m \leq r$. Let $p_m = \sqrt{1 - 2\lambda_m\rho_m + \lambda_m^2\nu_m^2 + 2\lambda_m\eta_m\nu_m^2}$ and $q_m = \sqrt{1 - 2\hat{\lambda}_m\hat{\rho}_m + \hat{\lambda}_m^2\hat{\nu}_m^2 + 2\hat{\lambda}_m\hat{\eta}_m\hat{\nu}_m^2}$, where $\{\lambda_m\}$ and $\{\hat{\lambda}_m\}$ are two positive sequences for each $1 \leq m \leq r$. Assume that $\Theta = \cap_{m=1}^{r} GVI(C, B_m, A_m) \neq \emptyset$, $\xi > 0, L > 0$, $0 < \mu < 2\xi/L^2$, $0 < \gamma < \mu(\xi - \mu L^2/2)/k = \pi/k$ and $p_m, q_m \in [0, \frac{1}{2})$,* *for each* $1 \leq m \leq r$*. Given the initial guess* $x_1 \in C$ *and* $\{x_n\}$ *is a sequence generated by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G)\Sigma_{m=1}^{r} \beta_{(m,n)} P_C(\hat{\lambda}_m B_m x_n - \lambda_m A_m x_n), \ \ \forall n \geq 1,$$

where $\{\alpha_n\}$ *and* $\{\beta_{(m,n)}\} \subset (0,1), \forall 1 \leq m \leq r$ *satisfying the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(C2) $\sum_{m=1}^{r} \beta_{(m,n)} = 1, \forall n \geq 1$,$\sum_{n=1}^{\infty} |\beta_{(m,n+1)} - \beta_{(m,n)}| < \infty$ *and* $\lim_{n\to\infty} \beta_{(m,n)} = \beta_m \in (0,1), \forall 1 \leq m \leq r$.

Then the sequence $\{x_n\}$ *converges strongly to a common element* $c \in \Theta$*, which is the unique solution of the HGVIP:* $\langle (\gamma f - \mu G)c, x - c\rangle \leq 0, \ \ \forall x \in \Theta$.

5.6 Some Application to Optimization Problems

From Theorem 5.1.1, we can dedude the following interesting corollary for solving the quadratic minimiztion problem.

Corollary 5.6.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* $f : C \to H$ *be a* ρ*-contraction (possibly nonself) with* $\rho \in (0,1)$*. Let* $S, T : C \to C$ *be two nonexpansive mappings with* $F(T) \neq \emptyset$*.*$\{\alpha_n\}$ *and* $\{\beta_n\}$ *are two sequences in* $(0,1)$*. Starting with an arbitrary initial guess* $x_0 \in C$ *and* $\{x_n\}$ *is a sequence generated by*

$$\begin{aligned} y_n &= \beta_n S x_n + (1-\beta_n) x_n, \\ x_{n+1} &= P_C[\alpha_n f(x_n) + (1-\alpha_n) T y_n], \quad \forall n \geq 1. \end{aligned} \tag{5.6.1}$$

Suppose that the following conditions are satisfied:

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

$(C2)$ $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = 0$*;*

$(C3)$ $\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$ *and* $\lim_{n\to\infty} \frac{|\beta_n - \beta_{n-1}|}{\beta_n} = 0$ *or*

$(C4)$ $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *and* $\sum_{n=1}^{\infty} |\beta_n - \beta n - 1| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to a point $\tilde{x} \in H$, which is the unique solution of the variational inequality:

$$\tilde{x} \in F(T), \qquad \langle (I - f)\tilde{x}, x - \tilde{x} \rangle \geq 0, \quad \forall x \in F(T). \tag{5.6.2}$$

Equivalently, we have $P_{F(T)}(f)\tilde{x} = \tilde{x}$. In particular, if we take $f = 0$, then the sequence $\{x_n\}$ converges in norm to the Minimum norm fixed point $\tilde{x}$ of T, namely, the point $\tilde{x}$ is the unique solution to the quadratic minimiztion problem:

$$z = \arg \min_{x \in F(T)} \|x\|^2. \tag{5.6.3}$$

Proof. As a matter of fact, if we take $A = I$ and $\gamma = 1$ in Theorem 5.1.1. This complete the proof. $\square$

CHAPTER 6 CONCLUSIONS

In this chapter, we conclude all the theorems obtained in this dissertation as follows:

Let Θ denote the class of those functions $\theta : (0,1]^5 \to [0,1]$ such that θ is continuous and

$$\theta(x, 1, 1, x, x) = x.$$

(1) Let $(X, M, *)$ be a fuzzy metric space and let f, g be self-mappings of X such that (f, g) is any one of the following:

(a) R-weakly commuting,

(b) R-weakly commuting of type (A_g),

(c) R-weakly commuting of type (A_f),

(d) R-weakly commuting of type (P).

If the following holds: (i) f and g satisfy the (CLRg) property;

(ii) $\int_0^{M(fx,fy,t)} \psi(s)ds \geq \int_0^{M(gx,gy,t)} \psi(s)ds$ for $x, y \in X$;

(iii) $\int_0^{M(fx,ffx,t)} \psi(s)ds > \int_0^{\eta(x)} \psi(s)ds$ for $fx \neq ffx$ and

$$\eta(x) = \theta(M(gx, gfx, t), M(fx, gx, t), M(ffx, gfx, t), M(fx, gfx, t), M(gx, ffx, t))$$

for some $\theta \in \Theta$,

whenever $\psi : \mathbb{R}_+ \to \mathbb{R}$ is a Lebesgue integrable mapping which is summable, nonnegative and such that

$$\int_0^{\epsilon} \psi(s)ds > 0$$

for each $\epsilon > 0$, then f and g have a common fixed point.

(2) Let $(X, M, *)$ be a fuzzy metric space and let f, g be self-mappings of X such that (f, g) is any one of the following:

(a) R-weakly commuting,

(b) R-weakly commuting of type (A_g),

(c) R-weakly commuting of type (A_f),

(d) R-weakly commuting of type (P).

If the following holds:

(i) f and g satisfy E.A. property and gX is a closed subspace of X;
(ii) $\int_0^{M(fx,fy,t)} \psi(s)ds \geq \int_0^{M(gx,gy,t)} \psi(s)ds$ for $x, y \in X$;
(iii) $\int_0^{M(fx,ffx,t)} \psi(s)ds > \int_0^{\eta(x)} \psi(s)ds$ for $fx \neq ffx$ and

$$\eta(x) = \theta(M(gx, gfx, t), M(fx, gx, t), M(ffx, gfx, t), M(fx, gfx, t), M(gx, ffx, t))$$

for some $\theta \in \Theta$,

whenever $\psi : \mathbb{R}_+ \to \mathbb{R}$ is a Lebesgue integrable mapping which is summable, nonnegative and such that

$$\int_0^{\epsilon} \psi(s)ds > 0$$

for each $\epsilon > 0$, then f and g have a common fixed point.

Let Δ denote the class of those functions $\delta : (0, 1]^4 \to [0, 1]$ such that δ is continuous and

$$\delta(x, 1, x, 1) = x.$$

(3) Let $(X, M, *)$ be a fuzzy metric space and let f, g be self-mappings of X such that (f, g) is any one of the following:

(a) R-weakly commuting,

(b) R-weakly commuting of type (A_g),

(c) R-weakly commuting of type (A_f),

(d) R-weakly commuting of type (P).

If the following holds:

(i) f and g satisfy the (CLRg) property;

(*ii*) $\int_0^{M(fx,fy,t)} \psi(s)ds \geq \int_0^{M(gx,gy,t)} \psi(s)ds$ for $x, y \in X$;
(*iii*) $\int_0^{M(fx,ffx,t)} \psi(s)ds > \int_0^{\eta(x)} \psi(s)ds$ for $fx \neq ffx$ and

$$\eta(x) = \delta(M(gx, gfx, t), M(fx, gx, t), M(fx, gfx, t), M(ffx, gfx, t))$$

for some $\delta \in \Delta$,

whenever $\psi : \mathbb{R}_+ \to \mathbb{R}$ is a Lebesgue integrable mapping which is summable, nonnegative and such that

$$\int_0^{\epsilon} \psi(s)ds > 0$$

for each $\epsilon > 0$, then f and g have a common fixed point.

(4) Let $(X, M, *)$ be a fuzzy metric space and let f, g be self-mappings of X such that (f, g) is any one of the following:

(a) R-weakly commuting,

(b) R-weakly commuting of type (A_g),

(c) R-weakly commuting of type (A_f),

(d) R-weakly commuting of type (P).

If the following holds:

(*i*) f and g satisfy the (E.A.) property and gX is a closed subspace of X;
(*ii*) $\int_0^{M(fx,fy,t)} \psi(s)ds \geq \int_0^{M(gx,gy,t)} \psi(s)ds$ for $x, y \in X$;
(*iii*) $\int_0^{M(fx,ffx,t)} \psi(s)ds > \int_0^{\eta(x)} \psi(s)ds$ for $fx \neq ffx$ and

$$\eta(x) = \delta(M(gx, gfx, t), M(fx, gx, t), M(fx, gfx, t), M(ffx, gfx, t))$$

for some $\delta \in \Delta$,

whenever $\psi : \mathbb{R}_+ \to \mathbb{R}$ is a Lebesgue integrable mapping which is summable, nonnegative and such that

$$\int_0^{\epsilon} \psi(s)ds > 0$$

for each $\epsilon > 0$, then f and g have a common fixed point.

(5) Let (X, d) be a K-metric space with a cone P having non-empty interior (normal or non-normal) and $F : X^3 \longrightarrow X$ and $g : X \to X$ be mappings satisfy CLR_g property. Suppose that for any $x, y, z, u, v, w \in X$, following condition

$$\begin{aligned} d(F(x,y,z), F(u,v,w)) \preceq\; & a_1 d(F(x,y,z), gx) + a_2 d(F(y,z,x), gy) \\ & + a_3 d(F(z,x,y), gz) + a_4 d(F(u,v,w), gu) \\ & + a_5 d(F(v,w,u), gv) + a_6 d(F(w,u,v), gw) \\ & + a_7 d(F(u,v,w), gx) + a_8 d(F(v,w,u), gy) \\ & + a_9 d(F(w,u,v), gz) + a_{10} d(F(x,y,z), gu) \\ & + a_{11} d(F(y,z,x), gv) + a_{12} d(F(z,x,y), gw) \\ & + a_{13} d(gx, gu) + a_{14} d(gy, gv) + a_{15} d(gz, gw), \end{aligned}$$

holds, where a_i, $i = 1, \cdots, 15$ are nonnegative real numbers such that $\sum_{i=1}^{15} a_i < 1$. Then F and g have a tripled coincidence point.

(6)Let (X, d) be a K-metric space with a cone P having non-empty interior (normal or non-normal) and $F : X^3 \longrightarrow X$ and $g : X \to X$ be mappings satisfy CLR_g property. Suppose that for any $x, y, z, u, v, w \in X$, following condition

$$\begin{aligned} d(F(x,y,z), F(u,v,w)) \preceq\; & a_1 d(F(x,y,z), gx) + a_2 d(F(y,z,x), gy) \\ & + a_3 d(F(z,x,y), gz) + a_4 d(F(u,v,w), gu) \\ & + a_5 d(F(v,w,u), gv) + a_6 d(F(w,u,v), gw) \\ & + a_7 d(F(u,v,w), gx) + a_8 d(F(v,w,u), gy) \\ & + a_9 d(F(w,u,v), gz) + a_{10} d(F(x,y,z), gu) \\ & + a_{11} d(F(y,z,x), gv) + a_{12} d(F(z,x,y), gw) \\ & + a_{13} d(gx, gu) + a_{14} d(gy, gv) + a_{15} d(gz, gw), \end{aligned}$$

holds, where a_i, $i = 1, \cdots, 15$ are nonnegative real numbers such that $\sum_{i=1}^{15} a_i < 1$. If F and g are W-compatible, then F and g have a unique common tripled fixed point. Moreover, common tripled fixed point of F and g is of the form (u, u, u) for some $u \in X$.

(7) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to H$ be a ρ-contraction with $\rho \in (0,1)$. Let $S, T : C \to C$ be two nonexpansive mappings with $F(T) \neq \emptyset$. Let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$. $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ and $0 < \gamma < \bar{\gamma}/\rho$. Starting with an arbitrary initial guess $x_0 \in C$ and $\{x_n\}$ is a sequence generated by

$$\begin{aligned} y_n &= \beta_n S x_n + (1-\beta_n)x_n, \\ x_{n+1} &= P_C[\alpha_n \gamma f(x_n) + (I - \alpha_n A)Ty_n], \quad \forall n \geq 1. \end{aligned}$$

Suppose that the following conditions are satisfied:

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

$(C2)$ $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = \tau = 0$;

$(C3)$ $\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$ and $\lim_{n\to\infty} \frac{|\beta_n - \beta_{n-1}|}{\beta_n} = 0$ or

$(C4)$ $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ and $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to a point $\tilde{x} \in H$, which is the unique solution of the variational inequality:

$$\tilde{x} \in F(T), \qquad \langle (A - \gamma f)\tilde{x}, x - \tilde{x} \rangle \geq 0, \quad \forall x \in F(T).$$

Equivalently, we have $P_{F(T)}(I - A + \gamma f)\tilde{x} = \tilde{x}$.

(8) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to H$ be a ρ-contraction (possibly nonself) with $\rho \in (0,1)$. Let $S, T : C \to C$ be two nonexpansive mappings with $F(T) \neq \emptyset$. Let A be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \bar{\gamma}/\rho$. $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$. Starting with an arbitrary initial guess $x_0 \in C$ nd $\{x_n\}$ is a sequence generated by

$$\begin{aligned} y_n &= \beta_n S x_n + (1-\beta_n)x_n, \\ x_{n+1} &= P_C[\alpha_n \gamma f(x_n) + (I - \alpha_n A)Ty_n], \quad \forall n \geq 1. \end{aligned}$$

Suppose that the following conditions are satisfied:

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

$(C2)$ $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = \tau \in (0, \infty)$;

$(C5)$ $\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|}{\alpha_n \beta_n} = 0$;

$(C6)$ there exists a constant $K > 0$ such that $\frac{1}{\alpha_n}|\frac{1}{\beta_n} - \frac{1}{\beta_{ni1}}| \leq K$.

Then the sequence $\{x_n\}$ converges strongly to a point $\tilde{x} \in H$, which is the unique solution of the variational inequality:

$$\tilde{x} \in F(T), \quad \langle \frac{1}{\tau}(A - \gamma f)\tilde{x} + (I - S)\tilde{x}, x - \tilde{x} \rangle \geq 0, \quad \forall x \in F(T).$$

(9) Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $f : C \to H$ be a ρ-contraction mapping, $S : C \to H$ be a nonexpansive mapping and $\{T_i\}_{i=1}^{\infty} : C \to C$ be a countable family of k_i-strict pseudo-contraction mappings and $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\alpha_0 = 1$, and $x_1 \in C$ and define the sequence $\{x_n\}$ by

$$\begin{cases} y_n = P_C[\beta_n S x_n + (1 - \beta_n)x_n], \\ x_{n+1} = P_C[\alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)V_i y_n], \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\alpha_n\}$ is a strictly decreasing sequence, $V_i = k_i I + (1 - k_i)T_i$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions (H2) with $\tau = 0$, (H3), either (H4) and (H5) , or (H6) and (H7). Then the sequence $\{x_n\}$ converges strongly to a point $z \in \mathcal{F}$, which is the unique solution of the variational inequality:

$$\langle (I - f)z, x - z \rangle \geq 0, \quad \forall x \in \mathcal{F}.$$

(10) Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $f : C \to H$ be a ρ-contraction mapping, $S : C \to C$ be a nonexpansive mapping and $\{T_i\}_{i=1}^{\infty} : C \to C$ be a countable family of k_i-strict pseudo-contraction mappings and $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\alpha_0 = 1$, and $x_1 \in C$ and define the sequence $\{x_n\}$ by

$$\begin{cases} y_n = P_C[\beta_n S x_n + (1 - \beta_n)x_n] = \beta_n S x_n + (1 - \beta_n)x_n \\ x_{n+1} = P_C[\alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)V_i y_n], \quad \forall n \geq 1 \end{cases}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\alpha_n\}$ is a strictly decreasing sequence, $V_i = k_i I + (1 - k_i)T_i$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions (H2) with

$\tau \in (0, \infty)$, (H3), (H8) and (H9). Then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \mathcal{F}$, which is the unique solution of the variational inequality:

$$\langle \frac{1}{\tau}(I - f)x^* + (I - S)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{F}.$$

(11) Let H be a real Hilbert space, C be a closed convex subset of H. Let $A : C \to H$ be a strongly positive linear bounded operator, $f : C \to H$ be a ρ-contraction, γ be a positive real number such that $\frac{\bar{\gamma}-1}{\rho} < \gamma < \frac{\bar{\gamma}}{\rho}$. Let $T : C \to C$ be a nonexpansive mapping, $B : C \to C$ be a β-strongly monotone and L-Lipschitz continuous. Let $\phi : C \to C$ be a k-contraction mapping with $k \in [0, 1)$. Suppose $\{x_n\}$ is a sequence generated by the following algorithm $x_0 \in C$ arbitrarily

$$\begin{cases} z_n = TP_C[I - \delta_n(A - \gamma f)]x_n, \\ y_n = (I - \mu\beta_n B)z_n, \\ x_{n+1} = \alpha_n \phi(x_n) + (1 - \alpha_n)y_n, \ \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}, \{\delta_n\} \subset [0, 1]$. If $\mu \in (0, \frac{2\beta}{L^2})$ is used and if $\{\beta_n\} \subset (0, 1]$ satisfy the following conditions:

(C1): $\Sigma_{n=1}^{\infty}|\delta_{n+1} - \delta_n| < \infty$, $\Sigma_{n=1}^{\infty}\delta_n = \infty$;

(C2): $\Sigma_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$;

(C3): $\Sigma_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, $\lim_{n\to\infty} \alpha_n = 0$;

(C4): $\delta_n \leq \beta_n$ and $\beta_n \leq \alpha_n$.

Then $\{x_n\}$ converges strongly to $x^* \in \Upsilon$, which is the unique solution of the variational inequality:

$$\text{Find } x^* \in \Upsilon \text{ such that } \langle (I - \phi)x^*, x - x^* \rangle \geq 0, \ \forall x \in \Upsilon,$$

where $\Upsilon := VI(\Omega, B) := VI\Big(VI(F(T), A - \gamma f), B\Big)$.

(12) Let C be a nonempty closed and convex subset of a real Hilbert space H such that $C \pm C \subset C$. Let $f : C \to C$ be a contraction with coefficient $k \in (0, 1)$. Let $G : C \to C$ be a ξ-strongly monotone and L-Lipschitz continuous mapping. let $A_m : C \to H$ be a relaxed (η_m, ρ_m)-cocoercive and ν_m-Lipschitz continuous mapping and $B_m : C \to H$ be a relaxed $(\hat{\eta}_m, \hat{\rho}_m)$-cocoercive and $\hat{\nu}_m$-Lipschitz continuous mapping for each $1 \leq m \leq r$. Let $p_m = \sqrt{1 - 2\lambda_m\rho_m + \lambda_m^2\nu_m^2 + 2\lambda_m\eta_m\nu_m^2}$ and $q_m = \sqrt{1 - 2\hat{\lambda}_m\hat{\rho}_m + \hat{\lambda}_m^2\hat{\nu}_m^2 + 2\hat{\lambda}_m\hat{\eta}_m\hat{\nu}_m^2}$, where $\{\lambda_m\}$ and $\{\hat{\lambda}_m\}$ are two positive

sequences for each $1 \leq m \leq r$. Assume that $\cap_{m=1}^{r} GVI(C, B_m, A_m) \neq \emptyset$, $\xi > 0, L > 0$, $0 < \mu < 2\xi/L^2$, $0 < \gamma < \mu(\xi - \mu L^2/2)/k = \pi/k$ and $p_m, q_m \in [0, \frac{1}{2})$, for each $1 \leq m \leq r$. Given the initial guess $x_1 \in C$ and $\{x_n\}$ is a sequence generated by

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G)\Sigma_{m=1}^{r} \beta_{(m,n)} P_C(\hat{\lambda}_m B_m x_n - \lambda_m A_m x_n), \ \ \forall n \geq 1,$$

where $\{\alpha_n\}, \{\beta_{(1,n)}\}, \{\beta_{(2,n)}\}, ..., \{\beta_{(r,n)}\}$ are sequences in $(0, 1)$, satisfying the following conditions:

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0, \Sigma_{n=1}^{\infty} \alpha_n = \infty$ and $\Sigma_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

$(C2)$ $\Sigma_{m=1}^{r} \beta_{(m,n)} = 1, \forall n \geq 1$,$\Sigma_{n=1}^{\infty} |\beta_{(m,n+1)} - \beta_{(m,n)}| < \infty$ and $\lim_{n\to\infty} \beta_{(m,n)} = \beta_m \in (0, 1), \forall 1 \leq m \leq r$.

Then the sequence $\{x_n\}$ converges strongly to $\tilde{x} \in \cap_{m=1}^{r} GVI(C, B_m, A_m)$, which is the unique solution of the HGVIP:

$$\langle (\gamma f - \mu G)\tilde{x}, x - \tilde{x} \rangle \leq 0, \ \ \forall x \in \cap_{m=1}^{r} GVI(C, B_m, A_m).$$

(13) Let $I = \{1, 2, ..., l\}$ be a finite index set. For each $i \in I$, let F_i be a bi-function from $C \times C$ into $\Re$ satisfying (A1)-(A4). Denote $T_{r_n}^i : H \to C$ by

$$T_{r_n}^i(x) \ = \ \left\{ z \in C : F_i(z, y) + \frac{1}{r_n}\langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}.$$

Let C be a nonempty closed and convex subset of a real Hilbert space H such that $C \pm C \subset C$. For each $i \in I$, let F_i be a bi-function from $C \times C$ into $\mathbb{R}$ satisfying

(A1) $F_i(x, x) = 0$ for all $x \in C$;

(A2) F_i is monotone, i.e., $F_i(x, y) + F_i(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} F_i(tz + (1 - t)x, y) \leq F_i(x, y)$;

(A4) for each $x \in C, y \mapsto F_i(x, y)$ is convex and lower semicontinuous.

Let $f : C \to C$ be a contraction with coefficient $k \in (0, 1)$. Let $G : C \to C$ be a ξ-strongly monotone and L-Lipschitz continuous mapping. let $A_m : C \to H$ be a relaxed (η_m, ρ_m)-cocoercive and ν_m-Lipschitz continuous mapping and $B_m : C \to H$ be a relaxed $(\hat{\eta}_m, \hat{\rho}_m)$-cocoercive and $\hat{\nu}_m$-Lipschitz continuous mapping for each $1 \leq m \leq r$. Let $p_m = \sqrt{1 - 2\lambda_m \rho_m + \lambda_m^2 \nu_m^2 + 2\lambda_m \eta_m \nu_m^2}$ and $q_m =$

$\sqrt{1-2\hat{\lambda}_m\hat{\rho}_m+\hat{\lambda}_m^2\hat{\nu}_m^2+2\hat{\lambda}_m\hat{\eta}_m\hat{\nu}_m^2}$, where $\{\lambda_m\}$ and $\{\hat{\lambda}_m\}$ are two positive sequences for each $1 \leq m \leq r$. Assume that $\Omega = (\cap_{i=1}^{l} EP(F_i)) \cap (\cap_{m=1}^{r} GVI(C, B_m, A_m))$ $\neq \emptyset$, $\xi > 0, L > 0$, $0 < \mu < 2\xi/L^2$, $0 < \gamma < \mu(\xi - \mu L^2/2)/k = \pi/k$ and $p_m, q_m \in [0, \frac{1}{2})$, for each $1 \leq m \leq r$. Given $\{x_n\}$ is a sequence generated by

$$\begin{cases} x_1 \in C, \\ u_n^i = T_{r_n}^i x_n, \qquad \forall i \in I, \\ v_n = \frac{u_n^1 + u_n^2 + \cdots + u_n^l}{l}, \\ y_n^m = P_C(\hat{\lambda}_m B_m v_n - \lambda_m A_m v_n), \qquad \forall m = 1, 2, \ldots, r, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G) \sum_{m=1}^{r} \beta_{(m,n)} y_n^m, \ \ \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_{(m,n)}\} \subset (0,1), \forall 1 \leq m \leq r$ and $\{r_n\} \subset (0, +\infty)$ satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(C2) $\sum_{m=1}^{r} \beta_{(m,n)} = 1, \forall n \geq 1$ $,\sum_{n=1}^{\infty} |\beta_{(m,n+1)} - \beta_{(m,n)}| < \infty$ and $\lim_{n\to\infty} \beta_{(m,n)} = \beta_m \in (0,1), \forall 1 \leq m \leq r$.

(C3) $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

Then the sequence $\{x_n\}$ converges strongly to a common element $c \in \Omega$, which is the unique solution of the HEGVIP:

$$\langle (\gamma f - \mu G)c, x - c \rangle \leq 0, \quad \forall x \in \Omega.$$

Assume that the following conditions hold:

(C1) $A_i : H \to H$ is an α_i-inverse-strongly monotone mapping and $VI(C, A_i)$ is the set of solutions to variational inequality problem with $A = A_i$, for all $i = 1, 2, 3$;

(C2) K_i and $K_{i,\beta}, \beta \in (0,1), i = 1, 2, 3$, are the mappings defined by

$$\begin{cases} K_i := P_{C_i}(I - \lambda A_i), \quad \lambda \in (0, 2\alpha_i], \\ K_{i,\beta} = (1-\beta)I + \beta K_i, \quad \beta \in (0,1), \end{cases}$$

respectively.

(14) Let A_i and $VI(C, A_i)$ satisfy the condition (C1) and let $f_i : H \to H$ be contractions with a contractive constant $h_i \in (0, 1)$, for all $i = 1, 2, 3$. Then there exists a unique element $(x^*, y^*, z^*) \in VI(C, A_1) \times VI(C, A_2) \times VI(C, A_3)$ such that the following three inequalities are satisfied

$$\begin{cases} \langle x^* - f_1(y^*), x - x^* \rangle \geq 0, & \forall x \in VI(C, A_1), \\ \langle y^* - f_2(z^*), y - y^* \rangle \geq 0, & \forall y \in VI(C, A_2), \\ \langle z^* - f_3(x^*), z - z^* \rangle \geq 0, & \forall z \in VI(C, A_3). \end{cases}$$

(15) Let $A_i, VI(C, A_i), K_i$ and $K_{i,\beta}$ satisfy the conditions (C1) and (C2), and let $f_i : H \to H$ be contractions with a contractive constant $h_i \in (0, 1)$, for all $i = 1, 2, 3$. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be three sequences defined by

$$\begin{cases} x_0, y_0, z_0 \in H, \\ x_{n+1} = (1 - \alpha_n)K_{1,\beta}x_n + \alpha_n f_1(K_{2,\beta}y_n), \\ y_{n+1} = (1 - \alpha_n)K_{2,\beta}y_n + \alpha_n f_2(K_{3,\beta}z_n), \\ z_{n+1} = (1 - \alpha_n)K_{3,\beta}z_n + \alpha_n f_3(K_{1,\beta}x_n), \qquad n = 0, 1, 2, \dots, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge to x^*, y^* and z^* respectively, where (x^*, y^*, z^*) is the unique element in $VI(C, A_1) \times VI(C, A_2) \times VI(C, A_3)$ such that the following three inequalities are satisfied

$$\begin{cases} \langle x^* - f_1(y^*), x - x^* \rangle \geq 0, & \forall x \in VI(C, A_1), \\ \langle y^* - f_2(z^*), y - y^* \rangle \geq 0, & \forall y \in VI(C, A_2), \\ \langle z^* - f_3(x^*), z - z^* \rangle \geq 0, & \forall z \in VI(C, A_3). \end{cases}$$

(16) Let $A_i, VI(C, A_i), K_i$ and $K_{i,\beta}$ satisfy the conditions (C1) and (C2) for each $i = 1, 2, 3$, and let $F : H \to H$ be a μ-Lipschitzian and r-strongly monotone mapping. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be three sequences defined by

$$\begin{cases} x_0, y_0, z_0 \in H, \\ x_{n+1} = (1 - \alpha_n)K_{1,\beta}x_n + \alpha_n f_1(K_{2,\beta}y_n), \\ y_{n+1} = (1 - \alpha_n)K_{2,\beta}y_n + \alpha_n f_2(K_{3,\beta}z_n), \\ z_{n+1} = (1 - \alpha_n)K_{3,\beta}z_n + \alpha_n f_3(K_{1,\beta}x_n), \qquad n = 0, 1, 2, \dots, \end{cases}$$

where $f_1 := I - \rho F, f_2 := I - \eta F, f_3 := I - \xi F$ with $\rho, \eta, \xi \in (0, \frac{2r}{\mu^2})$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequences

$\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge to x^*, y^* and z^* respectively, where (x^*, y^*, z^*) is the unique element in $VI(C, A_1) \times VI(C, A_2) \times VI(C, A_3)$ such that the following three inequalities are satisfied

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in VI(C, A_1), \\ \langle \eta F(z^*) + y^* - z^*, y - y^* \rangle \geq 0, & \forall y \in VI(C, A_2), \\ \langle \xi F(x^*) + z^* - x^*, z - z^* \rangle \geq 0 & \forall z \in VI(C, A_3). \end{cases}$$

Assume that the following conditions hold:

(C1) $M_i : H \to 2^H$ is a multi-valued maximal monotone mapping, $A_i : H \to H$ is an α_i-inverse-strongly monotone mapping and Ω_i is the set of solutions to variational inclusion problem with $A = A_i, M = M_i$ and $\Omega_i \neq \emptyset$, for all $i = 1, 2, 3$;

(C2) K_i and $K_{i,\beta}, \beta \in (0, 1), i = 1, 2, 3$, are the mappings defined by

$$\begin{cases} K_i := J_{M_i,\lambda}(I - \lambda A_i), & \lambda \in (0, 2\alpha_i], \\ K_{i,\beta} = (1 - \beta)I + \beta K_i, & \beta \in (0, 1), \end{cases}$$

respectively.

(17) Let A_i, M_i, Ω_i, K_i and $K_{i,\beta}$ satisfy the conditions (C1) and (C2), and let $f_i : H \to H$ be contractions with a contractive constant $h_i \in (0, 1)$, for all $i = 1, 2, 3$. Then there exists a unique element $(x^*, y^*, z^*) \in \Omega_1 \times \Omega_2 \times \Omega_3$ such that the following three inequalities are satisfied

$$\begin{cases} \langle x^* - f_1(y^*), x - x^* \rangle \geq 0, & \forall x \in \Omega_1, \\ \langle y^* - f_2(z^*), y - y^* \rangle \geq 0, & \forall y \in \Omega_2, \\ \langle z^* - f_3(x^*), z - z^* \rangle \geq 0, & \forall z \in \Omega_3. \end{cases}$$

(18) Let A_i, M_i, Ω_i, K_i and $K_{i,\beta}$ satisfy the conditions (C1) and (C2), and let $f_i : H \to H$ be contractions with a contractive constant $h_i \in (0, 1)$, for all $i = 1, 2, 3$. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be three sequences defined by

$$\begin{cases} x_0, y_0, z_0 \in H, \\ x_{n+1} = (1 - \alpha_n)K_{1,\beta}x_n + \alpha_n f_1(K_{2,\beta}y_n), \\ y_{n+1} = (1 - \alpha_n)K_{2,\beta}y_n + \alpha_n f_2(K_{3,\beta}z_n), \\ z_{n+1} = (1 - \alpha_n)K_{3,\beta}z_n + \alpha_n f_3(K_{1,\beta}x_n), & n = 0, 1, 2, \dots, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ generated be (4.2.3) converge to x^*, y^* and z^* respectively, where (x^*, y^*, z^*) is the unique element in $\Omega_1 \times \Omega_2 \times \Omega_3$ such that the following three inequalities are satisfied

$$\begin{cases} \langle x^* - f_1(y^*), x - x^* \rangle \geq 0, & \forall x \in \Omega_1, \\ \langle y^* - f_2(z^*), y - y^* \rangle \geq 0, & \forall y \in \Omega_2, \\ \langle z^* - f_3(x^*), z - z^* \rangle \geq 0, & \forall z \in \Omega_3. \end{cases}$$

(19) Let A_i, M_i, Ω_i, K_i and $K_{i,\beta}$ satisfy the conditions (C1) and (C2), and let $F : H \to H$ be a μ-Lipschitzian and r-strongly monotone mapping. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be three sequences defined by

$$\begin{cases} x_0, y_0, z_0 \in H, \\ x_{n+1} = (1 - \alpha_n)K_{1,\beta}x_n + \alpha_n f_1(K_{2,\beta}y_n), \\ y_{n+1} = (1 - \alpha_n)K_{2,\beta}y_n + \alpha_n f_2(K_{3,\beta}z_n), \\ z_{n+1} = (1 - \alpha_n)K_{3,\beta}z_n + \alpha_n f_3(K_{1,\beta}x_n), \qquad n = 0, 1, 2, \ldots, \end{cases}$$

where $f_1 := I - \rho F, f_2 := I - \eta F, f_3 := I - \xi F$ with $\rho, \eta, \xi \in (0, \frac{2r}{\mu^2})$ and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge to x^*, y^* and z^* respectively, where (x^*, y^*, z^*) is the unique element in $\Omega_1 \times \Omega_2 \times \Omega_3$ such that the following three inequalities are satisfied

$$\begin{cases} \langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in \Omega_1, \\ \langle \eta F(z^*) + y^* - z^*, y - y^* \rangle \geq 0, & \forall y \in \Omega_2, \\ \langle \xi F(x^*) + z^* - x^*, z - z^* \rangle \geq 0, & \forall z \in \Omega_3. \end{cases}$$

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