ON BEST PROXIMITY POINT THEOREMS IN METRIC SPACES

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
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Abstract

The purposes of this thesis are to introduce and investigate new mappings which are generalized contraction mappings for best proximity points in metric spaces. The proofs of the existence of best proximity point theorems which consist of best proximity points for a generalized proximal $\alpha-\psi-$contraction mapping, a generalized almost contraction mapping, a Kannan $\alpha-$admisisible weak $\phi-$contraction mapping and proximally coupled weak contraction mapping are also provided.

Keywords : Best Proximity Point / Coupled Best Proximity Point / $\alpha-$ Admissible

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บทคัดย่อ

วัถุประสงค์ของวิทยานิพนธ์นี้คือ เพื่อแนะนำและศึกษาการส่งชนิดใหม่ ซึ่งเป็นการส่งหดตัววางนัยทั่วไปสำหรับจุดใกล้เคียงที่ดีที่สุดในปริภูมิอิงระยะทาง ได้ทำการพิสูจน์ทฤษฎีบทการมีอยู่จริงของจุดใกล้เคียงที่ดีที่สุด ซึ่งประกอบไปด้วยปัญหาทฤษฎีบทจุดใกล้เคียงร่วมที่ดีที่สุด สำหรับการส่งหดตัวใกล้เคียงชนิด $\alpha-\psi$ นัยทั่วไป การส่งเกือบหดตัวนัยทั่วไป การส่งหดตัวบนคานนานแอดมิสซิเบิลแบบอ่อน ϕ และการส่งหดตัวใกล้เคียงคู่แบบอ่อน

คำสำคัญ : จุดใกล้เคียงคู่ที่ดีที่สุด / จุดใกล้เคียงที่ดีที่สุด / $\alpha-$ แอดมิสซิเบิล

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CHAPTER 1 INTRODUCTION

The best proximity point evolves as a generalization of the idea of the best approximation. The best approximation results provide an approximate solution to the fixed point equation $Tx = x$, when the non-self-mapping T has no fixed point. In respective, a well-known best approximation theorem, due to Fan [1], insists the fact that if K is a nonempty compact convex subset of a Hausdorff locally convex topological vector space E and $T : K \to E$ is a continuous mapping, then there exists an element x satisfying the condition $d(x, Tx) = \inf\{d(y, Ty) : y \in K\}$, where d is a metric on E. The best approximation theorem assures the existence of an approximate solution; the best proximity point theorem is considered for solving the problem to find an approximate solution which is optimal. Given nonempty closed subsets A and B of E, when a non-self-mapping $T : A \to B$ has not a fixed point, it is quite natural to find an element x^* such that $d(x^*, Tx^*)$ is minimum. The best proximity point theorems assure the existence of an element x^* such that $d(x^*, Tx^*) = d(A, B) := \inf\{d(x, y) : x \in A, y \in B\}$; this element is called the best proximity point of T. Moreover, if the mapping under discussion is a self-mapping, the best proximity point theorem becomes to a fixed point result. Some of interesting results regarding best proximity points can be found in [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]

In Chapter 1,we review the background of this thesis for best proximity point theorems.

In Chapter 2, we give the necessary notations, definitions, some useful lemmas which will be used in the later chapter.

In Chapter 3, we prove the existence of best proximity point theorems which is a generalized contraction for non-self mapping and also give some examples.

Finally, in Chapter 4, we give the summary of all the results and the conclusion of this dissertation.

CHAPTER 2 PRELIMINARIES

In this chapter, we give some basic concepts including with definitions, notations and some useful lemmas which are all necessary to the later chapters. Throughout this dissertation, let $\mathbb{R}$ and $\mathbb{N}$ stand for the set of all real numbers and the set of all natural numbers, respectively.

2.1 Some Definitions

Definition 2.1.1. Let X be a nonempty set. A *metric* on X is a real function $d : X \times X \to \mathbb{R}$ satisfying the following conditions:

(1) $d(x, y) \geq 0$ for all $x, y \in X$;

(2) $d(x, y) = 0 \Longleftrightarrow x = y$ for all $x, y \in X$;

(3) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(4) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

A set X with a metric d is called a *metric space*. The elements of X are called the *points* of the metric space (X, d).

Definition 2.1.2. A sequence $\{x_n\}$ is a metric space (X, d) is said to *converge* to $x \in X$ if, for every $\epsilon > 0$, there exist $N \in \mathbb{N}$ such that

$$d(x_n, x) < \epsilon,$$

for $n \geq N$. In such case, we write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$ and x is called the *limit* of a sequence $\{x_n\}$. If $\lim_{n\to\infty} x_n = x$ for some $x \in X$, the sequence $\{x_n\}$ is called *convergent*; otherwise it is called *divergent*.

Definition 2.1.3. A sequence $\{x_n\}$ in a metric space (X, d) is called *Cauchy sequence* if for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

Definition 2.1.4. A metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges.

Definition 2.1.5. A subset M of metric space (X,d) is *closed* if every sequence $\{x_n\}$ in M such that $\lim_{n\to\infty} x_n = x$ implies $x \in M$.

Definition 2.1.6. Let (X,d) be a metric space, $a \in X$ and $B \subseteq X$. The distance from a point a to $B \subseteq X$ is given by

$$d(a,B) = \inf\{d(a,b) : b \in B\}.$$

2.2 Best Proximity Point

Let A and B be nonempty subsets of a metric space (X,d), we recall the following notations and notions that will be used in what follows:

$$d(A,B) := \inf\{d(x,y) : x \in A, y \in B\},$$
$$A_0 := \{x \in A : d(x,y) = d(A,B) \text{ for some } y \in B\},$$
$$B_0 := \{y \in B : d(x,y) = d(A,B) \text{ for some } x \in A\}.$$

If $A \cap B \neq \emptyset$, then A_0 and B_0 are non-empty. Further, it is interesting to notice that A_0 and B_0 are contained in the boundaries of A and B respectively, provided A and B are closed subsets of a normed linear space such that $d(A,B) > 0$ [17].

Definition 2.2.1. A point x in A is said to be a *best proximity point* of the mapping $S : A \to B$ if it satisfies the condition that

$$d(x,Sx) = d(A,B).$$

It can be observed that a best proximity reduces to a fixed point if the underlying mapping is a self-mapping.

Definition 2.2.2. [18] A mapping $T : A \to B$ is called a *proximal contraction of the first kind* if there exists $k \in [0,1)$ such that

$$\left.\begin{array}{l} d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{array}\right\} \implies d(u,v) = kd(x,y),$$

for all $x, y, u, v \in A$.

It is easy to see that a self-mapping that is a proximal contraction of the first kind is precisely a contraction. However, a non-self-proximal contraction is not necessarily a contraction.

Definition 2.2.3. [18] A mapping $T : A \to B$ is called a *proximal contraction of the second kind* if there exists $k \in [0, 1)$ such that

$$\left.\begin{array}{l} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array}\right\} \implies d(Tu, Tv) = kd(Tx, Ty),$$

for all $x, y, u, v \in A$.

Definition 2.2.4. [19] A mapping $S : A \to B$ is called a *generalized proximal $\psi-$contraction of the first kind* if for all $x, y, u, v \in A$ satisfies

$$\left.\begin{array}{l} d(u, Sx) = d(A, B) \\ d(v, Sy) = d(A, B) \end{array}\right\} \implies d(u, v) = \psi d(x, y),$$

where $\psi : [0, \infty) \to [0, \infty)$ is an upper semicontinuous function from the right such that $\psi(t) < t$ for all $t > 0$.

Definition 2.2.5. [19] A mapping $S : A \to B$ is called a *generalized proximal $\psi-$contraction of the second kind* if for all $x, y, u, v \in A$ satisfies

$$\left.\begin{array}{l} d(u, Sx) = d(A, B) \\ d(v, Sy) = d(A, B) \end{array}\right\} \implies d(Su, Sv) = \psi d(Sx, Sy),$$

where $\psi : [0, \infty) \to [0, \infty)$ is an upper semicontinuous function from the right such that $\psi(t) < t$ for all $t > 0$.

It is easy to see that if we take $\psi(t) = \alpha(t)$, where $\alpha \in [0, 1)$, then a generalized proximal $\psi-$contraction of the first kind and generalized proximal $\psi-$contraction of the second kind reduce to a proximal contraction of the first kind and a proximal contraction of the second kind, respectively. Moreover, it is easy to see that a self-mapping generalized proximal $\psi-$contraction of the first kind and the second kind reduces to the condition of Boy and Wong' s fixed point theorem [20].

Definition 2.2.6. Let $S : A \to B$ and $T : B \to A$ be mappings. The pair (S, T) is called a *proximal cyclic contraction pair* if there exists $k \in [0, 1)$ such that

$$\left.\begin{array}{l} d(a, Sx) = d(A, B) \\ d(b, Ty) = d(A, B) \end{array}\right\} \implies d(a, b) \leq kd(x, y) + (1 - k)d(A, B),$$

for all $a, x \in A$ and $b, y \in B$.

Definition 2.2.7. Leting $S : A \to B$ and $g : A \to A$ be an isometry. The mapping S is said to preserve *isometric distance* with respect to g if

$$d(Sgx, Sgy) = d(Sx, Sy),$$

for all x, y in A.

Definition 2.2.8. A is said to be *approximatively compact with respect* to B if every sequence x_n in A satisfies the condition that $d(y, x_n) \to d(y, A)$ for some y in B has a convergent subsequence.

Definition 2.2.9. [21] Let (A, B) be a pair of nonempty subsets of X with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the *P-property* if and only if

$$\left.\begin{aligned} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{aligned}\right\} \quad \Longrightarrow \quad d(x_1, x_2) = (y_1, y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

It is easy to see that, for any nonempty subset A of X, the pair (A, A) has the P-property.

Definition 2.2.10. [22] Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the *weak P-property* if and only if

$$\left.\begin{aligned} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{aligned}\right\} \quad \Longrightarrow \quad d(x_1, x_2) \leq d(y_1, y_2),$$

for all $x_1, x_2 \in A$ and $y_1, y_2 \in B$

Definition 2.2.11. [23] A self mapping $T : X \to X$ is said to be α-*admissible*, where $\alpha : X \times X \to [0, \infty)$, if

$$x, y \in X, \;\; \alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1.$$

Definition 2.2.12. [24] Let $T : A \to B$ and $\alpha : A \times A \to [0, \infty)$. We say that T is $\alpha-$ *proximal admissible*, if

$$\left.\begin{aligned} &\alpha(x_1, x_2) \geq 1 \\ &d(u_1, Tx_1) = d(A, B) \\ &d(u_2, Tx_2) = d(A, B) \end{aligned}\right\} \quad \Longrightarrow \quad \alpha(u_1, u_2) \geq 1,$$

for all $x_1, x_2, u_1, u_2 \in A$.

Clearly, for self-mapping, T is α-proximal admissible implies T is α-admissible.

Definition 2.2.13. We say the function $\varphi : [0,\infty) \to [0,\infty)$ is a *(c)-comparison function* if and only if the following conditions hold:
(Φ_1) φ is a nondecreasing function,
(Φ_2) for any $t > 0$, $\sum_{n=1}^{\infty} \varphi^n(t)$ is a convergent series.
We denote the set of (c)-comparison function by Ψ.

It is easily proved that if φ is a (c)-comparison function, then $\varphi(t) < t$ for all $t > 0$.

Definition 2.2.14. [25] Let $\theta : [0,\infty)^4 \to [0,\infty)$ satisfies the following conditions:
(1) θ is continuous,
(2) $\theta(a,b,c,d) = 0$ if and only if the product $abcd = 0$.
We denote the class of function θ by Θ.

Definition 2.2.15. A set $\mathbb{P}$ is *partially ordered* by a relation $\preceq$ on $\mathbb{P}$ (we call $\preceq$ a *partial order*) provided the following are true:
(1) ($\preceq$ is *reflexive*) for each $x \in \mathbb{P}$, $x \preceq x$,
(2) ($\preceq$ is *transitive*) if $x \preceq y$ and $y \preceq z$, then $x \preceq z$,
(3) ($\preceq$ is *antisymmetric*) if $x \preceq y$ and $y \preceq x$, then $x = y$.

Definition 2.2.16. [26] Let $(X,\preceq)$ be a partially ordered set. The mapping $F : X \times X \to X$ is said to have the *mixed monotone property* if $F(x,y)$ is monotone nondecreasing in x and monotone nonincreasing in y; that is, for any $x, y \in X$,

$$x_1, x_2 \in X,\ x_1 \preceq x_2,\ \text{implies},\ F(x_1,y) \preceq F(x_2,y),$$

$$y_1, y_2 \in X,\ y_1 \preceq y_2,\ \text{implies},\ F(x,y_1) \succeq F(x,y_2).$$

Definition 2.2.17. Let $(X,d,\preceq)$ be a ordered metric space and A, B are nonempty subset of X. A mapping $F : A \times A \to B$ is said to be *proximal mixed monotone property* if $F(x,y)$ is proximally nondecreasing in x and is proximally non-increasing in y, that is, for all $x, y \in A$.

$$\left.\begin{aligned} x_1 &\preceq x_2 \\ d(u_1, F(x_1,y)) &= d(A,B) \\ d(u_2, F(x_2,y)) &= d(A,B) \end{aligned}\right\} \implies u_1 \preceq u_2$$

and

$$\left.\begin{array}{r} y_1 \preceq y_2 \\ d(u_3, F(x, y_1)) = d(A, B) \\ d(u_4, F(x, y_2)) = d(A, B) \end{array}\right\} \implies u_4 \preceq u_3,$$

where $x_1, x_2, y_1, y_2, u_1, u_2, u_3, u_4 \in A$.

One can see that, if $A = B$ in the above definition, the notion of proximal mixed monotone property reduces to that of mixed monotone property.

Lemma 2.2.18. *[26] Let $(X, d, \leq)$ be an ordered metric space and A, B are nonempty subset of X. Assume A_0 is nonempty. A mapping $F : A \times A \to B$ has proximal mixed monotone property with $F(A_0 \times A_0) \subseteq B_0$ then for any x_0, x_1, x_2, y_0 and y_1 are elements in A_0*

$$\left.\begin{array}{r} x_0 \leq x_1 \text{ and } y_0 \geq y_1 \\ d(x_1, F(x_0, y_0)) = d(A, B) \\ d(x_2, F(x_1, y_1)) = d(A, B) \end{array}\right\} \implies x_1 \leq x_2.$$

Lemma 2.2.19. *[26] Let $(X, d, \leq)$ be an ordered metric space and A, B are nonempty subset of X. Assume A_0 is nonempty. A mapping $F : A \times A \to B$ has proximal mixed monotone property with $F(A_0 \times A_0) \subseteq B_0$ then for any x_0, x_1, y_0, y_1 and y_2 are elements in A_0*

$$\left.\begin{array}{r} x_0 \leq x_1 \text{ and } y_0 \geq y_1 \\ d(y_1, F(y_0, x_0)) = d(A, B) \\ d(y_2, F(y_1, x_1)) = d(A, B) \end{array}\right\} \implies y_1 \geq y_2.$$

Definition 2.2.20. Let $(X, d, \leq)$ be an ordered metric space and A, B are nonempty subset of X. A mapping $F : A \times A \to B$ is said to be proximally coupled weak contraction if it satisfies the following condition:

$$\left.\begin{array}{r} x_1 \leq x_2 \text{and } y_1 \geq y_2 \\ d(u_1, F(x_1, y_1)) = d(A, B) \\ d(u_2, F(x_2, y_2)) = d(A, B) \end{array}\right\} \implies \begin{array}{r} \psi(d(u_1, u_2)) \leq \psi(\max(d(x_1, x_2), d(y_1, y_2))) \\ -\phi(\max(d(x_1, x_2), d(y_1, y_2))) \end{array}$$

for all $x_1, x_2, y_1, y_2, u_1, u_2 \in A$, where ψ is altering distance function, ϕ is nondecreasing function also $\phi(t) = 0$ iff $t = 0$.

One can see that, if $A = B$ in the above definition, the notion of proximally coupled weak contraction reduces to that coupled weak contraction

CHAPTER 3 BEST PROXIMITY POINT THEOREMS

The aim of this chapter is to introduce new mappings which is generalize contraction in non-self mapping and and prove best proximity point in metric spaces for these class.

3.1 Generalized Proximal α-ψ-Contraction Mappings and Best Proximity Points

In this section, we introduce the new class of generalized proximal $\alpha-\psi-$contraction mappings and prove best proximity theorems for this class and also give some examples to illustrate our main Theorem.

Definition 3.1.1. A mapping $S : A \to B$ is said to be a *generalized proximal $\alpha - \psi-$contraction of the first kind*, if satisfies

$$d(u, Sx) = d(v, Sy) = d(A, B) \Rightarrow \alpha(x, y)d(u, v) \leq \psi(d(x, y)),$$

for all u, v, x, y in A, where $\psi : [0, \infty) \to [0, \infty)$ is an upper semicontinuous function from the right such that $\psi(t) < t$ for all $t > 0$ and $\alpha : A \times A \longrightarrow [0, +\infty)$.

Example. Consider the complete metric space $\mathbb{R}^2$ with metric defined by

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|,$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Let

$$A = \{(x, 0) : 0 \leq x \leq 1\},$$

$$B = \{(x, 1) : 0 \leq x \leq 1\}.$$

Then $d(A, B) = 1$. Define the mappings $S : A \to B$ as follows:

$$S((x, 0)) = \left(\frac{x}{2} - \frac{x^2}{4}, 1\right).$$

First, we show that S is generalized proximal $\alpha - \psi-$contraction of the first kind.

Consider a function $\psi : [0, \infty) \to [0, \infty)$ defined by

$$\psi(t) = \begin{cases} t - \frac{t^2}{2}; 0 \leq t \leq 1 \\ t - 1; t \geq 1. \end{cases}$$

We define the mapping $\alpha : A \times A \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 2, & \forall x, y \in A \\ 0, & \text{otherwise.} \end{cases}$$

Let $(x_1, 0), (x_2, 0), (a_1, 0)$ and $(a_2, 0)$ be elements in A satisfying

$$d((x_1, 0), S(a_1, 0)) = d(A, B) = 1 \quad \text{and} \quad d((x_2, 0), S(a_2, 0)) = d(A, B) = 1.$$

It follows that

$$x_i = \frac{a_i}{2} - \frac{a_i^2}{4} \quad \text{for} \quad i = 1, 2.$$

Since α is commutative, we may assume that $a_1 - a_2 > 0$, so we have

$$\begin{aligned} \alpha((a_1, 0), (a_2, 0))d((x_1, 0), (x_2, 0)) &= 2d((x_1, 0), (x_2, 0)) \\ &= 2d\bigg(\left(\frac{a_1}{2} - \frac{a_1^2}{4}, 0 \right), \left(\frac{a_2}{2} - \frac{a_2^2}{4}, 0 \right) \bigg) \\ &= 2\bigg| \left(\frac{a_1}{2} - \frac{a_1^2}{4} \right) - \left(\frac{a_2}{2} - \frac{a_2^2}{4} \right) \bigg| \\ &= 2\left\{ \left(\frac{a_1}{2} - \frac{a_2}{2} \right) - \left(\frac{a_1^2}{4} - \frac{a_2^2}{4} \right) \right\} \\ &= 2\left\{ \frac{1}{2}(a_1 - a_2) - \frac{1}{4}\left(a_1^2 - a_2^2\right) \right\} \\ &\leq (a_1 - a_2) - \frac{1}{2}(a_1 - a_2)^2 \\ &= \psi\left(d\left((a_1, 0), (a_2, 0)\right)\right). \end{aligned}$$

Thus, S is a generalized proximal $\alpha-\psi-$contraction of the first kind. Next, we show that S is not a $\psi-$proximal contraction of the first kind. Suppose S is $\psi-$proximal contraction of the first kind then for each $(x, 0), (y, 0), (a, 0), (b, 0) \in A$ satisfying

$$d((x, 0), S(a, 0)) = d(A, B) = 1 \quad \text{and} \quad d((y, 0), S(b, 0)) = d(A, B) = 1.$$

The function $\psi : [0, \infty) \to [0, \infty)$ defined by

$$\psi(t) = \begin{cases} t - \frac{t^2}{2}; 0 \leq t \leq 1, \\ t - 1; t > 1. \end{cases}$$

It follows that

$$x = \frac{a}{2} - \frac{a^2}{4}, \quad y = \frac{b}{2} - \frac{b^2}{4}.$$

So, we have

$$d((x,0),(y,0)) \neq \psi\left(d\left((a,0),(b,0)\right)\right).$$

Therefore, S is not a $\psi-$proximal contraction of the first kind.

Definition 3.1.2. A mapping $S : A \to B$ is said to be a *generalized proximal $\alpha - \psi-$contraction of the second kind*, if satisfies

$$d(u,Sx) = d(v,Sy) = d(A,B) \Rightarrow \alpha(x,y)d(Su,Sv) \leq \psi(d(Sx,Sy)),$$

for all u, v, x, y in A, where $\psi : [0,\infty) \to [0,\infty)$ is an upper semicontinuous from the right such that $\psi(t) < t$ for all $t > 0$ and $\alpha : A \times A \longrightarrow [0,+\infty)$.

Definition 3.1.3. Let (X,d) be a metric space. A subset A of X satisfies property $\star$, if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x \in A$ for all $n \in \mathbb{N}$, then $\alpha(x,x_n) \geq 1$ for all $n \in \mathbb{N}$.

Definition 3.1.4. Let A and B be two subsets of metric space (X,d). Let $T : A \to B$ and $g : A \cup B \to A \cup B$. A mapping T satisfies condition $\square_T$, if $\forall x, y \in A$ such that $d(gx,Tx) = d(A,B)$ and $d(gy,Ty) = d(A,B)$, we have $\alpha(x,y) \geq 1$.

Theorem 3.1.5. *Let (X, d) be a complete metric space and let A and B be nonempty, closed subsets of X such that A_0 and B_0 are nonempty and A and B satisfies property $\star$. Let $S : A \to B$, $T : B \to A$ and $g : A \cup B \to A \cup B$ satisfy the following conditions:*

(a) S and T are generalized proximal $\alpha - \psi-$contraction of the first kind with $\alpha-$proximal admissible;

(b) g is an isometry;

(c) $S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$;

(d) $A_0 \subseteq g(A_0)$ and $B_0 \subseteq g(B_0)$;

(e) There exist elements x_0 and x_1 in A_0 such that $d(gx_1, Sx_0) = d(A,B)$ and $\alpha(x_0,x_1) \geq 1$.

(f) S and T satisfies property $\square_T$.

Then, there exists a unique point x in A and there exists a unique point $y \in B$ such *that*

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B).$$

Moreover, for any fixed x_0 *in* A_0*, the sequence* $\{x_n\}$*, defined by*

$$d(gx_{n+1}, Sx_n) = d(A, B),$$

converges to the element x*. For any fixed* y_0 *in* B_0*, the sequence* $\{y_n\}$*, defined by*

$$d(gy_{n+1}, Ty_n) = d(A, B),$$

converges to the element y*. On the other hand, a sequence* $\{u_n\}$ *in* A *converges to* x *with* $\alpha(x_n, u_n) \geq 1$*, if there is a sequence of positive numbers* $\{\epsilon_n\}$ *such that*

$$\lim_{n\to\infty} \epsilon_n = 0 \text{ and } d(u_{n+1}, z_{n+1}) \leq \epsilon_n,$$

where z_{n+1} *in* A *satisfies the condition that* $d(gz_{n+1}, Su_n) = d(A, B)$*.*

Proof. From condition (g), there exist elements x_0 and x_1 in A_0 such that

$$d(gx_1, Sx_0) = d(A, B), \text{ and } \alpha(x_0, x_1) \geq 1. \tag{3.1.1}$$

Again, since $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists an element x_2 in A_0 such that

$$d(gx_2, Sx_1) = d(A, B). \tag{3.1.2}$$

By (3.1.1), (3.1.2) and the α-proximal admissible, we get

$$\alpha(x_1, x_2) \geq 1.$$

Since $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists an element x_3 in A_0 such that

$$d(gx_3, Sx_2) = d(A, B). \tag{3.1.3}$$

Again, By (3.1.2), (3.1.3) and the α-proximal admissible, we get

$$\alpha(x_2, x_3) \geq 1.$$

By similar fashion, we can find x_n in A_0. Having chosen x_n, one can determine an element x_{n+1} in A_0 such that

$$d(gx_{n+1}, Sx_n) = d(A, B), \tag{3.1.4}$$

and

$$\alpha(x_n, x_{n+1}) \geq 1. \tag{3.1.5}$$

Because of the facts that $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, by a generalized proximal $\alpha-\psi-$contraction of the first kind of S, g is an isometry and property of ψ, for each n in $\mathbb{N}$, we have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \alpha(x_{n-1}, x_n) d(x_{n+1}, x_n) \\ &= \alpha(x_{n-1}, x_n) d(gx_{n+1}, gx_n) \\ &\leq \psi(d(x_{n-1}, x_n)) \\ &\leq d(x_{n-1}, x_n). \end{aligned}$$

Hence, that the sequence $\{d(x_{n+1}, x_n)\}$ is non-decreasing and bounded bolow. Hence, then exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = r.$$

We claim that $r = 0$.
If $r > 0$, then

$$\begin{aligned} r &= \lim_{n\to\infty} d(x_{n+1}, x_n) \\ &\leq \lim_{n\to\infty} \psi(d(x_n, x_{n-1})) \\ &= \psi(r) \\ &< r, \end{aligned}$$

which is a contradiction and hence $r = 0$. That is,

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.1.6}$$

Next we show that $\{x_n\}$ is a Cauchy sequence. Suppose the contrary, then there exists $\varepsilon > 0$ and subsequence $\{x_{m_k}\}, \{x_{n_k}\}$ of $\{x_n\}$ such that $n_k > m_k \geq k$ with

$$r_k := d(x_{m_k}, x_{n_k}) \geq \varepsilon \text{ and } d(x_{m_k}, x_{n_k-1}) < \varepsilon,$$

for $k \in \{1, 2, 3, \ldots\}$. Putting $\beta_n = d(x_{n+1}, x_n)$,

$$\begin{aligned} \varepsilon \leq r_k &\leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\ &< \varepsilon + \beta_{n_k-1}, \end{aligned}$$

it follows from, that

$$\lim_{k\to\infty} r_k = \varepsilon. \tag{3.1.7}$$

On the other hand, by constructing the sequence $\{x_n\}$, we have

$$d(gx_{m_k+1}, Sx_{m_k}) = d(A,B) \quad \text{and} \quad d(gx_{n_k+1}, Sx_{n_k}) = d(A,B).$$

By the transitive of α, we get $\alpha(x_{m_k}, x_{n_k}) \geq 1$. Since S is a generalized proximal $\alpha-\psi-$contraction of the first kind and g is an isometry, we have

$$\begin{aligned} d(x_{m_k+1}, x_{n_k+1}) &= d(gx_{m_k+1}, gx_{n_k+1}) \\ &\leq \alpha(x_{m_k}, x_{n_k}) d(gx_{m_k+1}, gx_{n_k+1}) \\ &= \alpha(x_{m_k}, x_{n_k}) d(x_{m_k}, x_{n_k}) \\ &\leq \psi(d(x_{m_k}, x_{n_k})), \end{aligned}$$

and we also have

$$\begin{aligned} \varepsilon \leq r_k &\leq d(x_{m_k}, x_{m_k+1}) + d(x_{n_k+1}, x_{n_k}) + d(x_{m_k+1}, x_{n_k+1}) \\ &= \beta_{m_k} + \beta_{n_k} + d(x_{m_k+1}, x_{n_k+1}) \\ &= \beta_{m_k} + \beta_{n_k} + d(gx_{m_k+1}, gx_{n_k+1}) \\ &\leq \beta_{m_k} + \beta_{n_k} + \alpha(x_{m_k}, x_{n_k}) d(gx_{m_k+1}, gx_{n_k+1}) \\ &\leq \beta_{m_k} + \beta_{n_k} + \psi(d(x_{m_k}, x_{n_k})). \end{aligned}$$

Taking $k \to \infty$ in above inequality, by (3.1.6), (3.1.7) and property of ψ, we get $\varepsilon \leq \psi(\varepsilon) < \varepsilon$, which is a contradiction, $\varepsilon = 0$. Thus $\{x_n\}$ is Cauchy sequence in A. Since A is subset of complete metric space X. Then the sequence $\{x_n\}$ is converges to some element x in A. Similarly, in view of the fact that $T(B_0) \subseteq A_0$ and $A_0 \subseteq g(A_0)$, we can conclude that there is a sequence $\{y_n\}$ such that $d(gy_{n+1}, Ty_n) = d(A,B)$ and converge to some element $y \in B$.

By g is an isometry, we have

$$d(x_{n+1}, y_{n+1}) = d(gx_{n+1}, gy_{n+1})$$

it follows that

$$d(x,y) = d(A,B),$$

so, we concluded that $x \in A_0$ and $y \in B_0$. Since $S(A_0) \subseteq B_0$ and $T(B_0) \subseteq A_0$, there is $u \in A$ and $v \in B$ such that

$$d(u, Sx) = d(A, B), \tag{3.1.8}$$

$$d(v, Ty) = d(A, B).$$

Since A satisfies property $\star$ and, we get $\alpha(x, x_n) \geq 1$ for all $n \in \mathbb{N}$.

From (3.1.4), (3.1.5) and (3.1.8), and the notion of generalized proximal $\alpha - \psi -$ contraction of first kind of S, we get

$$d(u, gx_{n+1}) \leq \alpha(x, x_n)d(u, gx_{n+1}) \leq \psi(d(x, x_n)).$$

Letting $n \to \infty$,we get $d(u, gx) \leq \psi(0) = 0$ and thus $u = gx$.

Therefore, we get

$$d(gx, Sx) = d(A, B). \tag{3.1.9}$$

Similarly, we can show that $v = gy$ and then

$$d(gy, Ty) = d(A, B). \tag{3.1.10}$$

From (3.1.9) and (3.1.10), we get

$$d(x, y) = d(gx, Sx) = d(gy, Ty) = d(A, B).$$

Next, to prove the uniqueness, let us suppose that there exist $x^* \in A$ and $y^* \in B$ with $x \neq x^*, y \neq y^*$ such that

$$d(gx^*, Sx^*) = d(A, B)$$

and

$$d(gy^*, Ty^*) = d(A, B).$$

Since g is an isometry, S and T are generalized proximal $\alpha - \psi -$contration of the first kind and from the (h); it follows that

$$d(x, x^\star) = d(gx, gx^\star) \leq \alpha(x, x^\star)d(gx, gx^\star) \leq \psi(d(x, x^\star)) < d(x, x^\star),$$

$$d(y, y^\star) = d(gy, gy^\star) \leq \alpha(y, y^\star)d(gy, gy^\star) \leq \psi(d(y, y^\star)) < d(y, y^\star).$$

which is a contradiction, so we have $x = x^*$ and $y = y^*$. On the other hand, let $\{u_n\}$ be a sequence in A and $\{\epsilon_n\}$ a sequence of positive real numbers such that

$$\lim_{n\to\infty} \epsilon_n = 0 \text{ and } d(u_{n+1}, z_{n+1}) \leq \epsilon_n,$$

where $z_{n+1} \in A$ satisfies the condition that $d(gz_{n+1}, Su_n) = d(A,B)$ and $\alpha(x_n, u_n) \geq 1$. Since S is a generalized proximal $\alpha - \psi-$contraction of first kind, we have

$$\begin{aligned} d(x_{n+1}, z_{n+1}) &\leq \alpha(x_n, u_n) d(x_n, z_n) \\ &\leq \psi(d(x_n, u_n)). \end{aligned}$$

Given $\epsilon > 0$, we choose a positive integer N such that $\epsilon_n \leq \epsilon$ for all $n \geq N$, we obtain that

$$\begin{aligned} d(u_{n+1}, x) &\leq d(u_{n+1}, x_{n+1}) + d(x_{n+1}, x) \\ &\leq d(u_{n+1}, z_{n+1}) + d(z_{n+1}, x_{n+1}) + d(x_{n+1}, x) \\ &\leq \psi(d(x_n, u_n)) + \epsilon_n + d(x_{n+1}, x). \end{aligned}$$

This claim that $d(u_n, x) \to 0$ as $n \to \infty$, suppose the contrary, by a inequality (3.1.10) and property of ψ, we get

$$\begin{aligned} \lim_{n\to\infty} d(u_{n+1}, x) &\leq \lim_{n\to\infty} (d(u_{n+1}, x_{n+1}) + d(x_{n+1}, x)) \\ &\leq \lim_{n\to\infty} (\psi(d(x_n, u_n)) + \epsilon_n + d(x_{n+1}, x)) \\ &< \lim_{n\to\infty} d(x_n, u_n) \\ &\leq \lim_{n\to\infty} (d(x_n, x) + d(x, u_n)) \\ &= \lim_{n\to\infty} d(x, u_n). \end{aligned}$$

Which is a contradiction, so we have $\{u_n\}$ is convergent and it converges to x. This completes the proof of the theorem. □

Example. Consider the complete metric space $\mathbb{R}^2$ with Euclidean metric. Let

$$A = \{(0, y) : y \in \mathbb{R}\}$$

and

$$B = \{(1, y) : y \in \mathbb{R}\}.$$

Define two mappings $S : A \to B$, $T : B \to A$ and $g : A \cup B \to A \cup B$ as follows:

$$S((0, y)) = \left(1, \frac{y}{8}\right),$$

$$T((1, y)) = \left(0, \frac{y}{8}\right),$$

$$g((x, y)) = (x, -y).$$

We define the mapping $\alpha : A \times A \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 2, & \forall x, y \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Then it is easy to see that $d(A, B) = 1$, $A_0 = A$, $B_0 = B$ and the mapping g is an isometry.

Next, we claim that S and T are generalized proximal $\alpha - \psi -$contractions of the first kind.

Consider a function $\psi : [0, \infty) \to [0, \infty)$ defined by $\psi(t) = t/2$ for all $t \geq 0$. If $(0, y_1), (0, y_2) \in A$ such that

$$\alpha((0, y_1), (0, y_2) \geq 1,$$
$$d(gu, S(0, y_1)) = d(A, B) = 1,$$
$$d(gv, S(0, y_2)) = d(A, B) = 1,$$

for all $u, v \in A$, then we have

$$gu = \left(0, \frac{y_1}{8}\right), \quad gv = \left(0, \frac{y_2}{8}\right).$$

We have,

$$\begin{aligned} \alpha((0, y_1), (0, y_2))d(gu, gv) &= 2d(gu, gv) \\ &= 2d\left(\left(0, \frac{y_1}{8}\right), \left(0, \frac{y_2}{8}\right)\right) \\ &= 2\left|\frac{y_1}{8} - \frac{y_2}{8}\right| \\ &= 2\left(\frac{1}{8}|y_1 - y_2|\right) \\ &= \frac{1}{4}|y_1 - y_2| \\ &\leq \frac{1}{4}d((0, y_1), (0, y_2)) \\ &= \psi d((0, y_1), (0, y_2)). \end{aligned}$$

Hence S is a generalized proximal $\alpha - \psi -$contraction of the first kind.

If $(1, y_1), (1, y_2) \in A$ such that

$$\alpha((1, y_1), (1, y_2) \geq 1,$$
$$d(ga, T(1, y_1)) = d(A, B) = 1,$$
$$d(gb, T(1, y_2)) = d(A, B) = 1.$$

for all $a, b \in A$, then, we get

$$ga = \left(1, \frac{y_1}{8}\right), \quad gb = \left(1, \frac{y_2}{8}\right).$$

In the same way, we can see that T is a generalized proximal $\alpha - \psi-$contraction of the first kind.

Further, it is easy to see that the unique element $(0, 0) \in A$ and $(1, 0) \in B$ such that

$$d(g(0, 0), S(0, 0)) = d(g(1, 0), T(1, 0)) = d((0, 0), (1, 0)) = d(A, B).$$

Theorem 3.1.6. *Let (X, d) be a complete metric space and A and B be nonempty, closed subsets of X. Further, suppose that A_0 and B_0 are nonempty and A and B satisfies property $\star$. Let $S : A \to B$ and $g : A \to A$ satisfy the following conditions:*

(a) S is a generalized proximal $\alpha - \psi-$contractions of first and second kinds with $\alpha-$proximal admissible;

(b) g is an isometry;

(c) S preserves isometric distance with respect to g;

(d) $S(A_0) \subseteq B_0$;

(e) $A_0 \subseteq g(A_0)$.

(f) There exist elements x_0 and x_1 in A_0 such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$;

(g) S and T satisfies property $\square_T$.

Then, there exists a unique point x in A such that

$$d(gx, Sx) = d(A, B).$$

Moreover, for any fixed x_0 in A_0, the sequence $\{x_n\}$, defined by

$$d(gx_{n+1}, Sx_n) = d(A, B),$$

converges to the element x.

On the other hand, a sequence $\{u_n\}$ in A converges to x with $\alpha(x_n, u_n) \geq 1$, if there is a sequence of positive numbers $\{\epsilon_n\}$ such that

$$\lim_{n\to\infty} \epsilon_n = 0 \text{ and } d(u_{n+1}, z_{n+1}) \leq \epsilon_n,$$

where z_{n+1} in A satisfies the condition that $d(gz_{n+1}, Su_n) = d(A, B)$.

Proof. Since $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, similarly in the proof of Theorem(3.1.5), we can construct the sequence $\{x_n\}$ of element in A_0 such that

$$d(gx_{n+1}, Sx_n) = d(A, B), \text{ and } \alpha(x_n, x_{n+1}) \geq 1, \tag{3.1.11}$$

for all non-negative number n. It follows from g is an isometry and the virtue of a generalized proximal $\alpha - \psi-$contraction of the first kind of S, we see that

$$d(x_n, x_{n+1}) = d(gx_n, gx_{n+1}) \leq \psi(d(x_n, x_{n-1})),$$

for all $n \in \mathbb{N}$. Similarly to the proof of Theorem(3.1.5), we can conclude that the sequence $\{x_n\}$ is a Cauchy sequence and converges to some x in A. Since S is a generalized proximal $\alpha - \psi-$contraction of the second kind and preserves isometric distance with respect to g that

$$\begin{aligned} d(Sx_n, Sx_{n+1}) &\leq \alpha(x_{n-1}, x_n) d(Sx_{n+1}, Sx_n) \\ &= \alpha(x_{n-1}, x_n) d(Sgx_{n+1}, Sgx_n) \\ &\leq \psi(d(Sx_{n-1}, Sx_n)) \\ &\leq d(Sx_{n-1}, Sx_n). \end{aligned}$$

Hence, that the sequence $\{d(Sx_{n+1}, Sx_n)\}$ is non-decreasing and bounded below.

Hence, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(Sx_{n+1}, Sx_n) = r.$$

We claim that $r = 0$. If $r > 0$ then by (3.1.12) and (3.1.12), we get

$$\begin{aligned} r &= \lim_{n\to\infty} d(Sx_{n+1}, Sx_n) \\ &\leq \lim_{n\to\infty} \psi(d(Sx_{n-1}, Sx_n)) \\ &= \psi(r) \\ &< r, \end{aligned}$$

which is a contradiction and hence

$$\lim_{n\to\infty} d(Sx_{n+1}, Sx_n) = 0. \tag{3.1.12}$$

Next, we show that $\{Sx_n\}$ is a Cauchy sequence. Suppose the contrary.
There exists $\varepsilon > 0$ and subsequence $\{Sx_{m_k}\}, \{Sx_{n_k}\}$ of $\{Sx_n\}$ such that $n_k > m_k \geq k$ with

$$r_k := d(Sx_{m_k}, Sx_{n_k}) \geq \varepsilon \quad \text{and} \quad d(Sx_{m_k}, Sx_{n_k-1}) < \varepsilon,$$

for $k \in \{1, 2, 3, \ldots\}$. Putting $\gamma_n = d(Sx_{n+1}, Sx_n)$, then

$$\begin{aligned} \varepsilon \leq r_k &\leq d(Sx_{m_k}, Sx_{n_k-1}) + d(Sx_{n_k-1}, Sx_{n_k}) \\ &< \varepsilon + \gamma_{n_k-1}, \end{aligned}$$

it follows from (3.1.12), that

$$\lim_{k\to\infty} r_k = \varepsilon. \tag{3.1.13}$$

On the other hand, by constructing the sequence $\{x_n\}$, we have

$$d(gx_{m_k+1}, Sx_{m_k}) = d(A, B) \quad \text{and} \quad d(gx_{n_k+1}, Sx_{n_k}) = d(A, B).$$

Using the transitive of α, we get $\alpha(x_{m_k}, x_{n_k}) \geq 1$. Since S is a generalized proximal $\alpha - \psi-$contraction of the second kind and g is an isometry, we have

$$\begin{aligned} d(Sx_{m_k+1}, Sx_{n_k+1}) &= d(Sgx_{m_k+1}, Sgx_{n_k+1}) \\ &\leq \alpha(x_{m_k}, x_{n_k}) d(Sgx_{m_k+1}, Sgx_{n_k+1}) \\ &= \alpha(x_{m_k}, x_{n_k}) d(Sx_{m_k+1}, Sx_{n_k+1}) \\ &\leq \psi(d(Sx_{m_k}, Sx_{n_k})). \end{aligned}$$

Notice also that

$$\begin{aligned} \varepsilon \leq r_k &\leq d(Sx_{m_k}, Sx_{m_k+1}) + d(Sx_{n_k+1}, Sx_{n_k}) + d(Sx_{m_k+1}, Sx_{n_k+1}) \\ &= \gamma_{m_k} + \gamma_{n_k} + d(Sx_{m_k+1}, Sx_{n_k+1}) \\ &\leq \gamma_{m_k} + \gamma_{n_k} + \psi(d(Sx_{m_k}, Sx_{n_k})). \end{aligned}$$

Taking $k \to \infty$ in above inequality, by (3.1.12), (3.1.13) and property of ψ, we get $\varepsilon \leq \psi(\varepsilon) < \epsilon$, which is a contradiction $\varepsilon = 0$. So we obtain the claim and then it converges to some y in B. Therefore, we can conclude that

$$d(gx, y) = \lim_{n\to\infty} d(gx_{n+1}, Sx_n) = d(A, B),$$

that is gx in A_0. Since $A_0 \subseteq g(A_0)$, we have $gx = gz$ for some z in A_0 and thus $d(gx, gz) = 0$. By the fact that g is an isometry, we have $d(x, z) = d(gx, gz) = 0$. Hence $x = z$ and so x becomes to a point in A_0. As $S(A_0) \subseteq B_0$ that

$$d(u, Sx) = d(A, B) \tag{3.1.14}$$

for some u in A. It follows from (3.1.11), (3.1.14), condition (c) and S is a generalized proximal $\alpha - \psi-$contraction of the first kind that

$$\begin{aligned} d(u, gx_{n+1}) &\leq \alpha(x_n, x)d(u, gx_{n+1}) \\ &\leq \psi(d(x_n, x)) \\ &\leq d(x_n, x), \end{aligned}$$

for all n in $\mathbb{N}$. Taking limit as $n \to \infty$, we get the sequence $\{gx_n\}$ converges to a point u. By the fact that x_n converges to x and g is continuous, we have

$$gx_n \to gx \text{ as } n \to \infty.$$

By the uniqueness of limit of the sequence, we conclude that $u = gx$. Therefore, it results that $d(gx, Sx) = d(u, Sx) = d(A, B)$. The uniqueness and the remaining part of the proof follows as in Theorem(3.1.5). This completes the proof of the theorem. □

3.2 Existence and Uniqueness of Best Proximity Points for Generalized Almost Contractions

In this section, we introduce the new class of generalized almost contraction mappings in metric spaces by using the $\alpha-$proximal admissible of Jleli et al. [24] and prove best proximity theorems for this class and also give some illustrative examples and applications to support our main results.

Definition 3.2.1. Let A and B be nonempty subsets of metric space X. A mapping $T : A \to B$ is said to be a *generalized almost* $(\varphi, \theta)_\alpha$ *contraction*, if and only if

$$\begin{aligned} \alpha(x, y)d(Tx, Ty) \leq\ & \varphi\big(M(x, y)) + \theta(d(y, Tx) - d(A, B), d(x, Ty) - d(A, B), \\ & d(x, Tx) - d(A, B), d(y, Ty) - d(A, B)\big), \end{aligned}$$

for all $x, y \in A$, where $\alpha : A \times A \to [0, \infty)$, $\varphi \in \Psi$, $\theta \in \Theta$ and

$$M(x,y) = \max\{d(x,y), d(x,Tx) - d(A,B), d(y,Ty) - d(A,B), \frac{1}{2}[d(x,Ty) + d(y,Tx)] - d(A,B)\}.$$

Clearly, if we take $\alpha(x,y) = 1$ for all $x, y \in A$ and $M(x,y) = d(x,y)$, the generalized almost $(\varphi,\theta)_\alpha$ contraction reduce to almost (φ,θ) contraction.

3.2.1 Existence of Best Proximity Points for Generalized Almost Contractions

Theorem 3.2.2. *Let A and B be nonempty closed subsets of a complete metric space X such that A_0 is nonempty and the pair (A,B) has the $P-$ property. Let $T : A \to B$ satisfy the following conditions:*

(a) T are α-proximal admissible and generalized almost $(\varphi,\theta)_\alpha-$contraction;

(b) T is continuous;

(c) there exist element x_0 and x_1 in A_0 such that $d(x_1, Tx_0) = d(A,B)$ and $\alpha(x_0, x_1) \geq 1$;

(d) $T(A_0) \subseteq B_0$.

Then there exists an element $x \in A$ such that

$$d(x,Tx) = d(A,B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Tx_n) = d(A,B),$$

converges to the element x.

Proof. By the hypothesis (c), there exist x_0 and x_1 in A_0 such that

$$d(x_1, Tx_0) = d(A,B) \quad \text{and} \quad \alpha(x_0, x_1) \geq 1. \tag{3.2.1}$$

From the fact that $T(A_0) \subseteq B_0$, there exists an element $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A,B). \tag{3.2.2}$$

By (3.2.1), (3.2.2) and the α-proximal admissible, we get

$$\alpha(x_1, x_2) \geq 1.$$

Since $T(A_0) \subseteq B_0$, we can find an element $x_3 \in A_0$ such that

$$d(x_3, Tx_2) = d(A, B). \tag{3.2.3}$$

Again, by (3.2.2), (3.2.3) and the α-proximal admissible, we have

$$\alpha(x_2, x_3) \geq 1.$$

By similar fashion, we can find x_n in A_0. Having chosen x_n, one can determine an element $x_{n+1} \in A_0$ such that

$$d(x_{n+1}, Tx_n) = d(A, B) \quad \text{and} \quad \alpha(x_n, x_{n+1}) \geq 1. \tag{3.2.4}$$

In view the facts that, the pair (A, B) has $P-$ property and generalized almost $(\varphi, \theta)_\alpha-$contraction of T, we have

$$\begin{aligned} d(x_1, x_2) &= d(Tx_0, Tx_1) \\ &\leq \alpha(x_0, x_1) d(Tx_0, Tx_1) \\ &\leq \varphi(M(x_0, x_1)) \\ &\quad +\theta(d(x_1, Tx_0) - d(A, B), d(x_0, Tx_1) - d(A, B), \\ &\quad d(x_0, Tx_0) - d(A, B), d(x_1, Tx_1) - d(A, B)) \qquad (3.2.5) \\ &= \varphi(M(x_0, x_1)) \\ &\quad +\theta(0, d(x_0, Tx_1) - d(A, B), d(x_0, Tx_0) - d(A, B), \\ &\quad d(x_1, Tx_1) - d(A, B)) \\ &= \varphi(M(x_0, x_1)). \end{aligned}$$

Since

$$\begin{aligned} M(x_0, x_1) &= \max\{d(x_0, x_1), d(x_0, Tx_0) - d(A, B), d(x_1, Tx_1) - d(A, B), \\ &\quad \tfrac{1}{2}[d(x_0, Tx_1) + d(x_1, Tx_0)] - d(A, B)\} \\ &\leq \max\{d(x_0, x_1), d(x_0, x_1) + d(x_1, Tx_0) - d(A, B), d(x_1, x_2) \\ &\quad +d(x_2, Tx_1) - d(A, B) \tfrac{1}{2}[d(x_0, x_1) + d(x_1, x_2) + d(x_2, Tx_1) \\ &\quad +d(A, B)] - d(A, B)\} \\ &= \max\{d(x_0, x_1), d(x_1, x_2), \tfrac{1}{2}[d(x_0, x_1) + d(x_1, x_2) + d(A, B) \\ &\quad +d(A, B)] - d(A, B)\} \\ &= \max\{d(x_0, x_1), d(x_1, x_2), \tfrac{1}{2}[d(x_0, x_1) + d(x_1, x_2)]\} \\ &= \max\{d(x_0, x_1), d(x_1, x_2)\}. \end{aligned}$$

(3.2.6)

By (3.2.5) and (3.2.6), we get

$$d(x_1, x_2) \leq \varphi(\max\{d(x_0, x_1), d(x_1, x_2)\}).$$

If there exist $n_0 \in \mathbb{N} \cup \{0\}$ such that $x_{n_0+1} = x_{n_0}$, by (3.2.4) we obtain the best proximity point. Suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$, then $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. If $\max\{d(x_0, x_1), d(x_1, x_2)\} = d(x_1, x_2)$, by the property $\varphi(t) < t$ for all $t > 0$, we get

$$d(x_1, x_2) \leq \varphi(\max\{d(x_0, x_1), d(x_1, x_2)\}) < d(x_1, x_2),$$

which is a contradiction and hence $\max\{d(x_0, x_1), d(x_1, x_2)\} = d(x_0, x_1)$. That is,

$$d(x_1, x_2) \leq \varphi(d(x_0, x_1)). \tag{3.2.7}$$

Again, since the pair (A, B) has $P-$ property, α-proximal admissible and generalized almost $(\varphi, \theta)_\alpha-$contraction of T, we have

$$\begin{aligned}
d(x_2, x_3) &= d(Tx_1, Tx_2) \\
&\leq \alpha(x_1, x_2)d(Tx_1, Tx_2) \\
&\leq \varphi(M(x_1, x_2)) \\
&\quad +\theta(d(x_2, Tx_1) - d(A, B), d(x_1, Tx_2) - d(A, B), \\
&\quad d(x_1, Tx_1) - d(A, B), d(x_2, Tx_2) - d(A, B)) \\
&= \varphi(M(x_1, x_2)) \\
&\quad +\theta(0, d(x_1, Tx_2) - d(A, B), d(x_1, Tx_1) - d(A, B), \\
&\quad d(x_2, Tx_2) - d(A, B)) \\
&= \varphi(M(x_1, x_2))
\end{aligned}$$

and since

$$\begin{aligned}
M(x_1, x_2) &= \max\{d(x_1, x_2), d(x_1, Tx_1) - d(A, B), d(x_2, Tx_2) - d(A, B), \\
&\quad \tfrac{1}{2}[d(x_1, Tx_2) + d(x_2, Tx_1)] - d(A, B)\} \\
&\leq \max\{d(x_1, x_2), d(x_1, x_2) + d(x_2, Tx_1) - d(A, B), d(x_2, x_3) \\
&\quad +d(x_3, Tx_2) - d(A, B)\tfrac{1}{2}[d(x_1, x_2) + d(x_2, x_3) + d(x_3, Tx_2) \\
&\quad +d(A, B)] - d(A, B)\} \\
&= \max\{d(x_1, x_2), d(x_2, x_3), \\
&\quad \tfrac{1}{2}[d(x_1, x_2) + d(x_2, x_3) + d(A, B) + d(A, B)] - d(A, B)\} \\
&= \max\{d(x_1, x_2), d(x_2, x_3), \tfrac{1}{2}[d(x_1, x_2) + d(x_2, x_3)]\} \\
&= \max\{d(x_1, x_2), d(x_2, x_3)\}.
\end{aligned}$$

By (3.2.5) and (3.2.6), we get

$$d(x_2, x_3) \leq \varphi(\max\{d(x_1, x_2), d(x_2, x_3)\}). \tag{3.2.8}$$

By similar argument as above, we can conclude that, $\max\{d(x_1, x_2), d(x_2, x_3)\} = d(x_1, x_2)$ and thus

$$d(x_2, x_3) \leq \varphi(d(x_1, x_2)). \tag{3.2.9}$$

Using (3.2.7) and (3.2.9) and the nondecreasing of φ, we get

$$d(x_2, x_3) \leq \varphi^2(d(x_0, x_1)).$$

Continuing this process, by induction, we have

$$d(x_n, x_{n+1}) \leq \varphi^n(d(x_0, x_1)), \tag{3.2.10}$$

for all $n \in \mathbb{N} \cup \{0\}$. Fix $\varepsilon > 0$ and let $h = h(\varepsilon)$ be a positive integer such that

$$\sum_{n \geq h} \varphi^n(d(x_0, x_1)) < \varepsilon. \tag{3.2.11}$$

Let $m > n > h$, using the triangular inequality, (3.2.10) and (3.2.11), we obtain that

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \varphi^k(d(x_0, x_1)) \leq \sum_{n \geq h} \varphi^n(d(x_0, x_1)) < \varepsilon.$$

This show that $\{x_n\}$ is a Cauchy sequence. Since A is a closed subset of complete metric spaces X, then there exists $x \in A$ such that

$$\lim_{n \to \infty} d(x_n, x) = 0.$$

By (3.2.4), (3.2.12) and the continuity of T, we get

$$d(x, Tx) = \lim_{n \to \infty} d(x_{n+1}, Tx_n) = d(A, B)$$

and the proof is completes. □

Next, we remove condition T is continuous in Theorem 3.2.2, by assuming the following condition which was defined by Jleli et al. [24] for proving the new best proximity point theorem.

(H) : If $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ for some $x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k.

Theorem 3.2.3. *Let A and B be nonempty closed subsets of a complete metric space X such that A_0 is nonempty and the pair (A, B) has the $P-$ property. Let $T : A \to B$ satisfy the following conditions:*

(a) T are α-proximal admissible and generalized almost $(\varphi, \theta)_\alpha-$contraction;

(b) A satisfies condition (H);

(c) there exist element x_0 and x_1 in A_0 such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha((x_0, x_1)) \geq 1$;

(d) $T(A_0) \subseteq B_0$.

Then there exists an element $x \in A$ such that

$$d(x, Tx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Tx_n) = d(A, B),$$

converges to the element x.

Proof. As in the proof of Theorem 3.2.2, we have

$$d(x_{n+1}, Tx_n) = d(A, B)$$

for all $n \geq 0$. Moreover, $\{x_n\}$ is a Cauchy sequence and converges to some point $x \in A$. By the $P-$ property and (3.2.10), we have

$$d(Tx_{n-1}, Tx_n) = d(x_n, x_{n+1}) \leq \varphi^n(d(x_0, x_1)) \tag{3.2.12}$$

for all $n \in \mathbb{N} \cup \{0\}$. That is $\lim_{n\to\infty} d(Tx_{n-1}, Tx_n) = 0$ and by the same argument as proof of Theorem 3.2.2, we obtain that $\{Tx_n\}$ is a Cauchy sequence. Since B is a closed subset of the complete metric space (X, d), there exists $x_\star \in B$ such that Tx_n converges to $x_\star$. Therefore,

$$d(x, x_\star) = \lim_{n\to\infty} d(x_{n+1}, Tx_n) = d(A, B).$$

On the other hand, from the condition (H) of T, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k. The pair (A, B) has $P-$ property and property of mapping T, we get

$$\begin{aligned} d(x_{n_k+1}, x) &= d(Tx_{n_k}, Tx) \\ &\leq \alpha(x_{n_k}, x)d(Tx_{n_k}, Tx) \\ &\leq \varphi(M(x_{n_k}, x)) \\ &\quad +\theta(d(x_{n_k}, Tx) - d(A, B), d(x, Tx_{n_k}) - d(A, B), \\ &\quad d(x, Tx) - d(A, B), d(x_{n_k}, Tx_{n_k}) - d(A, B)). \end{aligned}$$

Indeed,

$$\begin{aligned} M(x_{n_k}, x) &= \max\{d(x_{n_k}, x), d(x_{n_k}, Tx_{n_k}) - d(A, B), d(x, Tx) - d(A, B), \\ &\quad \tfrac{1}{2}[d(x_{n_k}, Tx) + d(x, Tx_{n_k})] - d(A, B)\} \\ &\leq \max\{d(x_{n_k}, x), d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, Tx_{n_k}) - d(A, B), \\ &\quad d(x, Tx) - d(A, B), \tfrac{1}{2}[d(x_{n_k}, x) + d(x, Tx) \\ &\quad +d(x, x_{n_k+1}) + d(x_{n_k+1}, Tx_{n_k})] - d(A, B)\} \\ &\leq \max\{d(x_{n_k}, x), d(x_{n_k}, x_{n_k+1}), d(x, Tx) - d(A, B), \\ &\quad \tfrac{1}{2}[d(x_{n_k}, x) + d(x, Tx) + d(x, x_{n_k+1}) + d(A, B)] - d(A, B)\} \\ &:= \mathcal{M}(x_{n_k}, x). \end{aligned}$$

From the definition of $\mathcal{M}(x_{n_k}, x)$, we get

$$\lim_{k\to\infty} \mathcal{M}(x_{n_k}, x) = d(x, Tx) - d(A, B). \tag{3.2.13}$$

Since

$$\begin{aligned} d(x, Tx) &\leq d(x, x_{n_k+1}) + d(x_{n_k+1}, Tx_{n_k}) + d(Tx_{n_k}, Tx) \\ &\leq d(x, x_{n_k+1}) + d(A, B) + d(Tx_{n_k}, Tx), \end{aligned}$$

it follows that

$$\begin{aligned} d(x, Tx) - d(x, x_{n_k+1}) - d(A, B) &\leq d(Tx_{n_k}, Tx) \\ &\leq \alpha(x_{n_k}, x)d(Tx_{n_k}, Tx) \\ &\leq \varphi(M(x_{n_k}, x)) \\ &\quad +\theta(d(x_{n_k}, Tx) - d(A, B), d(x, Tx_{n_k}) \\ &\quad -d(A, B), d(x, Tx) - d(A, B), d(x_{n_k}, Tx_{n_k}) \\ &\quad -d(A, B)) \\ &\leq \varphi(\mathcal{M}(x_{n_k}, x)) \\ &\quad +\theta(d(x_{n_k}, Tx) - d(A, B), d(x, Tx_{n_k}) \\ &\quad -d(A, B), d(x, Tx) - d(A, B), d(x_{n_k}, Tx_{n_k}) \\ &\quad -d(A, B)). \end{aligned}$$

Suppose that

$$d(x,Tx) - d(A,B) > 0,$$

then for k large enough, we have $\mathcal{M}(x_{n_k}, x) > 0$. Using the property $\varphi(t) < t$ for all $t > 0$, we get

$$\begin{aligned} d(x,Tx) - d(x,x_{n_k+1}) - d(A,B) &< \mathcal{M}(x_{n_k}, x) \\ &\quad +\theta(d(x_{n_k}, Tx) - d(A,B), d(x, Tx_{n_k}) \\ &\quad -d(A,B), d(x,Tx) - d(A,B), d(x_{n_k}, Tx_{n_k}) \\ &\quad -d(A,B)). \end{aligned} \tag{3.2.14}$$

Combining (3.2.13), (3.2.13) with (3.2.14) and the proprety of θ, we obtain that

$$\begin{aligned} d(x,Tx) - d(A,B) &= \lim_{k\to\infty} d(x,Tx) - d(x,x_{n_k+1}) - d(A,B) \\ &< \lim_{k\to\infty} \mathcal{M}(x_{n_k}, x) \\ &\quad + \lim_{k\to\infty} \theta(d(x_{n_k}, Tx) - d(A,B), d(x, Tx_{n_k}) - d(A,B), \\ &\quad d(x,Tx) - d(A,B), d(x_{n_k}, Tx_{n_k}) - d(A,B)) \\ &= \lim_{k\to\infty} \mathcal{M}(x_{n_k}, x) \\ &= d(x,Tx) - d(A,B), \end{aligned}$$

which is a contradiction and thus $d(x,Tx) - d(A,B) = 0$. Hence, $d(x,Tx) = d(A,B)$ and the proof is complete. □

3.2.2 The Uniqueness of Best Proximity Points for Generalized Almost Contractions

Next, we present an example where it can be appreciated that hypotheses in Theorems 3.2.2 and 3.2.3 do not guarantee uniqueness of the best proximity point.

Example. Let $X = \mathbb{R}^2$ with the Euclidean metric. Consider $A := \{(2,0),(0,2)\}$ and $B := \{(-2,0),(0,-2)\}$. Obviously, (A,B) satisfies the P-property and $d(A,B) = 2\sqrt{2}$, furthermore $A_0 = A$ and $B_0 = B$. Define $T : A \to B$ by $T(x,y) = (\frac{-y}{2}, \frac{-x}{2})$ for all $x, y \in A$, clearly T is continuous. Let $\alpha : A \times A \to [0,\infty)$ defined by

$$\alpha(x,y) = \begin{cases} 2 & ; x = y, \\ \frac{1}{2} & ; x \neq y. \end{cases}$$

We can show that T are α-proximal admissible and generalized almost $(\varphi,\theta)_\alpha-$contraction with $\varphi(t) = t/2$ for all $t \geq 0$ and for all $\theta \in \Theta$. Furthermore,

$$d((2,0), T(2,0)) = d((2,0),(0,-2)) = d((0,2),(-2,0)) = d((0,2), T(0,2)) = d(A,B).$$

Therefore, $(2,0)$ and $(0,2)$ are a best proximity point of mapping T.

Now, we need a sufficient condition for give uniqueness of the best proximity point as follows:

Definition 3.2.4. [24] Let $T : A \to B$ be a non-self mapping and $\alpha : A \times A \to [0, \infty)$. We say that T is $(\alpha, d)-$*regular* if for all $(x, y) \in \alpha^{-1}([0, 1))$, there exists $z \in A_0$ such that

$$\alpha((x, z)) \geq 1 \quad \text{and} \quad \alpha(y, z) \geq 1.$$

Theorem 3.2.5. *Adding condition $(\alpha, d)-$regular of T to the hypotheses of Theorem 3.2.2, then we obtain the uniqueness of the best proximity point of T.*

Proof. We shall only proof the part of uniqueness. Suppose that there exist x and x^* in A which are distinct best proximity points, that is

$$d(x, Tx) = d(A, B) \quad \text{and} \quad d(x^*, Tx^*) = d(A, B).$$

Using the pair (A, B) has $P-$ property, we have

$$d(x, x^*) = d(Tx, Tx^*). \tag{3.2.15}$$

Case I If $\alpha(x, x^*) \geq 1$. By (3.2.15) and generalized almost $(\varphi, \theta)_\alpha-$contraction of T, we have

$$\begin{aligned} d(x, x^*) &= d(Tx, Tx^*) \\ &\leq \alpha(x, x^*) d(Tx, Tx^*) \\ &\leq \varphi(M(x, x^*)) \\ &\quad +\theta(d(x^*, Tx) - d(A, B), d(x, Tx^*) - d(A, B), \\ &\quad d(x, Tx) - d(A, B), d(x^*, Tx^*) - d(A, B)) \\ &= \varphi(M(x, x^*)) \\ &\quad +\theta(d(x^*, Tx) - d(A, B), d(x, Tx^*) - d(A, B), 0, 0) \\ &= \varphi(M(x, x^*)), \end{aligned} \tag{3.2.16}$$

and since

$$\begin{aligned} M(x,x^*) &= \max\{d(x,x^*), d(x,Tx)-d(A,B), d(x^*,Tx^*)-d(A,B), \\ &\quad \tfrac{1}{2}[d(x,Tx^*)+d(x^*,Tx)]-d(A,B)\} \\ &= \max\{d(x,x^*),0,0,\tfrac{1}{2}[d(x,Tx^*)+d(x^*,Tx)]-d(A,B)\} \\ &\leq \max\{d(x,x^*),\tfrac{1}{2}[d(x,x^*)+d(x^*,Tx^*)+d(x^*,x)+d(x,Tx)] \\ &\quad -d(A,B)\} \\ &= \max\{d(x,x^*),\tfrac{1}{2}[d(x,x^*)+d(x^*,x)]\} \\ &= d(x,x^*). \end{aligned} \tag{3.2.17}$$

Combining (3.2.16) with (3.2.17) and using the property $\varphi(t) < t$ for all $t > 0$, we get

$$d(x,x^*) \leq \varphi(M(x,x^*)) = \varphi(d(x,x^*)) < d(x,x^*),$$

which is a contradiction and hence $x = x^*$.

Case II If $\alpha(x,x^*) < 1$. By the $(\alpha,d)-$regular of T, there exists $z \in A_0$ such that

$$\alpha((x,z)) \geq 1 \quad \text{and} \quad \alpha(x^*,z) \geq 1.$$

Since $T(A_0) \subseteq B_0$, there exists a point $v_0 \in A_0$ such that

$$d(v_0,Tz) = d(A,B).$$

From $\alpha((x,z)) \geq 1$, $d(x,Tx) = d(A,B)$ and $d(v_0,Tz) = d(A,B)$ and by the α-proximal admissible, we have

$$\alpha(x,v_0) \geq 1.$$

Since $T(A_0) \subseteq B_0$, there exists a point $v_1 \in A_0$ such that

$$d(v_1,Tv_0) = d(A,B).$$

By similar argument as above, we can conclude that $\alpha(x,v_1) \geq 1$. One can proceed further in a similar fashion to find v_n in A_0 with $v_{n+1} \in A_0$ such that

$$d(v_{n+1},Tv_n) = d(A,B) \quad \text{and} \quad \alpha(x,v_n) \geq 1, \tag{3.2.18}$$

for all $n \in \mathbb{N}$. By (3.2.18), the pair (A,B) has $P-$ property and property of mapping T, we get

$$d(x,v_{n+1}) = d(Tx,Tv_n). \tag{3.2.19}$$

Using the property of mapping T, we get

$$\begin{aligned} d(x, v_{n+1}) &= d(Tx, Tv_n) \\ &\leq \alpha(x, v_n) d(Tx, Tv_n) \\ &\leq \varphi(M(x, v_n)) \\ &\quad +\theta(d(v_n, Tx) - d(A, B), d(x, Tv_n) - d(A, B), \\ &\quad d(x, Tx) - d(A, B), d(v_n, Tv_n) - d(A, B)) \\ &= \varphi(M(x, v_n)) \\ &\quad +\theta(d(v_n, Tx) - d(A, B), d(x, Tv_n) - d(A, B), \\ &\quad 0, d(v_n, Tv_n) - d(A, B)) \\ &= \varphi(M(x, v_n)) \end{aligned}$$

and since

$$\begin{aligned} M(x, v_n) &= \max\{d(x, v_n), d(x, Tx) - d(A, B), d(v_n, Tv_n) - d(A, B), \\ &\quad \tfrac{1}{2}[d(x, Tv_n) + d(v_n, Tx)] - d(A, B)\} \\ &= \max\{d(x, v_n), 0, 0, \tfrac{1}{2}[d(x, Tv_n) + d(v_n, Tx)] - d(A, B)\} \\ &\leq \max\{d(x, v_n), \tfrac{1}{2}[d(x, v_{n+1}) + d(v_{n+1}, Tv_n) + d(v_n, x) + d(x, Tx)] \\ &\quad -d(A, B)\} \\ &= \max\{d(x, v_n), \tfrac{1}{2}[d(x, v_{n+1}) + d(v_n, x)]\} \\ &= \max\{d(x, v_n), d(x, v_{n+1})\}. \end{aligned}$$

Thus

$$d(x, v_{n+1}) \leq \varphi(M(x, v_n)) \leq \varphi(\max\{d(x, v_n), d(x, v_{n+1})\}).$$

If $v_N = x$, for some $N \in \mathbb{N}$. By (3.2.19), we get

$$d(x, v_{N+1}) = d(Tx, Tv_N) = 0$$

which implies that $v_{N+1} = x$. Moreover, we obtain $v_n = x$ for all $n \geq N$ and thus $v_n \to x$ as $n \to \infty$. Suppose that $v_n \neq x$ for all $n \in \mathbb{N}$, then $d(v_n, x) > 0$ for all n. If $\max\{d(x, v_n), d(x, v_{n+1})\} = d(x, v_{n+1})$, by the property $\varphi(t) < t$ for all $t > 0$, we get

$$d(x, v_{n+1}) \leq \varphi(M(x, v_n)) = \varphi(d(x, v_{n+1})) < d(x, v_{n+1})$$

which is a contradiction and hence $\max\{d(x, v_n), d(x, v_{n+1})\} = d(x, v_n)$. That is

$$d(x, v_{n+1}) \leq \varphi(M(x, v_n)) = \varphi(d(x, v_n)) \tag{3.2.20}$$

for all $n \geq N$. By induction of (3.2.20), we have

$$d(x, v_{n+1}) \leq \varphi^n(d(x, v_1)).$$

Taking $n \to \infty$, we obtain that $v_n \to x$ as $n \to \infty$. So, in all cases, we have $v_n \to x$ as $n \to \infty$. Similarly, we can prove that $v_n \to x^*$ as $n \to \infty$. By the uniqueness of limit, we conclude that $x = x^*$ and this completes the proof. □

Theorem 3.2.6. *Adding condition $(\alpha, d)-$regular of T to the hypotheses of Theorem 3.2.3, then we obtain the uniqueness of the best proximity point of T.*

Proof. Combine the proofs of Theorem 3.2.5 and Theorem 3.2.3. □

If we take $\varphi(t) = kt$, where $0 \leq k < 1$ and $\theta(t_1, t_2, t_3, t_4) = L\min\{t_1, t_2, t_3, t_4\}$, then Theorem 3.2.1 and Theorem 3.2.4, we get the following.

Theorem 3.2.7. *Let A and B be nonempty closed subsets of a complete metric space X such that A_0 is nonempty and the pair (A, B) has the $P-$ property. Let $T : A \to B$ satisfy the following conditions:*

(a) T is α-proximal admissible and

$$\begin{aligned} \alpha(x, y)d(Tx, Ty) \leq\ & kM(x, y) + L\min\{d(x, Ty) - d(A, B), d(y, Tx) - d(A, B) \\ & d(x, Tx) - d(A, B), d(y, Ty) - d(A, B)\} \end{aligned}$$

for all $x, y \in A$.

(b) T is continuous (or A satisfies condition (H));

(c) there exist element x_0 and x_1 in A_0 such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha((x_0, x_1)) \geq 1$;

(d) $T(A_0) \subseteq B_0$.

Then there exists an element $x \in A$ such that

$$d(x, Tx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Tx_n) = d(A, B),$$

converges to the element x.

If we add the condition that T is $(\alpha, d)-$regular in Theorem 3.2.7, therefore we can obtain the uniqueness of the best proximity point.

If we take $\alpha(x, y) = 1$, for all $x, y \in A$ in Theorem 3.2.2 and Theorem 3.2.3, we get the following Theorems.

Theorem 3.2.8. *Let A and B be nonempty closed subsets of a complete metric space X such that A_0 is nonempty and the pair (A, B)has the $P-$ property. Let $T : A \to B$ satisfy the following conditions:*

(a)

$$\begin{aligned} d(Tx, Ty) &\leq \varphi(M(x, y)) + \theta(d(x, Ty) - d(A, B), d(y, Tx) - d(A, B) \\ &\quad d(x, Tx) - d(A, B), d(y, Ty) - d(A, B)) \end{aligned}$$

for all $x, y \in A$.

(b) T is continuous (or A satisfies condition (H));

(c) $T(A_0) \subseteq B_0$.

Then there exists an element $x \in A$ such that

$$d(x, Tx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Tx_n) = d(A, B),$$

converges to the element x.

If $M(x, y) = d(x, y)$, then Theorem 3.2.8, include the following.

Theorem 3.2.9. *Let A and B be nonempty closed subsets of a complete metric space X such that A_0 is nonempty and the pair (A, B)has the $P-$ property. Let $T : A \to B$ satisfy the following conditions:*

(a)

$$\begin{aligned} d(Tx, Ty) &\leq \varphi(d(x, y)) + \theta(d(x, Ty) - d(A, B), d(y, Tx) - d(A, B) \\ &\quad d(x, Tx) - d(A, B), d(y, Ty) - d(A, B)) \end{aligned}$$

for all $x, y \in A$.

(b) T is continuous (or A satisfies condition (H));

(c) $T(A_0) \subseteq B_0$.

Then there exists an element $x \in A$ such that

$$d(x, Tx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Tx_n) = d(A, B),$$

converges to the element x.

If we take $\varphi(t) = kt$ and $\theta(t_1, t_2, t_3, t_4) = L \min\{t_1, t_2, t_3, t_4\}$, for all $x, y \in A$ in Theorem 3.2.9, we obtain the following theorem.

Theorem 3.2.10. *Let A and B be nonempty closed subsets of a complete metric space X such that A_0 is nonempty and the pair (A, B)has the $P-$ property. Let $T : A \to B$ satisfy the following conditions:*

(a)

$$\begin{aligned} d(Tx, Ty) &\leq kM(x, y) + L \min\{d(x, Ty) - d(A, B), d(y, Tx) - d(A, B) \\ &\quad d(x, Tx) - d(A, B), d(y, Ty) - d(A, B)\} \end{aligned}$$

for all $x, y \in A$.

(b) T is continuous (or A satisfies condition (H));

(c) $T(A_0) \subseteq B_0$.

Then there exists an element $x \in A$ such that

$$d(x, Tx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Tx_n) = d(A, B),$$

converges to the element x.

If $M(x, y) = d(x, y)$ and putting $L = 0$ in Theorem 3.2.10, we obtain the following.

Theorem 3.2.11. *Let A and B be nonempty closed subsets of a complete metric space X such that A_0 is nonempty and the pair (A, B)has the $P-$ property. Let $T : A \to B$ satisfy the following conditions:*

(a)

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in A$.

(b) T *is continuous (or* A *satisfies condition* (H) *);*

(c) there exist element x_0 *and* x_1 *in* A_0 *such that* $d(x_1, Tx_0) = d(A, B)$*;*

(d) $T(A_0) \subseteq B_0$.

Then there exists an element $x \in A$ *such that*

$$d(x, Tx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$*, the sequence* $\{x_n\}$*, defined by*

$$d(x_{n+1}, Tx_n) = d(A, B),$$

converges to the element x.

If $M(x, y) = \frac{k}{2}[d(x, Ty) + d(y, Tx)] - d(A, B)$ and putting $L = 0$ in Theorem 3.2.10, we obtain the following theorem:

Theorem 3.2.12. *Let* A *and* B *be nonempty closed subsets of a complete metric space* X *such that* A_0 *is nonempty and the pair* (A, B)*has the* $P-$ *property. Let* $T : A \to B$ *satisfy the following conditions:*

(a)

$$d(Tx, Ty) \leq \frac{k}{2}[d(x, Ty) + d(y, Tx)] - d(A, B)$$

for all $x, y \in A$.

(b) T *is continuous(or* A *satisfies condition* (H) *);*

(c) there exist element x_0 *and* x_1 *in* A_0 *such that* $d(x_1, Tx_0) = d(A, B)$*;*

(d) $T(A_0) \subseteq B_0$.

Then there exists an element $x \in A$ *such that*

$$d(x, Tx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$*, the sequence* $\{x_n\}$*, defined by*

$$d(x_{n+1}, Tx_n) = d(A, B),$$

converges to the element x.

It is easy to observe that for self-mappings, our results includes the following:

Theorem 3.2.13. *Let A be nonempty closed subsets of a complete metric space X and $T : A \to A$ such that*

$$d(Tx,Ty) \;\; \leq \;\; \varphi(M(x,y)) + \theta(\{d(x,Ty), d(y,Tx), d(x,Tx), d(y,Ty)\}),$$

for all $x, y \in A$, where $\varphi \in \Psi$ $\theta \in \Theta$. Then T has a unique fixed point $x \in A$. Moreover, for any fixed $x_0 \in A$, the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$, converges to the element x.

Theorem 3.2.14. *Let A be nonempty closed subsets of a complete metric space X and $T : A \to A$ such that*

$$d(Tx,Ty) \;\; \leq \;\; kM(x,y) + L\min\{d(x,Ty), d(y,Tx), d(x,Tx), d(y,Ty)\}.$$

Then T has a unique fixed point $x \in A$. Moreover, for any fixed $x_0 \in A$, the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$, converges to the element x.

Theorem 3.2.15. *Let A be nonempty closed subsets of a complete metric space X and $T : A \to A$ such that*

$$d(Tx,Ty) \;\; \leq \;\; kd(x,y) + L\min\{d(x,Ty), d(y,Tx), d(x,Tx), d(y,Ty)\}$$

for all $x, y \in A$. Then T has a unique fixed point $x \in A$.

Moreover, for any fixed $x_0 \in A$, the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$, converges to the element x.

We recall some preliminaries from (see, [24] also) as follows:

Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation over X. Denote

$$\mathcal{S} = \mathcal{R} \cup \mathcal{R}^{-1}$$

this is the symmetric relation attached to R. Clearly,

$$x, y \in X, x\mathcal{S}y \Longleftrightarrow x\mathcal{R}y \text{ or } y\mathcal{R}x.$$

Definition 3.2.16. [24] A mapping $T : A \to B$ is said to be *proximal comparative* if and only if

$$\left.\begin{array}{l} x_1\mathcal{S}x_2 \\ d(u_1, Tx_1) = d(A,B) \\ d(u_2, Tx_2) = d(A,B) \end{array}\right\} \quad \Longrightarrow \quad u_1\mathcal{S}u_2.$$

Corollary 3.2.17. *Let (X, d) be a complete metric space, $\mathcal{R}$ be a binary relation over X, and A and B be two non-empty, closed subsets of X such that A_0 are non-empty and the pair (A, B) has the $P-$ property. Let $T : A \to B$ such that the following conditions holds:*

(a) T is a continuous proximal comparative mapping;

(b) there exist element x_0 and x_1 in A_0 such that $d(x_1, Tx_0) = d(A, B)$ and $x_0 \mathcal{S} x_1$;

(c) there exist $\varphi \in \Psi$ and $\theta \in \Theta$ such that $x, y \in A, x\mathcal{S}y$ implies that

$$\begin{aligned} d(Tx, Ty) &\le \varphi\big(M(x, y)) + \theta(d(y, Tx) - d(A, B), d(x, Ty) - d(A, B), d(x, Tx) \\ &\quad -d(A, B), d(y, Ty) - d(A, B)\big); \end{aligned}$$

(d) $T(A_0) \subseteq B_0$.

Then there exists an element $x \in A$ such that

$$d(x, Tx) = d(A, B).$$

Proof. Define the mapping $\alpha : A \times A \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & ; x\mathcal{S}y, \\ 0 & ;\ \text{otherwise}. \end{cases} \tag{3.2.21}$$

Since T is proximal comparative, we have

$$\left.\begin{array}{l} x\mathcal{S}y \\ d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array}\right\} \Longrightarrow u\mathcal{S}v,$$

for all $u, v, x, y \in A$. Using the definition of α, we get

$$\left.\begin{array}{l} \alpha(x, y) \ge 1, \\ d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B) \end{array}\right\} \Longrightarrow \alpha(u, v) \ge 1,$$

for all $u, v, x, y \in A$ and hence T is α-proximal admissible. By the condition (b) implies that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \ge 1$. By the condition (c), we get

$$\begin{aligned} \alpha(x, y)d(Tx, Ty) &\le \varphi\big(M(x, y)) + \theta(d(y, Tx) - d(A, B), d(x, Ty) - d(A, B), \\ &\quad d(x, Tx) - d(A, B), d(y, Ty) - d(A, B)\big), \end{aligned}$$

that is, T is, generalized almost $(\varphi, \theta)_\alpha-$contraction. Therefore, all hypothesisses of Theorem 3.2.1 are satisfied, and the desired result follows immediately. □

Next, below we give an example to illustrate the main result Theorem 3.2.1.

Example. Consider $X = \mathbb{R}^4$ with the metric defined by

$$d((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| + |x_4 - y_4|$$

for all $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$. Let $A, B \subset X$ defined by

$$A := \left\{ \left(0, 0, \frac{1}{n}, \frac{-1}{n} \right) \right\} \cup \{(0, 0, 0, 0)\},$$

$$B := \left\{ \left(1, -1, \frac{1}{n}, \frac{-1}{n} \right) \right\} \cup \{(1, -1, 0, 0)\}.$$

Then A and B are nonempty closed subsets of X and $d(A, B) = 2$.
Moreover, $A_0 = A$ and $B_0 = B$. Suppose

$$d((0, 0, x_1, x_2), (1, -1, y_1, y_2)) = d(A, B) = 2$$

and

$$d((0, 0, x_1', x_2'), (1, -1, y_1', y_2')) = d(A, B) = 2,$$

then we get $x_1 = y_1, x_2 = y_2$ and $x_1' = y_1', x_2' = y_2'$. Hence, the pair (A, B) has the P-property. Let $T : A \to B$ be a mapping defined as

$$T\big(0, 0, x, y\big) = \left(0, 0, \frac{x}{2}, \frac{y}{2} \right)$$

for all $\big(0, 0, x, y\big) \in A$. We define the mapping $\alpha : A \times A \to [0, \infty)$ by

$$\alpha(x, y) = 1 \quad \text{for all}\, x, y \in A.$$

We can see that T is generalized almost $(\varphi, \theta)_\alpha-$contraction with $\varphi \in \Psi$ is given by $\varphi(t) = t/2$ for all $t \geq 0$ and for all $\theta \in \Theta$. Furthermore, $(0, 0, 0, 0) \in A$ is a best proximity point of mapping T.

3.3 Kannan $\alpha-$Admissible Weak $\phi-$Contraction

In this section, we introduce the existence of the best proximity points for Kannan $\alpha-$admissible weak $\phi-$contraction mapping in metric spaces.

Definition 3.3.1. A mapping $T : A \to B$ is said to be a *Kannan $\alpha-$admissible weak $\phi-$contraction*, if T satisfies

$$\alpha(x, y)d(Tx, Ty) \leq u(x, y) - \phi(u(x, y)),$$

for all x, y in A, where $u(x,y) = \frac{1}{2}[d(x,Tx) + d(y,Ty)]$, $\phi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$ and $\alpha : A \times A \longrightarrow [0,+\infty)$.

Theorem 3.3.2. *Let(X, d) be a complete metric space and let A and B be nonempty, closed subsets of X such that A_0 and B_0 are non-empty. Let $\alpha : A \times A \longrightarrow [0,+\infty)$ satisfy the following conditions:*

(a) $T(A_0) \subseteq B_0$ and (A,B) satisfies the weak P-property;

(b) T is α-proximal admissible;

(c) There exist elements x_0 and x_1 in A_0 such that $d(x_1,Tx_0) = d(A,B)$ and $\alpha(x_0,x_1) \geq 1$;

(d) T is a continuous Kannan $\alpha-$admissible weak $\phi-$contraction.

Then, there exists an element $x^\star \in A_0$ such that

$$d(x^\star, Tx^\star) = d(A,B).$$

Proof. From condition (c), there exist elements x_0 and x_1 in A_0 such that

$$d(x_1,Tx_0) = d(A,B) \quad \text{and} \quad \alpha(x_0,x_1) \geq 1.$$

Since $T(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that

$$d(x_2,Tx_1) = d(A,B).$$

Now, we have

$$\begin{aligned} \alpha(x_0,x_1) &\geq 1, \\ d(x_1,Tx_0) &= d(A,B), \\ d(x_2,Tx_1) &= d(A,B). \end{aligned}$$

Since T is $\alpha-$proximal admissible, this implies that $\alpha(x_1,x_2) \geq 1$.
Thus, we have

$$d(x_2,Tx_1) = d(A,B) \quad \text{and} \quad \alpha(x_1,x_2) \geq 1.$$

Again, Since $T(A_0) \subseteq B_0$, there exists $x_3 \in A_0$ such that

$$d(x_3,Tx_2) = d(A,B).$$

Now, we have

$$\alpha(x_1, x_2) \geq 1,$$
$$d(x_2, Tx_1) = d(A, B),$$
$$d(x_3, Tx_2) = d(A, B).$$

Since T is $\alpha-$proximal admissible, this implies that $\alpha(x_2, x_3) \geq 1$.

Thus, we have

$$d(x_3, Tx_2) = d(A, B) \quad \text{and} \quad \alpha(x_2, x_3) \geq 1.$$

Continuing this process, by induction, we can construct a sequence $\{x_n\} \subset A_0$ such that

$$d(x_{n+1}, Tx_n) = d(A, B) \quad \text{and} \quad \alpha(x_n, x_{n+1}) \geq 1, \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.3.1}$$

Since (A, B) satisfies the weak $P-$property, we conclude from (3.3.1) that

$$d(x_n, x_{n+1}) \leq d(Tx_{n-1}, Tx_n), \quad \forall n \in \mathbb{N}. \tag{3.3.2}$$

From condition (d), that is, T is a Kannan $\alpha-$admissible weak $\phi-$contraction, for all $\forall n \in \mathbb{N}$, we have

$$\alpha(x_{n-1}, x_n) d(Tx_{n-1}, Tx_n) \leq u(x_{n-1}, x_n) - \phi\big(u(x_{n-1}, x_n)\big).$$

On the other hand, from (3.3.1), we have $\alpha(x_{n-1}, x_n) \geq 1$ $\forall n \in \mathbb{N}$, which implies with the above inequality that

$$d(Tx_{n-1}, Tx_n) \leq u(x_{n-1}, x_n) - \phi\big(u(x_{n-1}, x_n)\big), \quad \forall n \in \mathbb{N}. \tag{3.3.3}$$

Combining (3.3.2) with (3.3.3) yields the following:

$$\begin{aligned} d(x_n, x_{n+1}) &\leq u(x_{n-1}, x_n) - \phi\big(u(x_{n-1}, x_n)\big), \\ &= \frac{1}{2}\Big[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)\Big] \\ &\quad -\phi\left\{\frac{1}{2}\Big[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)\Big]\right\} \\ &\leq \frac{1}{2}\Big[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\Big] \\ &\quad -\phi\left\{\frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\right\}, \end{aligned} \tag{3.3.4}$$

and so it follows that $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$, that is , the sequence $\{d(x_n, x_{n+1})\}$ is a nonnegative nonincreasing sequence. Then there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = r.$$

Taking $n \to \infty$ in (3.3.4), and using the continuity of ϕ

$$r \leq \frac{1}{2}(2r) - \phi\left(\frac{1}{2}(2r)\right)$$

and consequently, $\phi\big(\frac{1}{2}(2r)\big) = 0$. This gives us that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.3.5}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence. Suppose on the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ such that n_k is smallest index for which

$$m_k > n_k > k, \quad d(x_{m_k}, x_{n_k}) \geq \epsilon.$$

This means that

$$d(x_{m_{k-1}}, x_{n_k}) < \epsilon.$$

Then we have

$$\begin{aligned} \epsilon &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{m_{k-1}}) + d(x_{m_{k-1}}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{m_{k-1}}) + \epsilon. \end{aligned}$$

Letting $k \to \infty$ and using (3.3.6) we can conclude that

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon.$$

Since α is forward transitive and $n_k > m_k$, we can conclude that

$$\alpha(x_{m_{k-1}}, x_{n_{k-1}}) \geq 1. \tag{3.3.6}$$

Using the fact that T is Kannan $\alpha-$admissible weak $\phi-$contraction and (3.3.6), we have

$$\begin{aligned}
d(x_{m_k}, x_{n_k}) &= d(Tx_{m_{k-1}}, Tx_{n_{k-1}}) \\
&\leq \alpha(x_{m_{k-1}}, x_{n_{k-1}})d(Tx_{m_{k-1}}, Tx_{n_{k-1}}) \\
&\leq u(x_{m_{k-1}}, x_{n_{k-1}}) - \phi\Big(u(x_{m_{k-1}}, x_{n_{k-1}}\Big) \\
&= \frac{1}{2}\Big[d(x_{m_{k-1}}, Tx_{m_{k-1}}) + d(x_{n_{k-1}}, Tx_{n_{k-1}})\Big] \\
&\quad -\phi\left\{\frac{1}{2}\Big[d(x_{m_{k-1}}, Tx_{m_{k-1}}) + d(x_{n_{k-1}}, Tx_{n_{k-1}})\Big]\right\} \\
&\leq \frac{1}{2}\Big[d(x_{m_{k-1}}, x_{m_k}) + d(x_{n_{k-1}}, x_{n_k})\Big] \\
&\quad -\phi\left\{\frac{1}{2}\Big[d(x_{m_{k-1}}, x_{m_k}) + d(x_{n_{k-1}}, x_{n_k})\Big]\right\}
\end{aligned}$$

Letting $k \to \infty$ and by using (3.3.5), and the continuity of ϕ, we have

$$\epsilon \leq \frac{1}{2}(0) - \phi\Big(\frac{1}{2}(0)\Big) = 0.$$

Which is a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence in the metric space (X, d). Since A is a closed subset of the complete metric space X, there exists $X \in A$ such that

$$\lim_{n\to\infty} x_n) = x. \tag{3.3.7}$$

Letting $n \to \infty$ in (3.3.1), (3.3.7) and the continuity of T, we get

$$d(x, Tx) = d(A, B)$$

and the proof is completes. □

In the next result, we remove the continuity hypothesis, assuming the following condition in A: (H) If $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of x_n such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}$.

Theorem 3.3.3. *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\alpha : A \times A \to [0, \infty)$. Suppose that $T : A \to B$ is a non-self-mapping satisfying the following conditions:*

(a) $T(A_0) \subseteq B_0$ and (A, B) satisfies the weak $P-$property;

(b) T is $\alpha-$proximal admissible;

(c) There exist elements x_0 and x_1 in A_0 such that

$$d(x_1, Tx_0) = d(A, B) \quad and \quad \alpha(x_0, x_1) \geq 1;$$

(d) (H) holds and T is a Kannan $\alpha-$admissible weak $\phi-$contraction.

Then, there exists an element $x \in A_0$ such that

$$d(x, Tx) = d(A, B).$$

Proof. As in the proof of theorem (3.3.2), we have

$$d(x_{n+1}, Tx_n) = d(A, B).$$

for all $n \geq 0$. Moreover, $\{x_n\}$ is a Cauchy sequence and converges to some point $x \in A$. By the weak $P-$ property and (3.3.4), we have

$$\begin{aligned} d(Tx_{n-1}, Tx_n) &= d(x_n, x_{n+1}) \\ &\leq \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] - \phi\{\frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\} \end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. That is $\lim_{n\to\infty} d(Tx_{n-1}, Tx_n) = 0$ and by the same argument as proof of Theorem 3.1.5, we obtain that $\{Tx_n\}$ is a Cauchy sequence. Since B is a closed subset of the complete metric space (X, d), there exists $x_\star \in B$ such that Tx_n converges to $x_\star$. Therefore

$$d(x, x_\star) = \lim_{n\to\infty} d(x_{n+1}, Tx_n) = d(A, B) \tag{3.3.8}$$

On the other hand, from the condition (H) of T, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k. The pair (A, B) has weak $P-$ property and property of mapping T, we get

$$\begin{aligned} d(x_{n_{k+1}}, x) &= d(Tx_{n_k}, Tx) \\ &\leq \alpha(x_{n_k}, x)d(Tx_{n_k}, Tx) \\ &\leq u(x_{n_k}, x) - \phi(u(x_{n_k}, x)) \\ &= \frac{1}{2}\left[d(x_{n_k}, Tx_{n_k}) + d(x, Tx)\right] - \phi\left\{\frac{1}{2}\left[d(x_{n_k}, Tx_{n_k}) + d(x, Tx)\right]\right\} \\ &\leq \frac{1}{2}\left[d(x_{n_k}, x_{n_{k+1}}) + d(x, x)\right] - \phi\left\{\frac{1}{2}\left[d(x_{n_k}, x_{n_{k+1}}) + d(x, x)\right]\right\}. \end{aligned}$$

Since

$$\begin{aligned} d(x, Tx) &\leq d(x, x_{n_k+1}) + d(x_{n_k+1}, Tx_{n_k}) + d(Tx_{n_k}, Tx) \\ &\leq d(x, x_{n_k+1}) + d(A, B) + d(Tx_{n_k}, Tx) \end{aligned}$$

it follows that

$$\begin{aligned} d(x,Tx)-d(x,x_{n_k+1})-d(A,B) &\leq d(Tx_{n_k},Tx) \\ &\leq \frac{1}{2}\Big[d(x_{n_k},x_{n_{k+1}})+d(x,x)\Big] \\ &\quad -\phi\left\{\frac{1}{2}\Big[d(x_{n_k},x_{n_{k+1}})+d(x,x)\Big]\right\}. \end{aligned}$$

Letting $k\to\infty$, we get

$$d(x,Tx)-d(A,B)=0.$$

Hence, $d(x,Tx)=d(A,B)$ and the proof is complete. □

Definition 3.3.4. Let $T:A\to B$ be non-self-mapping and $\alpha:A\times A\to[0,\infty)$. We say that T is (α,d)-regular if for all $(x,y)\in\alpha^{-1}([0,1[)$, there exists $z\in A_0$ such that

$$\alpha(x,z)\geq 1 \quad \text{and} \quad \alpha(y,z)\geq 1.$$

Theorem 3.3.5. *In addition to the hypotheses of the theorem (3.3.2) (resp. Theorem(3.3.3)), suppose that T is $(\alpha,d)-$regular. Then T has a unique best proximity point.*

Proof. We shall only proof the part of uniqueness. Suppose that there exist x and x^* in A which are distinct best proximity points, that is

$$d(x,Tx)=d(A,B) \quad \text{and} \quad d(x^*,Tx^*)=d(A,B).$$

Using the weak $P-$property and (3.3.9), we get that

$$d(x,x^*)=d(Tx,Tx^*). \tag{3.3.9}$$

We distinguish two cases. Case 1. If $\alpha(x,x^*)\geq 1$.
Since T is a Kannan $\alpha-$admissible weak $\phi-$contraction, using (3.3.9), we obtain that

$$\begin{aligned} d(x,x^*) &= d(Tx,Tx^*) \\ &\leq \alpha(x,x^*)d(Tx,Tx^*) \\ &\leq u(x,x^*)-\phi(u(x,x^*)) \\ &= \frac{1}{2}\big[d(x,Tx)+d(x^*,Tx^*)\big]-\phi\left\{\frac{1}{2}\big[d(x,Tx)+d(x^*,Tx^*)\big]\right\}. \end{aligned}$$

Since $\phi(t) < t$ for all $t > 0$, the above inequality holds only if $d(x, x^*) = 0$, that is $x = x^*$.

Case 2. If $\alpha(x, x^*) < 1$.

By hypothesis, there exists $z \in A_0$ such that $\alpha(x, z) \geq 1$ and $\alpha(x^*, z) \geq 1$. Since $T(A_0) \subseteq B_0$, there exists $v_0 \in A_0$ such that

$$d(v_0, Tz) = d(A, B).$$

Now,we have

$$\begin{aligned} \alpha(x, z) &\geq 1, \\ d(x, Tx) &= d(A, B), \\ d(v_0, Tz) &= d(A, B). \end{aligned}$$

Since T is $\alpha-$proximal admissible, we get that $\alpha(x, v_0) \geq 1$.

Since $T(A_0) \subseteq B_0$, there exists $v_1 \in A_0$ such that

$$d(v_1, Tv_0) = d(A, B).$$

By similar argument as above, we can conclude that $\alpha(x, v_1) \geq 1$. One can proceed further in a similar fashion to find v_n in A_0 with $v_{n+1} \in A_0$ such that

$$d(v_{n+1}, Tv_n) = d(A, B) \quad \text{and} \quad \alpha(x, v_n) \geq 1 \quad \forall n \in \mathbb{N}. \tag{3.3.10}$$

Using the weak $P-$property and (3.3.10), we get that

$$d(x, v_{n+1}) = d(Tx_n, Tv_n).$$

Since T is $\alpha-$proximal admissible, we have

$$\begin{aligned} d(x, v_{n+1}) &= d(Tx, Tv_n) \\ &\leq \alpha(x, v_n) d(Tx, Tv_n) \\ &\leq u(x, v_n) - \phi(u(x, v_n)) \\ &= \frac{1}{2}[d(x, Tx) + d(v_n, Tv_n)] \\ &\quad -\phi\left\{\frac{1}{2}[d(x, Tx) + d(v_n, Tv_n)]\right\}. \end{aligned} \tag{3.3.11}$$

If $v_N = x$, for some $N \in \mathbb{N}$. By (3.3.11), we get

$$d(x, v_{N+1}) = d(Tx, Tv_N) = 0$$

which implies that $v_{N+1} = x$. Moreover, we obtain $v_n = x$ for all $n \geq N$ and thus $v_n \to x$ as $n \to \infty$. By the uniqueness of limit, we conclude that $x = x^*$ and this completes the proof. $\square$

3.4 Existence and Uniqueness of Coupled Best Proximity Point in Ordered Metric Spaces

In this section, we introduce the existence and uniqueness of coupled best proximity point for mappings satisfying proximally coupled weak contraction in a complete ordered metric space.

Let $(X, d, \preceq)$ be a partially ordered complete metric space. Further, we endow the product space $X \times X$ with the following partial order:

$$\text{for } (x, y), (u, v) \in X \times X, \;\; (u, v) \preceq (x, y) \Leftrightarrow x \preceq u, y \succeq v.$$

Theorem 3.4.1. *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let A and B be nonempty closed subsets of the metric space (X, d) such that $A_0 \neq \emptyset$. Let $F : A \times A \to B$ satisfy the following conditions.*

(i) F is continuous having the proximal mixed monotone property and proximally coupled weak contraction on A such that $F(A_0 \times A_0) \subseteq B_0$.

(ii) There exist elements (x_0, y_0) and (x_1, y_1) in $A_0 \times A_0$ such that

$$d(x_1, F(x_0, y_0)) = d(A, B) \text{ with } x_0 \preceq x_1 \text{ and}$$

$$d(y_1, F(y_0, x_0)) = d(A, B) \text{ with } y_0 \preceq y_1.$$

Then there exist $(x, y) \in A \times A$ such that $d(x, F(x, y)) = d(A, B)$ and $d(y, F(y, x)) = d(A, B)$.

Proof. By hypothesis there exist elements (x_0, y_0) and (x_1, y_1) in $A_0 \times A_0$ such that

$$d(x_1, F(x_0, y_0)) = d(A, B) \text{ with } x_0 \preceq x_1 \text{ and}$$

$$d(y_1, F(y_0, x_0)) = d(A, B) \text{ with } y_0 \succeq y_1.$$

Because of the fact that $F(A_0 \times A_0) \subseteq B_0$, there exists an element (x_2, y_2) in $A_0 \times A_0$ such that

$$d(x_2, F(x_1, y_1) = d(A, B) \text{ and}$$

$$d(y_2, F(y_1, x_1) = d(A, B).$$

Hence from Lemma 2.2.18 and Lemma 2.2.19, we obtain $x_1 \preceq x_2$ and $y_1 \succeq y_2$. Continuing this process, we can construct the sequences (x_n) and (y_n) in A_0 such that

$$d(x_{n+1}, F(x_n, y_n)) = d(A, B),\ \forall n \in \mathbb{N} \tag{3.4.1}$$

$$\text{with } x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \cdots \text{ and}$$

$$d(y_{n+1}, F(y_n, x_n)) = d(A, B),\ \forall n \in \mathbb{N} \tag{3.4.2}$$

$$\text{with } y_0 \succeq y_1 \succeq y_2 \succeq \cdots \succeq y_n \succeq y_{n+1} \cdots .$$

Since $d(x_n, F(x_{n-1}, y_{n-1})) = d(A, B)$, $d(x_{n+1}, F(x_n, y_n)) = d(A, B)$ and also we have $x_{n-1} \preceq x_n, y_{n-1} \succeq y_n,\ \forall\, n \in \mathbb{N}$. Now using F is proximally coupled weak contraction on A we get,

$$\psi(d(x_n, x_{n+1})) \leq \psi(\max(d(x_{n-1}, x_n), d(y_{n-1}, y_n))) - \phi(\max(d(x_{n-1}, x_n), d(y_{n-1}, y_n))). \tag{3.4.3}$$

As $\phi \geq 0$,

$$\psi(d(x_n, x_{n+1})) \leq \psi(\max(d(x_{n-1}, x_n), d(y_{n-1}, y_n)))$$

and, using the fact that ϕ is nondecreasing, we have

$$d(x_n, x_{n+1}) \leq \max(d(x_{n-1}, x_n), d(y_{n-1}, y_n)). \tag{3.4.4}$$

Similarly, since $x_{n-1} \leq x_n, y_{n-1} \geq y_n$, we get

$$\psi(d(y_n, y_{n+1})) \leq \psi(\max(d(y_{n-1}, y_n), d(x_{n-1}, x_n))) - \phi(\max(d(y_{n-1}, y_n), d(x_{n-1}, x_n))), \tag{3.4.5}$$

$$\leq \psi(\max(d(y_{n-1}, y_n), d(x_{n-1}, x_n)))$$

and consequently,

$$d(y_n, y_{n+1}) \leq \max(d(y_{n-1}, y_n), d(x_{n-1}, x_n)). \tag{3.4.6}$$

By (3.4.4) and (3.4.6), we get

$$\max(d(x_n, x_{n+1}), d(y_n, y_{n+1})) \leq \max(d(x_{n-1}, x_n), d(y_{n-1}, y_n)),$$

and, thus, the sequence $\{\max(d(x_n, x_{n+1}), d(y_n, y_{n+1}))\}$ is nonnegative decreasing. This implies that there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \max(d(x_n, x_{n+1}), d(y_n, y_{n+1})) = r. \tag{3.4.7}$$

One can see that if $\psi : [0, \infty] \to [0, \infty]$ is nondecreasing,

$$\psi(\max(a, b)) = \max(\psi(a), \psi(b))$$

for $a, b \in [0, \infty]$. Taking into account this and (3.4.3) and (3.4.5), we get

$$\begin{aligned}
\max(\psi(d(x_n, x_{n+1})), \psi(d(y_n, y_{n+1}))) &= \psi(\max(d(x_n, x_{n+1}), d(y_n, y_{n+1}))) \\
&\leq \psi(\max(d(x_{n-1}, x_n), d(y_{n-1}, y_n))) \\
&- \phi(\max(d(x_{n-1}, x_n), d(y_{n-1}, y_n))) \\
&\leq \psi(\max(d(x_{n-1}, x_n), d(y_{n-1}, y_n))).
\end{aligned}$$

Letting $n \to \infty$ and taking into account (3.4.7), we get

$$\psi(r) \leq \psi(r) - \lim_{n\to\infty} \phi(\max(d(x_{n-1}, x_n), d(y_{n-1}, y_n))) \leq \psi(r)$$

and this implies

$$\lim_{n\to\infty} \phi(\max(d(x_{n-1}, x_n), d(y_{n-1}, y_n))) = 0. \tag{3.4.8}$$

But, as $0 < r \leq \max(d(x_{n-1}, x_n), d(y_{n-1}, y_n))$ and ϕ is nondecreasing function,

$$0 < \phi(r) \leq \phi(\max(d(x_{n-1}, x_n), d(y_{n-1}, y_n))),$$

and this gives us $\lim_{n\to\infty} \phi(\max(d(x_{n-1}, x_n), d(y_{n-1}, y_n))) \geq \phi(r) > 0$ which contradicts to (3.4.8). Hence,

$$\lim_{n\to\infty} \max(d(x_n, x_{n+1}), d(y_n, y_{n+1})) = 0. \tag{3.4.9}$$

Now to prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequence. Assume that at least one of the sequences $\{x_n\}$ or $\{y_n\}$ is not a Cauchy sequence. This implies that $\lim_{n,m\to\infty} d(x_n, x_m) \nrightarrow 0$ or $\lim_{n,m\to\infty} d(y_n, y_m) \nrightarrow 0$, and, consequently,

$$\lim_{n,m\to\infty} \max(d(x_n, x_m), d(y_n, y_m)) \nrightarrow 0.$$

Then there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k)$ is smallest index for which $n(k) > m(k) > k$,

$$\max(d(x_{m(k)}, x_{n(k)}), d(y_{m(k)}, y_{n(k)})) \geq \epsilon. \tag{3.4.10}$$

This means that

$$\max(d(x_{m(k)}, x_{n(k)-1}), d(y_{m(k)}, y_{n(k)-1})) < \epsilon. \tag{3.4.11}$$

Since $x_{n(k)-1} \geq x_{m(k)-1}$ and $y_{n(k)-1} \leq y_{m(k)-1}$, using the proximally coupled weak contraction, we obtain

$$\begin{aligned}\psi(d(x_{n(k)}, x_{m(k)})) \leq \psi(&\max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})))\\ &- \phi(\max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})))\end{aligned} \tag{3.4.12}$$

and

$$\begin{aligned}\psi(d(y_{n(k)}, y_{m(k)})) \leq \psi(&\max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})))\\ &- \phi(\max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1}))).\end{aligned} \tag{3.4.13}$$

By (3.4.12) and (3.4.13), we get

$$\begin{aligned}\max(\psi(d(x_{n(k)}, x_{m(k)}), \psi(d(y_{n(k)}, y_{m(k)})) \leq \psi(&\max(d(x_{n(k)-1}, x_{m(k)-1}),\\ &d(y_{n(k)-1}, y_{m(k)-1})))\\ &- \phi(\max(d(x_{n(k)-1}, x_{m(k)-1}),\\ &d(y_{n(k)-1}, y_{m(k)-1}))).\end{aligned}$$

On the other hand, the triangular inequality and (3.4.11) give us

$$d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) < d(x_{n(k)}, x_{n(k)-1}) + \epsilon \tag{3.4.14}$$

and

$$d(y_{n(k)}, y_{m(k)}) \leq d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)}) < d(y_{n(k)}, y_{n(k)-1}) + \epsilon. \tag{3.4.15}$$

From (3.4.10),(3.4.14) and (3.4.15), we get

$$\epsilon \leq \max(d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})) \leq \max(d(x_{n(k)}, x_{n(k)-1}), d(y_{n(k)}, y_{n(k)-1})) + \epsilon.$$

Letting $k \to \infty$ in the last inequality and using (3.4.9), we have

$$\lim_{k\to\infty} \max(d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})) = \epsilon. \tag{3.4.16}$$

Again, the triangular inequality and (3.4.11) give us

$$d(x_{n(k)-1}, x_{m(k)-1}) \leq d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)-1}) < \epsilon + d(x_{m(k)}, x_{m(k)-1}) \tag{3.4.17}$$

and

$$d(y_{n(k)-1}, y_{m(k)-1}) \leq d(y_{n(k)-1}, y_{m(k)}) + d(y_{m(k)}, y_{m(k)-1}) < \epsilon + d(y_{m(k)}, y_{m(k)-1}). \tag{3.4.18}$$

By (3.4.17) and (3.4.18), we get

$$\begin{aligned} &\max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})) \\ &< \max(d(x_{m(k)}, x_{m(k)-1}), d(y_{m(k)}, y_{m(k)-1})) + \epsilon. \end{aligned} \tag{3.4.19}$$

Using the triangular inequality we have

$$d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)})$$

and

$$d(y_{n(k)}, y_{m(k)}) \leq d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)})$$

and by the two last inequalities and (3.4.10) we get

$$\begin{aligned} \epsilon &\leq \max(d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})) \\ &\leq \max(d(x_{n(k)}, x_{n(k)-1}), d(y_{n(k)}, y_{n(k)-1})) + \max(d(x_{n(k)-1}, x_{m(k)-1}), \\ &\quad d(y_{n(k)-1}, y_{m(k)-1})) + \max(d(x_{m(k)-1}, x_{m(k)}), d(y_{m(k)-1}, y_{m(k)})). \end{aligned} \tag{3.4.20}$$

By (3.4.19) and (3.4.20), we get

$$\begin{aligned} &\epsilon - \max(d(x_{n(k)}, x_{n(k)-1}), d(y_{n(k)}, y_{n(k)-1})) - \max(d(x_{m(k)-1}, x_{m(k)}), d(y_{m(k)-1}, y_{m(k)})) \\ &\leq \max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})) \\ &< \max(d(x_{m(k)}, x_{m(k)-1}), d(y_{m(k)}, y_{m(k)-1})) + \epsilon. \end{aligned}$$

Letting $k \to \infty$ in the last inequality and using (3.4.9), we have

$$\lim_{k\to\infty} \max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})) = \epsilon. \tag{3.4.21}$$

Finally, letting $k \to \infty$ in (3.4.18) and using (3.4.16),(3.4.21) and the continuity of ψ, we get

$$\psi(\epsilon) \leq \psi(\epsilon) - \lim_{k\to\infty} \phi(\max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1}))) \leq \psi(\epsilon)$$

and this implies

$$\lim_{k\to\infty} \phi(\max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1}))) = 0. \tag{3.4.22}$$

But, from $\lim_{k\to\infty} \max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})) = \epsilon$, we can find $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$

$$\frac{\epsilon}{2} \leq \max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1}))$$

and consequently,

$$0 < \phi(\frac{\epsilon}{2}) \leq \phi(\max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1}))) \text{ for } k \geq k_0.$$

Therefore, $0 < \phi(\frac{\epsilon}{2}) \leq \phi(\max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})))$ and this contradicts (3.4.22). Therefore, the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy.

Since A is closed subset of a complete metric space X, these sequences have limits. Thus, there exists $x, y \in A$ such that $x_n \to x$ and $y_n \to y$. Therefore $(x_n, y_n) \to (x, y)$ in $A \times A$. Since F is continuous, we have $F(x_n, y_n) \to F(x, y)$ and $F(y_n, x_n) \to F(y, x)$. Hence the continuity of the metric function d implies that $d(x_{n+1}, F(x_n, y_n)) \to d(x, F(x, y))$ and $d(y_{n+1}, F(y_n, x_n)) \to d(y, F(y, x))$.

But from equations (3.4.1) and (3.4.2) we get, the sequences $d(x_{n+1}, F(x_n, y_n))$ and $d(y_{n+1}, F(y_n, x_n))$ are constant sequences with the value $d(A, B)$. Therefore, $d(x, F(x, y)) = d(A, B)$ and $d(y, F(y, x) = d(A, B)$. This completes the proof of the theorem. □

Corollary 3.4.2. *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let A be nonempty closed subsets of the metric space (X, d) . Let $F : A \times A \to A$ satisfy the following conditions.*

(i) F is continuous having the proximal mixed monotone property and proximally coupled weak contraction on A.

(ii) There exist (x_0, y_0) and (x_1, y_1) in $A \times A$ such that $x_1 = F(x_0, y_0)$ with $x_0 \preceq x_1$ and $y_1 = F(y_0, x_0)$ with $y_0 \succeq y_1$.

Then there exist $(x, y) \in A \times A$ such that $d(x, F(x, y)) = 0$ and $d(y, F(y, x)) = 0$.

In what follows we prove that Theorem 3.4.1 is still valid for F not necessarily continuous, assuming the following hypothesis in A. A has the property that

$$\bullet (x_n) \text{ is a non-decreasing sequence in } A \text{ such that } x_n \to x, \text{ then } x_n \preceq x. \quad (3.4.23)$$

$$\bullet (y_n) \text{ is a non-increasing sequence in } A \text{ such that } y_n \to y, \text{ then } y \preceq y_n. \quad (3.4.24)$$

Theorem 3.4.3. *Assume the condition (3.4.23),(3.4.24) and A_0 is closed in X instead of continuity of F in the Theorem 3.4.1.*

Proof. Following the proof of Theorem 3.4.1, there exists sequences $\{x_n\}$ and $\{y_n\}$ in A satisfying the following condition

$$d(x_{n+1}, F(x_n, y_n)) = d(A, B) \text{ with } x_n \preceq x_{n+1}, \forall n \in \mathbb{N} \tag{3.4.25}$$

and

$$d(y_{n+1}, F(y_n, x_n)) = d(A, B) \text{ with } y_n \succeq y_{n+1}, \forall n \in \mathbb{N}. \tag{3.4.26}$$

Also, x_n converges to x and y_n converges to y in A. From (3.4.23) and (3.4.24), we get $x_n \leq x$ and $y_n \geq y$. Note that the sequences $\{x_n\}$ and $\{y_n\}$ are in A_0 and A_0 is closed. Therefore, $(x, y) \in A_0 \times A_0$. Since $F(A_0 \times A_0) \subseteq B_0$, we get $F(x, y)$ and $F(y, x)$ are in B_0. Therefore, there exists $(x^*, y^*) \in A_0 \times A_0$ such that

$$d(x^*, F(x, y)) = d(A, B) \tag{3.4.27}$$

and

$$d(y^*, F(y, x)) = d(A, B). \tag{3.4.28}$$

Since $x_n \preceq x$ and $y_n \succeq y$. By using F is proximally coupled weak contraction for (3.4.25) and (3.4.27), we get

$$\begin{aligned} \psi(d(x_{n+1}, x^*)) \leq \psi(&\max(d(x_n, x), d(y_n, y))) \\ &- \phi(\max(d(x_n, x), d(y_n, y))). \end{aligned} \tag{3.4.29}$$

Letting $n \to \infty$ in (3.4.29) and using continuity of ψ, we get

$$\psi(d(x, x^*)) \leq 0 - \lim_{n\to\infty} \phi(\max(d(y, y_n), d(x, x_n))) \leq 0.$$

Using $\psi(t) = 0$ iff $t = 0$, we get $d(x, x^*) = 0$, consequently, $x = x^*$. Similarly it can be proved that $y = y*$. Using these to (3.4.27) and (3.4.28),
we get $d(x, F(x, y)) = d(A, B)$ and $d(y, F(y, x)) = d(A, B)$. □

Corollary 3.4.4. *Assume the condition (3.4.23) and (3.4.24) instead of continuity of F in the Corollary 3.4.2.*

Now, we present an example where it can be appreciated that hypotheses in Theorem 3.4.1 and Theorem 3.4.3 do not guarantee uniqueness of the coupled best proximity point.

Example. Let $X = \{(0,1),(1,0),(-1,0),(0,-1)\} \subset \mathbb{R}^2$ and consider the usual order $(x,y) \preceq (z,t) \Leftrightarrow x \leq z$ and $y \leq t$.

Thus, $(X,\preceq)$ is a partially ordered set. Besides, (X,d_2) is a complete metric space when d_2 is the Euclidean metric. Let $A = \{(0,1),(1,0)\}$ and $B = \{(0,-1),(-1,0)\}$ be a closed subset of X. Then, $d(A,B) = \sqrt{2}$, $A = A_0$ and $B = B_0$. Let $F : A \times A \to B$ be defined by $F((x_1,x_2),(y_1,y_2)) = (-x_2,-x_1)$. Then, it can be seen that F is continuous such that $F(A_0 \times A_0) \subseteq B_0$. The only comparable pairs of points in A are $x \preceq x$ for $x \in A$, hence proximal mixed monotone property is satisfied trivially and also proximally coupled weak contraction is fulfilled for arbitrary control functions.

It can be shown that the other hypotheses of the theorem are also satisfied. However, F has three coupled best proximity points $((0,1),(0,1))$, $((0,1),(1,0))$ and $((1,0),(1,0))$.

One can prove that the coupled best proximity point is in fact unique, provided that the product space $A \times A$ endowed with the partial order mentioned earlier has the following property:

Every pair of elements has either a lower bound or an upper bound.

It is known that this condition is equivalent to :
For every pair of $(x,y),(x^*,y^*) \in A \times A$, there exists a (z_1,z_2) in $A \times A$, that is comparable to (x,y) and (x^*,y^*).

Theorem 3.4.5. *In addition to the hypothesis of Theorem 3.4.1(resp. Theorem 3.4.3), suppose that for every* (x,y) *and* (x^*,y^*) *in* $A_0 \times A_0$ *there exists* $(z_1,z_2) \in A_0 \times A_0$ *that is comparable to* (x,y) *and* (x^*,y^*) *then* F *has a unique coupled best proximity point of* F*.*

Proof. From Theorem 3.4.1(resp. Theorem 3.4.3), the set of coupled best proximity points of F is non-empty. Suppose that there exist (x,y) and (x^*,y^*) in A which are coupled best proximity points. That is,

$$d(x,F(x,y)) = d(A,B), d(y,F(y,x) = d(A,B)$$

and

$$d(x^*,F(x^*,y^*)) = d(A,B), d(y^*,F(y^*,x^*) = d(A,B).$$

We distinguish two cases:

Case:1 If (x, y) is comparable to (x^*, y^*) with respect to the ordering in $A \times A$. Using F is proximally coupled weak contraction to $d(x, F(x, y)) = d(A, B)$ and $d(x^*, F(x^*, y^*)) = d(A, B)$, we get

$$\psi(d(x, x^*)) \leq \psi(\max(d(x, x^*), d(y, y^*))) - \phi(\max(d(x, x^*), d(y, y^*))). \tag{3.4.30}$$

Similarly, one can prove that

$$\psi(d(y, y^*)) \leq \psi(\max(d(y, y^*), d(x, x^*))) - \phi(\max(d(y, y^*), d(x, x^*))). \tag{3.4.31}$$

From (3.4.30) and (3.4.31), we get

$$\max(\psi(d(x, x^*)), \psi(d(y, y^*)) \leq \psi(\max(d(y, y^*), d(x, x^*))) - \phi(\max(d(y, y^*), d(x, x^*))).$$

Using $\psi(\max(a, b)) = \max(\psi(a), \psi(b))$ for $a, b \in [0, \infty]$, we get

$$\psi(\max(d(x, x^*), d(y, y^*))) \leq \psi(\max(d(y, y^*), d(x, x^*))) - \phi(\max(d(y, y^*), d(x, x^*)))$$

this implies that $\phi(\max(d(y, y^*), d(x, x^*))) \leq 0$, using the property of ϕ, we get $\max(d(y, y^*), d(x, x^*)) = 0$. Hence, $x = x^*$ and $y = y*$.

Case:2 If (x, y) is not comparable to (x^*, y^*), then there exists $(u_1, v_1) \in A_0 \times A_0$ which is comparable to (x, y) and (x^*, y^*).

Since $F(A_0 \times A_0) \subseteq B_0$, there exists $(u_2, v_2) \in A_0 \times A_0$ such that $d(u_2, F(u_1, v_1)) = d(A, B)$ and $d(v_2, F(v_1, u_1)) = d(A, B)$. With out loss of generality assume that $(u_1, v_1) \leq (x, y)$(i.e., $x \geq u_1$ and $y \leq v_1$.) Note that $(u_1, v_1) \leq (x, y)$ implies that $(y, x) \leq (v_1, u_1)$. From Lemma 2.2.18 and Lemma 2.2.19, we get

$$\left.\begin{aligned} u_1 \leq x \text{ and } v_1 \geq y \\ d(u_2, F(u_1, v_1)) = d(A, B) \\ d(x, F(x, y)) = d(A, B) \end{aligned}\right\} \implies u_2 \leq x$$

and

$$\left.\begin{aligned} u_1 \leq x \text{ and } v_1 \geq y \\ d(v_2, F(v_1, u_1)) = d(A, B) \\ d(y, F(y, x)) = d(A, B) \end{aligned}\right\} \implies v_2 \geq y.$$

From the above to inequalities, we obtain $(u_2, v_2) \leq (x, y)$. Continuing this process, we get sequences $\{u_n\}$ and $\{v_n\}$ such that $d(u_{n+1}, F(u_n, v_n)) = d(A, B)$ and

$d(v_{n+1}, F(v_n, u_n)) = d(A, B)$ with $(u_n, v_n) \leq (x, y)$ $\forall n \in \mathbb{N}$. Using F is proximally coupled weak contraction, we get

$$\left.\begin{aligned} u_n \leq x \text{ and } v_n \geq y \\ d(u_n, F(u_{n-1}, v_{n-1})) = d(A, B) \\ d(x, F(x, y)) = d(A, B) \end{aligned}\right\} \implies \begin{aligned} \psi(d(u_n, x)) \leq \psi(\max(d(u_{n-1}, x), d(v_{n-1}, y))) \\ -\phi(\max(d(u_{n-1}, x), d(v_{n-1}, y))). \end{aligned} \tag{3.4.32}$$

Similarly, we can prove that

$$\left.\begin{aligned} y \leq v_n \text{ and } x \geq u_n \\ d(y, F(y, x)) = d(A, B) \\ d(v_n, F(v_{n-1}, u_{n-1})) = d(A, B) \end{aligned}\right\} \implies \begin{aligned} \psi(d(y, v_n)) \leq \psi(\max(d(y, v_{n-1}), d(x, u_{n-1}))) \\ -\phi(\max(d(y, v_{n-1}), d(x, u_{n-1}))). \end{aligned}$$

From(3.4.32) and (3.4.33), we obtain

$$\begin{aligned} \max(\psi(d(u_n, x)), \psi(d(y, v_n))) \leq \psi(\max(d(u_{n-1}, x), d(v_{n-1}, y))) \\ - \phi(\max(d(u_{n-1}, x), d(v_{n-1}, y))). \end{aligned}$$

But, $\psi(\max(a, b)) = \max(\psi(a), \psi(b))$ for $a, b \in [0, \infty]$, hence

$$\begin{aligned} \psi(\max(d(u_n, x), d(y, v_n))) &\leq \psi(\max(d(u_{n-1}, x), d(v_{n-1}, y))) \\ &\quad - \phi(\max(d(u_{n-1}, x), d(v_{n-1}, y))) \\ &\leq \psi(\max(d(u_{n-1}, x), d(v_{n-1}, y))). \end{aligned} \tag{3.4.33}$$

By using ψ is nondecreasing function, we get the sequence $\{\max(d(u_n, x), d(y, v_n))\}$ is nonnegative decreasing and bounded.

This implies that there exists $r \geq 0$ such that

$$\lim_{n \to \infty} \max(d(u_n, x), d(y, v_n)) = r \geq 0.$$

Suppose $\lim_{n \to \infty} \max(d(u_n, x), d(y, v_n)) = r > 0$.

Letting $n \to \infty$ in (3.4.33) and using the continuity of ψ, we get

$$\psi(r) \leq \psi(r) - \lim_{n \to \infty} \phi(\max(d(u_{n-1}, x), d(v_{n-1}, y))) \leq \psi(r).$$

This implies that

$$\lim_{n \to \infty} \phi(\max(d(u_{n-1}, x), d(v_{n-1}, y))) = 0. \tag{3.4.34}$$

But $0 < r \leq \max(d(u_n, x), d(y, v_n))$ and ϕ is nondecreasing function, hence

$$0 < \phi(r) \leq \phi(\max(d(u_n, x), d(y, v_n)))$$

and this gives us $\lim_{n\to\infty} \phi(\max(d(u_{n-1}, x), d(v_{n-1}, y))) \geq \phi(r) > 0$ which contradicts (3.4.34).

Hence,

$$\lim_{n\to\infty} \max(d(u_n, x), d(y, v_n)) = 0.$$

That is $u_n \to x$ and $v_n \to y$. Analogously, one can prove that $u_n \to x^*$ and $v_n \to y^*$. But the limit of the sequence is unique in metric space. Therefore, $x = x^*$ and $y = y^*$. Hence the proof. □

The following result, due to Harjani et.al in [12], is a corollary from the Theorem 3.4.5 by taking $A = B$.

Corollary 3.4.6. *In addition to the hypothesis of Corollary 3.4.2(resp. Corollary 3.4.4), suppose that for any two elements (x, y) and (x^*, y^*) in $A \times A$, there exists $(z_1, z_2) \in A \times A$ such that (z_1, z_2) is comparable to (x, y) and (x^*, y^*) then F has a unique coupled fixed point.*

CHAPTER 4 CONCLUSIONS

In this chapter, we conclude all the theorems obtained in this dissertation as follows:

(1) Let (X, d) be a complete metric space and let A and B be nonempty, closed subsets of X such that A_0 and B_0 are non-empty and A and B satisfies property $\star$. Let $S : A \to B$, $T : B \to A$ and $g : A \cup B \to A \cup B$ satisfy the following conditions:

(a) S and T are generalized proximal $\alpha - \psi-$contraction of the first kind with $\alpha-$proximal admissible;

(b) g is an isometry;

(c) $S(A_0) \subseteq B_0, T(B_0) \subseteq A_0$;

(d) $A_0 \subseteq g(A_0)$ and $B_0 \subseteq g(B_0)$;

(e) There exist elements x_0 and x_1 in A_0 such that $d(gx_1, Sx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$.

(f) S and T satisfies property $\Box_T$.

Then, there exists a unique point x in A and there exists a unique point $y \in B$ such that

$$d(gx, Sx) = d(gy, Ty) = d(x, y) = d(A, B).$$

Moreover, for any fixed x_0 in A_0, the sequence $\{x_n\}$, defined by

$$d(gx_{n+1}, Sx_n) = d(A, B),$$

converges to the element x. For any fixed y_0 in B_0, the sequence $\{y_n\}$, defined by

$$d(gy_{n+1}, Ty_n) = d(A, B),$$

converges to the element y. On the other hand, a sequence $\{u_n\}$ in A converges to x with $\alpha(x_n, u_n) \geq 1$, if there is a sequence of positive numbers $\{\epsilon_n\}$ such that

$$\lim_{n\to\infty} \epsilon_n = 0 \text{ and } d(u_{n+1}, z_{n+1}) \leq \epsilon_n,$$

where z_{n+1} in A satisfies the condition that $d(gz_{n+1}, Su_n) = d(A, B)$.

(2) Let (X, d) be a complete metric space and A and B be nonempty, closed subsets of X. Further, suppose that A_0 and B_0 are non-empty and A and B satisfies property $\star$. Let $S : A \to B$ and $g : A \to A$ satisfy the following conditions:

(a) S is a generalized proximal $\alpha - \psi -$contractions of first and second kinds with $\alpha-$proximal admissible;

(b) g is an isometry;

(c) S preserves isometric distance with respect to g;

(d) $S(A_0) \subseteq B_0$;

(e) $A_0 \subseteq g(A_0)$.

(f) There exist elements x_0 and x_1 in A_0 such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$;

(g) S and T satisfies property $\Box_T$.

Then, there exists a unique point x in A such that

$$d(gx, Sx) = d(A, B).$$

Moreover, for any fixed x_0 in A_0, the sequence $\{x_n\}$, defined by

$$d(gx_{n+1}, Sx_n) = d(A, B),$$

converges to the element x. On the other hand, a sequence $\{u_n\}$ in A converges to x with $\alpha(x_n, u_n) \geq 1$,, if there is a sequence of positive numbers $\{\epsilon_n\}$ such that

$$\lim_{n\to\infty} \epsilon_n = 0 \text{ and } d(u_{n+1}, z_{n+1}) \leq \epsilon_n,$$

where z_{n+1} in A satisfies the condition that $d(gz_{n+1}, Su_n) = d(A, B)$.

(3) Let A and B be nonempty closed subsets of a complete metric space X such that A_0 is nonempty and the pair (A, B) has the $P-$ property. Let $T : A \to B$ satisfy the following conditions:

(a) T are α-proximal admissible and generalized almost $(\varphi, \theta)_\alpha-$contraction;

(b) T is continuous;

(c) there exist element x_0 and x_1 in A_0 such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$;

(d) $T(A_0) \subseteq B_0$.

Then there exists an element $x \in A$ such that

$$d(x, Tx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Tx_n) = d(A, B),$$

converges to the element x.

(4) Let A and B be nonempty closed subsets of a complete metric space X such that A_0 is nonempty and the pair (A, B) has the $P-$ property. Let $T : A \to B$ satisfy the following conditions:

(a) T are α-proximal admissible and generalized almost $(\varphi, \theta)_\alpha-$contraction;

(b) If $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ for some $x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k;

(c) there exist element x_0 and x_1 in A_0 such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$;

(d) $T(A_0) \subseteq B_0$.

Then there exists an element $x \in A$ such that

$$d(x, Tx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Tx_n) = d(A, B),$$

converges to the element x.

(5) Let A and B be nonempty closed subsets of a complete metric space X such that A_0 is nonempty and the pair (A, B) has the $P-$ property. Let $T : A \to B$ satisfy the following conditions:

(a) T are α-proximal admissible and generalized almost $(\varphi, \theta)_\alpha-$contraction;

(b) T is continuous;

(c) there exist element x_0 and x_1 in A_0 such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$;

(d) $T(A_0) \subseteq B_0$.

(e) T is $(\alpha, d)-$regular.

Then there exists an element $x \in A$ such that

$$d(x, Tx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Tx_n) = d(A, B),$$

converges to the element x.

(6) Let A and B be nonempty closed subsets of a complete metric space X such that A_0 is nonempty and the pair (A, B) has the $P-$ property. Let $T : A \to B$ satisfy the following conditions:

(a) T are α-proximal admissible and generalized almost $(\varphi, \theta)_\alpha-$contraction;

(b) If $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ for some $x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k;

(c) there exist element x_0 and x_1 in A_0 such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$;

(d) $T(A_0) \subseteq B_0$,

(e) T is $(\alpha, d)-$regular.

Then there exists an element $x \in A$ such that

$$d(x, Tx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Tx_n) = d(A, B),$$

converges to the element x.

(7) Let (X, d) be a complete metric space and let A and B be nonempty, closed subsets of X such that A_0 and B_0 are non-empty. Let $\alpha : A \times A \longrightarrow [0, +\infty)$ satisfy the following conditions:

(a) $T(A_0) \subseteq B_0$ and (A, B) satisfies the weak P-property;

(b) T is α-proximal admissible;

(c) There exist elements x_0 and x_1 in A_0 such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$;

(d) T is a continuous Kannan weak $\alpha - \phi-$contraction.

Then, there exists an element $x^\star \in A_0$ such that

$$d(x^\star, Tx^\star) = d(A, B)$$

(8) Let (X, d) be a complete metric space and let A and B be nonempty, closed subsets of X such that A_0 and B_0 are non-empty. Let $\alpha : A \times A \longrightarrow [0, +\infty)$ satisfy the following conditions:

(a) $T(A_0) \subseteq B_0$ and (A, B) satisfies the weak P-property;

(b) T is α-proximal admissible;

(c) There exist elements x_0 and x_1 in A_0 such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$;

(d) T is a Kannan weak $\alpha - \phi-$contraction;

(e) If $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of x_n such that $\alpha(x_{n_k}, x) \geq 1$ for all k.

Then, there exists an element $x^\star \in A_0$ such that

$$d(x^\star, Tx^\star) = d(A, B)$$

(9) Let (X, d) be a complete metric space and let A and B be nonempty, closed subsets of X such that A_0 and B_0 are non-empty. Let $\alpha : A \times A \longrightarrow [0, +\infty)$ satisfy the following conditions:

(a) $T(A_0) \subseteq B_0$ and (A, B) satisfies the weak P-property;

(b) T is α-proximal admissible;

(c) There exist elements x_0 and x_1 in A_0 such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 1$;

(d) T is a Kannan weak $\alpha - \phi-$contraction;

(e) T is $(\alpha, d)-$regular;

(f) If $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of x_n such that $\alpha(x_{n_k}, x) \geq 1$ for all k.

Then, there exists an element $x^\star \in A_0$ such that

$$d(x^\star, Tx^\star) = d(A, B)$$

(10) Let $(X, \leq, d)$ be a partially ordered complete metric space. Let A and B be nonempty closed subsets of the metric space (X, d) such that $A_0 \neq \emptyset$. Let $F : A \times A \to B$ satisfy the following conditions.

(i) F is continuous having the proximal mixed monotone property and proximally coupled weak contraction on A such that $F(A_0 \times A_0) \subseteq B_0$.

(ii) There exist elements (x_0, y_0) and (x_1, y_1) in $A_0 \times A_0$ such that

$$d(x_1, F(x_0, y_0)) = d(A, B) \text{ with } x_0 \leq x_1$$

and

$$d(y_1, F(y_0, x_0)) = d(A, B) \text{ with } y_0 \geq y_1.$$

Then there exist $(x, y) \in A \times A$ such that $d(x, F(x, y)) = d(A, B)$ and $d(y, F(y, x)) = d(A, B)$.

(11) Let $(X, \leq, d)$ be a partially ordered complete metric space. Let A and B be nonempty closed subsets of the metric space (X, d) such that $A_0 \neq \emptyset$. Let $F : A \times A \to B$ satisfy the following conditions.

(i) F is continuous having the proximal mixed monotone property and proximally coupled weak contraction on A such that $F(A_0 \times A_0) \subseteq B_0$.

(ii) There exist elements (x_0, y_0) and (x_1, y_1) in $A_0 \times A_0$ such that

$$d(x_1, F(x_0, y_0)) = d(A, B) \text{ with } x_0 \leq x_1$$

and

$$d(y_1, F(y_0, x_0)) = d(A, B) \text{ with } y_0 \geq y_1.$$

(iii) (x_n) is a non-decreasing sequence in A such that $x_n \to x$, then $x_n \leq x$. (y_n) is a non-increasing sequence in A such that $y_n \to y$, then $y \leq y_n$.

Then, there exist $(x, y) \in A \times A$ such that $d(x, F(x, y)) = d(A, B)$ and $d(y, F(y, x)) = d(A, B)$.

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