CHAPTER 3 METHODOLOGY

In this chapter, we will be describing the finite difference method 13-point stencil, the finite difference method 25-point stencil and the integrated radial basis functions for solving the biharmonic problem. The details of these topics are following.

We consider biharmonic equation in two dimensions as following.

$$\frac{\partial^4 u(x,y)}{\partial x^4}+2\frac{\partial^4 u(x,y)}{\partial x^2 \partial y^2}+\frac{\partial^4 u(x,y)}{\partial y^4}=f(x,y). \tag{3.1}$$

Let uniform Cartesian grid $N \times N$, we defined $0 \le x \le 1$ and $0 \le y \le 1$. Subject to the boundary conditions are given by

$$u\big|_{\Gamma u}=\bar{u},\ \frac{\partial u}{\partial \boldsymbol{n}}\bigg|_{\Gamma q}=\bar{q}. \tag{3.2}$$

Where $\boldsymbol{n}$ is outward unit normal vector to the boundary Γ
$\bar{u}$ is the prescribed potential on essential Γ_u
$\bar{q}$ is normal flux on essential Γ_q.

3.1 Finite difference approximation of biharmonic equation with 2^{nd} order accuracy

3.1.1 The finite difference approximation of $\frac{\partial^4 u(x,y)}{\partial x^4}$ with 2^{nd} order accuracy

Consider a grid node (i,j) an intersection of the grid lines i and j. Let h is the gird size $\left(h=(x-x_i)=(y-y_i)\right)$ and $2 \le i \le N-2$ and $2 \le j \le N-2$. Take the Taylor series expansion of $u_{i-1,j}, u_{i+1,j}, u_{i-2,j}$ and $u_{i+2,j}$ at $u(x_i, y_j)$.

$$u(x,y_j)=u_{i,j}+(x-x_i)\frac{\partial u_{i,j}}{\partial x}+\frac{(x-x_i)^2}{2!}\frac{\partial^2 u_{i,j}}{\partial x^2}+\frac{(x-x_i)^3}{3!}\frac{\partial^3 u_{i,j}}{\partial x^3}+\frac{(x-x_i)^4}{4!}\frac{\partial^4 u_{i,j}}{\partial x^4}+\ldots$$

$$u_{i+1,j}=u_{i,j}+h\frac{\partial u_{i,j}}{\partial x}+\frac{h^2}{2!}\frac{\partial^2 u_{i,j}}{\partial x^2}+\frac{h^3}{3!}\frac{\partial^3 u_{i,j}}{\partial x^3}+\frac{h^4}{4!}\frac{\partial^4 u_{i,j}}{\partial x^4}+\frac{h^5}{5!}\frac{\partial^5 u_{i,j}}{\partial x^5}+\ldots \tag{3.3}$$

$$u_{i-1,j}=u_{i,j}-h\frac{\partial u_{i,j}}{\partial x}+\frac{h^2}{2!}\frac{\partial^2 u_{i,j}}{\partial x^2}-\frac{h^3}{3!}\frac{\partial^3 u_{i,j}}{\partial x^3}+\frac{h^4}{4!}\frac{\partial^4 u_{i,j}}{\partial x^4}-\frac{h^5}{5!}\frac{\partial^5 u_{i,j}}{\partial x^5}+\ldots \tag{3.4}$$

$$u_{i-1,j}+u_{i+1,j}=2u_{i,j}+\frac{2h^2}{2!}\frac{\partial^2 u_{i,j}}{\partial x^2}+\frac{2h^4}{4!}\frac{\partial^4 u_{i,j}}{\partial x^4}+\frac{2h^6}{6!}\frac{\partial^6 u_{i,j}}{\partial x^6}+\frac{2h^8}{8!}\frac{\partial^8 u_{i,j}}{\partial x^8}+\ldots \tag{3.5}$$

$$u_{i+2,j}=2u_{i,j}+2h\frac{\partial u_{i,j}}{\partial x}+\frac{(2h)^2}{2!}\frac{\partial^2 u_{i,j}}{\partial x^2}+\frac{(2h)^3}{3!}\frac{\partial^3 u_{i,j}}{\partial x^3}+\frac{(2h)^4}{4!}\frac{\partial^4 u_{i,j}}{\partial x^4}+\ldots \tag{3.6}$$

$$u_{i-2,j}=2u_{i,j}-2h\frac{\partial u_{i,j}}{\partial x}+\frac{(2h)^2}{2!}\frac{\partial^2 u_{i,j}}{\partial x^2}-\frac{(2h)^3}{3!}\frac{\partial^3 u_{i,j}}{\partial x^3}+\frac{(2h)^4}{4!}\frac{\partial^4 u_{i,j}}{\partial x^4}+\ldots \tag{3.7}$$

$$u_{i-2,j}+u_{i+2,j}=4u_{i,j}+\frac{2(2h)^2}{2!}\frac{\partial^2 u_{i,j}}{\partial x^2}+\frac{2(2h)^4}{4!}\frac{\partial^4 u_{i,j}}{\partial x^4}+\frac{2(2h)^6}{6!}\frac{\partial^6 u_{i,j}}{\partial x^5}+\ldots \tag{3.8}$$

Multiply 4 into equation (3.5)

$$4u_{i-1,j}+4u_{i+1,j}=8u_{i,j}+\frac{4\left(2h^2\right)}{2!}\frac{\partial^2 u_{i,j}}{\partial x^2}+\frac{4\left(2h^4\right)}{4!}\frac{\partial^4 u_{i,j}}{\partial x^4}+\frac{4\left(2h^6\right)}{6!}\frac{\partial^6 u_{i,j}}{\partial x^6}+\ldots \tag{3.9}$$

Subtract equation (3.9) from equation (3.8)

$$\begin{aligned}\left(u_{i-2,j}+u_{i+2,j}\right)-\left(4u_{i-1,j}-4u_{i+1,j}\right)\approx -6u_{i.j}&+\frac{2\left(2^4-4\right)h^4}{4!}\frac{\partial^4 u_{i.j}}{\partial x^4}\\ &+\frac{2\left(2^6-4\right)h^6}{6!}\frac{\partial^6 u_{i.j}}{\partial x^6}+\frac{2\left(2^8-4\right)h^8}{8!}\frac{\partial^8 u_{i.j}}{\partial x^8}+\ldots\end{aligned}$$

$$\begin{aligned}\left(u_{i-2,j}-4u_{i-1,j}+6u_{i,j}-4u_{i+1,j}+u_{i+2,j}\right)\approx h^4\frac{\partial^4 u_{i.j}}{\partial x^4}&+\frac{2\left(2^6-4\right)h^6}{6!}\frac{\partial^6 u_{i.j}}{\partial x^6}\\ &+\frac{2\left(2^8-4\right)h^8}{8!}\frac{\partial^8 u_{i.j}}{\partial x^8}+\ldots\end{aligned}$$

$$\begin{aligned}\frac{1}{h^4}\left(u_{i-2,j}-4u_{i-1,j}+6u_{i,j}-4u_{i+1,j}+u_{i+2,j}\right)\approx \frac{\partial^4 u_{i.j}}{\partial x^4}&+\left[\frac{2\left(2^6-4\right)}{6!}\right]h^2\frac{\partial^6 u_{i.j}}{\partial x^6}\\ &+\left[\frac{2\left(2^8-4\right)}{8!}\right]h^4\frac{\partial^8 u_{i.j}}{\partial x^8}+\ldots\end{aligned} \tag{3.10}$$

So the finite difference approximation of $\left.\frac{\partial^4 u}{\partial x^4}\right|_{i,j}$ with 2^{nd} order accuracy is

$$\left.\frac{\partial^4 u}{\partial x^4}\right|_{i,j}\approx\frac{1}{h^4}\left(u_{i-2,j}-4u_{i-1,j}+u_{i,j}-4u_{i+1,j}+u_{i+2,j}\right)+O\left(h^2\right) \tag{3.11}$$

where $O\left(h^2\right)$ is a truncation error term proportional to h^2.

3.1.2 The finite difference approximation of $2\left.\frac{\partial^4 u}{\partial x^2\partial y^2}\right|_{i,j}$ with 2^{nd} order accuracy

$$\begin{aligned}2\left.\frac{\partial^4 u}{\partial x^2\partial y^2}\right|_{i,j}&\approx\frac{2}{h^2}\left[\frac{\partial^2}{\partial x^2}\left(u_{i,j-1}-2u_{i,j}+u_{i,j+1}\right)\right]+O\left(h^2\right)\\ &\approx\frac{2}{h^4}\Big[\left(u_{i-1,j-1}-2u_{i,j-1}+u_{i+1,j-1}\right)-2\left(u_{i-1,j}-2u_{i,j}+u_{i+1,j}\right)\\ &\quad+\left(u_{i-1,j+1}-2u_{i,j+1}+u_{i+1,j+1}\right)\Big]+O\left(h^2\right)\\ &\approx\frac{1}{h^4}\Big[2\left(u_{i-1,j-1}+u_{i+1,j-1}+u_{i-1,j+1}+u_{i+1,j+1}\right)\\ &\quad-4\left(u_{i-1,j}+u_{i+1,j}+u_{i,j-1}+u_{i-1,j+1}\right)+8u_{i,j}\Big]+O\left(h^2\right).\end{aligned} \tag{3.12}$$

3.1.3 The finite difference approximation of $\left.\frac{\partial^4 u}{\partial y^4}\right|_{i,j}$ with 2^{nd} order accuracy

Substitution $\left.\frac{\partial^4 u}{\partial y^4}\right|_{i,j}$ into equation (3.10) by holding i fixed and differentiating with respect to j.

$$\left.\frac{\partial^4 u}{\partial y^4}\right|_{i,j} \approx \frac{1}{h^4}(u_{i,j-2} - 4u_{i,j-1} + u_{i,j} - 4u_{i,j+1} + u_{i,j+2}) + O\left(h^2\right). \tag{3.13}$$

Adding equations (3.11), (3.12) and (3.13), so the finite difference approximation of biharmonic equation with 2^{nd} order accuracy is expressed as following

$$\begin{aligned}&\left(u_{i-2,j} + u_{i+2,j} + u_{i,j-2} + u_{i,j+2}\right) - 8\left(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right)\\ &+2\left(u_{i-1,j-1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i+1,j+1}\right) + 20u_{i,j} = h^4 f_{ij}\end{aligned} \tag{3.14}$$

$$\left\{\begin{matrix} & & 1 & & \\ & 2 & -8 & 2 & \\ 1 & -8 & 20 & -8 & 1 \\ & 2 & -8 & 2 & \\ & & 1 & & \end{matrix}\right\} u_{i,j} + O\left(h^2\right) = h^4 f_{i,j}.$$

Figure 3.1 Diagram of biharmonic equation with 2^{nd} order accuracy.

3.2 Finite difference approximation of biharmonic equation with 4^{th} order accuracy

3.2.1 The finite difference approximation of $\left.\frac{\partial^4 u}{\partial x^4}\right|_{i,j}$ with 4^{th} order accuracy

From Taylor series in equation (3.10) uniform of finite difference approximation of $\frac{\partial^4 u}{\partial x^4}$ with 4^{th} order accuracy is following

$$\frac{1}{h^4}(u_{i-2,j} - 4u_{i-1,j} + 6u_{i,j} - 4u_{i+1,j} + u_{i+2,j}) \approx \frac{\partial^4 u_{i.j}}{\partial x^4} + \frac{12h^2}{72}\frac{\partial^6 u_{i.j}}{\partial x^6} + \frac{504h^4}{8!}\frac{\partial^8 u_{i.j}}{\partial x^8} + \ldots$$

$$\left.\frac{\partial^4 u}{\partial x^4}\right|_{i,j} \approx \frac{1}{h^4}\left(u_{i-2,j} - 4u_{i-1,j} + 6u_{i,j} - 4u_{i+1,j} + u_{i+2,j}\right) - \frac{12h^2}{72}\frac{\partial^6 u_{i.j}}{\partial x^6} + O\left(h^4\right)$$

$$\left.\frac{\partial^4 u}{\partial x^4}\right|_{i,j} \approx \frac{1}{h^4}\left(u_{i-2,j} - 4u_{i-1,j} + 6u_{i,j} - 4u_{i+1,j} + u_{i+2,j}\right) - \frac{12h^2}{72}\frac{\partial^2}{\partial x^2}\left(\frac{\partial^4 u_{i.j}}{\partial x^4}\right) + O\left(h^4\right)$$

$$\left.\frac{\partial^4 u}{\partial x^4}\right|_{i,j} \approx \frac{1}{h^4}\left(u_{i-2,j}-4u_{i-1,j}+6u_{i,j}-4u_{i+1,j}+u_{i+2,j}\right)$$
$$-\frac{12h^2}{72}\frac{\partial^2}{\partial x^2}\left(f_{i,j}-\frac{2\partial^4 u_{i.j}}{\partial x^2\partial y^2}-\frac{\partial^4 u_{i.j}}{\partial y^4}\right)+O\left(h^4\right). \tag{3.15}$$

Consider term of $\frac{\partial^2}{\partial x^2}\left(f_{i,j}-\frac{2\partial^4 u_{i.j}}{\partial x^2\partial y^2}-\frac{\partial^4 u_{i.j}}{\partial y^4}\right)$ in equation (3.15) we get

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{i,j} \approx \frac{1}{12h^2}\left(-f_{i-2,j}+16f_{i-1,j}-30f_{i,j}+16f_{i+1,j}-f_{i+2,j}\right) \tag{3.16}$$

$$2\left.\frac{\partial^6 u}{\partial x^4\partial y^2}\right|_{i,j} \approx \frac{2}{12h^2}\left(\frac{\partial^4}{\partial x^4}\left(-u_{i,j-2}+16u_{i,j-1}-30u_{i,j}+16u_{i,j+1}-u_{i,j+2}\right)\right)$$
$$\approx \frac{1}{12h^6}-\left(2\left(u_{i-2,j-2}-4u_{i-1,j-2}+6u_{i,j-2}-4u_{i+1,j-2}+u_{i+2,j-2}\right)\right.$$
$$+32\left(u_{i-2,j-1}-4u_{i-1,j-1}+6u_{i,j-1}-4u_{i+1,j-1}+u_{i+2,j-1}\right)$$
$$-60\left(u_{i-2,j}-4u_{i-1,j}+6u_{i,j}-4u_{i+1,j}+u_{i+2,j}\right)$$
$$+32\left(u_{i-2,j+1}-4u_{i-1,j+1}+6u_{i,j+1}-4u_{i+1,j+1}+u_{i+2,j+1}\right)$$
$$\left.-2\left(u_{i-2,j+2}-4u_{i-1,j+2}+6u_{i,j+2}-4u_{i+1,j+2}+u_{i+2,j+2}\right)\right) \tag{3.17}$$

$$\left.\frac{\partial^6 u}{\partial x^2\partial y^4}\right|_{i,j} \approx \frac{1}{h^4}\left(\frac{\partial^2}{\partial x^2}\left(u_{i,j-2}-4u_{i,j-1}+6u_{i,j}-4u_{i,j+1}+u_{i,j-2}\right)\right)$$
$$\approx \frac{1}{12h^6}\left(\left(-u_{i-2,j-2}+16u_{i-1,j-2}-30u_{i,j-2}+16u_{i+1,j-2}-u_{i+2,j-2}\right)\right.$$
$$-4\left(-u_{i-2,j-1}+16u_{i-1,j-1}-30u_{i,j-1}+16u_{i+1,j-1}-u_{i+2,j-1}\right)$$
$$+6\left(-u_{i-2,j}+16u_{i-1,j}-30u_{i,j}+16u_{i+1,j}-u_{i+2,j}\right)$$
$$-4\left(-u_{i-2,j+1}+16u_{i-1,j+1}-30u_{i,j+1}+16u_{i+1,j+1}-u_{i+2,j+1}\right)$$
$$\left.+\left(-u_{i-2,j+2}+16u_{i-1,j+2}-30u_{i,j+2}+16u_{i+1,j+2}-u_{i+2,j+2}\right)\right). \tag{3.18}$$

Substitution equation (3.16), (3.17) and (3.18) into equation (3.15), so the finite difference approximation of $\frac{\partial^4 u}{\partial x^4}$ with 4^{th} order accuracy is

$$
\begin{aligned}
\left.\frac{\partial^4 u}{\partial x^4}\right|_{i,j} &\approx \frac{1}{h^4}\left(u_{i-2,j}-4u_{i-1,j}+6u_{i,j}-4u_{i+1,j}+u_{i+2,j}\right)-\frac{12h^2}{72}\frac{\partial^2}{\partial x^2}\left(f_{i,j}-2\frac{\partial^4 u_{i.j}}{\partial x^2\partial y^2}-\frac{\partial^4 u_{i.j}}{\partial y^4}\right)\\
&\approx \frac{1}{72}\left(f_{i-2,j}-16f_{i-1,j}+30f_{i,j}-16f_{i+1,j}+f_{i+2,j}\right)\\
&+\frac{1}{12h^4}\Big(\left(-3u_{i-2,j-2}+24u_{i-1,j-2}-42u_{i,j-2}+24u_{i+1,j-2}-3u_{i+2,j-2}\right)\\
&+\left(36u_{i-2,j-1}-192u_{i-1,j-1}+312u_{i,j-1}-192u_{i+1,j-1}+36u_{i+2,j-1}\right)\\
&+\left(6u_{i-2,j}+48u_{i-1,j}-108u_{i,j}+48u_{i+1,j}+6u_{i+2,j}\right)\\
&+\left(36u_{i-2,j+1}-192u_{i-1,j+1}+312u_{i,j+1}-192u_{i+1,j+1}+36u_{i+2,j+1}\right)\\
&+\left(-3u_{i-2,j+2}+24u_{i-1,j+2}-42u_{i,j+2}+24u_{i+1,j+2}-3u_{i+2,j+2}\right)\Big)+O\left(h^4\right)
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
\left.\frac{\partial^4 u}{\partial x^4}\right|_{i,j} &\approx \left\{\begin{matrix}
-\frac{3}{72h^4} & \frac{24}{72h^4} & -\frac{42}{72h^4} & \frac{24}{72h^4} & -\frac{3}{72h^4}\\
\frac{36}{72h^4} & -\frac{192}{72h^4} & \frac{312}{72h^4} & -\frac{192}{72h^4} & \frac{36}{72h^4}\\
\frac{6}{72h^4} & \frac{48}{72h^4} & -\frac{108}{72h^4} & \frac{48}{72h^4} & \frac{6}{72h^4}\\
\frac{36}{72h^4} & -\frac{192}{72h^4} & \frac{312}{72h^4} & -\frac{192}{72h^4} & \frac{36}{72h^4}\\
-\frac{3}{72h^4} & \frac{24}{72h^4} & -\frac{42}{72h^4} & \frac{24}{72h^4} & -\frac{3}{72h^4}
\end{matrix}\right\} u_{i,j}\\
&+\left\{\begin{matrix}\frac{1}{72} & -\frac{16}{72} & \frac{30}{72} & -\frac{16}{72} & \frac{1}{72}\end{matrix}\right\} f_{i,j}+O\left(h^4\right).
\end{aligned}
$$

Figure 3.2 Diagram of $\left.\frac{\partial^4 u}{\partial x^4}\right|_{i,j}$ with 4^{th} order accuracy.

3.2.2 The finite difference approximation of $2\left.\frac{\partial^4 u}{\partial x^2 \partial y^2}\right|_{i,j}$ with 4^{th} order accuracy

$$
\begin{aligned}
2\left.\frac{\partial^4 u}{\partial x^2 \partial y^2}\right|_{i,j} &\approx \frac{2}{12h^2}\left(\frac{\partial^2}{\partial y^2}\left(-u_{i-2,j}+16u_{i-1,j}-30u_{i,j}+16u_{i+1,j}-u_{i+2,j}\right)\right)\\
&\approx \frac{1}{72h^4}\Big(\left(u_{i-2,j-2}-16u_{i-1,j-2}+30u_{i,j-2}-16u_{i+1,j-2}+u_{i+2,j-2}\right)\\
&+\left(-16u_{i-2,j-1}+256u_{i-1,j-1}-480u_{i,j-1}+256u_{i+1,j-1}-16u_{i+2,j-1}\right)\\
&+\left(30u_{i-2,j}-480u_{i-1,j}+900u_{i,j}-480u_{i+1,j}+30u_{i+2,j}\right)\\
&+\left(-16u_{i-2,j+1}+256u_{i-1,j+1}-480u_{i,j+1}+256u_{i+1,j+1}-16u_{i+2,j+1}\right)\\
&+\left(u_{i-2,j+2}-16u_{i-1,j+2}+30u_{i,j+2}-16u_{i+1,j+2}+u_{i+2,j+2}\right)\Big)+O\left(h^4\right)
\end{aligned}
\tag{3.20}
$$

$$
2\left.\frac{\partial^4 u}{\partial x^2 \partial y^2}\right|_{i,j} \approx \left\{\begin{matrix}
\frac{1}{72h^4} & -\frac{16}{72h^4} & \frac{30}{72h^4} & -\frac{16}{72h^4} & \frac{1}{72h^4}\\
-\frac{16}{72h^4} & \frac{256}{72h^4} & -\frac{480}{72h^4} & \frac{256}{72h^4} & -\frac{16}{72h^4}\\
\frac{30}{72h^4} & -\frac{480}{72h^4} & \frac{900}{72h^4} & -\frac{480}{72h^4} & \frac{30}{72h^4}\\
-\frac{16}{72h^4} & \frac{256}{72h^4} & -\frac{480}{72h^4} & \frac{256}{72h^4} & -\frac{16}{72h^4}\\
\frac{1}{72h^4} & -\frac{16}{72h^4} & \frac{30}{72h^4} & -\frac{16}{72h^4} & \frac{1}{72h^4}
\end{matrix}\right\} u_{i,j} + O\left(h^4\right)
$$

Figure 3.3 Diagram of $2\left.\frac{\partial^4 u}{\partial x^2 \partial y^2}\right|_{i,j}$ with 4^{th} order accuracy.

3.2.3 The finite difference approximation of $\left.\frac{\partial^4 u}{\partial y^4}\right|_{i,j}$ with 4^{th} order accuracy

From Taylor series in equation (3.15) we substitution $\left.\frac{\partial^4 u}{\partial y^4}\right|_{i,j}$ by holding i fixed and differentiating with respect to j and consider $\frac{\partial^2}{\partial y^2}\left(f_{i,j}-\frac{2\partial^4 u_{i.j}}{\partial x^2 \partial y^2}-\frac{\partial^4 u_{i.j}}{\partial x^4}\right)$ in term of equation (3.15) we get

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{i,j} \approx \frac{1}{12h^2}\left(-f_{i,j-2}+16f_{i,j-1}-30f_{i,j}+16f_{i,j+1}-f_{i,j+2}\right) \tag{3.21}$$

$$\begin{aligned}2\left.\frac{\partial^6 u}{\partial x^2 \partial y^4}\right|_{i,j} \approx \frac{1}{12h^6}\Big(&2\left(-u_{i-2,j-2}+16u_{i-1,j-2}-30u_{i,j-2}+16u_{i+1,j-2}-u_{i+2,j-2}\right)\\ &-8\left(-u_{i-2,j-1}+16u_{i-1,j-1}-30u_{i,j-1}+16u_{i+1,j-1}-u_{i+2,j-1}\right)\\ &+12\left(-u_{i-2,j}+16u_{i-1,j}-30u_{i,j}+16u_{i+1,j}-u_{i+2,j}\right)\\ &-8\left(-u_{i-2,j+1}+16u_{i-1,j+1}-30u_{i,j+1}+16u_{i+1,j+1}-u_{i+2,j+1}\right)\\ &+2\left(-u_{i-2,j+2}+16u_{i-1,j+2}-30u_{i,j+2}+16u_{i+1,j+2}-u_{i+2,j+2}\right)\Big)\end{aligned} \tag{3.22}$$

$$\begin{aligned}\left.\frac{\partial^6 u}{\partial x^4 \partial y^2}\right|_{i,j} \approx \frac{1}{12h^6}\Big(&-\left(u_{i-2,j-2}-4u_{i-1,j-2}+6u_{i,j-2}-4u_{i+1,j-2}+u_{i+2,j-2}\right)\\ &+16\left(u_{i-2,j-1}-4u_{i-1,j-1}+6u_{i,j-1}-4u_{i+1,j-1}+u_{i+2,j-1}\right)\\ &-30\left(u_{i-2,j}-4u_{i-1,j}+6u_{i,j}-4u_{i+1,j}+u_{i+2,j}\right)\\ &+16\left(u_{i-2,j+1}-4u_{i-1,j+1}+6u_{i,j+1}-4u_{i+1,j+1}+u_{i+2,j+1}\right)\\ &-\left(u_{i-2,j+2}-4u_{i-1,j+2}+6u_{i,j+2}-4u_{i+1,j+2}+u_{i+2,j+2}\right)\Big).\end{aligned} \tag{3.23}$$

Substitution equation (3.21), (3.22) and (3.23) into equation (3.15), so the finite difference approximation of $\left.\frac{\partial^4 u}{\partial y^4}\right|_{i,j}$ with 4^{th} order accuracy is

$$\begin{aligned}\left.\frac{\partial^4 u}{\partial y^4}\right|_{i,j} \approx{}& \frac{1}{72}\left(f_{i,j-2}-16f_{i,j-1}+30f_{i,j}-16f_{i,j+1}+f_{i,j+2}\right)\\ &+\frac{1}{12h^4}\Big(\left(-3u_{i-2,j-2}+24u_{i-1,j-2}-42u_{i,j-2}+24u_{i+1,j-2}-3u_{i+2,j-2}\right)\\ &+\left(36u_{i-2,j-1}-192u_{i-1,j-1}+312u_{i,j-1}-192u_{i+1,j-1}+36u_{i+2,j-1}\right)\\ &+\left(6u_{i-2,j}+48u_{i-1,j}-108u_{i,j}+48u_{i+1,j}+6u_{i+2,j}\right)\\ &+\left(36u_{i-2,j+1}-192u_{i-1,j+1}+312u_{i,j+1}-192u_{i+1,j+1}+36u_{i+2,j+1}\right)\\ &+\left(-3u_{i-2,j+2}+24u_{i-1,j+2}-42u_{i,j+2}+24u_{i+1,j+2}-3u_{i+2,j+2}\right)\Big)+O\left(h^4\right).\end{aligned} \tag{3.24}$$

$$\left.\frac{\partial^4 u}{\partial y^4}\right|_{i,j} \approx \begin{Bmatrix} -\frac{3}{72h^4} & \frac{36}{72h^4} & \frac{6}{72h^4} & \frac{36}{72h^4} & -\frac{3}{72h^4} \\ \frac{24}{72h^4} & -\frac{192}{72h^4} & \frac{48}{72h^4} & -\frac{192}{72h^4} & \frac{24}{72h^4} \\ -\frac{42}{72h^4} & \frac{312}{72h^4} & -\frac{108}{72h^4} & \frac{312}{72h^4} & -\frac{42}{72h^4} \\ \frac{24}{72h^4} & -\frac{192}{72h^4} & \frac{48}{72h^4} & -\frac{192}{72h^4} & \frac{24}{72h^4} \\ -\frac{3}{72h^4} & \frac{36}{72h^4} & \frac{6}{72h^4} & \frac{36}{72h^4} & -\frac{3}{72h^4} \end{Bmatrix} u_{i,j} + \begin{Bmatrix} \frac{1}{72} \\ -\frac{16}{72} \\ \frac{30}{72} \\ -\frac{16}{72} \\ \frac{1}{72} \end{Bmatrix} f_{i,j} + O\left(h^4\right).$$

Figure 3.4 Diagram of $\left.\frac{\partial^4 u}{\partial y^4}\right|_{i,j}$ with 4^{th} order accuracy.

Substitution equations (3.19), (3.20) and (3.24) into biharmonic equation, so the finite difference approximation of biharmonic equation at 4^{th} order accuracy is

$$\begin{aligned}
&-5\left(u_{i-2,j-2} + u_{i+2,j-2} + u_{i-2,j+2} + u_{i+2,j+2}\right) \\
&+44\left(u_{i-1,j-2} + u_{i+1,j-2} + u_{i-1,j+2} + u_{i+1,j+2} + u_{i-2,j-1} + u_{i+2,j-1} + u_{i-2,j+1} + u_{i+2,j+1}\right) \\
&-6\left(u_{i,j-2} + u_{i,j+2} + u_{i-2,j} + u_{i+2,j}\right) - 128\left(u_{i-1,j-1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i+1,j+1}\right) \\
&-120\left(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right) + 684u_{i,j} \\
&= h^4\left(12 f_{i,j} - \left(f_{i-2,j} + f_{i+2,j} + f_{i,j-2} + f_{i,j+2}\right) + 16\left(f_{i-1,j} + f_{i+1,j} + f_{i,j-1} + f_{i,j+1}\right)\right).
\end{aligned} \tag{3.25}$$

$$\begin{Bmatrix} -5 & 44 & -6 & 44 & -5 \\ 44 & -128 & -120 & -128 & 44 \\ -6 & -120 & 684 & -120 & -6 \\ 44 & -128 & -120 & -128 & 44 \\ -5 & 44 & -6 & 44 & -5 \end{Bmatrix} u_{i,j} + O\left(h^4\right) = \begin{Bmatrix} & & -h^4 & & \\ & & 16h^4 & & \\ -h^4 & 16h^4 & 12h^4 & 16h^4 & -h^4 \\ & & 16h^4 & & \\ & & -h^4 & & \end{Bmatrix} f_{i,j}$$

Figure 3.5 Diagram of biharmonic equation with 4^{th} order accuracy.

3.3 Integrated Radial Basis Functions (IRBF$_s$)

From biharmonic equation in equation (1) we let

$$v=\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}. \tag{3.26}$$

The second order derivative of v is decomposed into finite difference method

$$\frac{\partial^2 v(x,y)}{\partial x^2}+\frac{\partial^2 v(x,y)}{\partial y^2}=\frac{\partial^4 u(x,y)}{\partial x^4}+2\frac{\partial^4 u(x,y)}{\partial x^2\partial y^2}+\frac{\partial^4 u(x,y)}{\partial y^4}=f(x,y). \tag{3.27}$$

The figure 3.6 show that a schematic outline of a five-point stencil associated with a node(i,j). Nodes are locally numbered from left to right and from bottom to top $((i,j)\equiv 3)$. We construct the approximations for the field variable v and its derivatives on two lines defined by 1-3-5 and 2-3-4 in a separate manner.

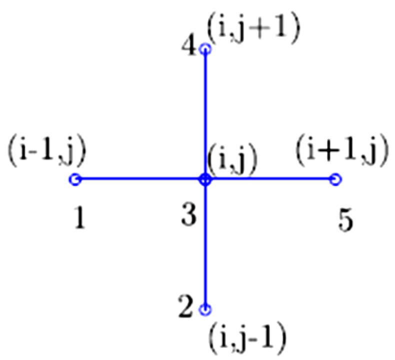

Figure 3.6 Schematic outline of a 5-point stencil.

On a line 1-3-5, the second-order derivative of v is decomposed into RBFs with respect to x is expressed as following

$$\frac{\partial^2 v(x)}{\partial x^4}=w_1G_1(x)+w_3G_3(x)+w_5G_5(x) \tag{3.28}$$

where $x_1\le x_3\le x_5, G_k(x)$ and w_k are the RBF$_s$ and the associated weight of node $k(k=\{1,3,5\})$. If the multiquadric (MQ) function is chosen, so $G_k(x)$ is expressed as following

$$G_k(x)=\sqrt{(x-x_k)^2+a_k^{\,2}} \tag{3.29}$$

where $a_k^{\,2}$ is the multiquadric width(free parameter). It is noted that another form of the free parameter $\varepsilon=1/a_k$ is also used in some case to enable comparison with published results in the literature.

Approximate expressions for the variable v is obtained by integrating (3.28) with respect to x

$$\frac{\partial v}{\partial x} = w_1 H_1 + w_3 H_3 + w_5 H_5 + C_1 \tag{3.30}$$

$$v(x) = w_1 \overline{H}_1 + w_3 \overline{H}_3 + w_5 \overline{H}_5 + C_1 x + C_2 \tag{3.31}$$

where $H_k(x) = \int G_k(x)dx,\ \overline{H}_k(x) = \int H(x)dx, k = \{1,3,5\}$ and C_1, C_2 are two constants of integration. With the presence of C_1 and C_2 there are 5 unknowns, instead of the usual 3, over a straight line of 3 nodes. This feature will be exploited here to incorporate the values of $\partial^2 v/\partial x^2$ at nodes 1 and 5 into the RBF approximations. We construct the system that represents the relation between the RBFs space and the physical space as following

$$\begin{pmatrix} v_1 \\ v_3 \\ v_5 \\ \dfrac{\partial^2 v_1}{\partial x^2} \\ \dfrac{\partial^2 v_5}{\partial x^2} \end{pmatrix} = \underbrace{\begin{bmatrix} \overline{H}_1(x_1) & \overline{H}_3(x_1) & \overline{H}_5(x_1) & x_1 & 1 \\ \overline{H}_1(x_3) & \overline{H}_3(x_3) & \overline{H}_5(x_3) & x_3 & 1 \\ \overline{H}_1(x_5) & \overline{H}_3(x_5) & \overline{H}_5(x_5) & x_5 & 1 \\ G_1(x_1) & G_3(x_1) & G_5(x_1) & 0 & 0 \\ G_1(x_5) & G_3(x_5) & G_5(x_5) & 0 & 0 \end{bmatrix}}_{C} \begin{pmatrix} w_1 \\ w_3 \\ w_5 \\ C_1 \\ C_2 \end{pmatrix} \tag{3.32}$$

where C is a 9×9 matrix that will hereafter be called the conversion matrix. Solving equation (3.32) leads to

$$\begin{pmatrix} w_1 \\ w_3 \\ w_5 \\ C_2 \\ C_2 \end{pmatrix} = C^{-1} \begin{bmatrix} v_1 & v_3 & v_5 & \dfrac{\partial^2 v_1}{\partial x^2} & \dfrac{\partial^2 v_5}{\partial x^2} \end{bmatrix}^T . \tag{3.33}$$

Approximate expressions for v and its derivatives in the physical space are obtained by substituting (3.33) into (3.31), (3.30) and (3.28)

$$v(x) = \begin{bmatrix} \overline{H}_1(x) & \overline{H}_3(x) & \overline{H}_5(x) & x & 1 \end{bmatrix} C^{-1} \begin{bmatrix} v_1 & v_3 & v_5 & \dfrac{\partial^2 v_1}{\partial x^2} & \dfrac{\partial^2 v_5}{\partial x^2} \end{bmatrix}^T \tag{3.34}$$

$$\frac{\partial v(x)}{\partial x} = \begin{bmatrix} H_1(x) & H_3(x) & H_5(x) & 1 & 0 \end{bmatrix} C^{-1} \begin{bmatrix} v_1 & v_3 & v_5 & \dfrac{\partial^2 v_1}{\partial x^2} & \dfrac{\partial^2 v_5}{\partial x^2} \end{bmatrix}^T \tag{3.35}$$

$$\frac{\partial^2 v(x)}{\partial x^2} = \begin{bmatrix} G_1(x) & G_3(x) & G_5(x) & 0 & 0 \end{bmatrix} C^{-1} \begin{bmatrix} v_1 & v_3 & v_5 & \dfrac{\partial^2 v_1}{\partial x^2} & \dfrac{\partial^2 v_5}{\partial x^2} \end{bmatrix}^T \tag{3.36}$$

where $x_1 \le x \le x_5$ They can be rewritten in the following form

$$v(x) = \phi_1(x) v_1 + \phi_3(x) v_3 + \phi_5(x) v_5 + \overline{\phi}_1(x) \frac{\partial^2 v_1}{\partial x^2} + \overline{\phi}_5(x) \frac{\partial^2 v_5}{\partial x^2} \tag{3.37}$$

$$\frac{\partial v(x)}{\partial x} = \frac{d\phi_1(x)}{dx} v_1 + \frac{d\phi_3(x)}{dx} v_3 + \frac{d\phi_5(x)}{dx} v_5 + \frac{d\overline{\phi}_1(x)}{dx} \frac{\partial^2 v_1}{\partial x^2} + \frac{d\overline{\phi}_5(x)}{dx} \frac{\partial^2 v_5}{\partial x^2} \tag{3.38}$$

$$\frac{\partial^2 v(x)}{\partial x^2} = \frac{d^2\phi_1(x)}{dx^2} v_1 + \frac{d^2\phi_3(x)}{dx^2} v_3 + \frac{d^2\phi_5(x)}{dx^2} v_5 + \frac{d^2\overline{\phi}_1(x)}{dx^2} \frac{\partial^2 v_1}{\partial x^2} + \frac{d^2\overline{\phi}_5(x)}{dx^2} \frac{\partial^2 v_5}{\partial x^2} \tag{3.39}$$

where $\left\{\phi_1(x),\phi_3(x),\phi_5(x),\bar{\phi}_1(x),\bar{\phi}_5(x)\right\}$ are the set of IRBFs in the physical space. At $x=x_3$, for brevity, we rewrite expressions (3.38) and (3.39) as

$$\frac{\partial v_3}{\partial x}=\mu_1 v_1+\mu_3 v_3+\mu_5 v_5+\bar{\mu}_1\frac{\partial^2 v_1}{\partial x^2}+\bar{\mu}_5\frac{\partial^2 v_5}{\partial x^2} \tag{3.40}$$

$$\frac{\partial^2 v_3}{\partial x^2}=\eta_1 v_1+\eta_3 v_3+\eta_5 v_5+\bar{\eta}_1\frac{\partial^2 v_1}{\partial x^2}+\bar{\eta}_5\frac{\partial^2 v_5}{\partial x^2} \tag{3.41}$$

where $\mu_k=\frac{d\phi_k(x_3)}{dx},\bar{\mu}_1=\frac{d\bar{\phi}_1(x_3)}{dx},\bar{\mu}_5=\frac{d\bar{\phi}_5(x_3)}{dx},\eta_k=\frac{d^2\phi_k(x_3)}{dx^2},\bar{\eta}_1=\frac{d^2\bar{\phi}_1(x_3)}{dx^2},$ $\bar{\eta}_5=\frac{d^2\bar{\phi}_5(x_3)}{dx^2}$ and $k=\{1,3,5\}$.

On a line 2-3-4, in a similar form line 1-3-5, which they are expressed as following

$$\frac{\partial v_3}{\partial y}=\varphi_2 v_2+\varphi_3 v_3+\varphi_4 v_4+\bar{\varphi}_2\frac{\partial^2 v_2}{\partial y^2}+\bar{\varphi}_4\frac{\partial^2 v_4}{\partial y^2} \tag{3.42}$$

$$\frac{\partial^2 v_3}{\partial y^2}=\theta_2 v_2+\theta_3 v_3+\theta_4 v_4+\bar{\theta}_2\frac{\partial^2 v_2}{\partial y^2}+\bar{\theta}_4\frac{\partial^2 v_4}{\partial y^2} \tag{3.43}$$

where $\varphi_k=\frac{d\phi_k(y_3)}{dy},\bar{\varphi}_2=\frac{d\bar{\phi}_2(y_3)}{dy},\bar{\varphi}_4=\frac{d\bar{\phi}_4(y_3)}{dy},\theta_k=\frac{d^2\phi_k(y_3)}{dy^2},\bar{\theta}_2=\frac{d^2\bar{\phi}_2(y_3)}{dy^2},$ $\bar{\theta}_4=\frac{d^2\bar{\phi}_4(y_3)}{dy^2}$ and $k=\{2,3,4\}$.

The bihamornic equation is solved using a Picard-type iteration scheme and we can be written in the following equation associated with node 3, i.e. (i,j)

$$\frac{\partial^2 v_3^{\,n}}{\partial x^2}=\eta_1 v_1^{\,n}+\eta_3 v_3^{\,n}+\eta_5 v_5^{\,n}+\bar{\eta}_1\frac{\partial^2 v_1^{\,n-1}}{\partial x^2}+\bar{\eta}_5\frac{\partial^2 v_5^{\,n-1}}{\partial x^2} \tag{3.44}$$

$$\frac{\partial^2 v_3^{\,n}}{\partial y^2}=\theta_2 v_2^{\,n}+\theta_3 v_3^{\,n}+\theta_4 v_4^{\,n}+\bar{\theta}_2\frac{\partial^2 v_2^{\,n-1}}{\partial y^2}+\bar{\theta}_4\frac{\partial^2 v_4^{\,n-1}}{\partial y^2} \tag{3.45}$$

where the superscript n is used to indicate a current iteration (i,j). The discretization equation associated with node 3, i.e. (i,j), is obtained by substituting (3.44) and (3.45) into (3.27)

$$\begin{aligned}\left(\eta_1 v_1^{\,n}+\eta_3 v_3^{\,n}+\eta_5 v_5^{\,n}\right)+\left(\theta_2 v_2^{\,n}+\theta_3 v_3^{\,n}+\theta_4 v_4^{\,n}\right)&=f_3-\left(\bar{\eta}_1\frac{\partial^2 v_1^{\,n-1}}{\partial x^2}+\bar{\eta}_5\frac{\partial^2 v_5^{\,n-1}}{\partial x^2}\right)\\&\quad-\left(\bar{\theta}_2\frac{\partial^2 v_2^{\,n-1}}{\partial y^2}+\bar{\theta}_4\frac{\partial^2 v_4^{\,n-1}}{\partial y^2}\right).\end{aligned} \tag{3.46}$$

Solve this linear system by Gauss-Seidel iteration is expressed as following

$$\begin{aligned}v_3^{\,n,k+1}=\Big(f_3-\bar{\eta}_1\frac{\partial^2 v_1^{\,n-1}}{\partial x^2}-\bar{\eta}_5\frac{\partial^2 v_5^{\,n-1}}{\partial x^2}-\bar{\theta}_2\frac{\partial^2 v_2^{\,n-1}}{\partial y^2}-\bar{\theta}_4\frac{\partial^2 v_4^{\,n-1}}{\partial y^2}-\eta_1 v_1^{\,n,k+1}-\eta_5 v_5^{\,n,k}\\-\theta_2 v_2^{\,n,k+1}-\theta_4 v_4^{\,n,k}\Big)/\left(\eta_3+\theta_3\right)\end{aligned} \tag{3.47}$$

with initial values is
$v_i^n = v_i^{n-1}; i = 1, 2, 3, 4, 5$.
Where the superscript k is used to indicate Gauss-Seidel iteration.
Consider equation (3.26), we construct the approximations for the field variable u and its derivatives on the x and y grid lines. On line x and y the second-order derivative of u are decomposed into IRBFs and the equations are solved using a Picard-type iteration scheme.

$$\frac{\partial^2 u_3^n}{\partial x^2} = \eta_1 u_1^n + \eta_3 u_3^n + \eta_5 u_5^n + \bar{\eta}_1 \frac{\partial^2 u_1^{n-1}}{\partial x^2} + \bar{\eta}_5 \frac{\partial^2 u_5^{n-1}}{\partial x^2} \tag{3.48}$$

$$\frac{\partial^2 u_3^n}{\partial y^2} = \theta_2 u_2^n + \theta_3 u_3^n + \theta_4 u_4^n + \bar{\theta}_2 \frac{\partial^2 u_2^{n-1}}{\partial y^2} + \bar{\theta}_4 \frac{\partial^2 u_4^{n-1}}{\partial y^2}. \tag{3.49}$$

The discretization equation associated with node 3, i.e. (i, j), is obtained by substituting (3.48) and (3.49) into (3.26)

$$\begin{aligned} v_3^n = {} & \eta_1 u_1^n + \eta_3 u_3^n + \eta_5 u_5^n + \bar{\eta}_1 \frac{\partial^2 u_1^{n-1}}{\partial x^2} + \bar{\eta}_5 \frac{\partial^2 u_5^{n-1}}{\partial x^2} \\ & + \theta_2 u_2^n + \theta_3 u_3^n + \theta_4 u_4^n + \bar{\theta}_2 \frac{\partial^2 u_2^{n-1}}{\partial y^2} + \bar{\theta}_4 \frac{\partial^2 u_4^{n-1}}{\partial y^2}. \end{aligned} \tag{3.50}$$

Solve this linear system by Gauss-Seidel iteration is expressed as following

$$\begin{aligned} u_3^{n,k+1} = {} & \left(v_3^{n,k+1} - \bar{\eta}_1 \frac{\partial^2 u_1^{n-1}}{\partial x^2} - \bar{\eta}_5 \frac{\partial^2 u_5^{n-1}}{\partial x^2} - \bar{\theta}_2 \frac{\partial^2 u_2^{n-1}}{\partial y^2} - \bar{\theta}_4 \frac{\partial^2 u_4^{n-1}}{\partial y^2} \right. \\ & \left. - \eta_1 u_1^{n,k+1} - \eta_5 u_5^{n,k} - \theta_2 u_2^{n,k+1} - \theta_4 u_4^{n,k} \right) / \left(\eta_3 + \theta_3 \right) \end{aligned} \tag{3.51}$$

with initial values is
$u_i^n = u_i^{n-1}; i = 1, 2, 3, 4, 5.$

3.4 Iteration Method

3.4.1 Iteration method of finite difference method (FDM) with 2^{nd} order accuracy

From equation (3.14) we will have iteration method of biharmonic equation at 2^{nd} order accuracy as following

$$\begin{aligned} u_{i,j}^{n+1} = u_{i,j}^{n+1} + \Big(& \tau h^4 f_{i,j} - \left(u_{i-2,j} + u_{i+2,j} + u_{i,j-2} + u_{i,j+2} \right) \\ & + 8\left(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} \right) - 2\left(u_{i-1,j-1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i+1,j+1} \right) \\ & - 20 u_{i,j} \Big) \end{aligned} \tag{3.52}$$

where τ is time step, which the operation count for each iteration is 20 times for each node.

$$u_{i,j}^{n+1} = u_{i,j}^{n} + \tau h^4 f_{i,j} + \left\{ \begin{array}{ccccc} & & -\tau & & \\ & -2\tau & 8\tau & -2\tau & \\ -\tau & 8\tau & -20\tau & 8\tau & -\tau \\ & -2\tau & 8\tau & -2\tau & \\ & & -\tau & & \end{array} \right\} u_{i,j} + O\left(h^2\right)$$

Figure 3.7 Diagram of iteration method biharmonic equation at 2^{nd} order accuracy.

3.4.2 Iteration method of finite difference method (FDM) with 4^{th} order accuracy

From equation (3.25) we will have iteration method of biharmonic equation at 4^{th} order accuracy as following

$$\begin{aligned} u_{i,j}^{n+1} = u_{i,j}^{n} + \tau\Big(h^4 \Big(12 f_{i,j} - \big(f_{i-2,j} + f_{i+2,j} + f_{i,j-2} + f_{i,j+2}\big) \\ +16\big(f_{i-1,j} + f_{i+1,j} + f_{i,j-1} + f_{i,j+1}\big)\Big) - 5\big(u_{i-2,j-2} + u_{i+2,j-2} + u_{i-2,j+2} + u_{i+2,j+2}\big) \\ +44\big(u_{i-1,j-2} + u_{i+1,j-2} + u_{i-1,j+2} + u_{i+1,j+2} + u_{i-2,j-1} + u_{i+2,j-1} + u_{i-2,j+1} + u_{i+2,j+1}\big) \\ -6\big(u_{i,j-2} + u_{i,j+2} + u_{i-2,j} + u_{i+2,j}\big) - 128\big(u_{i-1,j-1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i+1,j+1}\big) \\ -120\big(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\big) + 684 u_{i,j}\Big) \end{aligned} \tag{3.53}$$

the operation count for each iteration is 44 times for each node.

$$\begin{aligned} u_{i,j}^{n+1} = u_{i,j}^{n} + \left\{ \begin{array}{ccccc} & & -\tau h^4 & & \\ & & 16\tau h^4 & & \\ -\tau h^4 & 16\tau h^4 & 12\tau h^4 & 16\tau h^4 & -\tau h^4 \\ & & 16\tau h^4 & & \\ & & \tau h^4 & & \end{array} \right\} f_{i,j} \\ + \left\{ \begin{array}{ccccc} 5\tau & -44\tau & 6\tau & -44\tau & 5\tau \\ -44\tau & 128\tau & 120\tau & 128\tau & 44\tau \\ 6\tau & 120\tau & -684\tau & 120\tau & -6\tau \\ -44\tau & 128\tau & 120\tau & 128\tau & 44\tau \\ 5\tau & 44\tau & 6\tau & 44\tau & 5\tau \end{array} \right\} u_{i,j} + O\left(h^4\right) \end{aligned}$$

Figure 3.8 Diagram of iteration method biharmonic equation with 4^{th} order accuracy.

3.4.3 Iteration method of integrated radial basis functions (IRBF$_s$)

$$
\begin{aligned}
u_{i,j}^{n,k+1} = \Big[\Big(f_{i,j} &- \frac{1}{h^2}\Big[\bar{\eta}_{i-1,j}\left(v_{i-2,j} - 2v_{i-1,j} + v_{i,j}\right) + \bar{\eta}_{i+1,j}\left(v_{i,j} - 2v_{i+1,j} + v_{i+2,j}\right) \\
&+ \bar{\theta}_{i,j-1}\left(v_{i,j-2} - 2v_{i,j-1} + v_{i,j}\right) + \bar{\theta}_{i,j+1}\left(v_{i,j} - 2v_{i,j+1} + v_{i,j+2}\right)\Big] \\
&- \bar{\eta}_{i-1,j} v_{i-1,j}^{n,k+1} - \bar{\eta}_{i+1,j} v_{i+1,j}^{n,k} - \bar{\theta}_{i-1,j} v_{i,j-1}^{n,k+1} - \bar{\theta}_{i-1,j} v_{i,j+1}^{n,k}\Big) / \left(\eta_{i,j} + \theta_{i,j}\right) \\
&- \frac{1}{h^2}\Big[\bar{\eta}_{i-1,j}\left(u_{i-2,j} - 2u_{i-1,j} + u_{i,j}\right) + \bar{\eta}_{i+1,j}\left(u_{i,j} - 2u_{i+1,j} + u_{i+2,j}\right) \\
&+ \bar{\theta}_{i,j-1}\left(u_{i,j-2} - 2u_{i,j-1} + u_{i,j}\right) + \bar{\theta}_{i,j+1}\left(u_{i,j} - 2u_{i,j+1} + u_{i,j+2}\right)\Big] - \bar{\eta}_{i-1,j} u_{i-1,j}^{n,k+1} \\
&- \bar{\eta}_{i+1,j} u_{i+1,j}^{n,k} - \bar{\theta}_{i-1,j} u_{i,j-1}^{n,k+1} - \bar{\theta}_{i-1,j} u_{i,j+1}^{n,k}\Big] / \left(\eta_{i,j} + \theta_{i,j}\right)
\end{aligned}
\tag{3.54}
$$

the operation count for each iteration is 62 times for each node.

3.5 Boundary Condition

Consider boundary condition $\left.\frac{\partial u}{\partial n}\right|_{\Gamma q} = \bar{q}$, we using backward and forward difference approximation for solving boundary.

3.5.1 Backward difference approximation of $\left.\frac{\partial u}{\partial x}\right|_{i=N,j}$ with 2^{nd} order accuracy

$$
\begin{aligned}
\left.\frac{\partial u}{\partial x}\right|_{i=N,j} &\approx \frac{1}{2h}\left(u_{N-2,j} - 4u_{N-1,j} + 3u_{N,j}\right) = \bar{q} \\
-u_{N-1,j} &= \frac{1}{4}\left(2h\bar{q} - u_{N-2,j} - 3u_{N,j}\right) \\
u_{N-1,j}^{n+1} &= \frac{1}{4}\left(-2h\bar{q} + 3u_{N,j} + u_{N-2,j}\right) \\
u_{N-1,j}^{n+1} &= \frac{1}{4}\Big[-2h\bar{q} + 3\bar{u}_{N,j} + u_{N-2,j}^{n} + \tau\Big[h^4 f_{N-2,j} \\
&\quad -\left(u_{N-4,j} + u_{N,j} + u_{N-2,j-2} + u_{N-2,j+2}\right) + 8\left(u_{N-3,j} + u_{N-1,j} + u_{N-2,j-1} + u_{N-2,j+1}\right) \\
&\quad -2\left(u_{N-3,j-1} + u_{N-1,j-1} + u_{N-3,j+1} + u_{N-1,j+1}\right) - 20u_{N-2,j}\Big]\Big]
\end{aligned}
\tag{3.55}
$$

3.5.2 Backward difference approximation of $\left.\frac{\partial u}{\partial y}\right|_{i,j=N}$ with 2^{nd} order accuracy

$$\left.\frac{\partial u}{\partial y}\right|_{i,j=N} \approx \frac{1}{2h}\left(u_{i,N-2} - 4u_{i,N-1} + 3u_{i,N}\right) = \bar{q}$$

$$-u_{i,N-1} = \frac{1}{4}\left(2h\bar{q} - u_{i,N-2} - 3u_{i,N}\right)$$

$$u_{i,N-1}^{n+1} = \frac{1}{4}\left(-2h\bar{q} + 3u_{i,N} + u_{i,N-2}\right)$$

$$\begin{aligned} u_{i,N-1}^{n+1} = \frac{1}{4}\Big[&-2h\bar{q} + 3\bar{u}_{i,N} + u_{i,N-2}^{n} + \tau\Big[h^4 f_{i,N-2} \\ &-\left(u_{i-2,N-2} + u_{i+2,N-2} + u_{i,N-4} + u_{i,N}\right) + 8\left(u_{i-1,N-2} + u_{i+1,N-2} + u_{i,N-3} + u_{i,N-1}\right) \\ &-2\left(u_{i-1,N-3} + u_{i+1,N-3} + u_{i-1,N-1} + u_{i+1,N-1}\right) - 20u_{i,N-2}\Big]\Big] \end{aligned} \tag{3.56}$$

3.5.3 Forward difference approximation of $\left.\frac{\partial u}{\partial x}\right|_{i=N,j}$ with 2^{nd} order accuracy

$$\left.\frac{\partial u}{\partial x}\right|_{i=N,j} \approx \frac{1}{2h}\left(-3u_{N,j} + 4u_{N+1,j} - u_{N+2,j}\right) = \bar{q}$$

$$u_{N+1,j} = \frac{1}{4}\left(2h\bar{q} + 3u_{N,j} + u_{N+2,j}\right)$$

$$u_{N+1,j}^{n+1} = \frac{1}{4}\left(2h\bar{q} + 3u_{N,j} + u_{N+2,j}\right)$$

$$\begin{aligned} u_{N+1,j}^{n+1} = \frac{1}{4}\Big[&2h\bar{q} + 3\bar{u}_{N,j} + u_{N+2,j}^{n} + \tau\Big[h^4 f_{N+2,j} \\ &-\left(u_{N,j} + u_{N+2,j} + u_{N+2,j-2} + u_{N+2,j+2}\right) + 8\left(u_{N+1,j} + u_{N+1,j} + u_{N+2,j-1} + u_{N+2,j+1}\right) \\ &-2\left(u_{N+1,j-1} + u_{N+3,j-1} + u_{N+1,j+1} + u_{N+3,j+1}\right) - 20u_{N+2,j}\Big]\Big] \end{aligned} \tag{3.57}$$

3.5.4 Forward difference approximation of $\left.\frac{\partial u}{\partial y}\right|_{i,j=N}$ with 2^{nd} order accuracy

$$\left.\frac{\partial u}{\partial y}\right|_{i,j=N} \approx \frac{1}{2h}\left(-3u_{i,N} + 4u_{i,N+1} - u_{i,N+2}\right) = \bar{q}$$

$$u_{i,N+1} = \frac{1}{4}\left(2h\bar{q} + 3u_{i,N} + u_{i,N+2}\right)$$

$$u^{n+1}_{i,N+1} = \frac{1}{4}\left(2h\bar{q} + 3u_{i,N} + u_{i,N-2}\right)$$

$$\begin{aligned} u^{n+1}_{i,N+1} = \frac{1}{4}\Big[& 2h\bar{q} + 3\bar{u}_{i,N} + u^{n}_{i,N+2} + \tau\Big[h^4 f_{i,N+2} \\ & -\left(u_{i-2,N+2} + u_{i+2,N+2} + u_{i,N} + u_{i,N+2}\right) + 8\left(u_{i-1,N+2} + u_{i+1,N+2} + u_{i,N+1} + u_{i,N+3}\right) \\ & -2\left(u_{i-1,N+1} + u_{i+1,N+1} + u_{i-1,N+3} + u_{i+1,N+3}\right) - 20u_{i,N+2}\Big]\Big] \end{aligned} \quad (3.58)$$

3.5.5 Backward difference approximation of $\left.\frac{\partial u}{\partial x}\right|_{i=N,j}$ with 4^{th} order accuracy

$$
\begin{aligned}
\left.\frac{\partial u}{\partial x}\right|_{i=N,j} &\approx \frac{1}{12h}\left(-u_{N-4,j}+6u_{N-3,j}-18u_{N-2,j}+10u_{N-1,j}+3u_{N,j}\right)=\bar{q}\\
u_{N-1,j} &= \frac{1}{10}\left(12h\bar{q}+u_{N-4,j}-6u_{N-3,j}+18u_{N-2,j}-3u_{N,j}\right)\\
u_{N-1,j}^{n+1} &= \frac{1}{10}\left(12h\bar{q}-3u_{N,j}+18u_{N-2,j}-6u_{N-3,j}+u_{N-4,j}\right)\\
u_{N-1,j}^{n+1} &= \frac{1}{10}\Big[12h\bar{q}-3\bar{u}_{N,j}+\Big[18u_{N-2,j}^{n}+\tau\Big(h^4\big(12f_{N-2,j}\\
&\quad -\left(f_{N-4,j}+f_{N,j}+f_{N-2,j-2}+f_{N-2,j+2}\right)+16\left(f_{N-3,j}+f_{N-1,j}+f_{N-2,j-1}+f_{N-2,j+1}\right)\big)\\
&\quad -5\left(u_{N-4,j-2}+u_{N,j-2}+u_{N-4,j+2}+u_{N,j+2}\right)\\
&\quad +44\left(u_{N-3,j-2}+u_{N-1,j-2}+u_{N-3,j+2}+u_{N-1,j+2}+u_{N-4,j-1}+u_{N,j-1}+u_{N-4,j+1}+u_{N,j+1}\right)\\
&\quad -6\left(u_{N-2,j-2}+u_{N-2,j+2}+u_{N-4,j}+u_{N,j}\right)-128\left(u_{N-3,j-1}+u_{N-1,j-1}+u_{N-3,j+1}+u_{N-1,j+1}\right)\\
&\quad -120\left(u_{N-3,j}+u_{N-1,j}+u_{N-2,j-1}+u_{N-2,j+1}\right)+68u_{N-2,j}\Big)\Big]\\
&\quad -6\Big[u_{N-3,j}^{n}+\tau\Big(h^4\big(12f_{N-3,j}-\left(f_{N-5,j}+f_{N-1,j}+f_{N-3,j-2}+f_{N-3,j+2}\right)\\
&\quad +16\left(f_{N-4,j}+f_{N-2,j}+f_{N-3,j-1}+f_{N-3,j+1}\right)\big)-5\left(u_{N-5,j-2}+u_{N-1,j-2}+u_{N-5,j+2}+u_{N-1,j+2}\right)\\
&\quad +44\left(u_{N-4,j-2}+u_{N-2,j-2}+u_{N-4,j+2}+u_{N-2,j+2}+u_{N-5,j-1}+u_{N-1,j-1}+u_{N-5,j+1}+u_{N-1,j+1}\right)\\
&\quad -6\left(u_{N-3,j-2}+u_{N-3,j+2}+u_{N-5,j}+u_{N-1,j}\right)-128\left(u_{N-4,j-1}+u_{N-2,j-1}+u_{N-4,j+1}+u_{N-2,j+1}\right)\\
&\quad -120\left(u_{N-4,j}+u_{N-2,j}+u_{N-3,j-1}+u_{N-3,j+1}\right)+68u_{N-3,j}\Big)\Big]\\
&\quad +\Big[u_{N-4,j}^{n}+\tau\Big(h^4\big(12f_{N-4,j}-\left(f_{N-6,j}+f_{N-2,j}+f_{N-4,j-2}+f_{N-4,j+2}\right)\\
&\quad +16\left(f_{N-5,j}+f_{N-3,j}+f_{N-4,j-1}+f_{N-4,j+1}\right)\big)\\
&\quad -5\left(u_{N-6,j-2}+u_{N-2,j-2}+u_{N-6,j+2}+u_{N-2,j+2}\right)\\
&\quad +44\left(u_{N-5,j-2}+u_{N-3,j-2}+u_{N-5,j+2}+u_{N-3,j+2}+u_{N-6,j-1}+u_{N-2,j-1}+u_{N-6,j+1}+u_{N-2,j+1}\right)\\
&\quad -6\left(u_{N-4,j-2}+u_{N-4,j+2}+u_{N-6,j}+u_{N-2,j}\right)-128\left(u_{N-5,j-1}+u_{N-3,j-1}+u_{N-5,j+1}+u_{N-3,j+1}\right)\\
&\quad -120\left(u_{N-5,j}+u_{N-3,j}+u_{N-4,j-1}+u_{N-4,j+1}\right)+68u_{N-4,j}\Big)\Big]
\end{aligned}
\tag{3.59}
$$

3.5.6 Backward difference approximation of $\left.\frac{\partial u}{\partial y}\right|_{i,j=N}$ with 4^{th} order accuracy

$$\left.\frac{\partial u}{\partial y}\right|_{i,j=N} \approx \frac{1}{12h}\left(-u_{i,N-4}+6u_{i,N-3}-18u_{i,N-2}+10u_{i,N-1}+3u_{i,N}\right)=\bar{q}$$

$$u_{i,N-1}=\frac{1}{10}\left(12h\bar{q}+u_{i,N-4}-6u_{i,N-3}+18u_{i,N-2}-3u_{i,N}\right)$$

$$u_{i,N-1}^{n+1}=\frac{1}{10}\left(12h\bar{q}-3u_{i,N}+18u_{i,N-2}-6u_{i,N-3}+u_{i,N-4}\right)$$

$$\begin{aligned}
u_{i,N-1}^{n+1}=\frac{1}{10}\Big[&12h\bar{q}-3\bar{u}_{i,N}+\Big[18u_{i,N-2}^{n}+\tau\Big(h^4\big(12f_{i,N-2}-\big(f_{i-2,N-2}+f_{i+2,N-2}+f_{i,N-4}+f_{i,N}\big)\\
&+16\big(f_{i-1,N-2}+f_{i+1,N-2}+f_{i,N-3}+f_{i,N-1}\big)\big)-5\big(u_{i-2,N-4}+u_{i+2,N-4}+u_{i-2,N}+u_{i+2,N}\big)\\
&+44\big(u_{i-1,N-4}+u_{i+1,N-4}+u_{i-1,N}+u_{i+1,N}+u_{i-2,N-3}+u_{i+2,N-3}+u_{i-2,N-1}+u_{i+2,N-1}\big)\\
&-6\big(u_{i,N-4}+u_{i,N}+u_{i-2,N-2}+u_{i+2,N-2}\big)-128\big(u_{i-1,N-3}+u_{i+1,N-3}+u_{i-1,N-1}+u_{i+1,N-1}\big)\\
&-120\big(u_{i-1,N-2}+u_{i+1,N-2}+u_{i,N-3}+u_{i,N-1}\big)+68u_{i,N-2}\Big)\Big]\\
&-6\Big[u_{i,N-3}^{n}+\tau\Big(h^4\big(12f_{i,N-3}-\big(f_{i-2,N-3}+f_{i+2,N-3}+f_{i,N-5}+f_{i,N-1}\big)\\
&+16\big(f_{i-1,N-3}+f_{i+1,N-3}+f_{i,N-4}+f_{i,N-2}\big)\big)-5\big(u_{i-2,N-5}+u_{i+2,N-5}+u_{i-2,N-1}+u_{i+2,N-1}\big)\\
&+44\big(u_{i-1,N-5}+u_{i+1,N-5}+u_{i-1,N-1}+u_{i+1,N-1}+u_{i-2,N-4}+u_{i+2,N-4}+u_{i-2,N-2}+u_{i+2,N-2}\big)\\
&-6\big(u_{i,N-5}+u_{i,N-1}+u_{i-2,N-3}+u_{i+2,N-3}\big)-128\big(u_{i-1,N-4}+u_{i+1,N-4}+u_{i-1,N-2}+u_{i+1,N-2}\big)\\
&-120\big(u_{i-1,N-3}+u_{i+1,N-3}+u_{i,N-4}+u_{i,N-2}\big)+68u_{i,N-3}\Big)\Big]\\
&+\Big[u_{i,N-4}^{n}+\tau\Big(h^4\big(12f_{i,N-4}-\big(f_{i-2,N-4}+f_{i+2,N-4}+f_{i,N-6}+f_{i,N-2}\big)\\
&+16\big(f_{i-1,N-4}+f_{i+1,N-4}+f_{i,N-5}+f_{i,N-3}\big)\big)-5\big(u_{i-2,N-6}+u_{i+2,N-6}+u_{i-2,N-2}+u_{i+2,N-2}\big)\\
&+44\big(u_{i-1,N-6}+u_{i+1,N-6}+u_{i-1,N-2}+u_{i+1,N-2}+u_{i-2,N-5}+u_{i+2,N-5}+u_{i-2,N-3}+u_{i+2,N-3}\big)\\
&-6\big(u_{i,N-6}+u_{i,N-2}+u_{i-2,N-4}+u_{i+2,N-4}\big)-128\big(u_{i-1,N-5}+u_{i+1,N-5}+u_{i-1,N-3}+u_{i+1,N-3}\big)\\
&-120\big(u_{i-1,N-4}+u_{i+1,N-4}+u_{i,N-5}+u_{i,N-3}\big)+68u_{i,N-4}\Big)\Big]
\end{aligned}$$

(3.60)

3.5.7 Forward difference approximation of $\left.\frac{\partial u}{\partial x}\right|_{i=N,j}$ with 4^{th} order accuracy

$$\left.\frac{\partial u}{\partial x}\right|_{i=N,j} \approx \frac{1}{12h}\left(-3u_{N,j} - 10u_{N+1,j} + 18u_{N+2,j} - 6u_{N+3,j} + u_{N+4,j}\right) = \bar{q}$$

$$u_{N+1,j} = \frac{1}{10}\left(-12h\bar{q} - 3u_{N,j} + 18u_{N+2,j} - 6u_{N+3,j} + u_{N+4,j}\right)$$

$$u^{n+1}_{N+1,j} = \frac{1}{10}\left(-12h\bar{q} - 3u_{N,j} + 18u_{N+2,j} - 6u_{N+3,j} + u_{N+4,j}\right)$$

$$\begin{aligned}
u^{n+1}_{N+1,j} = \frac{1}{10}\Big[&-12h\bar{q} - 3\bar{u}_{N,j} + \Big[18u^{n}_{N+2,j} + \tau\Big(h^4\Big(12 f_{N+2,j} \\
&-\left(f_{N,j} + f_{N+4,j} + f_{N+2,j-2} + f_{N+2,j+2}\right) + 16\left(f_{N+1,j} + f_{N+3,j} + f_{N+2,j-1} + f_{N+2,j+1}\right)\Big) \\
&-5\left(u_{N,j-2} + u_{N+4,j-2} + u_{N,j+2} + u_{N+4,j+2}\right) \\
&+44\left(u_{N+1,j-2} + u_{N+3,j-2} + u_{N+1,j+2} + u_{N+3,j+2} + u_{N,j-1} + u_{N+4,j-1} + u_{N,j+1} + u_{N+4,j+1}\right) \\
&-6\left(u_{N+2,j-2} + u_{N+2,j+2} + u_{N,j} + u_{N+4,j}\right) - 128\left(u_{N+1,j-1} + u_{N+3,j-1} + u_{N+1,j+1} + u_{N+3,j+1}\right) \\
&-120\left(u_{N+1,j} + u_{N+3,j} + u_{N+2,j-1} + u_{N+2,j+1}\right) + 68u_{N+2,j}\Big)\Big] \\
&-6\Big[u^{n}_{N+3,j} + \tau\Big(h^4\Big(12 f_{N+3,j} - \left(f_{N+1,j} + f_{N+5,j} + f_{N+3,j-2} + f_{N+3,j+2}\right) \\
&+16\left(f_{N+2,j} + f_{N+4,j} + f_{N+3,j-1} + f_{N+3,j+1}\right)\Big) \\
&-5\left(u_{N+1,j-2} + u_{N+5,j-2} + u_{N+1,j+2} + u_{N+5,j+2}\right) \\
&+44\left(u_{N+2,j-2} + u_{N+4,j-2} + u_{N+2,j+2} + u_{N+4,j+2} + u_{N+1,j-1} + u_{N+5,j-1} + u_{N+1,j+1} + u_{N+5,j+1}\right) \\
&-6\left(u_{N+3,j-2} + u_{N+3,j+2} + u_{N+1,j} + u_{N+5,j}\right) - 128\left(u_{N+2,j-1} + u_{N+4,j-1} + u_{N+2,j+1} + u_{N+4,j+1}\right) \\
&-120\left(u_{N+2,j} + u_{N+4,j} + u_{N+3,j-1} + u_{N+3,j+1}\right) + 68u_{N+3,j}\Big)\Big] \\
&+\Big[u^{n}_{N+4,j} + \tau\Big(h^4\Big(12 f_{N+4,j} - \left(f_{N+2,j} + f_{N+6,j} + f_{N+4,j-2} + f_{N+4,j+2}\right) \\
&+16\left(f_{N+3,j} + f_{N+5,j} + f_{N+4,j-1} + f_{N+4,j+1}\right)\Big) \\
&-5\left(u_{N+2,j-2} + u_{N+6,j-2} + u_{N+2,j+2} + u_{N+6,j+2}\right) \\
&+44\left(u_{N+3,j-2} + u_{N+5,j-2} + u_{N+3,j+2} + u_{N+5,j+2} + u_{N+2,j-1} + u_{N+6,j-1} + u_{N+2,j+1} + u_{N+6,j+1}\right) \\
&-6\left(u_{N+4,j-2} + u_{N+4,j+2} + u_{N+2,j} + u_{N+6,j}\right) \\
&-128\left(u_{N+3,j-1} + u_{N+5,j-1} + u_{N+3,j+1} + u_{N+5,j+1}\right) \\
&-120\left(u_{N+3,j} + u_{N+5,j} + u_{N+4,j-1} + u_{N+4,j+1}\right) + 68u_{N+4,j}\Big)\Big]
\end{aligned} \tag{3.61}$$

3.5.8 Forward difference approximation of $\left.\frac{\partial u}{\partial y}\right|_{i,j=N}$ with 4^{th} order accuracy

$$
\begin{aligned}
\left.\frac{\partial u}{\partial y}\right|_{i,j=N} &\approx \frac{1}{12h}\left(-3u_{i,N}-10u_{i,N+1}+18u_{i,N+2}-6u_{i,N+3}+u_{i,N+4}\right)=\bar{q}\\
u_{i,N+1} &= \frac{1}{10}\left(-12h\bar{q}-3u_{i,N}+18u_{i,N+2}-6u_{i,N+3}+u_{i,N+4}\right)\\
u_{i,N+1}^{n+1} &= \frac{1}{10}\left(-12h\bar{q}-3u_{i,N}+18u_{i,N+2}-6u_{i,N+3}+u_{i,N+4}\right)\\
u_{i,N+1}^{n+1} &= \frac{1}{10}\Big[-12h\bar{q}-3\bar{u}_{i,N}+\Big[18u_{i,N+2}^{n}+\tau\Big(h^{4}\Big(12f_{i,N+2}\\
&\quad \left(f_{i-2,N+2}+f_{i+2,N+2}+f_{i,N}+f_{i,N+4}\right)+16\left(f_{i-1,N+2}+f_{i+1,N+2}+f_{i,N+1}+f_{i,N+3}\right)\Big)\\
&\quad -5\left(u_{i-2,N}+u_{i+2,N}+u_{i-2,N+4}+u_{i+2,N+4}\right)\\
&\quad +44\left(u_{i-1,N}+u_{i+1,N}+u_{i-1,N+4}+u_{i+1,N+4}+u_{i-2,N+1}+u_{i+2,N+1}+u_{i-2,N+3}+u_{i+2,N+3}\right)\\
&\quad -6\left(u_{i,N}+u_{i,N+4}+u_{i-2,N+2}+u_{i+2,N+2}\right)-128\left(u_{i-1,N+1}+u_{i+1,N+1}+u_{i-1,N+3}+u_{i+1,N+3}\right)\\
&\quad -120\left(u_{i-1,N+2}+u_{i+1,N+2}+u_{i,N+1}+u_{i,N+3}\right)+68u_{i,N+2}\Big)\Big]\\
&\quad -6\Big[u_{i,N+3}^{n}+\tau\Big(h^{4}\Big(12f_{i,N+3}-\left(f_{i-2,N+3}+f_{i+2,N+3}+f_{i,N+1}+f_{i,N+5}\right)\\
&\quad +16\left(f_{i-1,N+3}+f_{i+1,N+3}+f_{i,N+2}+f_{i,N+4}\right)\Big)\\
&\quad -5\left(u_{i-2,N+1}+u_{i+2,N+1}+u_{i-2,N+5}+u_{i+2,N+5}\right)\\
&\quad +44\left(u_{i-1,N+1}+u_{i+1,N+1}+u_{i-1,N+5}+u_{i+1,N+5}+u_{i-2,N+2}+u_{i+2,N+2}+u_{i-2,N+4}+u_{i+2,N+4}\right)\\
&\quad -6\left(u_{i,N+1}+u_{i,N+1}+u_{i-2,N+3}+u_{i+2,N+3}\right)\\
&\quad -128\left(u_{i-1,N+2}+u_{i+1,N+2}+u_{i-1,N+4}+u_{i+1,N+4}\right)\\
&\quad -120\left(u_{i-1,N+3}+u_{i+1,N+3}+u_{i,N+2}+u_{i,N+4}\right)+68u_{i,N+3}\Big)\Big]\\
&\quad +\Big[u_{i,N+4}^{n}+\tau\Big(h^{4}\Big(12f_{i,N+4}-\left(f_{i-2,N+4}+f_{i+2,N+4}+f_{i,N+2}+f_{i,N+6}\right)\\
&\quad +16\left(f_{i-1,N+4}+f_{i+1,N+4}+f_{i,N+3}+f_{i,N+5}\right)\Big)-5\left(u_{i-2,N+2}+u_{i+2,N+2}+u_{i-2,N+6}+u_{i+2,N+6}\right)\\
&\quad +44\left(u_{i-1,N+2}+u_{i+1,N+2}+u_{i-1,N+6}+u_{i+1,N+6}+u_{i-2,N+3}+u_{i+2,N+3}+u_{i-2,N+5}+u_{i+2,N+5}\right)\\
&\quad -6\left(u_{i,N+2}+u_{i,N+6}+u_{i-2,N+4}+u_{i+2,N+4}\right)-128\left(u_{i-1,N+3}+u_{i+1,N+3}+u_{i-1,N+5}+u_{i+1,N+5}\right)\\
&\quad -120\left(u_{i-1,N+4}+u_{i+1,N+4}+u_{i,N+3}+u_{i,N+5}\right)+68u_{i,N+4}\Big)\Big]
\end{aligned}
\tag{3.62}
$$