CHAPTER 2 THEOREM

In the chapter 2, we will be present radial basis functions method, integrated radial basis functions and biharmonic equation, which they are described as following.

2.1 Radial Basis Functions (RBF$_s$) Method

(Baxter, 1992; Wright, 2003) the RBF method is now one for the primary tools for interpolating multidimensional scattered data. It is simple form and ability to accurately approximate an underlying function has made the method particularly popular. In this section we review the conditions on the basis functions to guarantee a nonsingular method. This is followed by summary of some of some important theoretical and computational results, as a summary of some of the applications the RBF method has been successfully applied to.

A radial basis function approximation takes the form as follows:

$$s(x) = \sum_{i \in I} \lambda_i \phi\left(\|x - i\|\right), \qquad x \in R^d \tag{2.1}$$

where $\phi : [0, \infty) \to R$, is a fixed univariate function, $\|\cdot\|$ denote the Euclidean norm and the coefficients $(\lambda_i)\, i \in I$ are real numbers. Consequently our approximation s is a linear combination of translates of a fixed function $x \to \phi(\|x\|)$ which is “radially symmetric” with respect to the given norm, in the sense that it clearly possesses the symmetries of the unit ball.

If I is a finite set, say $I = (x_j)_{j=1}^{n}$, the interpolation conditions provide the symmetric linear system as following

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} \\ \lambda \\ \\ \end{bmatrix} = \begin{bmatrix} \\ f \\ \\ \end{bmatrix} \tag{2.2}$$

where $A = \left(\phi\left(\|x_j - x_k\|\right)\right)_{j,k=1}^{n}$, $\lambda = (\lambda)_{j=1}^{n}$, $f = (f)_{j=1}^{n}$ and $(x_j)_{j=1}^{n}$ are the centres of radial basis function interpolant . Furthermore, it is usual to refer to ϕ as the radial basis function, if the norm is understood.

(Micchelli, 1986) gave sufficient conditions for $\phi(r)$ in (2.1.1) to guarantee that the A matrix in (2.1.2) is unconditionally nonsingular, and thus that the basis RBF method is uniquely solvable. Table 1.1 lists a few of the many available choices for $\phi(r)$.

Table 1.1 Some commonly used radial basis functions.

Type of basis function	$\phi(r)\ \ (r \geq 0)$
Infinitely smooth RBFs	
Gaussian (GA)	$e^{-(\varepsilon r)^2}$
Inverse quadratic (IQ)	$\dfrac{1}{1+(\varepsilon r)^2}$
Inverse multiquadric (IMQ)	$\dfrac{1}{\sqrt{1+(\varepsilon r)^2}}$
Multiquadric (MQ)	$\sqrt{1+(\varepsilon r)^2}$
Piecewise smooth RBFs	
Linear	r
Cubic	r^3
Thin Plate Spline (TPS)	$r^2 \log r$

Note: in all cases, $\varepsilon > 0$.

$\phi(r)$ that lead to a uniquely solvable method are given in the first five entries of Table 1.1 The parameter ε in the infinitely smooth RBFs from the table is a free parameter for controlling the shape of functions. At this point, assume that it is some fixed non-zero real value.

Gaussian (GA)

(Baxter, 1992) here we choose Gaussian $\varphi(r) = e^{-(\varepsilon r)^2}$, where ε is shape parameter. Gaussian method is very important to the choice of parameter ε, as we might expect (Franke, 1982). Furthermore, Gaussian method cannot even reproduce constants when interpolating function values given on an infinite regular grid (Buhmann, 1990) so, its potential for practical computer calculations seems to be small. However, it possesses many properties which continue to win admirers in spite of these problems. Especially, it seems that users are seduced by its smoothness and rapid decay. Moreover the Gaussian interpolation matrix is positive definite if the centers are distinct, as well as being suited to iterative techniques.

Multiquadric (MQ)

(Baxter, 1992) here the multiquadric can be written in the form $\varphi(r)=\sqrt{1+(\varepsilon r)^2}$. The interpolation matrix A is invertible provided only that the points are all different and there are at least two of them. Further, this matrix has an important spectral property: it is almost negative definite.

This radial basis function provided the most accurate interpolation surfaces of all the methods tried for interpolation in two dimensions (Frank, 1982). His centers were mildly irregular in the sense that the range of distances between centers was not so large that the average distance became useless. Frank found that the method worked best when ε was chosen to be close to this average distance. It is still true to say that we do not know how to choose ε for a general function. (Buhmann and Dyn, 1991) derived error estimates which indicated that a large value of ε should provide excellent accuracy. This was borne out by some calculations and an analysis of (Powell, 1991) in the case when the centre formed a regular grid in one dimension.

Inverse Multiquadric (IMQ)

(Baxter, 1992) here we can be write Inverse multiquadric in form $\varphi(r)=\dfrac{1}{\sqrt{1+(\varepsilon r)^2}}$

(Frank, 1982) found that this radial basis function can provide excellent approximations, even when the number of centre is small. As for the multiquadric, there is no good choice of ε known at present (Franke, 1982).

Thin Plate Spline (TPS)

The name thin plate spline refers to the physical analogy involving the bending of a thin sheet of metal. In the physical setting, deflection is in the z direction, orthogonal to the plane. In order to apply this idea to the problem of coordinate transformation, one interprets the lifting of the plate as a displacement of the x or y coordinates within the plate. The thin plate spline is the two-dimensional analog of the cubic spline in one dimension. It is the fundamental solution to the biharmonic equation, and has the form $\varphi(r)=r^2\log r$. Thin plate spline has been widely used as the non-rigid transformation model in image alignment and shape matching. The popularity of Thin plate spline comes from a number of advantages: (i) the interpolation is smooth with derivatives of any order, (ii) the model has no free parameters that need manual tuning, (iii) it has closed-form solutions for both warping and parameter estimation, (iv) there is a physical explanation for its energy function.

2.2 Integrated Radial Basis Functions (IRBFs)

Integrated radial basis functions (IRBFs) are developed from radial basis function (RBFs). If the second-order derivative $u''(x)$ is approximated by the original radial basis functions, i.e.

$$u''(x) = \sum_{k=1}^{N} \mu_k \varphi_k(x). \tag{2.3}$$

Then the first-order derivative $u'(x)$ can be obtained by integration, which is written as

$$u'(x) = \int u''(x)\,dx = \sum_{k=1}^{N} \mu_k H_k(x) + C_1. \tag{2.4}$$

Similarly, the function u is obtained as

$$u(x) = \int u'(x)\,dx = \sum_{k=1}^{N} \mu_k \overline{H}_k(x) + C_1 x + C_2. \tag{2.5}$$

Here $H_k(x)$ and $\overline{H}_k(x)$ are the IRBFs and μ_k is the coefficient for $H_k(x)$ and $\overline{H}_k(x)$, C_1 and C_2 are the constants of integration.

2.3 Biharmonic Equation

The biharmonic equation is the governing equations that describe thin plate bending problems. Thin plate structures are widely used in engineering practice for the design of aircraft, ship, and ground structures. Numerical study behavior under various loadings condition is, therefore, essential. Apart from a few thin plates bending problem with simple transverse load or simple boundary condition, a general solution is difficult to obtain analytically. Some numerical method such as finite element method (FEM), boundary element method (BEM), hybrid-Trefftz finite element method (HT-FEM), and method of fundamental solution (MFS) are developed to analyze bending deformation of thin plate structures under various transverse load and boundary conditions.The biharmonic equation is encountered in plane problems of elasticity. It is also used to describe slow flows of viscous incompressible fluids. Several phenomena in Engineering and Mathematical physics are modeled as biharmonic equations.

The general biharmonic equation is written as

$$\nabla^4 u = 0. \tag{2.6}$$

For a function $u = (u_1, u_2)$ in two dimensions is expressed as in the form

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{2.7}$$

∇^2 is called two-dimensional Laplace operator.
Thus, the bihamonic equation is expressed as following

$$\nabla^4 u = \nabla^2(\nabla^2)u. \tag{2.8}$$

In two dimensions, the biharmonic equation is expressed as in the form

$$\frac{\partial^4 u(x,y)}{\partial x^4} + 2\frac{\partial^4 u(x,y)}{\partial x^2 \partial y^2} + \frac{\partial^4 u(x,y)}{\partial y^4} = f(x,y) \tag{2.9}$$

where $(x,y) \in [a,b] \times [c,d]$.