CHAPTER 4 NUMERICAL EXPERIMENTS

In this chapter, some numerical results are presented to verify this approach which compares to an exact solution by investigating the accuracy and convergence as well as computational efficiency of presented formulation and technique. The analyzed domain is $\Omega = [0,1] \times [0,1]$. The error of u and v ,which are presented in the numerical results, are represented by maximum relative error (MRE) and root mean square of relative error (RMSRE) of u and v, respectively where

$$MRE_u = \max \left| \frac{u_i - \hat{u}_i}{u_i} \right|, i = 1,2, \dots , N$$

$$RMSRE_u = \sqrt{\frac{1}{N} \sum_{i=1}^{N} \left(\frac{u_i - \hat{u}_i}{u_i} \right)^2} \ , i = 1,2, \dots , N$$

u_i and $\hat{u}_i$ are the exact and computed values of u at point $\boldsymbol{x}_{\mathrm{i}}$,respectively, and N is the number of nodes.

4.1 Example 1

For the first example consider the following system of nonlinear PDEs in the region $\Omega = [0,1] \times [0,1]$;

$$\frac{\partial u}{\partial t} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + u - u^2 v^2 + g_1(x, y, t),$$

$$\frac{\partial v}{\partial t} = \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + v - u^2 + uv + g_2(x, y, t),$$

where

$$g_1(x, y, t) = e^{4x-6t} - 5e^{-2t+x+y} \ ,$$

$$g_2(x, y, t) = -4e^{-t+x-y} + e^{-4t+2x+2y} - e^{2x-3t},$$

The initial and Dirichlet boundary conditions are chosen in such a way that the exact solution is

$$u(x, y, t) = e^{-2t+x+y} \ , \ \ v(x, y, t) = e^{-t+x-y}.$$

In this example, the governing equations resemble (2.1) with $\beta = 0$ and $D_1 = D_2$=1.The show results have been obtained using N = 25, 81, 256 and 441 nodal points, respectively. In case of $\Delta t = 0.01$ at time instant t =0.1, by the MLPG5 based on MKA with temporal discretization using the Euler, Runge-Kutta and Crank-Nicolson method , the MRE of u and v are rather low and the RMSRE of u and v decrease by increasing the number of nodal points(see in Fig.(4.1),(4.2) and (4.3)). In addition, the cubic polynomial basis is the best for constructing the nodal shape function.

Fig.4.4 (a) and (b) show the comparison of the accuracy among the Euler, Runge-Kutta and Crank-Nicolson methods; It shows that by increasing the number of nodal points, the MRE of u and v, using the Euler method, increases and the MRE of u and v using the Crank-Nicolson method increases a little bit. Meanwhile, by increasing the number of nodal points the MRE of u and v, using the Runge-Kutta method decreases and then they do not change. The RMSRE of u and v using the Euler, Runge-Kutta and Crank-Nicolson methods decrease by increasing the number of nodal points (see in Fig.4.5 (a) and (b)). However, the errors using the Runge-Kutta and Crank-Nicolson methods are less than the errors using the Euler method. Fig.4.6 shows the exact solution of u and v and Fig.4.7, 4.8 and 4.9 reveal the trial solutions and the corresponding error profile of u and v. From these figure, the trial solution of u and v using the Euler, Runge-Kutta and Crank-Nicolson methods are similar to the exact solution. In addition, there are no the errors at the boundary area because of moving Kringing satisfies the Kronecker delta property (see in Fig 4.7(b) and (d), Fig.4.8 (b) and (d) and Fig.4.9 (b) and (d)). In case of $\Delta t = 0.1$ at time instant t=1, the solution of u and v using the Euler and Runge-Kutta methods are unstable. Meanwhile, the developed formulation, using the Crank-Nicolson method, works well and has accuracy of the estimation. From Fig.4.10 (a) and (b), the MRE of u and v are rather low and the RMSRE of u and v decrease by increasing the number of nodal points. Fig.4.11 and 4.12 confirm the accuracy of the proposed method.

Fig.4.13-4.18 shows the results by the MLPG5 method based on the radial point interpolation method with temporal discretization by the Runge-Kutta and Crank-Nicolson methods. From these figures, the developed formulation based on the RPIM works well as same as the MLPG5 method based on MKA and the experimental results similar to the MLPG5 method based on the MKA. In case of $\Delta t = 0.01$, the MRE of by u and v are rather low, and the RMSRE decrease by increasing the number of nodal points (see in Fig.4.13, Fig.4.14, Fig.4.16 and Fig.4.17). In case of $\Delta t = 0.1$ at time instant $t = 1$, the solution of u and v by the MLPG5 based on RPIM with temporal discretization by the Runge-Kutta method are unstable. On the other hand, the developed method using the Crank-Nicolson method has accuracy of the estimation and satisfies the Kronecker delta property (see in Fig.4.15 and Fig.4.18).

Fig.4.19 represents the MLPG4 method based on MKA with temporal discretization by the Euler method. This developed formulation works well. However, the computational cost using this method is rather high.

4.1.1 The Results of Example 1 by MLPG5 Method Based on MKA

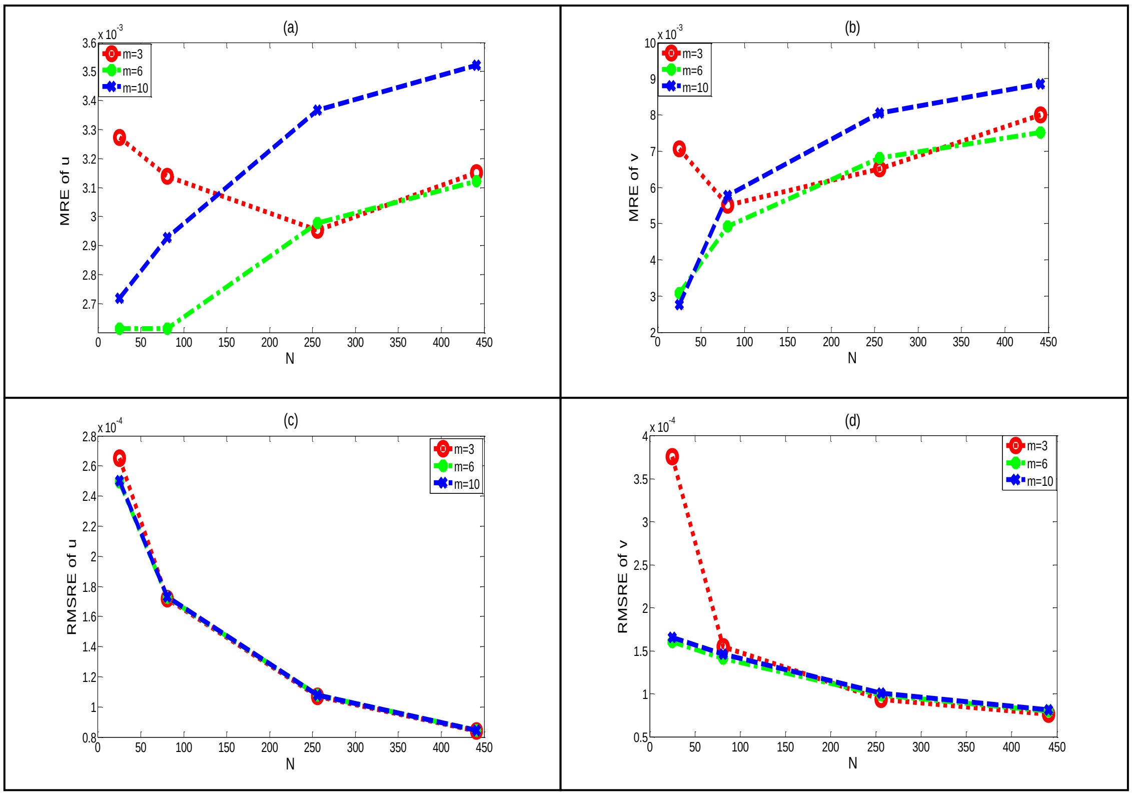
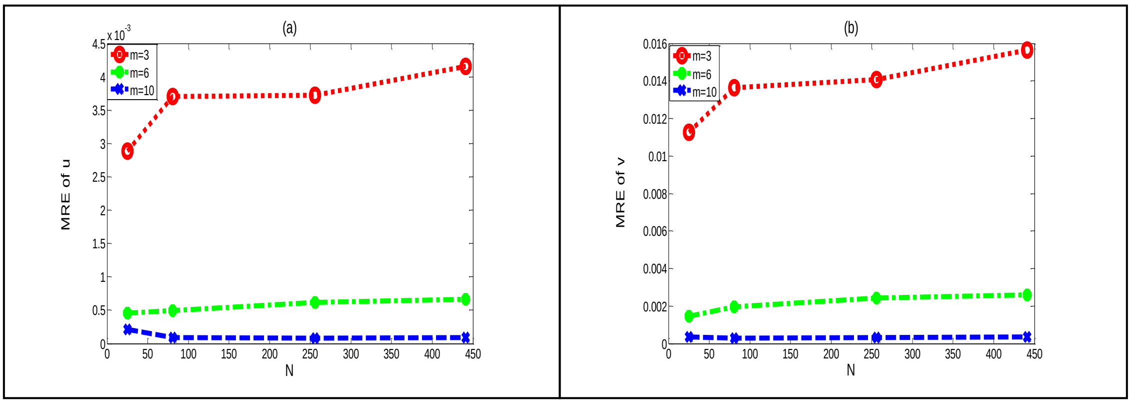


Figure 4.1 Both error of u and v using the MKA with temporal discretization by the Euler method and $\Delta t = 0.01$: (a) MRE of u; (b) MRE of v; (c) RMSRE of u; (d) RMSRE of v.



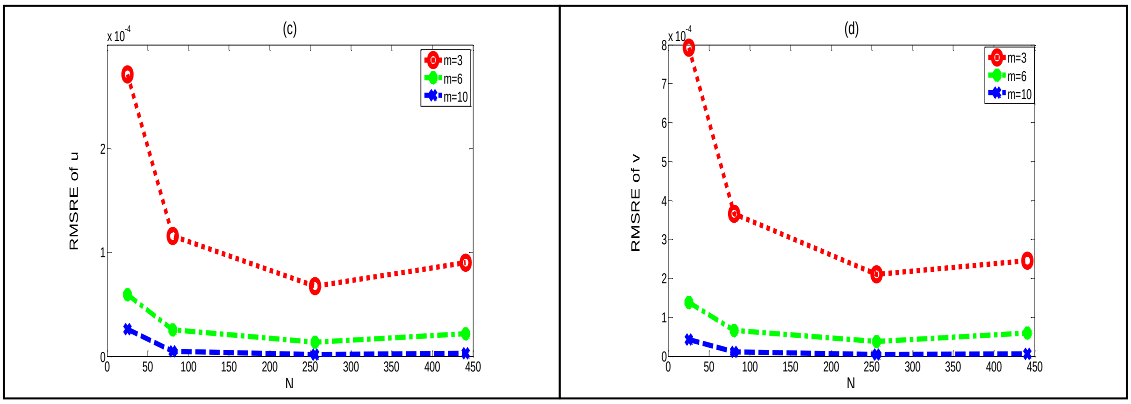


Figure 4.2 Both error of u and v using the MKA with temporal discretization by the Rung-Kutta method and $\Delta t = 0.01$: (a) MRE of u; (b) MRE of v; (c) RMSRE of u; (d) RMSRE of v.

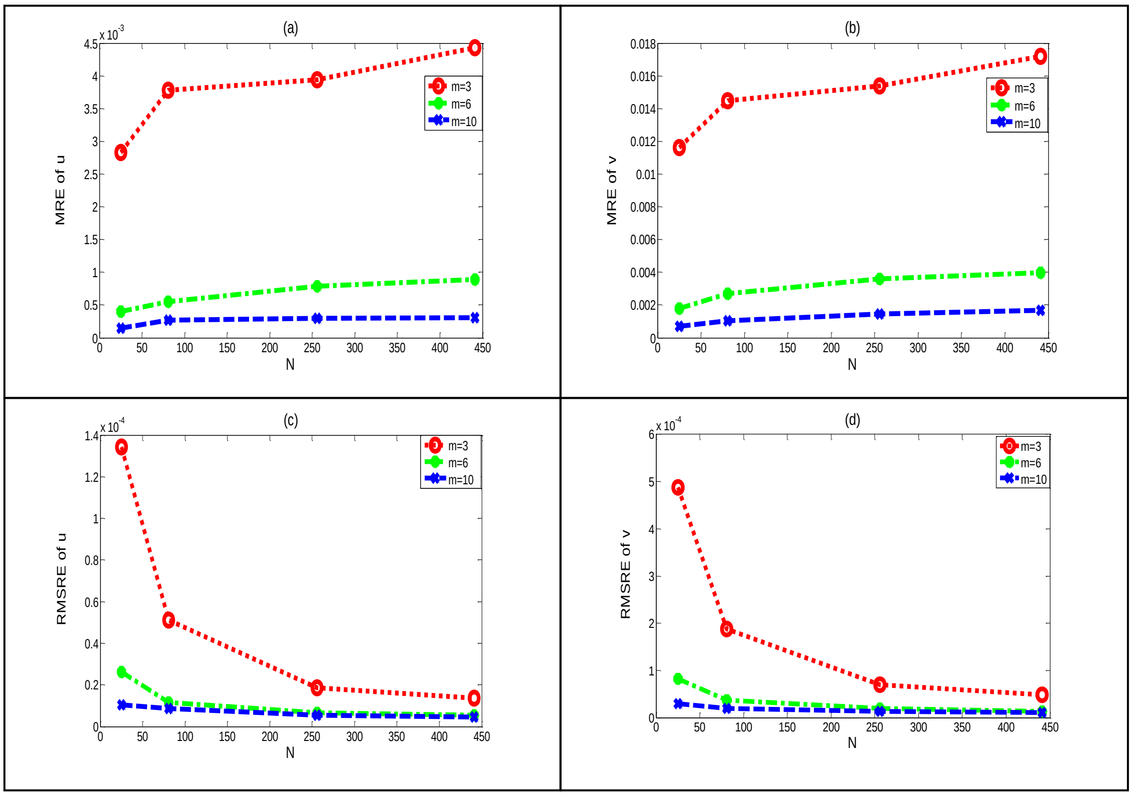


Figure 4.3 Both errors of u and v using the MKA with temporal discretization by the Crank-Nicolson method and $\Delta t = 0.01$: (a) MRE of u; (b) MRE of v; (c) RMSRE of u; (d) RMSRE of v.

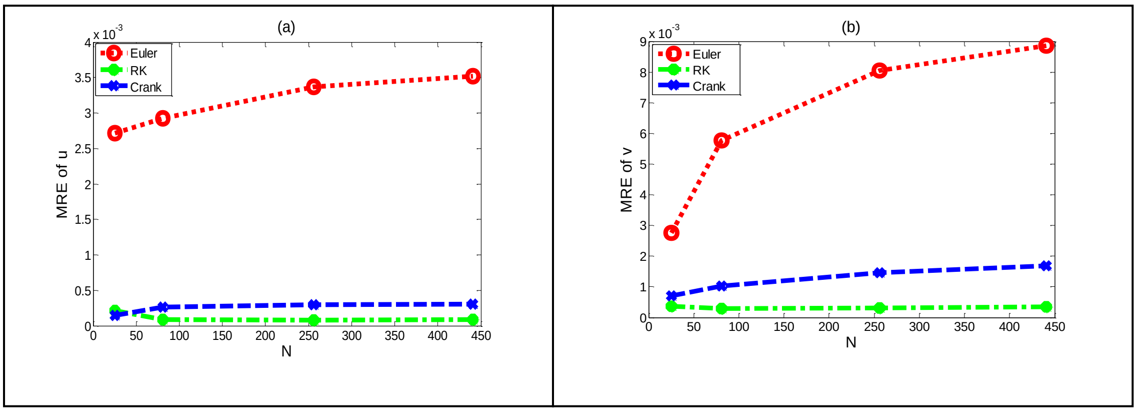


Figure 4.4 The comparison of MRE of u and v based on the MKA among the Euler, Runge-Kutta and Crank-Nicolson methods, $m = 10, t = 0.1$ and $\Delta t = 0.01$: (a) MRE of u; (b) MRE of v.

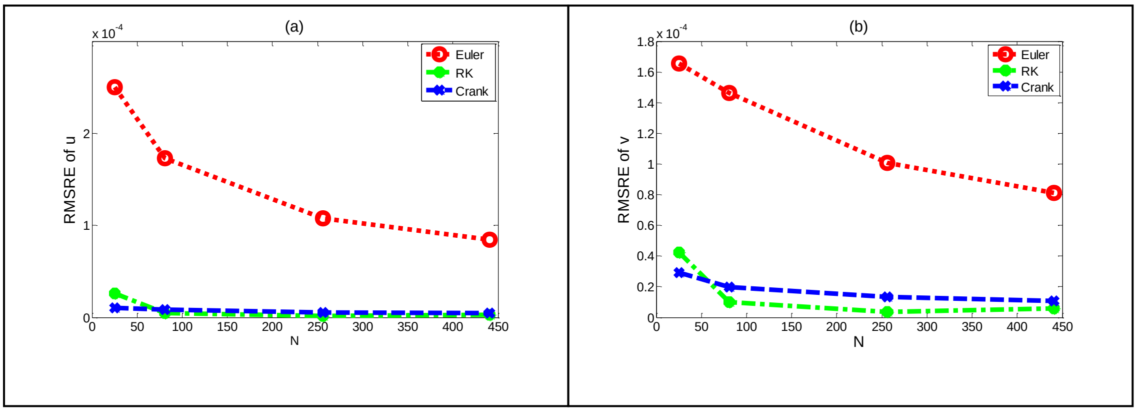


Figure 4.5 The comparison of RMSRE of u and v based on the MKA among the Euler, Runge-Kutta and Crank-Nicolson methods, $m = 10, t = 0.1$ and $\Delta t = 0.01$:
(a) RMSRE of u; (b) RMSRE of v.

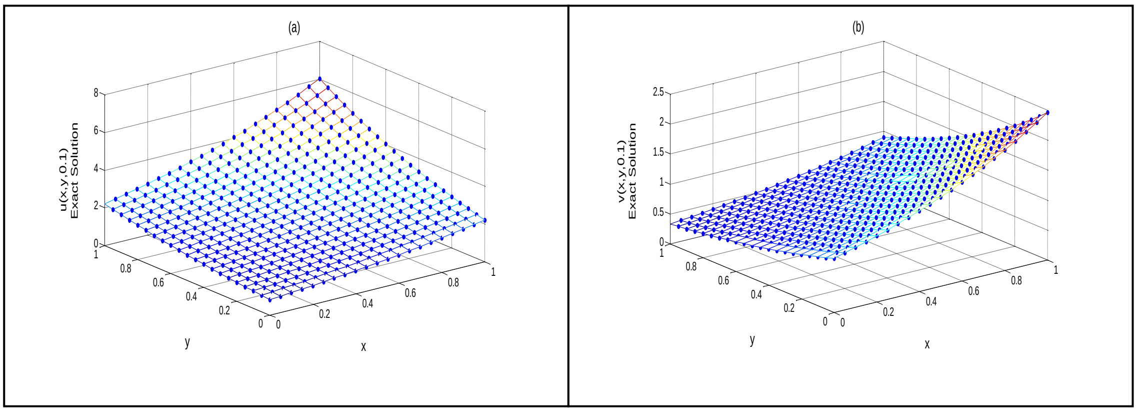


Figure 4.6 Exact solutions of u and v using N=441, $m = 10, t = 0.1$ and $\Delta t = 0.01$: (a) exact solution of u; (b) exact solution of v

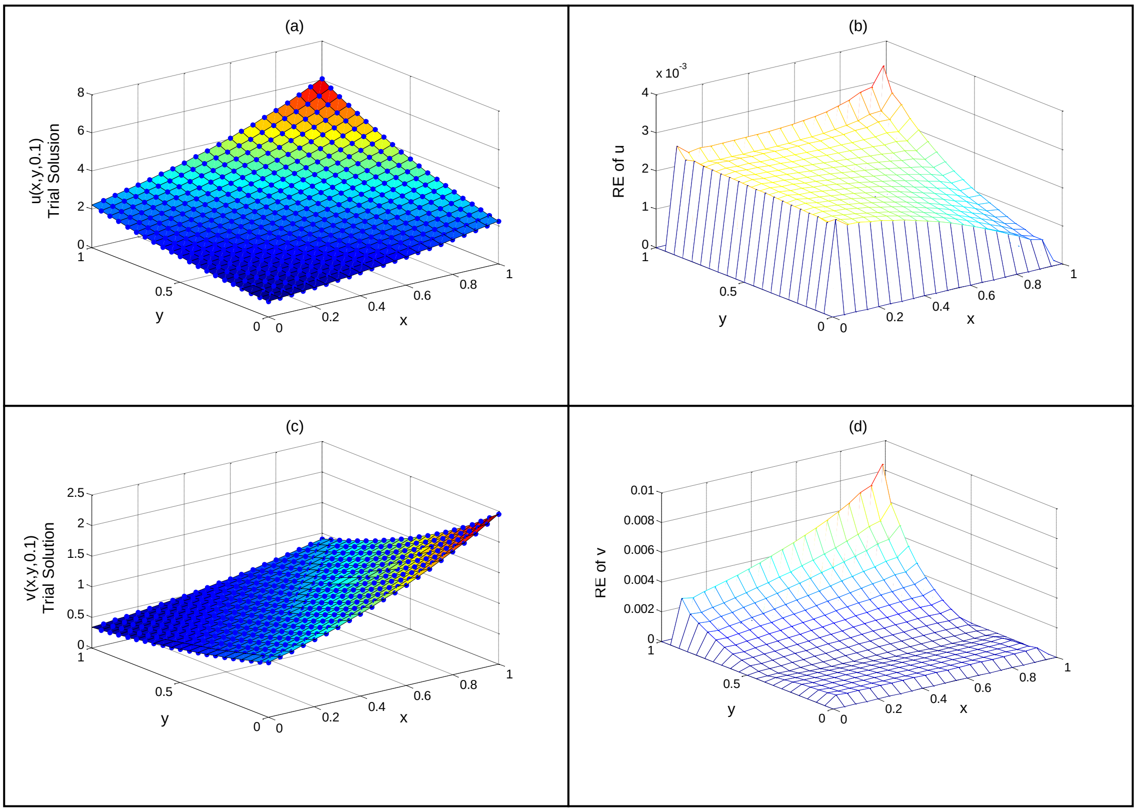
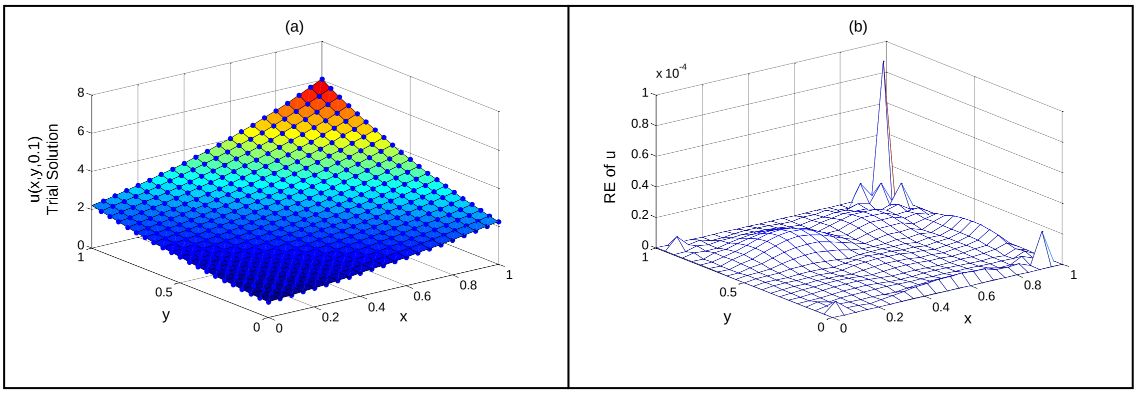


Figure 4.7 Trial solutions and errors of u and v by the MLPG5 method based on MKA with temporal discretization by the Euler method, N=441, $m = 10, t = 0.1$ and $\Delta t = 0.01$; (a) trial solution of u; (b) corresponding error profile of u; (c) trial solution of v; (d) corresponding error profile of v.



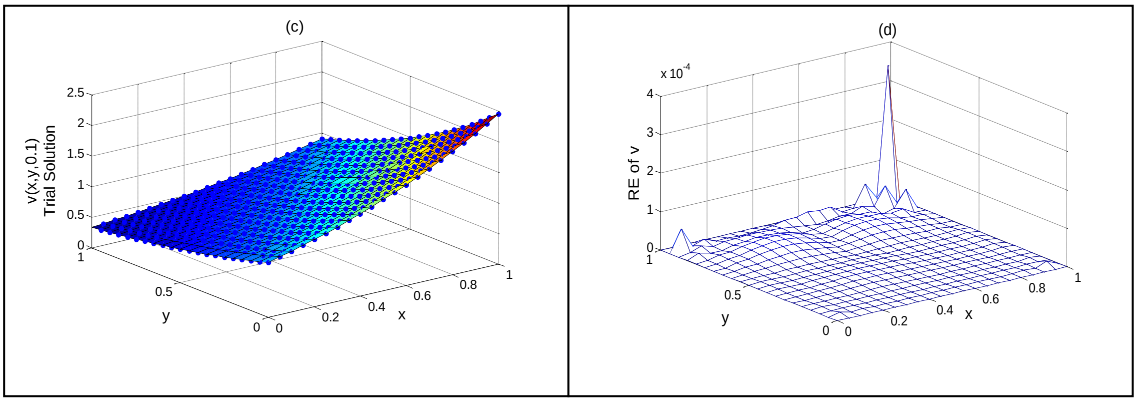


Figure 4.8 Trial solutions and errors of u and v by the MLPG5 method based on MKA with temporal discretization by the Runge-Kutta method, N=441, $m = 10, t = 0.1$ and $\Delta t = 0.01$: (a) trial solution of u; (b) corresponding error profile of u; (c) trial solution of v; (d) corresponding error profile of v.

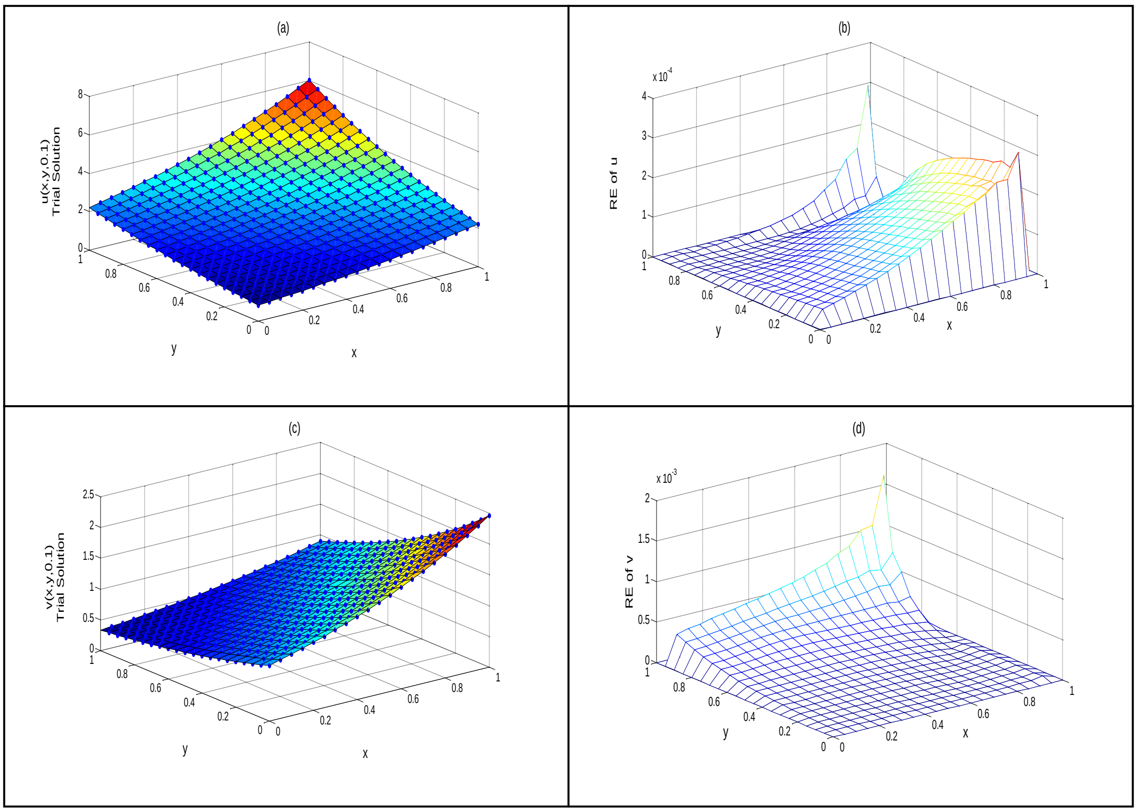


Figure 4.9 Trial solutions and errors of u and v by the MLPG5 method based on MKA with temporal discretization by the Crank-Nicolson method, N=441, $m = 10, t = 0.1$ and $\Delta t = 0.01$: (a) trial solution of u; (b) corresponding error profile of u;
(c) trial solution of v; (d) corresponding error profile of v.

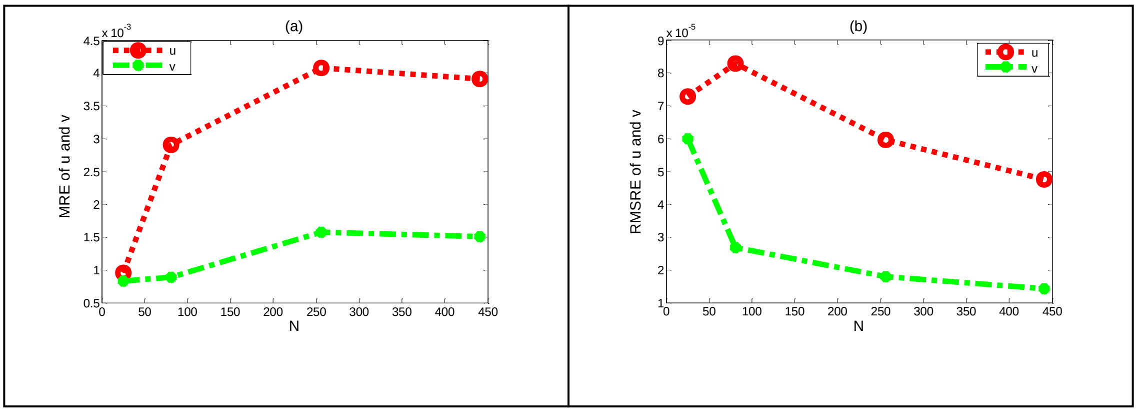


Figure 4.10 Both errors of u and v by the MLPG5 method based on MKA with temporal discretization by the Crank-Nicolson methods against the number of nodal points, $m = 10, t = 1$ and $\Delta t = 0.1$: (a) MRE of u and v; (b) RMSRE of u and v.

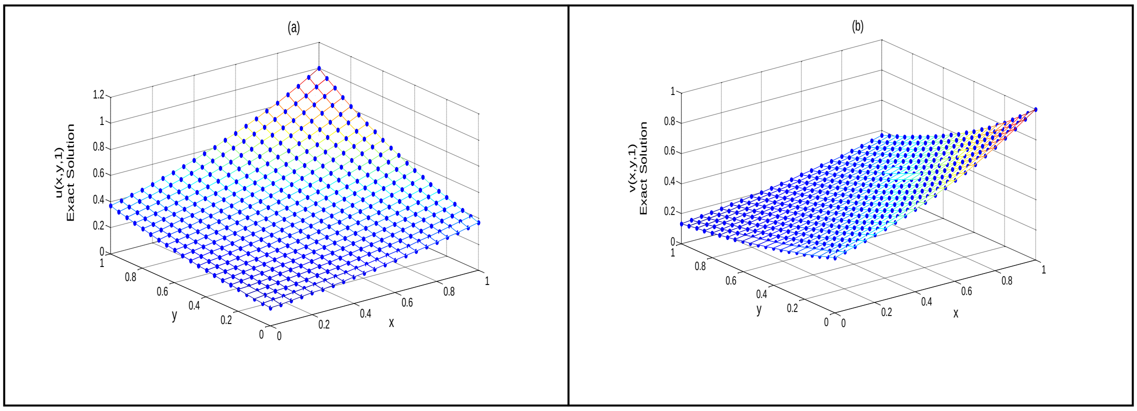
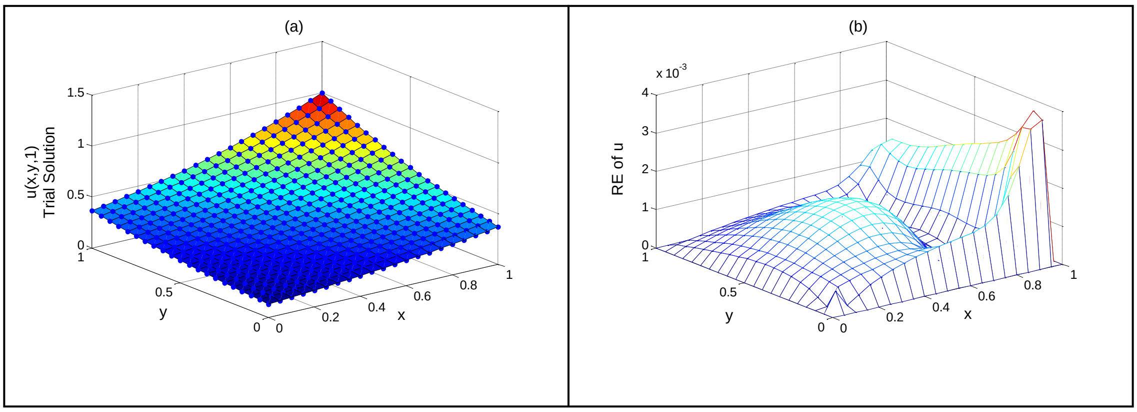


Figure 4.11 Exact solutions of u and v using N=441, $m = 10, t = 1$ and $\Delta t = 0.1$: (a) exact solution of u; (b) exact solution of v.



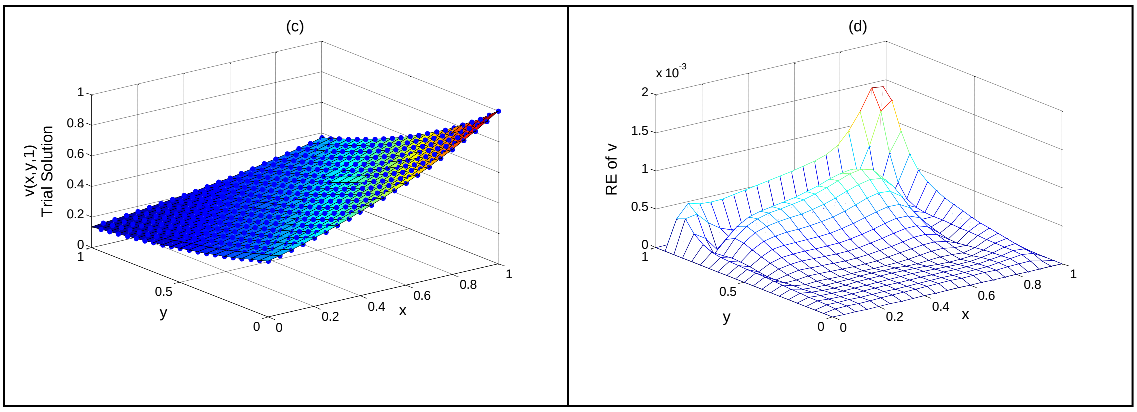


Figure 4.12 Trial solutions and errors of u and v by the MLPG5 method based on MKA with temporal discretization by the Crank-Nicolson methods, N=441, $m = 10$, $t = 1$ and $\Delta t = 0.1$: (a) trial solution of u; (b) corresponding error profile of u; (c) trial solution of v; (d) corresponding error profile of v.

4.1.2 The Results of Example 1 by The MLPG5 Method Based on RPIM

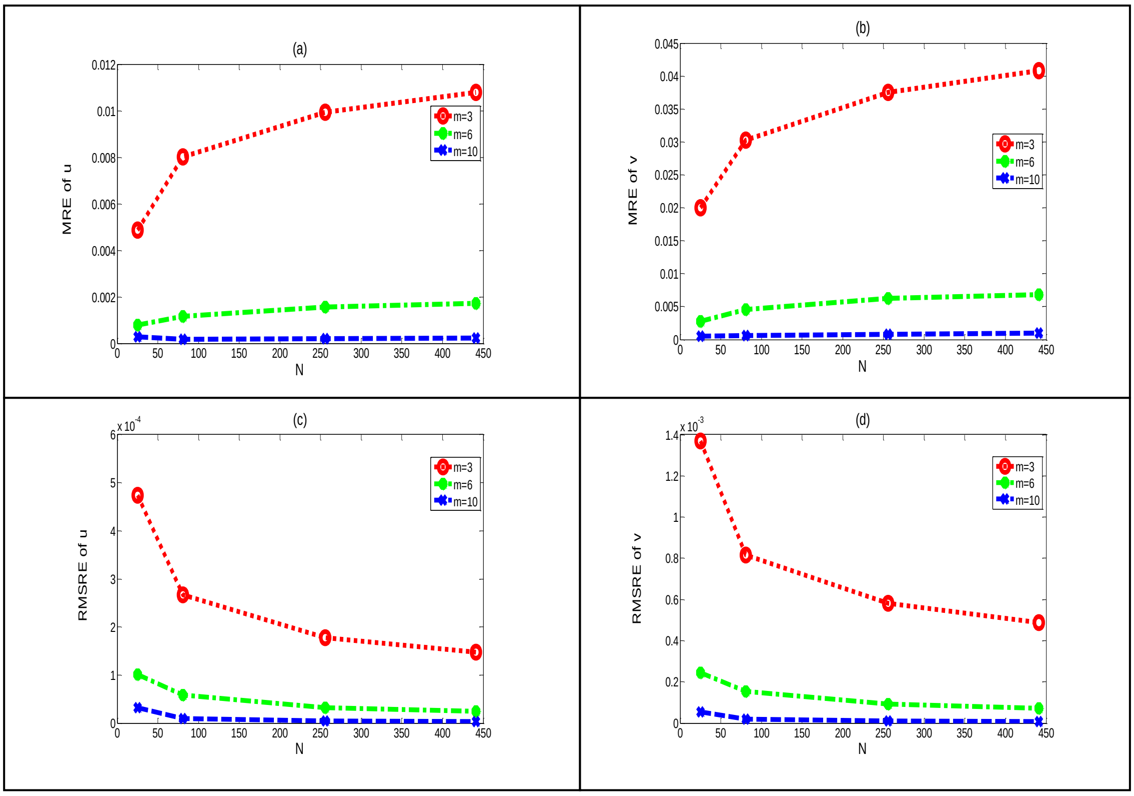


Figure 4.13 Both error of u and v by the MLPG5 method based on the RPIM with temporal discretization by the Runge-Kutta method and $\Delta t = 0.01$: (a) MRE of u; (b) MRE of v; (c) RMSRE of u; (d) RMSRE of v.

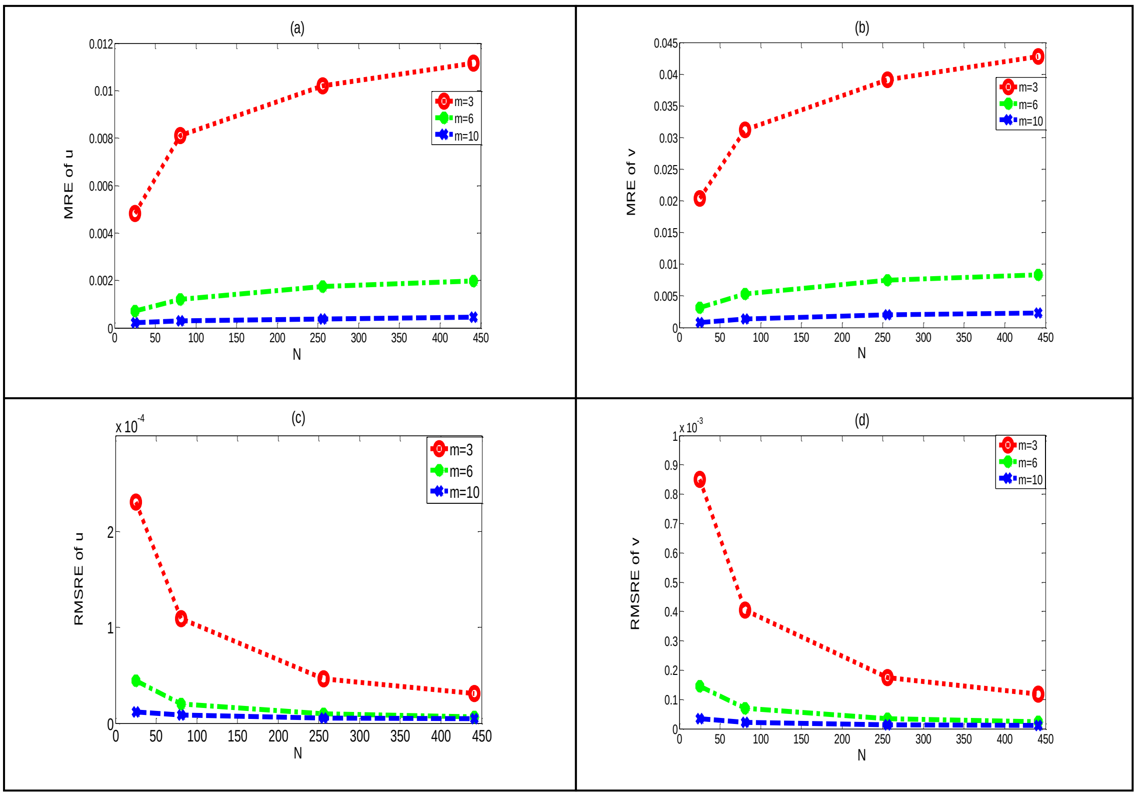
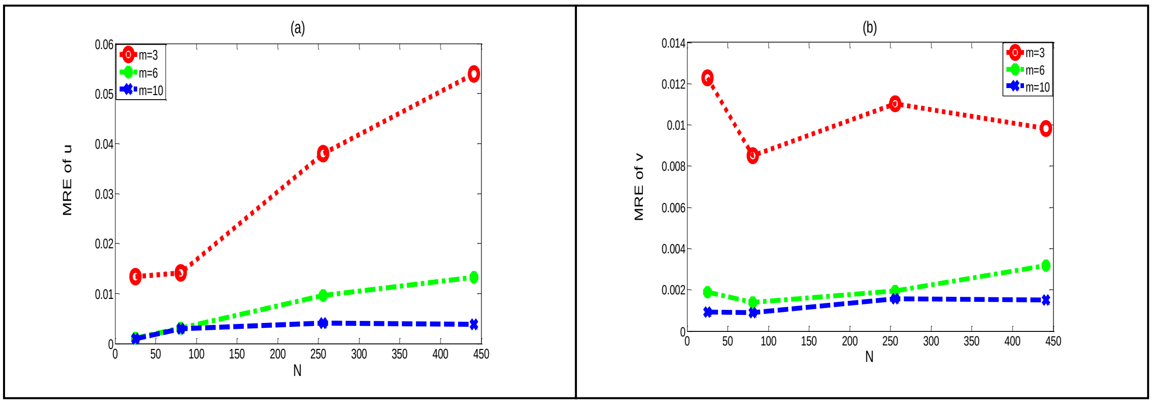


Figure 4.14 Both error of u and v the MLPG5 method based on the RPIM with temporal discretization by the Crank-Nicolson method and $\Delta t = 0.01$: (a) MRE of u; (b) MRE of v; (c) RMSRE of u; (d) RMSRE of v.



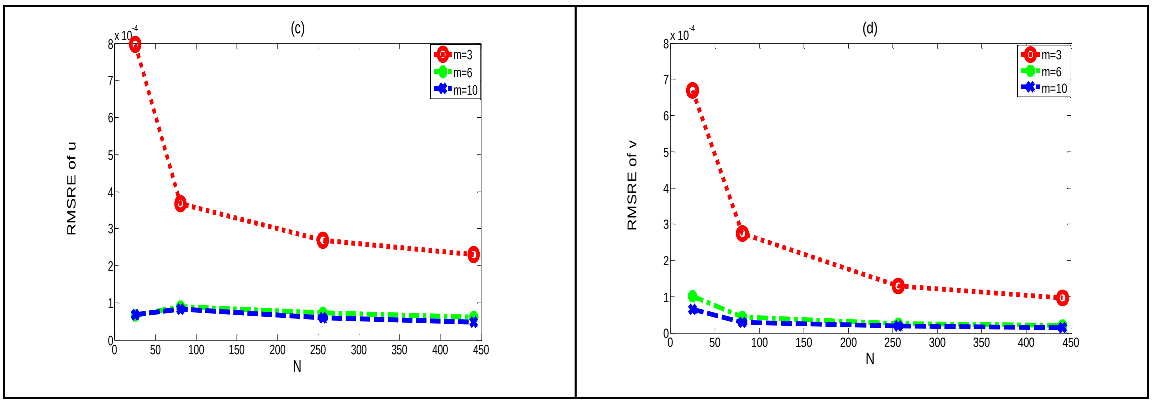


Figure 4.15 Both error of u and v by the MLPG5 method based on the RPIM with temporal discretization by the Crank-Nicolson method and $\Delta t = 0.1$: (a) MRE of u; (b) MRE of v; (c) RMSRE of u; (d) RMSRE of v.

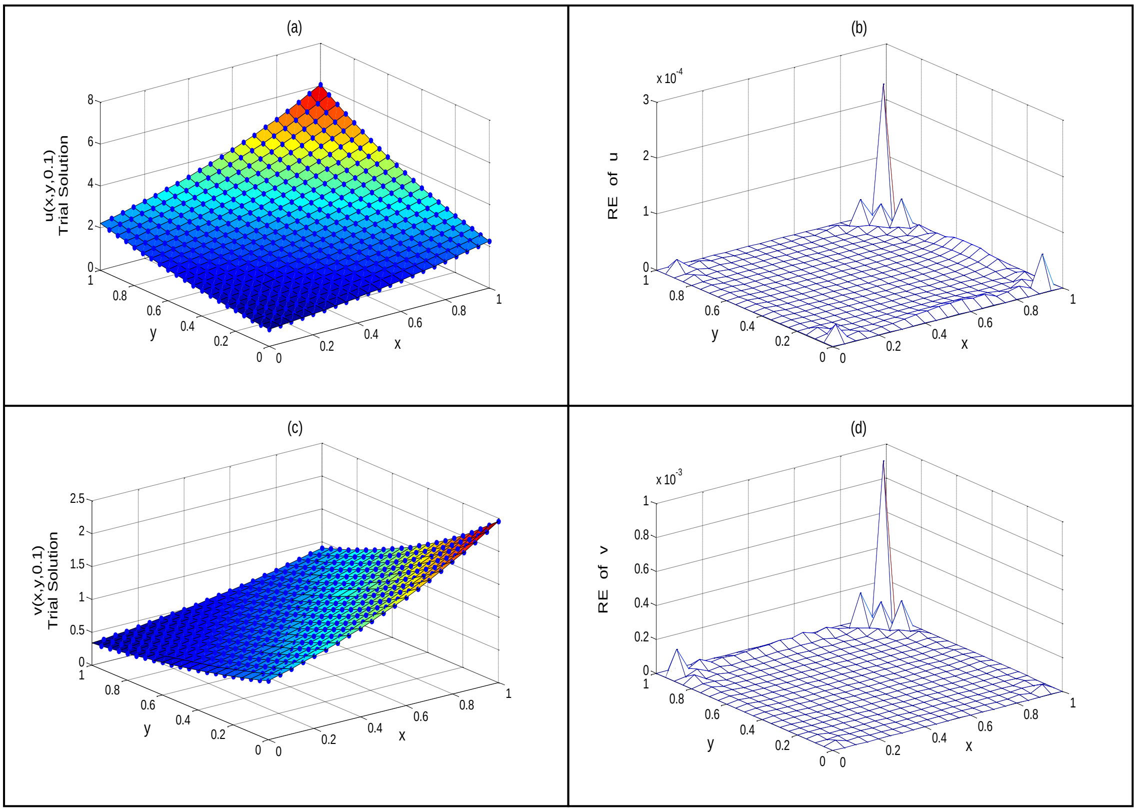


Figure 4.16 Trial solutions and errors of u and v by the MLPG5 method based on the RPIM with temporal discretization by the Runge-Kutta method, N=441, $m = 10$, $t = 0.1$ and $\Delta t = 0.01$: (a) trial solution of u; (b) corresponding error profile of u; (c) trial solution of v; (d) corresponding error profile of v.

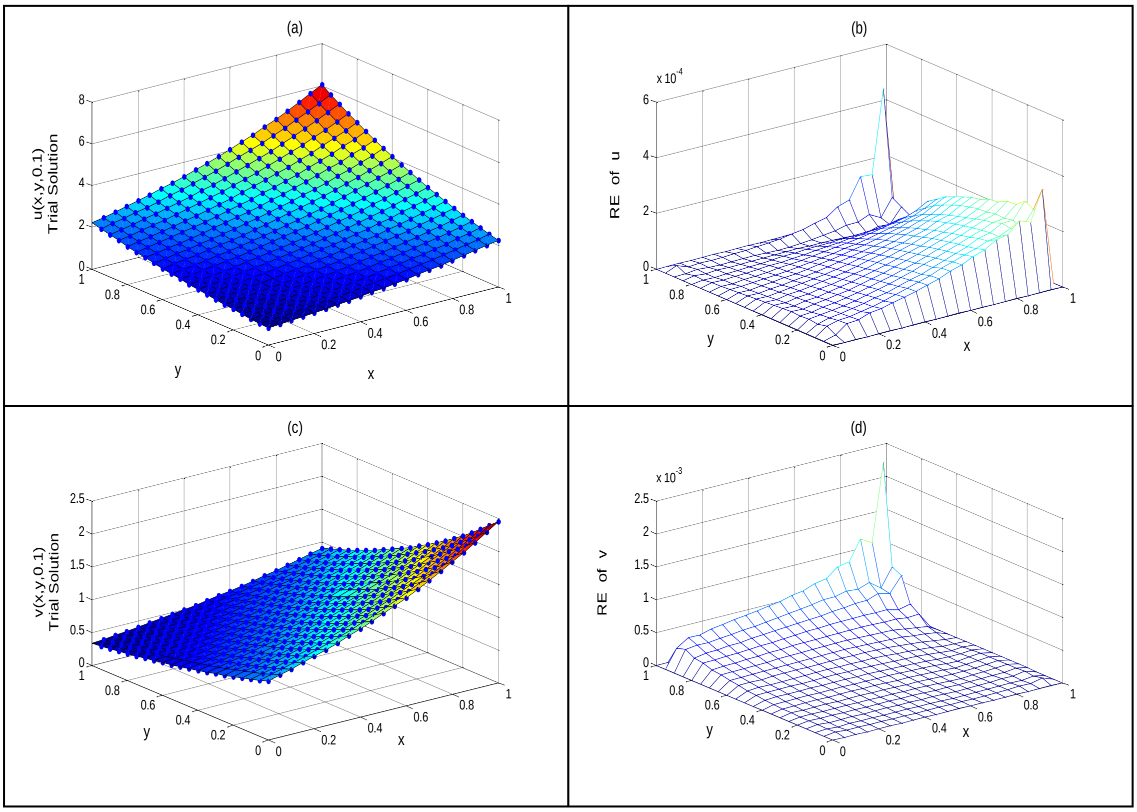
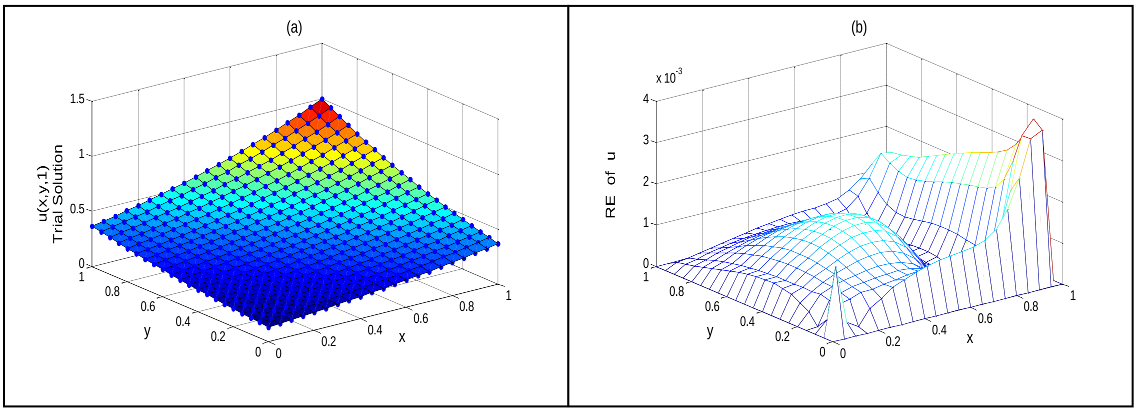


Figure 4.17 Trial solutions and errors of u and v by the MLPG5 method based on the RPIM with temporal discretization by the Crank-Nicolson method, N=441, $m = 10$, $t = 0.1$ and $\Delta t = 0.01$: (a) trial solution of u; (b) corresponding error profile of u; (c) trial solution of v; (d) corresponding error profile of v.



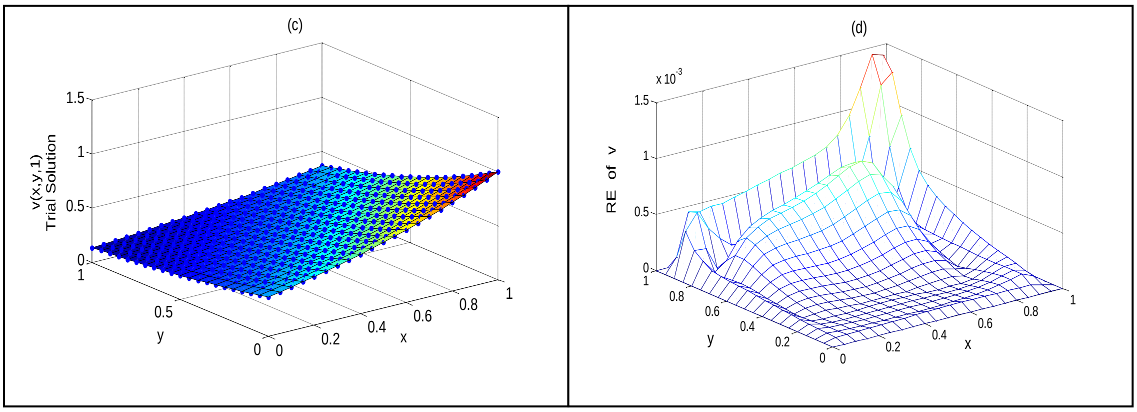
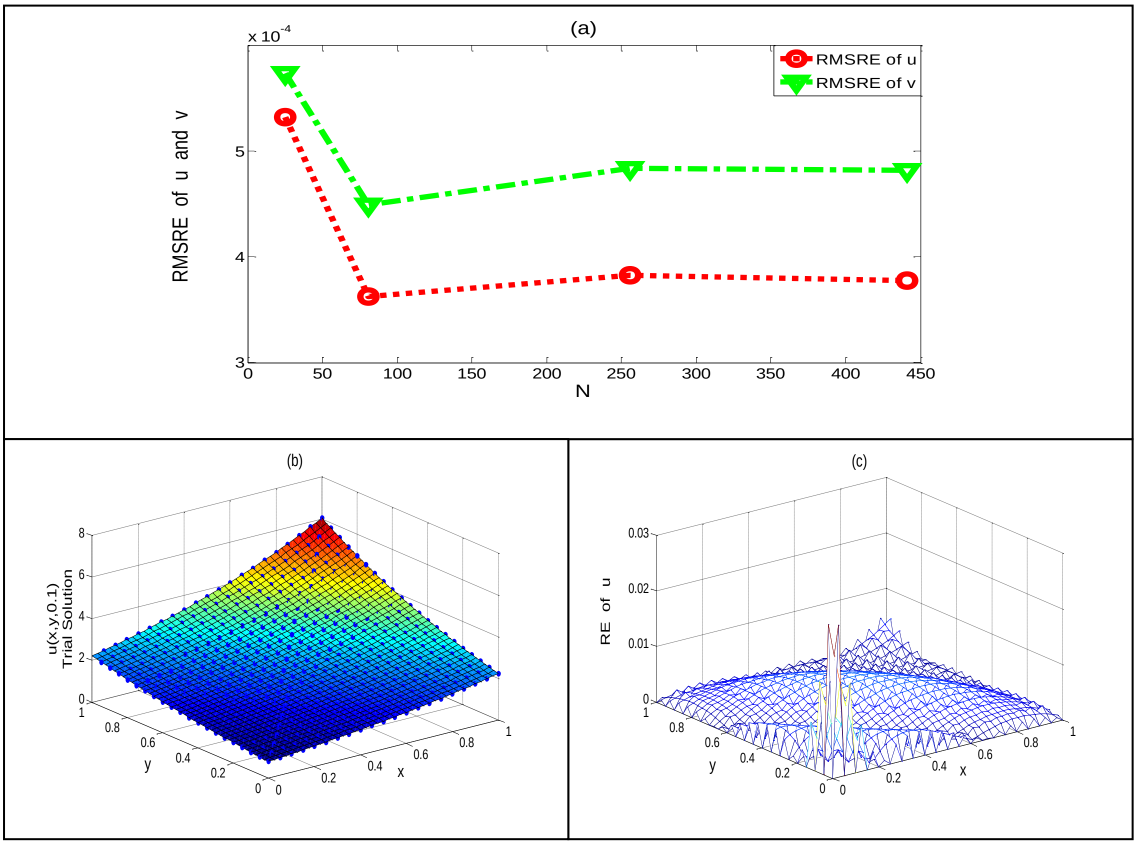


Figure 4.18 Trial solutions and errors of u and v by the MLPG5 method based on the RPIM with temporal discretization by the Crank-Nicolson method, N=441, $m = 10$, $t = 1$ and $\Delta t = 0.1$: (a) trial solution of u; (b) corresponding error profile of u; (c) trial solution of v ; (d) corresponding error profile of v.

4.1.3 The Results of Example1 by The MLPG4 Method Based on MKA



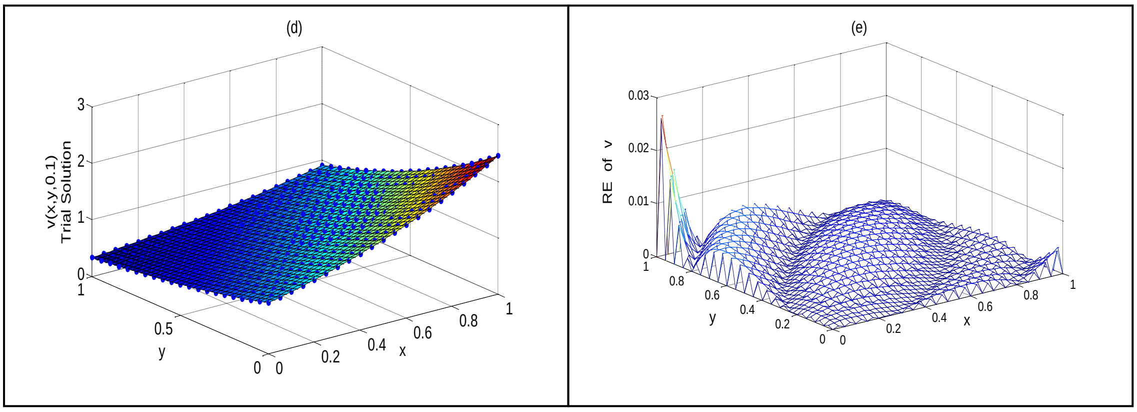


Figure 4.19 Trial solutions and errors of u and v by MLPG4 method based on MKA against the number of nodal points at time instant $t = 0.1$ using $\Delta t = 0.01$ and $N = 441$: (a) RMSRE of u and v; (b) trial solution of u; (c) corresponding error profile of u; (d) trial solution of v; (e) corresponding error profile of v.

4.2 Example 2, application for Brusselator system

To confirm that the newly- developed formulation from this research can solve a real word application example well, let us consider the nonlinear reaction-diffusion Brusselator system in the two- dimensional region $\Omega = [0,1] \times [0,1]$ (Twizell, 1999).

$$\frac{\partial u}{\partial t} = D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - (A+1)u + u^2 v + B,$$

$$\frac{\partial v}{\partial t} = D\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + Au - u^2 v,$$

with D=0.002 , $A = \frac{1}{2}$, B=1 , initial conditions

$$u(x,y,0) = \frac{1}{2}x^2 - \frac{1}{3}x^3, \qquad v(x,y,0) = \frac{1}{2}y^2 - \frac{1}{3}y^3,$$

and Neuman boundary conditions:

$$\frac{\partial u(0,y,t)}{\partial x} = \frac{\partial u(1,y,t)}{\partial x} = \frac{\partial u(x,0,t)}{\partial y} = \frac{\partial u(x,1,t)}{\partial y} = 0,$$

$$\frac{\partial v(0,y,t)}{\partial x} = \frac{\partial v(1,y,t)}{\partial x} = \frac{\partial v(x,0,t)}{\partial y} = \frac{\partial v(x,1,t)}{\partial y} = 0,$$

for which the exact solution is unknown. For small values of the diffusion coefficient D, if $1 - A + B^2 > 0$ then the numerical solution of the Brusselator system converges to an equilibrium point$(B, A/B)$ and the exact solution is unknown. The experimental results for maximum and minimum values of the exact solutions using the MLPG5 method based on the MKA with temporal discretization by the Euler, Runge-Kutta and Crank-Nicolson methods are presented in table 4.1, respectively. Table 4.1 shows the results of the developed method in comparison with Shirzadi (2013).

From this table, it is found that the approximate solutions tend to the steady state values of $(u^*, v^*) = (B, A/B) = \left(1, \frac{1}{2}\right)$. Fig.4.20-4.21 show the viewpoint of the changing solution from initial condition to the steady state as t tends to infinity. The experimental results are similar to those previously reported (Shirzadi,2013 ; Ul-Islam, 2010; Twizell, 1999).

For the developed method based on RPIM, the approximate solutions still tends to 1 and 0.5, respectively (see in table 4.2). The experimental results are similar to the before results. Fig.4.22 confirms the high accuracy of the proposed method.

4.2.1 The Results of Example 2 by The MLPG5 Method Based on MKA

Table 4.1 The solution by the MLPG5 method based on MKA and using $m = 10$, $N = 441$ and $\Delta t = 0.1$ obtained at some different points.

t	(0.2,0.2)							
	Shirzadi(2013)		Euler		Runge-Kutta		Crank-Nicolson	
	u	v	u	v	u	v	u	v
1	0.5324	0.1698	0.5489	0.1668	0.5327	0.1709	0.5318	0.1577
2	0.7035	0.3723	0.7120	0.3739	0.7043	0.3728	0.7003	0.3593
3	0.8182	0.4956	0.8243	0.4987	0.8191	0.4954	0.8125	0.4861
4	0.9109	0.5366	0.9172	0.5384	0.9116	0.5361	0.9040	0.5321
5	0.9723	0.5308	0.9781	0.5300	0.9726	0.5304	0.9665	0.5301
6	1.0000	0.5144	1.0037	0.5124	1.0000	0.5142	0.9966	0.5153
7	1.0064	0.5036	1.0076	0.5020	1.0063	0.5036	1.0050	0.5046
8	1.0047	0.4995	1.0045	0.4987	1.0046	0.4995	1.0045	0.5001
9	1.0020	0.4988	1.0015	0.4986	1.0020	0.4989	1.0022	0.4991
10	1.0004	0.4992	1.0001	0.4993	1.0005	0.4993	1.0007	0.4993
↓	↓	↓	↓	↓	↓	↓	↓	↓
∞	1	0.5	1	0.5	1	0.5	1	0.5

t	(0.4,0.6)							
	Shirzadi(2013)		Euler		Runge-Kutta		Crank-Nicolson	
	u	v	u	v	u	v	u	v
1	0.5509	0.2566	0.5668	0.2539	0.5513	0.2574	0.5507	0.2464
2	0.7281	0.4301	0.7371	0.4313	0.7289	0.4304	0.7256	0.4199
3	0.8461	0.5218	0.8529	0.5239	0.8470	0.5215	0.8417	0.5149
4	0.9340	0.5407	0.9404	0.5411	0.9346	0.5402	0.9289	0.5376
5	0.9852	0.5264	0.9904	0.5249	0.9854	0.5261	0.9811	0.5260
6	1.0044	0.5102	1.0071	0.5082	1.0044	0.5102	1.0021	0.5109
7	1.0066	0.5016	1.0072	0.5003	1.0066	0.5017	1.0058	0.5023
8	1.0036	0.4989	1.0035	0.4984	1.0039	0.4990	1.0038	0.4994
9	1.0014	0.4988	1.0009	0.4988	1.0014	0.4989	1.0016	0.4990
10	1.0002	0.4993	0.9999	0.4995	1.0002	0.4994	1.0003	0.4994
↓	↓	↓	↓	↓	↓	↓	↓	↓
∞	1	0.5	1	0.5	1	0.5	1	0.5

Table 4.1 (Cont.)

t	(0.5,0.5)							
	Shirzadi(2013)		Euler		Runge-Kutta		Crank-Nicolson	
	u	v	u	v	u	v	u	v
1	0.5552	0.2401	0.5703	0.2377	0.5556	0.2409	0.5550	0.2302
2	0.7260	0.4196	0.7345	0.4211	0.7268	0.4199	0.7236	0.4095
3	0.8420	0.5170	0.8484	0.5194	0.8428	0.5167	0.8375	0.5100
4	0.9302	0.5398	0.9365	0.5406	0.9307	0.5393	0.9250	0.5366
5	0.9830	0.5271	0.9883	0.5358	0.9832	0.5268	0.9788	0.5267
6	1.0036	0.5109	1.0065	0.5089	1.0036	0.5108	1.0012	0.5116
7	1.0066	0.5019	1.0072	0.5006	1.0065	0.5020	1.0056	0.5027
8	1.0040	0.4990	1.0037	0.4984	1.0040	0.4991	1.0039	0.4995
9	1.0015	0.4988	1.0010	0.4988	1.0015	0.4989	1.0017	0.4990
10	1.0003	0.4993	0.9999	0.4995	1.0002	0.4994	1.0004	0.4994
↓	↓	↓	↓	↓	↓	↓	↓	↓
∞	1	0.5	1	0.5	1	0.5	1	0.5

t	(0.8,0.9)							
	Shirzadi(2013)		Euler		Runge-Kutta		Crank-Nicolson	
	u	v	u	v	u	v	u	v
1	0.5814	0.3176	0.5951	0.3159	0.5818	0.3182	0.5817	0.3101
2	0.7531	0.4685	0.7618	0.4698	0.7539	0.4685	0.7518	0.4611
3	0.8697	0.5361	0.8767	0.5377	0.8705	0.5357	0.8669	0.5311
4	0.9510	0.5404	0.9574	0.5400	0.9515	0.5399	0.9476	0.5380
5	0.9934	0.5223	0.9980	0.5204	0.9936	0.5221	0.9906	0.5218
6	1.0066	0.5073	1.0086	0.5053	1.0066	0.5073	1.0049	0.5076
7	1.0064	0.5004	1.0066	0.4993	1.0064	0.5004	1.0057	0.5009
8	1.0033	0.4986	1.0028	0.4983	1.0033	0.4987	1.0032	0.4990
9	1.0010	0.4989	1.0005	0.4989	1.0010	0.4989	1.0011	0.4991
10	1.0000	0.4994	0.9998	0.4996	1.0000	0.4995	1.0001	0.4995
↓	↓	↓	↓	↓	↓	↓	↓	↓
∞	1	0.5	1	0.5	1	0.5	1	0.5

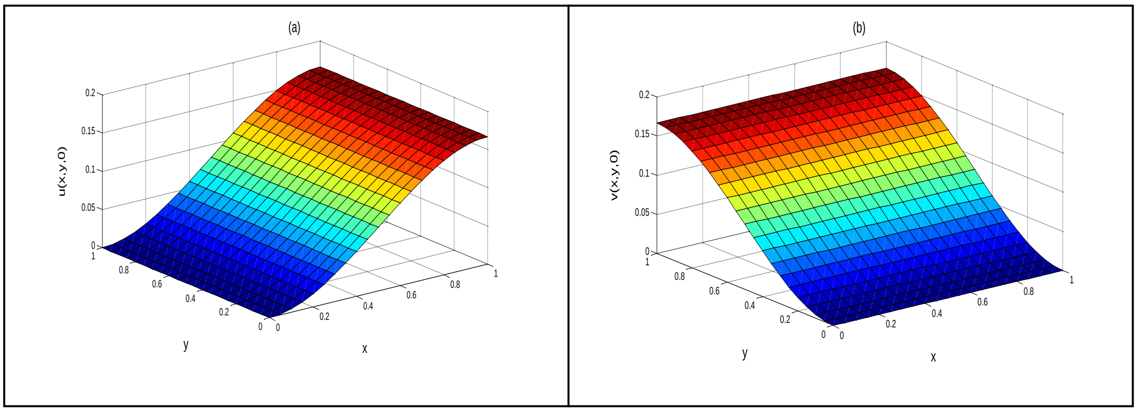


Figure 4.20 Concentration profiles of u and v: (a) u at $t = 0$; (b) v at $t = 0$

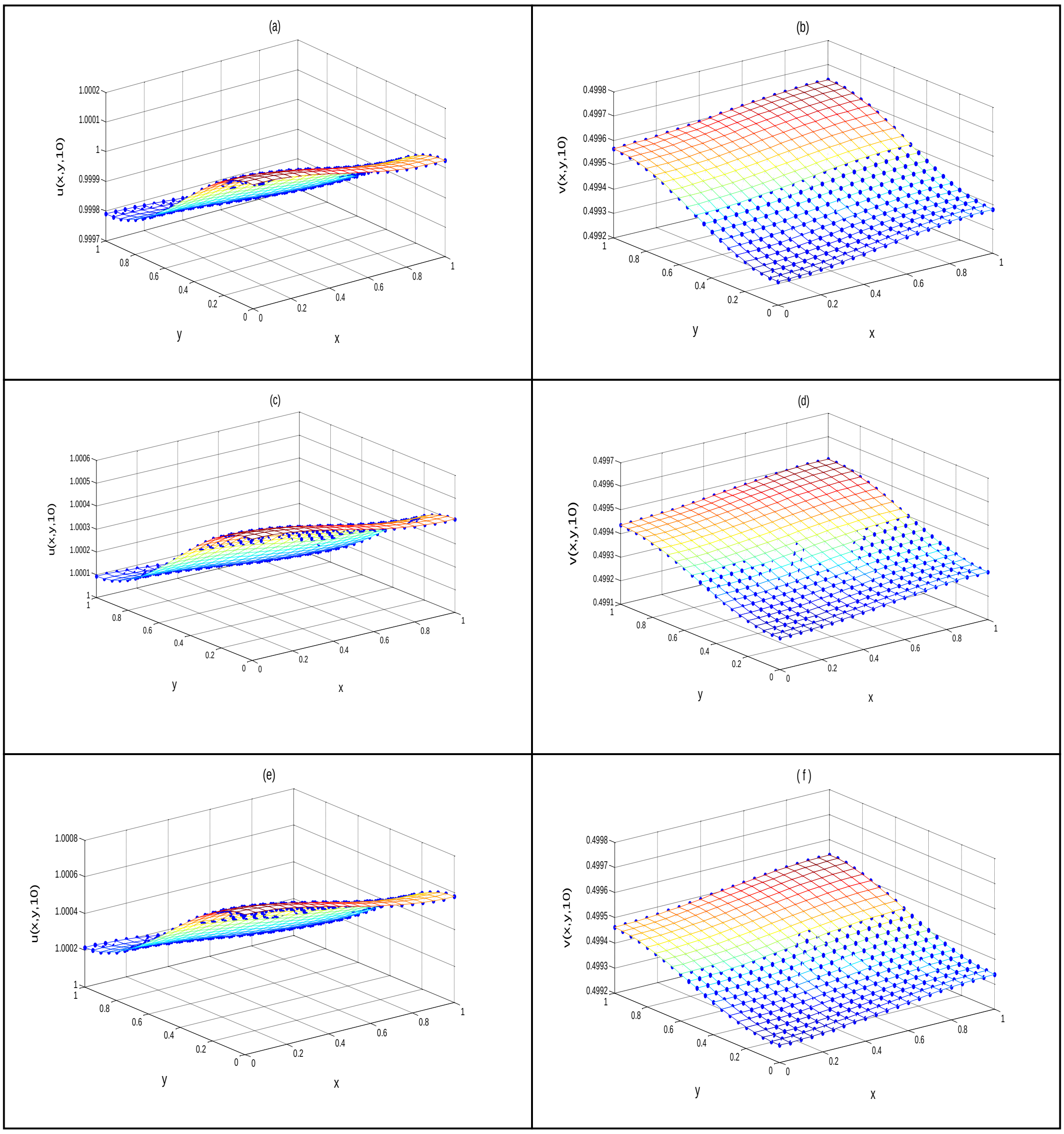


Figure 4.21 The approximate solutions of u and v at $t = 10$ obtained by the MLPG5 method based on MKA using $N = 441, m = 10$ and $\Delta t = 0.1$: (a) u using the Euler method; (b) v using the Euler method; (c) u using the Runge-Kutta method; (d) v using the Runge-Kutta method; (e) u using the Crank-Nicolson method; (f) v using the Crank-Nicolson method.

4.2.2 The Results of Example 2 by The MLPG5 Method Based on RPIM

Table 4.2 The solution by the MLPG5 method based on RPIM and using $m = 10$, $N = 441$ and $\Delta t = 0.1$ obtained at some different points.

t	(0.2,0.2)					
	Shirzadi(2013)		Runge-Kutta		Crank-Nicolson	
	u	v	u	v	u	v
1	0.5324	0.1698	0.5327	0.1709	0.5318	0.1577
2	0.7035	0.3723	0.7043	0.3728	0.7003	0.3593
3	0.8182	0.4956	0.8191	0.4954	0.8125	0.4861
4	0.9109	0.5366	0.9116	0.5361	0.9040	0.5321
5	0.9723	0.5308	0.9726	0.5304	0.9665	0.5301
6	1.0000	0.5144	1.0000	0.5143	0.9966	0.5153
7	1.0064	0.5036	1.0063	0.5036	1.0050	0.5046
8	1.0047	0.4995	1.0046	0.4995	1.0045	0.5001
9	1.0020	0.4988	1.0020	0.4989	1.0022	0.4991
10	1.0004	0.4992	1.0005	0.4993	1.0007	0.4993
↓	↓	↓	↓	↓	↓	↓
∞	1	0.5	1	0.5	1	0.5

t	(0.4,0.6)					
	Shirzadi(2013)		Runge-Kutta		Crank-Nicolson	
	u	v	u	v	u	v
1	0.5509	0.2566	0.5513	0.2574	0.5507	0.2464
2	0.7281	0.4301	0.7289	0.4304	0.7256	0.4199
3	0.8461	0.5218	0.8470	0.5215	0.8417	0.5149
4	0.9340	0.5407	0.9346	0.5402	0.9289	0.5376
5	0.9852	0.5264	0.9854	0.5261	0.9811	0.5260
6	1.0044	0.5102	1.0044	0.5102	1.002	0.5109
7	1.0066	0.5016	1.0066	0.5017	1.0058	0.5023
8	1.0036	0.4989	1.0040	0.4990	1.0038	0.4994
9	1.0014	0.4988	1.0014	0.4989	1.0016	0.4990
10	1.0002	0.4993	1.0002	0.4994	1.0003	0.4994
↓	↓	↓	↓	↓	↓	↓
∞	1	0.5	1	0.5	1	0.5

Table4.2 (Cont).

t	(0.5,0.5)					
	Shirzadi(2013)		Runge-Kutta		Crank-Nicolson	
	u	v	u	v	u	v
1	0.5552	0.2401	0.5556	0.2409	0.5550	0.2302
2	0.7260	0.4196	0.7268	0.4199	0.7236	0.4095
3	0.8420	0.5170	0.8428	0.5167	0.8375	0.5100
4	0.9302	0.5398	0.9307	0.5393	0.9250	0.5366
5	0.9830	0.5271	0.9832	0.5268	0.9788	0.5267
6	1.0036	0.5109	1.0036	0.5109	1.0012	0.5116
7	1.0066	0.5019	1.0065	0.5020	1.0056	0.5027
8	1.0040	0.4990	1.0040	0.4991	1.0039	0.4995
9	1.0015	0.4988	1.0015	0.4989	1.0017	0.4990
10	1.0003	0.4993	1.0003	0.4994	1.0004	0.4994
↓	↓	↓	↓	↓	↓	↓
∞	1	0.5	1	0.5	1	0.5

t	(0.8,0.9)					
	Shirzadi(2013)		Runge-Kutta		Crank-Nicolson	
	u	v	u	v	u	v
1	0.5814	0.3176	0.5818	0.3182	0.5817	0.3101
2	0.7531	0.4685	0.7539	0.4685	0.7518	0.4611
3	0.8697	0.5361	0.8705	0.5357	0.8669	0.5311
4	0.9510	0.5404	0.9515	0.5399	0.9476	0.5380
5	0.9934	0.5223	0.9936	0.5221	0.9906	0.5218
6	1.0066	0.5073	1.0066	0.5072	1.0049	0.5076
7	1.0064	0.5004	1.0064	0.5004	1.0057	0.5009
8	1.0033	0.4986	1.0033	0.4987	1.0032	0.4990
9	1.0010	0.4989	1.0010	0.4989	1.0011	0.4991
10	1.0000	0.4994	1.0000	0.4995	1.0001	0.4995
↓	↓	↓	↓	↓	↓	↓
∞	1	0.5	1	0.5	1	0.5

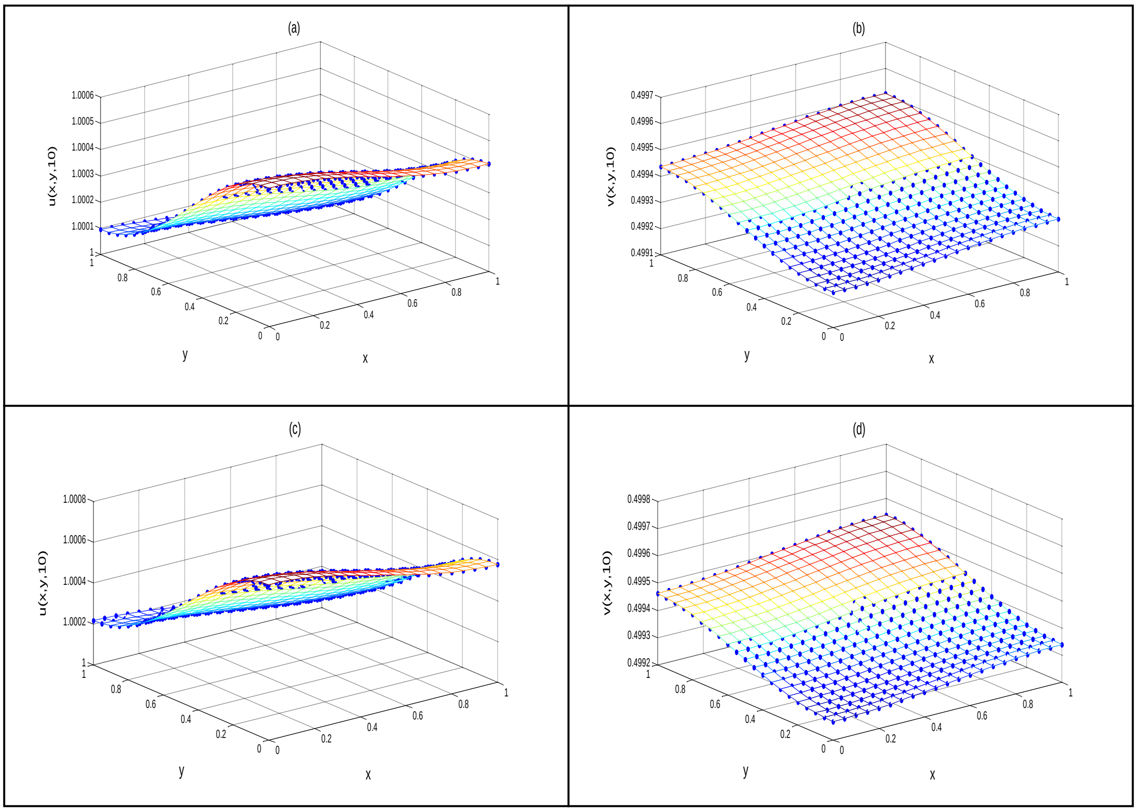


Figure 4.22 The approximate solutions at $t = 10$ obtained by the MLPG5 method based on RPIM using $N = 441, m = 10$ and $\Delta t = 0.1$: (a) u using the Runge-Kutta method; (b) v using the Runge-Kutta method; (c) u using the Crank-Nicolson method; (d) v using the Crank-Nicolson method.