CHAPTER 3 METHODOLOGY

This chapter reveals the space discretization by the MLPG5 and MLPG4 methods, which are developed based on MKA and RPIM, subjected to the Dirichlet and Neumann boundary conditions. The Euler, Runge-Kutta and Crank-Nicolson methods are used for temporal discretization.

3.1 Space Discretization by MLPG Method with a Heaviside Step Test Function (MLPG5)

The local integral formulation of equation (2.1) can be written as:

$$\int_{\Omega_s^i} \frac{\partial u}{\partial t} w_i d\Omega = D_1 \int_{\Omega_s^i} (\nabla^2 u) w_i d\Omega + \alpha_1 \int_{\Omega_s^i} u\, w_i d\Omega + \int_{\Omega_s^i} A(u,v)\, w_i d\Omega + \int_{\Omega_s^i} f_1\,(\boldsymbol{x},t) w_i d\Omega, \tag{3.1}$$

$$\int_{\Omega_s^i} \frac{\partial v}{\partial t} w_i d\Omega = D_2 \int_{\Omega_s^i} (\nabla^2 v) w_i d\Omega + \alpha_2 \int_{\Omega_s^i} v\, w_i d\Omega + \beta \int_{\Omega_s^i} u\, w_i d\Omega + \int_{\Omega_s^i} B(u,v)\, w_i d\Omega + \int_{\Omega_s^i} f_2\,(\boldsymbol{x},t) w_i d\Omega, \tag{3.2}$$

where w_i is a Heaviside step used as the test function:

$$w(\boldsymbol{x}) = \begin{cases} 1\,, & \boldsymbol{x} \in \Omega_s^i \\ 0\,, & \boldsymbol{x} \notin \Omega_s^i, \end{cases} \tag{3.3}$$

u and v are trial functions, and instead of the entire domain Ω_s we have considered a sub-domain Ω_s^i located entirely within domain Ω, $\boldsymbol{x} = (x,y)^T \in \Omega \subset \mathbb{R}^2$ .The domain Ω is enclosed by $\Gamma = \Gamma_D \cup \Gamma_N$, with boundary conditions:

$$u = \bar{u} \quad \text{and} \quad v = \bar{v} \quad \text{on } \Gamma_D\ , \tag{3.4}$$

$$\boldsymbol{n} \cdot \nabla u = \bar{q}_u \quad \text{and} \quad \boldsymbol{n} \cdot \nabla v = \bar{q}_v \quad \text{on} \quad \Gamma_N\ , \tag{3.5}$$

where $\boldsymbol{n} = (n_1, n_2)^{\,T}$ is an outward unit normal of the boundary and $\nabla u \equiv \frac{\partial u}{\partial \boldsymbol{n}}$. The condition (3.4) is often referred to as the Dirichlet boundary condition; and (3.5) as the Neumann boundary condition.

We can integrate it by part, by using

$$\nabla^2 u w_i = \nabla \cdot (\nabla u w_i) - \nabla u \cdot \nabla w_i,$$

$\int_{\Omega_s^i} (\nabla^2 u) w_i d\Omega$ in Eq.(3.1) can be written as:

$$\int_{\Omega_s^i} (\nabla^2 u) w_i d\Omega = \int_{\Omega_s^i} \big(\nabla \cdot (\nabla u w_i)\big) d\Omega - \int_{\Omega_s^i} (\nabla u) \cdot (\nabla w_i) d\,\Omega.$$

By divergence theorem, the first term on the right hand of equation (3.1) becomes

$$\int_{\Omega_s^i} (\nabla^2 u) w_i d\Omega = \int_{\partial\Omega_s^i} \frac{\partial u}{\partial \boldsymbol{n}} w_i d\Gamma - \int_{\Omega_s^i} (\nabla u) \cdot (\nabla w_i) d\,\Omega\,,$$

Similarly, the first term on the right hand of equation (3.2) becomes

$$\int_{\Omega_s^i} (\nabla^2 v) w_i d\Omega = \int_{\partial\Omega_s^i} \frac{\partial v}{\partial \boldsymbol{n}} w_i d\Gamma - \int_{\Omega_s^i} (\nabla u) \cdot (\nabla w_i) d\,\Omega\,,$$

where $\partial\Omega_s^i = L_s^i \cup \Gamma_{sD}^i \cup \Gamma_{sN}^i$ is the boundary of Ω_s^i, $\boldsymbol{n} = (n_1, n_2)^T$ is the outward unit normal to the boundary $\partial\Omega_s^i$, and $\partial u/\partial\boldsymbol{n}$ is the normal derivative, i.e., the derivative in the outward normal direction to the boundary $\partial\Omega_s$. Recall that derived local integral equations do not involve any gradients of the field variables. This is very appropriate from the point of view of meshless approximations, because of saving the computational time and decreasing the inaccuracy due to derivatives of field variables.

Let $\tilde{u}(\boldsymbol{x}, t)$ and $\tilde{v}(\boldsymbol{x}, t)$, which substitute $u(\boldsymbol{x}, t)$ and $v(\boldsymbol{x}, t)$ respectively, be the trial solutions.

$$\tilde{u}(\boldsymbol{x}, t) = \sum_{j=1}^{N} \phi_j\,(\boldsymbol{x}) \hat{u}_j(t), \quad \tilde{v}_j(\boldsymbol{x}, t) = \sum_{j=1}^{N} \phi_j\,(\boldsymbol{x}) \hat{v}_j(t), \tag{3.6}$$

$$\tilde{A} = A\big(\tilde{u}(\boldsymbol{x}, t), \tilde{v}(\boldsymbol{x}, t)\big), \qquad \tilde{B} = B\big(\tilde{u}(\boldsymbol{x}, t), \tilde{v}(\boldsymbol{x}, t)\big). \tag{3.7}$$

For internal nodes, from local integral equations (3.1) and (3.2), $\nabla w_i = 0$ and using the MKA or RPIM (Eqs.(2.9) or (2.24)), we have the following nonlinear equations:

$$\begin{aligned} \sum_{j=1}^{N} \left[\int_{\Omega_s^i} \phi_j(\boldsymbol{x}) d\Omega \right] \frac{\partial \hat{u}_j}{\partial t} &= \sum_{j=1}^{N} \left[D_1 \left(\int_{L_s^i} \phi_{j,n}(\boldsymbol{x}) d\Gamma + \int_{\Gamma_{sD}^i} \phi_{j,n}(\boldsymbol{x}) d\Gamma \right) \right. \\ &\quad \left. + \left(\alpha_1 \int_{\Omega_s^i} \phi_j(\boldsymbol{x}) d\Omega \right) \right] \hat{u}_j + D_1 \int_{L_{sN}^i} \bar{q}_u d\Gamma + \int_{\Omega_s^i} \tilde{A}\, d\Omega \\ &\quad + \int_{\Omega_s^i} f_1\,(\boldsymbol{x}, t) d\Omega, \end{aligned} \tag{3.8}$$

$$\sum_{j=1}^{N}\left[\int_{\Omega_s^i}\phi_j(\boldsymbol{x})d\Omega\right]\frac{\partial\hat{v}_j}{\partial t}$$
$$=\sum_{j=1}^{N}\left[D_2\left(\int_{L_s^i}\phi_{j,n}(\boldsymbol{x})d\Gamma+\int_{\Gamma_{sD}^i}\phi_{j,n}(\boldsymbol{x})d\Gamma\right)\right.$$
$$\left.+\left(\alpha_2\int_{\Omega_s^i}\phi_j(\boldsymbol{x})d\Omega\right)\right]\hat{v}_j+\sum_{j=1}^{N}\left[\beta\int_{\Omega_s^i}\phi_j(\boldsymbol{x})d\Omega\right]\hat{u}_j$$
$$+D_2\int_{L_{sN}^i}\bar{q}_v d\Gamma+\int_{\Omega_s^i}\tilde{B}\,d\Omega+\int_{\Omega_s^i}f_2\,(\boldsymbol{x},t)d\Omega. \tag{3.9}$$

The following abbreviations have been used for the integral term:

$$k_{ij}=\int_{\Omega_s^i}\phi_j(\boldsymbol{x})\,d\Omega,$$

$$h_{ij}^1=D_1\left(\int_{L_s^i}\left(\frac{\partial\phi_j(\boldsymbol{x})}{\partial\boldsymbol{n}}\right)d\Gamma+\int_{L_{sD}^i}\left(\frac{\partial\phi_j(\boldsymbol{x})}{\partial\boldsymbol{n}}\right)d\Gamma\right)+\alpha_1\int_{\Omega_s^i}\phi_j(\boldsymbol{x})d\Omega,$$

$$h_{ij}^2=D_2\left(\int_{L_s^i}\left(\frac{\partial\phi_j(\boldsymbol{x})}{\partial\boldsymbol{n}}\right)d\Gamma+\int_{L_{sD}^i}\left(\frac{\partial\phi_j(\boldsymbol{x})}{\partial\boldsymbol{n}}\right)d\Gamma\right)+\alpha_2\int_{\Omega_s^i}\phi_j(\boldsymbol{x})d\Omega,$$

$$l_{ij}=\beta\int_{\Omega_s^i}\phi_j(\boldsymbol{x})d\Omega,$$

$$f_{1i}=D_1\int_{L_{sN}^i}\bar{q}_u d\Gamma+\int_{\Omega_s^i}\tilde{A}\,d\Omega\;+\int_{\Omega_s^i}f_1\,(\boldsymbol{x},t)d\Omega,$$

$$f_{2i}=D_2\int_{L_{sN}^i}\bar{q}_v d\Gamma+\int_{\Omega_s^i}\tilde{B}\,d\Omega\;+\int_{\Omega_s^i}f_2\,(\boldsymbol{x},t)d\Omega.$$

The boundary and domain integrals are calculated by using the Gauss-Legendre quadrature method. We can rewrite Eqs. (3.8) and (3.9) as:

$$\sum_{j=1}^{N}k_{ij}\frac{\partial\hat{u}_j}{\partial t}=\sum_{\boldsymbol{j=1}}^{N}h_{ij}^1\,\hat{u}_j+f_{1i}\,, \tag{3.10}$$

$$\sum_{j=1}^{N} k_{ij}\frac{\partial \hat{v}_j}{\partial t} = \sum_{j=1}^{N}\left(h_{ij}^{2}\hat{v}_j + l_{ij}\hat{u}_j\right) + f_{2i}. \tag{3.11}$$

Temporal discretization

Eqs. (3.10) and (3.11) can be transformed into vector form as:

$$\boldsymbol{K}\frac{\partial \widehat{\boldsymbol{U}}}{\partial t} = \boldsymbol{H_1}\widehat{\boldsymbol{U}} + \boldsymbol{F_1}. \tag{3.12}$$

Similarly, we have

$$\boldsymbol{K}\frac{\partial \widehat{\boldsymbol{V}}}{\partial t} = \boldsymbol{H_2}\widehat{\boldsymbol{V}} + \boldsymbol{L}\widehat{\boldsymbol{U}} + \boldsymbol{F_2}, \tag{3.13}$$

where

$\boldsymbol{K} = \left[k_{ij}\right]_{N\times N}, \boldsymbol{H_1} = \left[h_{ij}^{1}\right]_{N\times N}, \ \boldsymbol{H_2} = \left[h_{ij}^{2}\right]_{N\times N}, \boldsymbol{L} = \left[l_{ij}\right]_{N\times N}\ , \ \boldsymbol{F_1} \quad = [f_{1i}]_{N\times 1}$

$\boldsymbol{F_2} = [f_{2i}]_{N\times 1}\ , \widehat{\boldsymbol{U}} = [\hat{u}_1 \ \ \hat{u}_2 \ \ \cdots \ \ \hat{u}_N]',$ and $\widehat{\boldsymbol{V}} = [\hat{v}_1 \ \ \hat{v}_2 \ \ \cdots \ \ \hat{v}_N]'.$

The finite-difference approximation of the time derivatives of Eqs. (3.12) and (3.13) in the Euler method is given as follows:

$$\boldsymbol{K}\widehat{\boldsymbol{U}}^{k+1} = (\boldsymbol{K} + \Delta t\boldsymbol{H}_1)\widehat{\boldsymbol{U}}^{k} + \Delta t\boldsymbol{F}_1^{k}, \tag{3.14}$$

$$\boldsymbol{K}\widehat{\boldsymbol{V}}^{k+1} = (\boldsymbol{K} + \Delta t\boldsymbol{H}_2)\widehat{\boldsymbol{V}}^{k} + \Delta t\left(\boldsymbol{L}\widehat{\boldsymbol{U}}^{k} + \boldsymbol{F}_2^{k}\right), \tag{3.15}$$

In case of the RK's method, Eqs.(3.12) and (3.13) can be rewritten

$$\boldsymbol{K}\frac{\partial \widehat{\boldsymbol{U}}}{\partial t} = \boldsymbol{H_1}\widehat{\boldsymbol{U}} + \boldsymbol{F_1} = \boldsymbol{G_1}\left(\widehat{\boldsymbol{U}}, t\right), \tag{3.16}$$

$$\boldsymbol{K}\frac{\partial \widehat{\boldsymbol{V}}}{\partial t} = \boldsymbol{H_2}\widehat{\boldsymbol{V}} + \boldsymbol{L}\widehat{\boldsymbol{U}} + \boldsymbol{F_2} = \boldsymbol{G_2}\left(\widehat{\boldsymbol{U}}, \widehat{\boldsymbol{V}}, t\right), \tag{3.17}$$

hence

$$\boldsymbol{K}\widehat{\boldsymbol{U}}^{k+1} = \boldsymbol{K}\widehat{\boldsymbol{U}}^{k} + \frac{\Delta t}{6}(S_{11} + 2S_{21} + 2S_{31} + S_{41}), \tag{3.18}$$

$$\boldsymbol{K}\widehat{\boldsymbol{V}}^{k+1} = \boldsymbol{K}\widehat{\boldsymbol{V}}^{k} + \frac{\Delta t}{6}(S_{12} + 2S_{22} + 2S_{32} + S_{42}), \tag{3.19}$$

where

$$S_{11} = \boldsymbol{G_1}\left(\widehat{\boldsymbol{U}}_{\boldsymbol{1}}, t_1\right) \quad , \quad S_{12} = \boldsymbol{G_2}\left(\widehat{\boldsymbol{U}}_1, \widehat{\boldsymbol{V}}_1, t_1\right),$$

$$S_{21} = \boldsymbol{G_1}\left(\widehat{\boldsymbol{U}}_{\boldsymbol{1}} + \frac{\Delta \boldsymbol{t}}{2} S_{11}, t_1 + \frac{\Delta t}{2}\right) \quad , \quad S_{22} = \boldsymbol{G_2}\left(\widehat{\boldsymbol{U}}_1 + \frac{\Delta \boldsymbol{t}}{2} S_{12}, \widehat{\boldsymbol{V}}_{\boldsymbol{1}} + \frac{\Delta \boldsymbol{t}}{2} S_{12}, t_1 + \frac{\Delta t}{2}\right),$$

$$S_{31} = \boldsymbol{G_1}\left(\widehat{\boldsymbol{U}}_{\boldsymbol{1}} + \frac{\Delta \boldsymbol{t}}{2} S_{21}, t_1 + \frac{\Delta t}{2}\right) \quad , \quad S_{32} = \boldsymbol{G_2}\left(\widehat{\boldsymbol{U}}_1 + \frac{\Delta \boldsymbol{t}}{2} S_{22}, \widehat{\boldsymbol{V}}_{\boldsymbol{1}} + \frac{\Delta \boldsymbol{t}}{2} S_{22}, t_1 + \frac{\Delta t}{2}\right),$$

$$S_{41} = \boldsymbol{G_1}\left(\widehat{\boldsymbol{U}}_{\boldsymbol{1}} + \Delta t S_{31}, t_1 + \Delta t\right) \quad , \quad S_{42} = \boldsymbol{G_2}\left(\widehat{\boldsymbol{U}}_1 + \Delta t S_{32}, \widehat{\boldsymbol{V}}_{\boldsymbol{1}} + \Delta t S_{32}, t_1 + \Delta t\right),$$

In case of the Crank-Nicloson method, Eqs.(3.12) and (3.13) can be rewritten as:

$$(2\boldsymbol{K} - \Delta t \boldsymbol{H_1})\widehat{\boldsymbol{U}}^{\boldsymbol{k+1}} = (2\boldsymbol{K} + \Delta t \boldsymbol{H_1})\widehat{\boldsymbol{U}}^{\boldsymbol{k}} + \Delta t\left(\boldsymbol{F}_1^{k+1} + \boldsymbol{F}_1^{k}\right), \tag{3.20}$$

$$\begin{aligned}(2\boldsymbol{K} - \Delta t \boldsymbol{H_2})\widehat{\boldsymbol{V}}^{\boldsymbol{k+1}} = (2\boldsymbol{K} + \Delta t \boldsymbol{H_2})\widehat{\boldsymbol{V}}^{\boldsymbol{k}} + \Delta t(\boldsymbol{LU}^{k+1} + \boldsymbol{LU}^{k}) \\ + \Delta t\left(\boldsymbol{F}_2^{k+1} + \boldsymbol{F}_2^{k}\right).\end{aligned} \tag{3.21}$$

In case of the Euler and the Runge-Kutta methods, $\widehat{\boldsymbol{U}}^{\boldsymbol{k+1}}$ and $\widehat{\boldsymbol{V}}^{\boldsymbol{k+1}}$ can be solve in Eqs. (3.14) , (3.15) ,(3.16) and(3.17), respectively. For the Crank-Nicolson method, because of $\boldsymbol{A}$ and $\boldsymbol{B}$ are nonlinear functions of u and v**,** we solve them iteratively in each time step with replacing $\boldsymbol{A}^{k+1}$ and $\boldsymbol{B}^{k+1}$by $\boldsymbol{A}^{k}$ and $\boldsymbol{B}^{k}$, respectively, at zeroth iteration. Eqs.(3.20) and (3.21) are converted into a set of nonlinear algebraic equation for unknowns $\widehat{\boldsymbol{U}}^{\boldsymbol{k+1}}$and $\widehat{\boldsymbol{V}}^{\boldsymbol{k+1}}$.

3.2 Space Discretization by MLPG Method with The Fundamental Solution Test Function (MLPG4)

Let w_i of Eqs.(3.1) and (3.2) is surrogated u_i^*, where u_i^* is a test function, u and v are trial functions, and instead of the entire domain Ω_s we have considered a sub-domain Ω_s^i located entirely domain Ω which is circle of radius r_0 and centered at node.
$\boldsymbol{x} = (x, y) \in \Omega \subset \mathbb{R}^2$.

It is well known that $u^* = -(1/2\pi)\ln(r) + C$ is fundamental solution corresponding to Poisson's equation, i.e., $\nabla^2 u^* + \delta(x, y) = 0$, where C is an arbitrary constant, $\delta(x, y)$ is the Dirac delta function and r is the distance the field and source points, i.e., $r_i = \|\boldsymbol{x} - \boldsymbol{x}_i\|$. If we choose $C = (1/2\pi)\, ln(r_0)$ where r_0 is the radius of circular sub-domain Ω_s^i center at point $\boldsymbol{x}_i$, then the modified fundamental solution to Poisson's equation can be given by

$$u_i^* = -\frac{1}{2\pi} ln\left(\frac{r_i}{r_0}\right).$$

We can integrate it by part, by using

$$\nabla^2 u u^* = \nabla \cdot (\nabla u u_i^*) - \nabla u \cdot \nabla u_i^*, \tag{3.22}$$

$\int_{\Omega_s^i} (\nabla^2 u) u_i^* d\Omega$ in Eq.(3.1) can be written as:

$$\int_{\Omega_s^i} (\nabla^2 u) u_i^* d\Omega = \int_{\Omega_s^i} \big(\nabla \cdot (\nabla u u_i^*)\big) d\Omega - \int_{\Omega_s^i} (\nabla u) \cdot (\nabla u_i^*) d\,\Omega. \tag{3.23}$$

By divergence theorem, the first term on the right hand of equation (3.23) becomes

$$\int_{\Omega_s^i} (\nabla^2 u) u_i^* d\Omega = \int_{\partial\Omega_s^i} \frac{\partial u}{\partial \boldsymbol{n}} u_i^* d\Gamma - \int_{\Omega_s^i} (\nabla u) \cdot (\nabla u_i^*) d\,\Omega\,, \tag{3.24}$$

where $\partial\Omega_s^i = L_s^i \cup \Gamma_{sD}^i \cup \Gamma_{sN}^i$ is the boundary of Ω_s^i, $\boldsymbol{n} = (n_1, n_2)^T$ is the outward unit normal to the boundary $\partial\Omega_s^i$, and $\partial u / \partial \boldsymbol{n}$ is the normal derivative, i.e., the derivative in the outward normal direction to the boundary $\partial\Omega_s$. Recall that derived local integral equations do not involve any gradients of the field variables. This is very appropriate from the point of view of meshless approximation because of saving the computational time and decreasing the inaccuracy due to derivatives of field variables. Since u_i^* vanishes on L_s^i , $\boldsymbol{n} \cdot \nabla u = \bar{q}_u$.

We can integrate it by part, by using

$$(\nabla u) \cdot (\nabla u_i^*) = \nabla \cdot (u \nabla u_i^*) - u \nabla^2 u_i^*, \tag{3.25}$$

Eq. (3.24) becomes

$$\begin{aligned} \int_{\Omega_s^i} (\nabla^2 u) u_i^* d\Omega = \int_{\Gamma_{sD}^i} \frac{\partial u}{\partial \boldsymbol{n}} u_i^* d\Gamma + \int_{\Gamma_{sN}^i} \bar{q}_u u_i^* d\Gamma - \int_{\Omega_s^i} \nabla \cdot (u \nabla u_i^*) d\,\Omega \\ + \int_{\Omega_s^i} u \nabla^2 u_i^* d\,\Omega. \end{aligned} \tag{3.26}$$

Since $\nabla^2 u^* + \delta(\boldsymbol{x}, \boldsymbol{y}) = 0$, the equation (3.26) becomes

$$\begin{aligned} \int_{\Omega_s^i} (\nabla^2 u) u_i^* d\Omega = \int_{\Gamma_{sD}^i} \frac{\partial u}{\partial \boldsymbol{n}} u_i^* d\Gamma + \int_{\Gamma_{sN}^i} \bar{q}_u u_i^* d\Gamma - \int_{\Omega_s^i} \nabla \cdot (u \nabla u_i^*) d\,\Omega \\ - u(\boldsymbol{x}_i, t). \end{aligned} \tag{3.27}$$

By divergence theorem, Eq. (3.27) can be written as:

$$\begin{aligned} \int_{\Omega_s^i} (\nabla^2 u) u_i^* d\Omega = -u(\boldsymbol{x}_i, t) + \int_{\Gamma_{sD}^i} \frac{\partial u}{\partial \boldsymbol{n}} u_i^* d\Gamma + \int_{\Gamma_{sN}^i} \bar{q}_u u_i^* d\Gamma \\ - \int_{L_s^i} u \frac{\partial u_i^*}{\partial \boldsymbol{n}} d\Gamma - \int_{\Gamma_{sD}^i} \bar{u} \frac{\partial u_i^*}{\partial \boldsymbol{n}} d\,\Gamma - \int_{\Gamma_{sN}^i} u \frac{\partial u_i^*}{\partial \boldsymbol{n}} d\,\Gamma. \end{aligned} \tag{3.28}$$

Similarly, we have

$$\int_{\Omega_s^i} (\nabla^2 v) u_i^* d\Omega = -v(\boldsymbol{x}_i, t) + \int_{\Gamma_{sD}^i} \frac{\partial v}{\partial \boldsymbol{n}} u_i^* d\Gamma + \int_{\Gamma_{sN}^i} \bar{q}_v u_i^* d\Gamma$$
$$- \int_{L_s^i} v \frac{\partial u_i^*}{\partial \boldsymbol{n}} d\Gamma - \int_{\Gamma_{sD}^i} \bar{v} \frac{\partial u_i^*}{\partial \boldsymbol{n}} d\Gamma - \int_{\Gamma_{sN}^i} v \frac{\partial u_i^*}{\partial \boldsymbol{n}} d\Gamma. \quad (3.29)$$

The local integration equation of (3.28) and (3.29) can be written as:

$$\int_{\Omega_s^i} \frac{\partial u}{\partial t} u_i^* d\Omega = D_1 \Bigg[-u(\boldsymbol{x}_i, t) + \int_{\Gamma_{sD}^i} \frac{\partial u}{\partial \boldsymbol{n}} u_i^* d\Gamma$$
$$+ \int_{\Gamma_{sN}^i} \bar{q}_u u_i^* d\Gamma - \int_{L_s^i} u \frac{\partial u_i^*}{\partial \boldsymbol{n}} d\Gamma$$
$$- \int_{\Gamma_{sD}^i} \bar{u} \frac{\partial u_i^*}{\partial \boldsymbol{n}} d\Gamma - \int_{\Gamma_{sN}^i} u \frac{\partial u_i^*}{\partial \boldsymbol{n}} d\Gamma \Bigg] \quad (3.30)$$
$$+ \alpha_1 \int_{\Omega_s^i} u \, u_i^* d\Omega + \int_{\Omega_s^i} A(u, v) \, u_i^* d\Omega$$
$$+ \int_{\Omega_s^i} f_1 (\boldsymbol{x}, t) u_i^* d\Omega,$$

$$\int_{\Omega_s^i} \frac{\partial v}{\partial t} u_i^* d\Omega = D_2 [-v(\boldsymbol{x}_i, t)$$
$$+ \int_{\Gamma_{sD}^i} \frac{\partial v}{\partial \boldsymbol{n}} u_i^* d\Gamma + \int_{\Gamma_{sN}^i} \bar{q}_v u_i^* d\Gamma$$
$$- \int_{L_s^i} v \frac{\partial u_i^*}{\partial \boldsymbol{n}} d\Gamma - \int_{\Gamma_{sD}^i} \bar{v} \frac{\partial u_i^*}{\partial \boldsymbol{n}} d\Gamma - \int_{\Gamma_{sN}^i} v \frac{\partial u_i^*}{\partial \boldsymbol{n}} d\Gamma. \Bigg] \quad (3.31)$$
$$+ \alpha_2 \int_{\Omega_s^i} v \, u_i^* d\Omega + \beta \int_{\Omega_s^i} u \, u_i^* d\Omega + \int_{\Omega_s^i} B(u, v) \, u_i^* d\Omega$$
$$+ \int_{\Omega_s^i} f_2 (\boldsymbol{x}, t) u_i^* d\Omega,$$

Let $\tilde{u}(\boldsymbol{x}, t)$ and $\tilde{v}(\boldsymbol{x}, t)$ are the trial solution and. We define $u(\boldsymbol{x}, t) = \tilde{u}(\boldsymbol{x}, t)$ and $v(\boldsymbol{x}, t) = \tilde{v}(\boldsymbol{x}, t)$. The Eqs. (3.30) and (3.31) can be written as:

$$\int_{\Omega_s^i} \frac{\partial \tilde{u}}{\partial t} u_i^* d\Omega = D_1 \Bigg[-\tilde{u}(\boldsymbol{x}_i, t) + \int_{\Gamma_{sD}^i} \frac{\partial \tilde{u}}{\partial \boldsymbol{n}} u_i^* d\Gamma + \int_{\Gamma_{sN}^i} \bar{q}_u u_i^* d\Gamma$$
$$- \int_{L_s^i} \tilde{u} \frac{\partial u_i^*}{\partial \boldsymbol{n}} d\Gamma - \int_{\Gamma_{sD}^i} \bar{u} \frac{\partial u_i^*}{\partial \boldsymbol{n}} d\Gamma - \int_{\Gamma_{sN}^i} \tilde{u} \frac{\partial u_i^*}{\partial \boldsymbol{n}} d\Gamma \Bigg] \quad (3.32)$$
$$+ \alpha_1 \int_{\Omega_s^i} \tilde{u} \, u_i^* d\Omega + \int_{\Omega_s^i} A(\tilde{u}, \tilde{v}) \, u_i^* d\Omega + \int_{\Omega_s^i} f_1 (\boldsymbol{x}, t) u_i^* d\Omega,$$

$$\int_{\Omega_s^i} \frac{\partial \tilde{v}}{\partial t} u_i^* d\Omega = D_2 \Bigg[-\tilde{v}(\boldsymbol{x}_i, t) + \int_{\Gamma_{sD}^i} \frac{\partial \tilde{v}}{\partial \boldsymbol{n}} u_i^* d\Gamma \int_{\Gamma_{sN}^i} \bar{q}_v u_i^* d\Gamma - \int_{L_s^i} \tilde{v} \frac{\partial u_i^*}{\partial \boldsymbol{n}} d\Gamma - \int_{\Gamma_{sD}^i} \bar{v} \frac{\partial u_i^*}{\partial \boldsymbol{n}} d\,\Gamma - \int_{\Gamma_{sN}^i} \tilde{v} \frac{\partial u_i^*}{\partial \boldsymbol{n}} d\,\Gamma. \Bigg] + \alpha_2 \int_{\Omega_s^i} \tilde{v}\, u_i^* d\Omega + \beta \int_{\Omega_s^i} \tilde{u}\, u_i^* d\Omega + \int_{\Omega_s^i} B(\tilde{u}, \tilde{v})\, u_i^* d\Omega + \int_{\Omega_s^i} f_2\,(\boldsymbol{x}, t) u_i^* d\Omega, \tag{3.33}$$

For internal nodes, from local integral equations (3.1) and (3.2), and using the MKA or RPIM (Eqs.(2.9) or (2.24)), we have the following nonlinear equations:

$$\sum_{j=1}^{N} \left(\int_{\Omega_s^i} \phi_j\,(\boldsymbol{x}) u_i^* d\Omega \right) \frac{\partial \hat{u}_j}{\partial t} = \sum_{j=1}^{N} \Bigg[D_1 \Bigg[-\phi_j(\boldsymbol{x}) + \left(\int_{\Gamma_{sD}^i} \frac{\partial \phi_j(\boldsymbol{x})}{\partial \boldsymbol{n}} u_i^* d\Gamma \right) - \left(\int_{L_s^i} \phi_j(\boldsymbol{x}) \frac{\partial u_i^*}{\partial \boldsymbol{n}} d\Gamma \right) - \left(\int_{\Gamma_{sN}^i} \phi_j(\boldsymbol{x}) \frac{\partial u_i^*}{\partial \boldsymbol{n}} d\Gamma \right) \Bigg] + \left(\alpha_1 \int_{\Omega_s^i} \phi_j(\boldsymbol{x})\, u_i^* d\Omega \right) \Bigg] \hat{u}_j + D_1 \left[\left(\int_{\Gamma_{sN}^i} \bar{q}_u u_i^* d\Gamma \right) - \int_{\Gamma_{sD}^i} \bar{u} \frac{\partial u_i^*}{\partial \boldsymbol{n}} d\,\Gamma \right] + \int_{\Omega_s^i} \tilde{A}\, u_i^* d\Omega + \int_{\Omega_s^i} f_1\,(\boldsymbol{x}, t) u_i^* d\Omega, \tag{3.34}$$

Similarly, we have

$$\sum_{j=1}^{N}\left(\int_{\Omega_s^i}\phi_j(\boldsymbol{x})u_i^* d\Omega\right)\frac{\partial\hat{v}_j}{\partial t} = \sum_{j=1}^{N}\Bigg[D_2\Bigg[-\phi_j(\boldsymbol{x})+\left(\int_{\Gamma_{sD}^i}\frac{\partial\phi_j(\boldsymbol{x})}{\partial\boldsymbol{n}}u_i^* d\Gamma\right) - \left(\int_{L_s^i}\phi_j(\boldsymbol{x})\frac{\partial u_i^*}{\partial\boldsymbol{n}}d\Gamma\right)-\left(\int_{\Gamma_{sN}^i}\phi_j(\boldsymbol{x})\frac{\partial u_i^*}{\partial\boldsymbol{n}}d\Gamma\right)\Bigg] + \left(\alpha_2\int_{\Omega_s^i}\phi_j(\boldsymbol{x})\,u_i^* d\Omega\right)\Bigg]\hat{v}_j + \sum_{j=1}^{n}\left(\beta\int_{\Omega_s^i}\phi_j(\boldsymbol{x})\,u_i^* d\Omega\right)\hat{u}_j + D_2\left[\left(\int_{\Gamma_{sN}^i}\bar{q}_v u_i^* d\Gamma\right)-\int_{\Gamma_{sD}^i}\bar{v}\frac{\partial u_i^*}{\partial\boldsymbol{n}}d\Gamma\right] + \int_{\Omega_s^i}\tilde{B}\,u_i^* d\Omega + \int_{\Omega_s^i}f_2(\boldsymbol{x},t)u_i^* d\Omega. \qquad (3.35)$$

The following abbreviations have been used for the integral terms:

$$a_{ij} = \int_{\Omega_s^i}\phi_j(\boldsymbol{x})u_i^* d\Omega,$$

$$b_{ij}^1 = D_1\left[-\phi_j(\boldsymbol{x})+\left(\int_{\Gamma_{sD}^i}\frac{\partial\phi_j(\boldsymbol{x})}{\partial\boldsymbol{n}}u_i^* d\Gamma\right)-\left(\int_{L_s^i}\phi_j(\boldsymbol{x})\frac{\partial u_i^*}{\partial\boldsymbol{n}}d\Gamma\right) - \left(\int_{\Gamma_{sN}^i}\phi_j(\boldsymbol{x})\frac{\partial u_i^*}{\partial\boldsymbol{n}}d\Gamma\right)\right]+\left(\alpha_1\int_{\Omega_s^i}\phi_j(\boldsymbol{x})\,u_i^* d\Omega\right),$$

$$b_{ij}^2 = D_2\left[-\phi_j(\boldsymbol{x})+\left(\int_{\Gamma_{sD}^i}\frac{\partial\phi_j(\boldsymbol{x})}{\partial\boldsymbol{n}}u_i^* d\Gamma\right)-\left(\int_{L_s^i}\phi_j(\boldsymbol{x})\frac{\partial u_i^*}{\partial\boldsymbol{n}}d\Gamma\right) - \left(\int_{\Gamma_{sN}^i}\phi_j(\boldsymbol{x})\frac{\partial u_i^*}{\partial\boldsymbol{n}}d\Gamma\right)\right]+\left(\alpha_2\int_{\Omega_s^i}\phi_j(\boldsymbol{x})\,u_i^* d\Omega\right),$$

$$c_{ij} = \beta\int_{\Omega_s^i}\phi_j(\boldsymbol{x})\,u_i^* d\Omega,$$

$$d_{1i} = D_1\left[\left(\int_{\Gamma_{sN}^i}\bar{q}_u u_i^* d\Gamma\right)-\int_{\Gamma_{sD}^i}\bar{u}\frac{\partial u_i^*}{\partial\boldsymbol{n}}d\Gamma\right]+\int_{\Omega_s^i}\tilde{A}\,u_i^* d\Omega + \int_{\Omega_s^i}f_1(\boldsymbol{x},t)u_i^* d\Omega,$$

$$d_{2i} = D_2\left[\left(\int_{\Gamma_{sN}^i}\bar{q}_v u_i^* d\Gamma\right)-\left(\int_{\Gamma_{sD}^i}\bar{v}\frac{\partial u_i^*}{\partial\boldsymbol{n}}d\Gamma\right)\right]+\int_{\Omega_s^i}\tilde{B}\,u_i^* d\Omega + \int_{\Omega_s^i}f_2(\boldsymbol{x},t)u_i^* d\Omega.$$

The boundary and domain integrals are calculated by using the Gauss-Legendre quadrature method. We can rewrite the Eqs. (3.34) and (3.35) as:

$$\sum_{j=1}^{N} a_{ij} \frac{\partial \hat{u}_j}{\partial t} = \sum_{\boldsymbol{j=1}}^{N} b_{ij}^{1}\, \hat{u}_j + d_{1i}, \tag{3.36}$$

$$\sum_{j=1}^{N} a_{ij} \frac{\partial \hat{v}_j}{\partial t} = \sum_{\boldsymbol{j=1}}^{N} \left(b_{ij}^{2} \hat{v}_j + c_{ij} \hat{u}_j \right) + d_{2i}. \tag{3.37}$$

The Eqs. (3.36) and (3.37) can be written in vector form as:

$$\boldsymbol{A} \frac{\partial \widehat{\boldsymbol{U}}}{\partial t} = \boldsymbol{B_1} \widehat{\boldsymbol{U}} + \boldsymbol{D}_1, \tag{3.38}$$

$$\boldsymbol{A} \frac{\partial \widehat{\boldsymbol{V}}}{\partial t} = \boldsymbol{B_2} \widehat{\boldsymbol{V}} + \boldsymbol{C} \widehat{\boldsymbol{U}} + \boldsymbol{D}_2, \tag{3.39}$$

where

$$\boldsymbol{A} = \left[a_{ij}\right]_{n\times n}, \boldsymbol{B_1} = \left[b_{ij}^{1}\right]_{n\times n}, \boldsymbol{B_2} = \left[b_{ij}^{2}\right]_{n\times n}, \boldsymbol{C} = \left[c_{ij}\right]_{n\times n}, \boldsymbol{D_1} = [d_{1i}]_{n\times 1}$$

$$\boldsymbol{D_2} = [d_{2i}]_{n\times 1}, \widehat{\boldsymbol{U}} = [\hat{u}_1 \;\; \hat{u}_2 \;\; \cdots \;\; \hat{u}_N]', \text{and } \widehat{\boldsymbol{V}} = [\hat{v}_1 \;\; \hat{v}_2 \;\; \cdots \;\; \hat{v}_N]'.$$

In case of Euler's method, the Eqs.(3.38) and (3.39) become

$$\boldsymbol{A}\widehat{\boldsymbol{U}}^{k+1} = (\boldsymbol{A} + \Delta t \boldsymbol{B}_1) \widehat{\boldsymbol{U}}^{k} + \Delta t \boldsymbol{D}_1^{k}, \tag{3.40}$$

$$\boldsymbol{A}\widehat{\boldsymbol{V}}^{k+1} = (\boldsymbol{A} + \Delta t \boldsymbol{B}_2) \widehat{\boldsymbol{V}}^{k} + \Delta t \left(\boldsymbol{C} \widehat{\boldsymbol{U}}^{k} + \boldsymbol{D}_2^{k} \right). \tag{3.41}$$

$\widehat{\boldsymbol{U}}^{k+1}$ and $\widehat{\boldsymbol{V}}^{k+1}$ can be solved in Eqs. (3.40) and (3.41)