CHAPTER 4

TWO-ECONOMY MODEL

Why is it needed to study in the case of two-country? In the case of small country, the solution technique adopted is the integrated world approach which basically implies similar conclusions to those derived from a closed economy. All variables depend on the world capital-labor ratio – in other words, the world resource. In the case of two-country, each economy's characteristics can influence the price of the world market since the prices are solved from market clearing conditions. With different method, this provides somewhat different results from the small economy model. However, most studies in the context of overlapping generations and trade in two-country model usually accompany with two goods. Distinctively, this study is comprised of two intermediate goods and one final good. All of them are tradable. Therefore, this chapter, a self-contained study, states firstly on the model setting, and then solving equilibrium. The steady state of price and its analysis are explained in the last section.

4.1 The Environment and the Model

In the case of two countries, on the other hand, the world economy consists of only two big countries, namely country N and S. Each consumer lives for two periods, young and old. The young inelastically supplies a fixed amount of labor to the firm and receives a wage at the competitive rate in order to consume and save. The old retires from work and consumes final good by using their saving and returns on saving from the previous period. People are born N_t^j in each period with growth rate n^j. In the consumption side, each economy is assumed to have identical technology, but different in saving rate, population growth (as well as dependency ratio), and initial factor endowments. There are three goods in the economy; two of them are intermediate goods *i*, where *i* = 1, 2, and the other is a final good *y*, produced by combining both intermediate goods. This setting follows Ventura (1997), but assuming different technologies. The productions of intermediate goods use capital and labor in the economy with technology of intermediate good 1 using less capital intensive than technology of intermediate good 2. Intermediate goods can be traded on world markets at world prices, as well as final good *y*. Moreover, only the final good y is used for consumption and expansion of capital. No world credit market so that each country's sale or purchase of intermediate good 1 must equal its purchase or sale of intermediate good 2. Capital and labor are perfectly mobile across sectors but immobile across borders in the first period.

4.1.1 Consumer

Each individual lives for two-period in the overlapping-generation world with perfect foresight. In period *t*, N_t^j individuals are born with n^j growth rate of population; that is, $N_t^j = (1+n^j)N_{t-1}^j$. In their first period of life, the young inelastically supply labor to earn the competitive wage and then allocate their wage income between consumption and saving, which is equal to investment to acquire capital in the next period. In their second period of life, the young become the old and retire from work, consuming all saving and its returns from the previous period. There is no international borrowing and lending since allowing it would result in indeterminacy of production and trade in equilibrium. Let c_{yt} and c_{ot} be the consumption in period *t* of each young and old individual. Then, the budget constraints in accordance with the behaviors of country *j*'s representative individual in overlapping generations are

$$p_{yt}^j c_{yt}^j + s_t^j = w_t^j \qquad (4.1.1)$$

$$p_{yt+1}^j c_{ot+1}^j = R_{t+1}^j s_t^j \qquad (4.1.2)$$

where s_t^j is individual saving, and $R_{t+1}^j = 1 + r_{t+1}^j - \delta$ is the return on saving held from period *t* to *t+1*. Since it is assumed that the depreciation rate is zero, then $R_{t+1}^j = 1 + r_{t+1}^j$. p_{yt}^j and p_{yt+1}^j are country *j*'s prices of final good in each period.

The utility function of an individual born at *t*, $U(c_{yt}, c_{ot+1})$, follows Cobb Douglas utility function, which is monotonically increasing, twice continuously differentiable and strictly quasi-concave, $u'(\cdot) > 0, u''(\cdot) < 0$, satisfying the Inada conditions. That is, by monotonic transformation in the logarithmic form,

$$U(c_{yt}, c_{ot+1}) = \beta^j \log c_{yt}^j + (1-\beta^j)\log c_{ot+1}^j, \qquad (4.1.3)$$

where β^j stands for how individual deciding to allocate their consumption between young and old. Thus, consumer's problem is that the individual born in period t chooses (c_{yt}, c_{ot+1}, s_t) to maximize utility (4.1.3) subject to the budget constraints (4.1.1) and (4.1.2), which can be rearranged to obtain the intertemporal budget constraint as

$$p_{yt}^j c_{yt}^j + \frac{p_{yt+1}^j c_{ot+1}^j}{(1+r_{t+1}^j)} = w_t^j \,. \qquad (4.1.4)$$

Thus, the Lagrange equation is

$$L = \beta^j \log c_{yt}^j + (1-\beta^j)\log c_{ot+1}^j + \lambda\left[w_t^j - p_{yt}^j c_{yt}^j - \frac{p_{yt+1}^j c_{ot+1}^j}{(1+r_{t+1}^j)} \right].$$

Then, the first order condition is

$$\frac{\beta^j}{c_{yt}^j} = \lambda p_{yt}^j, \qquad (4.1.5)$$

$$\frac{1-\beta^j}{c_{0t+1}^j} = \frac{\lambda p_{yt+1}^j}{(1+r_{t+1}^j)}. \qquad (4.1.6)$$

Combining together both (3.1.5) and (3.1.6) gives us the standard intertemporal Euler equation,

$$\frac{U'_{c_{yt}}(\cdot)}{U'_{c_{ot+1}}(\cdot)} = \frac{c_{ot+1}^j}{c_{yt}^j} = \frac{1-\beta^j}{\beta^j}\frac{p_{yt}^j}{p_{yt+1}^j}(1+r_{t+1}^j), \qquad (4.1.7)$$

which is the marginal rate of substitution between consumption of the young and old. Finally, the optimization obtains the following results,

$$c_{yt}^j = \beta^j \frac{w_t^j}{p_{yt}^j}, \qquad (4.1.8)$$

$$c_{ot+1}^j = (1-\beta^j)(1+r_{t+1}^j)\frac{w_t^j}{p_{yt+1}^j}, \qquad (4.1.9)$$

$$s_t^j = (1-\beta^j) w_t^j \,. \qquad (4.1.10)$$

As shown by the above three equations, the optimal consumption is a fraction of real wage, as well as the optimal saving. Furthermore, the optimal consumption in both periods is an increasing function in wage, as well as the optimal saving, $\frac{\partial s_t}{\partial w_t} > 0$, under the assumption that consumption in both periods is a normal good. According to transformed logarithmic utility function, $\frac{\partial s_t}{\partial r_{t+1}} = 0$, which implies that the elasticity of substitution between consumption in the two periods is equal to one (Bianconi, 1995).

In equilibrium, total saving in the present period, $s_t^j N_t^j$, is equal total capital in the next period,

$$K_{t+1}^j = s_t^j N_t^j . \qquad (4.1.11)$$

Substituting equation (4.1.10) into (4.1.11), and $N_{t+1}^j / N_t^j = (1+n^j)$, then the equilibrium condition becomes

$$(1+n^j)k_{t+1}^j = (1-\beta^j)w_t^j , \qquad (4.1.12)$$

which implies that the next period capital-labor ratio in each economy depends on a wage rate, a saving proportion, and a population growth.

4.1.2 Production in Two-economy world

The model setting is similar to a small economy model, except that there are two large countries, country N and country S. There are still three tradable goods in the economy; two of them are intermediate goods and the other is a final good y, produced by combining both intermediate goods. Technology of intermediate good 1 requires less capital intensive than technology of intermediate good 2. The production functions are all Cobb Douglas. In the case of two-economy world, goods and factors are represented in capital letters, which mean total amount.

a) Intermediate goods

The intermediate goods 1 and 2 produced in country *j*, where *j* = *N*, *S*, at time *t*, given by X_{1t}^{j}, X_{2t}^{j} respectively, are expressed as follows,

$$X_{1t}^{j} = (K_{1t}^{j})^{\alpha_1}(L_{1t}^{j})^{1-\alpha_1}, \quad 0 < \alpha_1 < 1, \tag{4.1.13}$$

$$X_{2t}^{j} = (K_{2t}^{j})^{\alpha_2}(L_{2t}^{j})^{1-\alpha_2}, \quad 0 < \alpha_2 < 1, \tag{4.1.14}$$

Since intermediate good 1 production is relatively less capital intensive than intermediate good 2 production, then $\alpha_1 < \alpha_2$.

Full employment of both factors requires that factors needed to produce intermediate goods 1 and 2 are equal to total factor supplies in each period. That is,

for labor,
$$L_{1t}^{Dj} + L_{2t}^{Dj} = L_t^{j}, \tag{4.1.15}$$

and for capital,
$$K_{1t}^{Dj} + K_{2t}^{Dj} = K_t^{j}, \tag{4.1.16}$$

Competitive intermediate goods firms decide how to allocate the total capital and labor available in that country across the two goods, taking each country's prices of intermediate goods, p_{1t}^{j} and p_{2t}^{j}, and factor prices, w_t^{j} and r_t^{j}, as given. Then, each intermediate good firm solves the following profit maximization problems:

$$\max\ \pi_{1t}^{j} = p_{1t}^{j}(k_{1t}^{j})^{\alpha_1}(l_{1t}^{j})^{1-\alpha_1} - r_t^{j}k_{1t}^{j} - w_t^{j}l_{1t}^{j}, \tag{4.1.17}$$

$$\max\ \pi_{2t}^{j} = p_{2t}^{j}(k_{2t}^{j})^{\alpha_2}(l_{2t}^{j})^{1-\alpha_2} - r_t^{j}k_{2t}^{j} - w_t^{j}l_{2t}^{j}. \tag{4.1.18}$$

The first order conditions of profit maximizing are

$$w_t^{j} = (1-\alpha_1)p_{1t}^{j}(K_{1t}^{j})^{\alpha_1}(L_{1t}^{j})^{-\alpha_1} = (1-\alpha_2)p_{2t}^{j}(K_{2t}^{j})^{\alpha_2}(L_{2t}^{j})^{-\alpha_2} \tag{4.1.19}$$

$$r_t^{j} = \alpha_1 p_{1t}^{j}(K_{1t}^{j})^{\alpha_1-1}(L_{1t}^{j})^{1-\alpha_1} = \alpha_2 p_{2t}^{j}(K_{2t}^{j})^{\alpha_2-1}(L_{2t}^{j})^{1-\alpha_2}, \tag{4.1.20}$$

Equation (4.1.19) and (4.1.20) represents that each factor price is equal its marginal product in both two intermediate sectors. As a result of solving profit maximization problems, the demands for factors are obtained as

$$L_{1t}^{j} = \left(\frac{\alpha_1}{1-\alpha_1}\frac{w_t^{j}}{r_t^{j}}\right)^{-\alpha_1} X_{1t}^{j}, \quad L_{2t}^{j} = \left(\frac{\alpha_2}{1-\alpha_2}\frac{w_t^{j}}{r_t^{j}}\right)^{-\alpha_2} X_{2t}^{j}, \tag{4.1.21}$$

$$K_{1t}^{j} = \left(\frac{\alpha_1}{1-\alpha_1}\frac{w_t^j}{r_t^j}\right)^{1-\alpha_1} X_{1t}^{j} \ , \quad K_{2t}^{j} = \left(\frac{\alpha_2}{1-\alpha_2}\frac{w_t^j}{r_t^j}\right)^{1-\alpha_2} X_{2t}^{j} . \quad (4.1.22)$$

Thus, *factor market clearing conditions* require that factor supplies are equal to factor demands, according to equation (4.1.21) and (4.1.22),

$$L_t^j = L_{1t}^{Dj} + L_{2t}^{Dj} = \left(\frac{\alpha_1}{1-\alpha_1}\frac{w_t^j}{r_t^j}\right)^{-\alpha_1} X_{1t}^{j} + \left(\frac{\alpha_2}{1-\alpha_2}\frac{w_t^j}{r_t^j}\right)^{-\alpha_2} X_{2t}^{j} , \quad (4.1.23)$$

$$K_t^j = K_{1t}^{Dj} + K_{2t}^{Dj} = \left(\frac{\alpha_1}{1-\alpha_1}\frac{w_t^j}{r_t^j}\right)^{1-\alpha_1} X_{1t}^{j} + \left(\frac{\alpha_2}{1-\alpha_2}\frac{w_t^j}{r_t^j}\right)^{1-\alpha_2} X_{2t}^{j} . \quad (4.1.24)$$

Solve for w_t^j and r_t^j from *zero-profit conditions*, then obtaining the same wage and rental function as small economy.

$$w_t^j = (1-\alpha_1)A^{\alpha_1}\left(\frac{p_{1t}^{\alpha_2}}{p_{2t}^{\alpha_1}}\right)^{\frac{1}{\alpha_2-\alpha_1}} = (1-\alpha_2)B^{\alpha_2}\left(\frac{p_{1t}^{\alpha_2}}{p_{2t}^{\alpha_1}}\right)^{\frac{1}{\alpha_2-\alpha_1}} , \quad (4.1.25)$$

$$r_t^j = \alpha_1 A^{\alpha_1-1}\left(\frac{p_{1t}^{\alpha_2-1}}{p_{2t}^{\alpha_1-1}}\right)^{\frac{1}{\alpha_2-\alpha_1}} = \alpha_2 B^{\alpha_2-1}\left(\frac{p_{1t}^{\alpha_2-1}}{p_{2t}^{\alpha_1-1}}\right)^{\frac{1}{\alpha_2-\alpha_1}} , \quad (4.1.26)$$

Since all wage and rental rates in each country are given by prices of intermediate goods, as well as there is no factor intensity reversal, the factor prices must be equalized so that wage is equal between countries, as well as rental rate.

The method of solving intermediate good supplies and factor demands are corresponding small economy method due to both economies face similar conditions. Thus, supplies of intermediate good 1 and 2 as a function of intermediate good prices, total capital and labor possessed by that country, $X_{it}^{j} = X_i(p_{1t}^j, p_{2t}^j, K_t^j, L_t^j)$ for $i = 1, 2$, are given by

$$X_{1t}^{j} = \frac{A^{\alpha_1}}{B-A}\left[BL_t^j - K_t^j\left(\frac{p_{1t}}{p_{2t}}\right)^{\frac{1}{\alpha_1-\alpha_2}}\right]\left(\frac{p_{1t}}{p_{2t}}\right)^{\frac{\alpha_1}{\alpha_2-\alpha_1}} , \quad (4.1.27)$$

$$X_{2t}^{j} = \frac{B^{\alpha_2}}{B-A}\left[K_t^j\left(\frac{p_{1t}}{p_{2t}}\right)^{\frac{1}{\alpha_1-\alpha_2}} - AL_t^j\right]\left(\frac{p_{1t}}{p_{2t}}\right)^{\frac{\alpha_2}{\alpha_2-\alpha_1}} . \quad (4.1.28)$$

Consequently, the demands for factors of each good as a function of intermediate good prices and total capital and labor possessed by that country in period t, $L_{it}^{Dj} = L_i(p_{1t}^j, p_{2t}^j, K_t^j, L_t^j)$ and $K_{it}^{Dj} = K_i(p_{1t}^j, p_{2t}^j, K_t^j, L_t^j)$, for $i = 1, 2$, are expressed as

$$L_{1t}^{Dj} = \frac{1}{B-A}\left[BL_t^j - K_t^j \left(\frac{p_{1t}}{p_{2t}} \right)^{\frac{1}{\alpha_1-\alpha_2}} \right], \qquad (4.1.29)$$

$$L_{2t}^{Dj} = \frac{1}{B-A}\left[-AL_t^j + K_t^j \left(\frac{p_{1t}}{p_{2t}} \right)^{\frac{1}{\alpha_1-\alpha_2}} \right], \qquad (4.1.30)$$

$$K_{1t}^{Dj} = \frac{A}{B-A}\left[BL_t^j \left(\frac{p_{1t}}{p_{2t}} \right)^{\frac{1}{\alpha_2-\alpha_1}} - K_t^j \right], \qquad (4.1.31)$$

$$K_{2t}^{Dj} = \frac{B}{B-A}\left[-AL_t^j \left(\frac{p_{1t}}{p_{2t}} \right)^{\frac{1}{\alpha_2-\alpha_1}} + K_t^j \right]. \qquad (4.1.32)$$

In terms of total amount of factor endowment, it is quite difficult to compare resource allocation between sectors in each country. However, if these factor demands are considered in per labor at period t, the results would be similarly to the case of small economy model, equation (3.1.34) to (3.1.37). That is, countries that have a higher capital-labor ratio devote a larger fraction of capital and labor to the capital-intensive sector 2 whereas countries that have a lower capital-labor ratio allocate a larger fraction of capital and labor to the labor-intensive sector 1.

b) Final good

The technology of final good also follows Cobb Douglas production function, $y_t^j = (z_{1t}^j)^{\alpha_3}(z_{2t}^j)^{1-\alpha_3}$, where y_t^j is final good per worker produced by country *j* at time *t*, z_{it}^j is per worker intermediate good 1 and 2 needed to produce final good *y* by country *j* at time *t*. Competitive final good firms choose how to allocate intermediate good 1 and 2, produced domestically or bought from abroad, taking the current world prices of intermediate goods, p_{1t}, p_{2t}, and final good, p_{yt}. Thus, firm problem of profit maximization is

$$\max \ \pi_y^j = p_{yt}^j (Z_{1t}^j)^{\alpha_3} (Z_{2t}^j)^{1-\alpha_3} - p_{1t}^j Z_{1t}^j - p_{2t}^j Z_{2t}^j . \qquad (4.1.33)$$

Then, first order conditions for profit maximizing problem are

$$p_{1t}^j = \alpha_3 p_{yt}^j (Z_{1t}^j)^{\alpha_3 - 1} (Z_{2t}^j)^{1-\alpha_3} , \qquad (4.1.34)$$

$$p_{2t}^j = (1-\alpha_3) p_{yt}^j (Z_{1t}^j)^{\alpha_3} (Z_{2t}^j)^{-\alpha_3} . \qquad (4.1.35)$$

Solving equation (4.1.34) and (4.1.35) to obtain demands for intermediate goods express as follows,

$$Z_{1t}^j = \left(\frac{\alpha_3}{1-\alpha_3} \frac{p_{2t}^j}{p_{1t}^j} \right)^{1-\alpha_3} Y_t^j , \qquad (4.1.36)$$

$$Z_{2t}^j = \left(\frac{\alpha_3}{1-\alpha_3} \frac{p_{2t}^j}{p_{1t}^j} \right)^{-\alpha_3} Y_t^j . \qquad (4.1.37)$$

Thus, demands for intermediate goods are given by the equilibrium prices of the integrated economy and the final good needed in that country.

4.2 Equilibrium at World Market

Although the countries' prices of the two-country model before trade may be different (each country takes its domestic price as given), after trade occurs, all prices must be equated in order to clear market. Thus, the symbol representation of prices in this chapter will be the same for both economies. The study starts from the final good in order to find the final good demand which finally determine the final good supply. Then, consider the intermediate good clearing where the demand for intermediate good is conditional on final good needed to produce. Thus, price ratio of intermediate good is important to determine the whole economy.

4.2.1 Final good market clearing

In the two-country model, the capital letters are used to represent total amounts. The amount of final good produced depends on the final good demand[1], which is constituted by consumption and expansion of capital in each country. That is,

$$Y_{dt}^{N} + Y_{dt}^{S} = C_{t}^{N} + C_{t}^{S} + K_{t+1}^{N} + K_{t+1}^{S}$$

$$Y_{dt}^{N} + Y_{dt}^{S} = c_{yt}^{N} N_{t}^{N} + c_{ot}^{N} N_{t-1}^{N} + c_{yt}^{S} N_{t}^{S} + c_{ot}^{S} N_{t-1}^{S} + s_{t}^{N} N_{t}^{N} + s_{t}^{S} N_{t}^{S} \qquad (4.2.1)$$

where $K_{t+1}^{j} = s_{t}^{j} N_{t}^{j}$ at the equilibrium of saving and investment transformation and all consumption variables are optimal consumptions of each generation individual.

At the world final good market clearing, demand must be equal to supply, $Y_{st}^{N} + Y_{st}^{S} = Y_{dt}^{N} + Y_{dt}^{D}$. Thus, the price of final good between the two countries must be the same in order to clear the market. Similarly to the case of small economy, the final good price can be normalized to be 1 for simplicity.

In the case of CRS production function, zero-profit conditions are hold. From equation (4.1.25) and (4.1.26) show that wage and rental rates are the function of intermediate good prices. That is, $w_t = \delta_w \left(\frac{p_{1t}^{\alpha_2}}{p_{2t}^{\alpha_1}} \right)^{\frac{1}{\alpha_2 - \alpha_1}}$ and $r_t = \delta_r \left(\frac{p_{1t}^{\alpha_2 - 1}}{p_{2t}^{\alpha_1 - 1}} \right)^{\frac{1}{\alpha_2 - \alpha_1}}$,

where $\delta_w = \left(\frac{\alpha_2^{\alpha_1 \alpha_2} (1-\alpha_2)^{\alpha_1 (1-\alpha_2)}}{\alpha_1^{\alpha_1 \alpha_2} (1-\alpha_1)^{\alpha_2 (1-\alpha_1)}} \right)^{\frac{1}{\alpha_1 - \alpha_2}} = (1-\alpha_1) A^{\alpha_1}$, and

$$\delta_r = \left(\frac{\alpha_1^{\alpha_1 (1-\alpha_2)} (1-\alpha_1)^{(1-\alpha_1)(1-\alpha_2)}}{\alpha_2^{(1-\alpha_1)\alpha_2} (1-\alpha_2)^{(1-\alpha_1)(1-\alpha_2)}} \right)^{\frac{1}{\alpha_1 - \alpha_2}} = \alpha_1 A^{\alpha_1 - 1}$$

If price of good 2 is normalized to be equal to 1, then $w_t = \delta_w p_{1t}^{\frac{\alpha_2}{\alpha_2 - \alpha_1}}$ and $r_t = \delta_r p_{1t}^{\frac{\alpha_2 - 1}{\alpha_2 - \alpha_1}}$. Note that the optimal consumptions and saving are governed by the wage and rental rates. Correspondingly, the wage and rental rates also rely on intermediate good prices. Hence, the optimal consumptions and saving are also

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[1] If the final good produced totally depends on world resources at each period, the result of solving equilibrium price ratio is the same as a small economy model. This solving can be seen in Appendix D.

determined by intermediate good prices. By substituting wage and rental rates in terms of intermediate prices, the optimal consumptions and saving become

$$c_{yt}^{j} = \beta^{j} w_{t}^{j} = \beta^{j} \delta_{w} p_{1t}^{\frac{\alpha_2}{\alpha_2-\alpha_1}}, \qquad (4.2.2)$$

$$c_{ot+1}^{j} = (1+r_{t+1}^{j})(1-\beta^{j}) w_{t}^{j} = (1+\delta_{r} p_{1t+1}^{\frac{\alpha_2-1}{\alpha_2-\alpha_1}})(1-\beta^{j}) \delta_{w} p_{1t}^{\frac{\alpha_2}{\alpha_2-\alpha_1}} \qquad (4.2.3)$$

And $$K_{t+1}^{j} = S_{t}^{j} = s_{t}^{j} N_{t}^{j} = (1+n^{j}) k_{t+1}^{j} = (1-\beta^{j}) w_{t}^{j} = (1-\beta^{j}) \delta_{w} p_{1t}^{\frac{\alpha_2}{\alpha_2-\alpha_1}}. \qquad (4.2.4)$$

As a result, the final good demand expressed in terms of intermediate good prices is

$$\begin{aligned} Y_{st}^{W} = Y_{dt}^{N} + Y_{dt}^{S} = {} & \beta^{N} \delta_{w} p_{1t}^{\frac{\alpha_2}{\alpha_2-\alpha_1}} N_{t}^{N} + (1+\delta_{r} p_{1t}^{\frac{\alpha_2-1}{\alpha_2-\alpha_1}})(1-\beta^{N}) \delta_{w} p_{1t-1}^{\frac{\alpha_2}{\alpha_2-\alpha_1}} N_{t-1}^{N} \\ & + \beta^{S} \delta_{w} p_{1t}^{\frac{\alpha_2}{\alpha_2-\alpha_1}} N_{t}^{S} + (1+\delta_{r} p_{1t}^{\frac{\alpha_2-1}{\alpha_2-\alpha_1}})(1-\beta^{S}) \delta_{w} p_{1t-1}^{\frac{\alpha_2}{\alpha_2-\alpha_1}} N_{t-1}^{S}, \qquad (4.2.5) \\ & + (1-\beta^{N}) \delta_{w} p_{1t}^{\frac{\alpha_2}{\alpha_2-\alpha_1}} N_{t}^{N} + (1-\beta^{S}) \delta_{w} p_{1t}^{\frac{\alpha_2}{\alpha_2-\alpha_1}} N_{t}^{S} \end{aligned}$$

where Y_{st}^{W} represents final good supply and Y_{dt}^{j} is interpreted as final good demand of country *j*. Note that $N_{t}^{j} = (1+n^{j}) N_{t-1}^{j}$, then $N_{t-1}^{j} = \frac{N_{t}^{j}}{(1+n^{j})}$. For the total world population, N_{t}^{W}, there are $N_{t}^{W} = N_{t}^{N} + N_{t}^{S} = \eta^{N} N_{t}^{W} + \eta^{S} N_{t}^{W}$, where $\eta^{N} + \eta^{S} = 1$, total share of world population is equal to 1. (See further details in Appendix E) Thus, the final good demand is

$$\begin{aligned} Y_{st}^{W} = Y_{dt}^{N} + Y_{dt}^{S} = {} & \left[(\beta^{N} \eta^{N} + \beta^{S} \eta^{S}) + \left((1-\beta^{N}) \eta^{N} + (1-\beta^{S}) \eta^{S} \right) \right] N_{t}^{W} \delta_{w} p_{1t}^{\frac{\alpha_2}{\alpha_2-\alpha_1}} \\ & + \left[\frac{(1-\beta^{N})}{(1+n^{N})} \eta^{N} + \frac{(1-\beta^{S})}{(1+n^{S})} \eta^{S} \right] N_{t}^{W} \delta_{w} p_{1t-1}^{\frac{\alpha_2}{\alpha_2-\alpha_1}} \qquad (4.2.6) \\ & + \left[\frac{(1-\beta^{N})}{(1+n^{N})} \eta^{N} + \frac{(1-\beta^{S})}{(1+n^{S})} \eta^{S} \right] N_{t}^{W} \delta_{r} \delta_{w} p_{1t}^{\frac{\alpha_2-1}{\alpha_2-\alpha_1}} p_{1t-1}^{\frac{\alpha_2}{\alpha_2-\alpha_1}} \end{aligned}$$

Consider that $\left[(\beta^{N} \eta^{N} + \beta^{S} \eta^{S}) + \left((1-\beta^{N}) \eta^{N} + (1-\beta^{S}) \eta^{S} \right) \right] = 1$, and let

$\Theta = \left[\frac{(1-\beta^{N})}{(1+n^{N})} \eta^{N} + \frac{(1-\beta^{S})}{(1+n^{S})} \eta^{S} \right]$. Then,

$$Y_{st}^{W} = Y_{dt}^{N} + Y_{dt}^{S} = N_{t}^{W} \delta_{w} p_{1t}^{\frac{\alpha_2}{\alpha_2-\alpha_1}} + \Theta N_{t}^{W} \delta_{w} p_{1t-1}^{\frac{\alpha_2}{\alpha_2-\alpha_1}} + \Theta N_{t}^{W} \delta_{r} \delta_{w} p_{1t}^{\frac{\alpha_2-1}{\alpha_2-\alpha_1}} p_{1t-1}^{\frac{\alpha_2}{\alpha_2-\alpha_1}} \qquad (4.2.7)$$

That is, final good clearing can be expressed in term of intermediate good's prices. Since the prices are in different periods, we have to consider further in the intermediate good market to get the whole market clearing.

4.2.2 Intermediate good market clearing

Recall from equation (4.1.36) and (4.1.37), that demands for intermediate goods are $Z_{1t}^{j} = \left(\frac{\alpha_3}{1-\alpha_3} \frac{p_{2t}}{p_{1t}} \right)^{1-\alpha_3} Y_t^j$, and $Z_{2t}^{j} = \left(\frac{\alpha_3}{1-\alpha_3} \frac{p_{2t}}{p_{1t}} \right)^{-\alpha_3} Y_t^j$. With normalization of p_{2t}, they become $Z_{1t}^{j} = \left(\frac{1-\alpha_3}{\alpha_3} p_{1t} \right)^{\alpha_3-1} Y_t^j$, and $Z_{2t}^{j} = \left(\frac{1-\alpha_3}{\alpha_3} p_{1t} \right)^{\alpha_3} Y_t^j$.

Moreover, recall from equation (4.1.27) and (4.1.28) that supplies of intermediate goods are, $X_{1t}^{j} = L_{1t}^{j} A^{\alpha_1} \left(\frac{p_{2t}}{p_{1t}} \right)^{\frac{\alpha_1}{\alpha_1-\alpha_2}}$, and $X_{2t}^{j} = L_{2t}^{j} B^{\alpha_2} \left(\frac{p_{2t}}{p_{1t}} \right)^{\frac{\alpha_2}{\alpha_1-\alpha_2}}$. With normalization of p_{2t}, they become $X_{1t}^{j} = L_{1t}^{j} A^{\alpha_1} p_{1t}^{\frac{\alpha_1}{\alpha_2-\alpha_1}}$, and $X_{2t}^{j} = L_{2t}^{j} B^{\alpha_2} p_{1t}^{\frac{\alpha_2}{\alpha_2-\alpha_1}}$.

Hence, international market clearing of intermediate good 1 is

$$Z_1^N + Z_1^S = X_1^N + X_1^S,$$

and that of intermediate good 2 is

$$Z_2^N + Z_2^S = X_2^N + X_2^S.$$

According to Walras' Law, the market for good 2 is also cleared simultaneously at every period *t* when the market for good 1 is cleared. Consequently, intermediate good 1 is chosen to represent the clearing of intermediate good market. Substituting the demand and supply of intermediate good 1 into the market clearing condition then obtains

$$\left(\frac{1-\alpha_3}{\alpha_3} p_{1t} \right)^{\alpha_3-1} Y_{dt}^N + \left(\frac{1-\alpha_3}{\alpha_3} p_{1t} \right)^{\alpha_3-1} Y_{dt}^S = L_{1t}^N A^{\alpha_1} p_{1t}^{\frac{\alpha_1}{\alpha_2-\alpha_1}} + L_{1t}^S A^{\alpha_1} p_{1t}^{\frac{\alpha_1}{\alpha_2-\alpha_1}},$$

$$\left(\frac{1-\alpha_3}{\alpha_3} \right)^{\alpha_3-1} p_{1t}^{\alpha_3-1} \left(Y_{dt}^N + Y_{dt}^S \right) = \left(L_{1t}^N + L_{1t}^S \right) A^{\alpha_1} p_{1t}^{\frac{\alpha_1}{\alpha_2-\alpha_1}}, \qquad (4.2.8)$$

where $\left(Y_t^N + Y_t^S\right)$ is the derived demand of final good from equation (4.2.7) and $\left(L_{1t}^N + L_{1t}^S\right)$ is the demand for labor used in production of intermediate good 1 from equation (4.1.29) and (4.1.30). That is,

$$\left(\frac{1-\alpha_3}{\alpha_3}\right)^{\alpha_3-1} \frac{B-A}{A^{\alpha_1}} p_{1t}^{\alpha_3-1} p_{1t}^{\frac{\alpha_1}{\alpha_1-\alpha_2}} N_t^W \left(\delta_w p_{1t}^{\frac{\alpha_2}{\alpha_2-\alpha_1}} + \Theta\delta_w p_{1t-1}^{\frac{\alpha_2}{\alpha_2-\alpha_1}} + \Theta\delta_r\delta_w p_{1t}^{\frac{\alpha_2-1}{\alpha_2-\alpha_1}} p_{1t-1}^{\frac{\alpha_2}{\alpha_2-\alpha_1}}\right) = B\left(L_t^N + L_t^S\right) - \left(K_t^N + K_t^S\right) p_{1t}^{\frac{1}{\alpha_1-\alpha_2}} \tag{4.2.9}$$

By notation, in fact, $L_t^N + L_t^S = L_t^W = N_t^W$ is the total world supply of labor in period *t*. Make it move forward for one period. Then,

$$\left(\frac{1-\alpha_3}{\alpha_3}\right)^{\alpha_3-1} \left(\frac{B-A}{A^{\alpha_1}}\right) p_{1t+1}^{\frac{\alpha_1\alpha_3-\alpha_2(1-\alpha_3)}{\alpha_1-\alpha_2}} \left(\delta_w p_{1t+1}^{\frac{\alpha_2}{\alpha_2-\alpha_1}} + \Theta\delta_w p_{1t}^{\frac{\alpha_2}{\alpha_2-\alpha_1}} + \Theta\delta_r\delta_w p_{1t+1}^{\frac{\alpha_2-1}{\alpha_2-\alpha_1}} p_{1t}^{\frac{\alpha_2}{\alpha_2-\alpha_1}}\right) = B - \frac{\left(K_{t+1}^N + K_{t+1}^S\right)}{N_{t+1}^W} p_{1t+1}^{\frac{1}{\alpha_1-\alpha_2}} \tag{4.2.10}$$

Similar to the case of small economy, define $\gamma = \alpha_1\alpha_3 + \alpha_2(1-\alpha_3)$, and consider that

$$\frac{\left(K_{t+1}^N + K_{t+1}^S\right)}{N_{t+1}^W} = \frac{N_{t+1}^N}{N_{t+1}^W}\frac{K_{t+1}^N}{N_{t+1}^N} + \frac{N_{t+1}^S}{N_{t+1}^W}\frac{K_{t+1}^N}{N_{t+1}^S} = \eta^N k_{t+1}^N + \eta^S k_{t+1}^S . \tag{4.2.11}$$

Recall that $k_{t+1}^j = \frac{(1-\beta^j)}{(1+n^j)} w_t^j = \frac{(1-\beta^j)}{(1+n^j)} \delta_w p_{1t}^{\frac{\alpha_2}{\alpha_2-\alpha_1}}$. Then, at the market equilibrium of intermediate good 1,

$$\left(\frac{1-\alpha_3}{\alpha_3}\right)^{\alpha_3-1} \left(\frac{B-A}{A^{\alpha_1}}\right) p_{1t+1}^{\frac{\gamma}{\alpha_1-\alpha_2}} \left(\delta_w p_{1t+1}^{\frac{\alpha_2}{\alpha_2-\alpha_1}} + \Theta\delta_w p_{1t}^{\frac{\alpha_2}{\alpha_2-\alpha_1}} + \Theta\delta_r\delta_w p_{1t+1}^{\frac{\alpha_2-1}{\alpha_2-\alpha_1}} p_{1t}^{\frac{\alpha_2}{\alpha_2-\alpha_1}}\right) = B - \left(\frac{(1-\beta^N)}{(1+n^N)}\eta^N\delta_w p_{1t}^{\frac{\alpha_2}{\alpha_2-\alpha_1}} + \frac{(1-\beta^S)}{(1+n^S)}\eta^S\delta_w p_{1t}^{\frac{\alpha_2}{\alpha_2-\alpha_1}}\right) p_{1t+1}^{\frac{1}{\alpha_1-\alpha_2}} \tag{4.2.12}$$

There are prices in two periods as a result of demand for final good from old people depends on rental rate from different period. Thus, to clear the market, it is needed to be analyzed at a steady state.

4.2.3 Price at Steady state

At the steady state, $p_{1t} = p_{1t+1} = p_1$.

$$\left(\frac{1-\alpha_3}{\alpha_3}\right)^{\alpha_3-1}\left(\frac{B-A}{A^{\alpha_1}}\right)p_1^{\frac{\gamma}{\alpha_1-\alpha_2}}p_1^{\frac{\alpha_2}{\alpha_2-\alpha_1}}\left(\delta_w+\Theta\delta_w+\Theta\delta_r\delta_w p_1^{\frac{\alpha_2-1}{\alpha_2-\alpha_1}}\right) = B-\Theta\delta_w p_1^{\frac{\alpha_2}{\alpha_2-\alpha_1}}p_1^{\frac{1}{\alpha_1-\alpha_2}} \quad (4.2.13)$$

Let $\Phi = \left(\frac{1-\alpha_3}{\alpha_3}\right)^{\alpha_3-1}\left(\frac{B-A}{A^{\alpha_1}}\right)$, then the excess demand function at the steady state follows

$$\Phi\left(1+\Theta\right)\delta_w p_1^{\frac{\alpha_2-\gamma}{\alpha_2-\alpha_1}} + \Phi\Theta\delta_r\delta_w p_1^{\frac{\alpha_2-\gamma}{\alpha_2-\alpha_1}}p_1^{\frac{\alpha_2-1}{\alpha_2-\alpha_1}} + \Theta\delta_w p_1^{\frac{\alpha_2-1}{\alpha_2-\alpha_1}} - B = 0. \quad (4.2.14)$$

That is, equation (4.2.14) is characterized in terms of all parameters and only price variable. Unfortunately, this equation is unable to express in closed form solution since it cannot be factorized due to the powers of variable p_1.

4.3 Sensitivity Analysis: Numerical Examples

What would it happen to the price at the steady state when some parameter changes? In order to analyze at the steady state, sensitivity analysis is chosen together with using Implicit Function Theorem. Let $G\left(\Theta,\Phi,\delta_r,\delta_w,p_1\right)=B$, then p_1 is a function of other parameters, $p_1(\Theta,\Phi,\delta_r,\delta_w)$. By Implicit Function Theorem,

$$\frac{\partial p_1}{\partial\Theta}\left(\Theta,\Phi,\delta_r,\delta_w\right) = -\frac{\frac{\partial G}{\partial\Theta}\left(\Theta,\Phi,\delta_r,\delta_w\right)}{\frac{\partial G}{\partial p_1}\left(\Theta,\Phi,\delta_r,\delta_w\right)} \quad (4.3.1)$$

$$\frac{\partial G}{\partial\Theta} = \Phi\delta_w p_1^{\frac{\alpha_2-\gamma}{\alpha_2-\alpha_1}} + \Phi\delta_r\delta_w p_1^{\frac{2\alpha_2-\gamma-1}{\alpha_2-\alpha_1}} + \delta_w p_1^{\frac{\alpha_2-1}{\alpha_2-\alpha_1}}$$

$$\frac{\partial G}{\partial p_1} = \left(\frac{\alpha_2-\gamma}{\alpha_2-\alpha_1}\right)\Phi\left(1+\Theta\right)\delta_w p_1^{\frac{\alpha_1-\gamma}{\alpha_2-\alpha_1}} + \left(\frac{2\alpha_2-\gamma-1}{\alpha_2-\alpha_1}\right)\Phi\Theta\delta_r\delta_w p_1^{\frac{\alpha_1+\alpha_2-\gamma-1}{\alpha_2-\alpha_1}} + \left(\frac{\alpha_2-1}{\alpha_2-\alpha_1}\right)\Theta\delta_w p_1^{\frac{\alpha_1-1}{\alpha_2-\alpha_1}}$$

Thus,

$$\frac{\partial p_1}{\partial \Theta} = -\frac{\Phi\delta_w p_1^{\alpha_3} + \Phi\delta_r\delta_w p_1^{\frac{-(1+\gamma)}{\alpha_2-\alpha_1}} + \delta_w p_1^{\frac{\alpha_2-1}{\alpha_2-\alpha_1}}}{\alpha_3\Phi\left(1+\Theta\right)\delta_w p_1^{-(1-\alpha_3)} - \left(\frac{1+\gamma}{\alpha_2-\alpha_1}\right)\Phi\Theta\delta_r\delta_w p_1^{\frac{\alpha_1+\alpha_2-\gamma-1}{\alpha_2-\alpha_1}} - \left(\frac{1-\alpha_2}{\alpha_2-\alpha_1}\right)\Theta\delta_w p_1^{\frac{\alpha_1-1}{\alpha_2-\alpha_1}}} \quad (4.3.2)$$

Equation (4.3.2) cannot exactly point out the sign of any change since the sign depends on the value of parameters, as well as the value of price at steady state. Hence, the numerical examples are used to investigate the change of parameters.

4.3.1 Numerical setting up for Price at a steady state

I assume the values of the parameter based on the possibility on the real world and the assumptions of the model at the beginning. However, those values are not the exact real world situation, but just closed to it.

The parameters that must be assumed the values are $\alpha_1, \alpha_2, \alpha_3, \beta^N, \beta^S, n^N, n^S, \eta^N$, and η^S. According to the assumption that intermediate good 1 is labor intensive and intermediate good 2 is capital intensive, $\alpha_1 < \alpha_2$. α_3, however, is the parameter of a final good, which is arbitrary value. No assumption on the factor intensive of final good. Thus, I assume α_3 to be basically equal to 0.5. Since country N averagely represent the developed country and country S averagely represent developing or less-developed country, β^N is reasonable to be smaller than β^S. This implies developed country tends to have a propensity to save, $(1-\beta^j)$, more than developing or less-developed country. Furthermore, the growth rate of population in country N is normally lower than country S. (And thus, the country N's population ratio to the world population should be less than the country S's). All Matlab codes are in Appendix G.

4.3.2 Numerical example results for price(s) at a steady state and interpretation

Numerical example 1: change in α_1 and α_2, given α_3=0.5, β^N = 0.8, β^S = 0.9, n^N =0.02, n^S =0.04, η^N =0.4, and η^S =0.6

The value of α_1 varies between 0.1 – 0.5 while the value of α_2 varies between 0.5 – 0.9, with each 0.1 interval. Note that α_1 and α_2 cannot be equal to 0.5 at the same time. The results are shown in table F1, Appendix F. That is, given the value of α_2, when the α_1 increases, prices at steady state also increase. (There are 2 real number values of price at a steady state.) Given the value of α_1, when the α_2 increases, prices at steady state change in the opposite direction. When α_2 is 0.9, a small value of price is zero. Moreover, the larger differences between α_1 and α_2, the smaller values of prices at steady state will be.

The price considered at a steady state is price of intermediate good 1 which is labor intensive. Hence, the above result means that when intermediate good 1 is much less capital intensive compared to intermediate good 2, the prices of intermediate good 1 become smaller.

Numerical example 2: change in α_3, given β^N = 0.7, β^S = 0.9, n^N =0.02, n^S =0.04, η^N =0.4, and η^S =0.6, with two cases of α_1 and α_2.

The value of α_3 varies between 0.1-0.9. The results are shown in table F2, Appendix F. Given parameters' value, a smaller price at steady state is quite constant while a larger price decreases substantially when α_3 increases in the range of less capital intensive of final good. This implies that price of intermediate good 1 is quite high when final good is more labor intensive.

Numerical example 3: change in β^N and β^S, given α_1 = 0.2, α_2 = 0.7, α_3=0.5, n^N =0.02, n^S =0.04, with three cases of η^N and η^S.

The value of β^N varies between 0.3-0.7 while the value of β^S varies between 0.5-0.9, with each 0.1 interval. The results are shown in the table F.1 and F.2, Appendix F. That is, given the value of β^N, when the β^S increases, a smaller value of *p* decreases while a larger value of *p* increases. Given the value of β^S, when β^N increases, a smaller value of *p* decreases while a larger value of *p* increases.

In table F3.3 and F3.4, the value of η^N and η^S have been changed to compare the chance of steady state prices. There are two directions in both small and large value of prices.

It can be considered only a larger value since a smaller value is almost zero. The result is that when marginal propensity to save of country *j* decreases, which implies finally that capital stocks increase, it leads to an increase in the relative price (or the price of intermediate good 1).

Numerical example 4: change in n^N and n^S, given α_1 = 0.2, α_2 = 0.7, α_3 =0.5, β^N =0.8, β^S = 0.9, η^N =0.3, η^S =0.7.

The value of n^N varies between 0.01-0.05 while the value of n^S varies between 0.03-0.08, with each 0.01 interval. The results are shown in the table F4, Appendix F. That is, given the value of n^S, when the n^N increases, p-steady state with small value slightly increases while p-steady state with larger value decreases. Given the value of n^N, when n^S increases, a smaller value of *p* increases while a larger value of *p* decreases.

Similar to example 3, a larger value should be considered since a smaller value is almost zero. The result is that when the population growth rate of country *j* increases, which implies finally that labors increase, it leads to a decrease in the relative price (or the price of intermediate good 1).

To conclude, the two-economy can describe the interaction between country's specific parameters on price at steady state. Since the solving method is different from the small economy model, the results on explanation the world are also distinguished. With no closed form solution, numerical examples provide some useful explanation on effects of saving rate and population growth as stated above. These numerical examples provide additional describing factor-intensive parameters on prices at steady state.