CHAPTER 3

SMALL ECONOMY MODEL

Many studies in trade and growth in small economy model use Ramsey model with tradable intermediate goods and nontradable final good following Ventura (1997). Unlike those studies, this thesis uses Overlapping Generations model with tradable intermediate and final goods in order to capture the real world. However, an integrated economy approach to see the equilibrium prices and dynamic economy is adopted similarly to the previous studies. This chapter is self-contained study that can be considered as one paper. Thus, the organization starts with the environment and setting up the model in each small economy for both consumer and production. The next section is the integrated approach, followed by resource allocations in the trade environment, and, the last section, the dynamic equilibrium and analysis.

3.1 The Environment and the Model

In the case of small economy, the world economy is composed of many small economies which the population size of country *j* is a share of the world population; $\eta_t^j = N_t^j / N_t^w$, *j=1,2,…,J*. Each consumer lives for two periods, young and old. The young inelastically supplies a fixed amount of labor to the firm and receives a wage at the competitive rate in order to consume and save. The old retires from work and consumes final good by using their saving and returns on saving from the previous period. People are born N_t^j in each period with growth rate n^j. In the consumption side, each economy is assumed to have identical technology, but different in saving rate, population growth (as well as dependency ratio), and initial factor endowments. There are three goods in the economy; two of them are intermediate goods *i*, where *i* = 1, 2, and the other is a final good *y*, produced by combining both intermediate goods. This setting follows Ventura (1997), with different technologies. The productions of intermediate goods use capital and labor in the economy with technology of intermediate good 1 using less capital intensive than technology of intermediate good 2. Both intermediate and final goods can be traded in world markets at world prices. Moreover, only the final good y is used for consumption and expansion of capital. No world credit market, so each country's sale or purchase of intermediate good 1 must equal its purchase or sale of intermediate good 2. Capital and labor are perfectly mobile across sectors but immobile across borders in the first period.

3.1.1 Consumer

Each individual lives for two-period in the overlapping-generation world with perfect foresight. In period *t*, N_t^j individuals are born with n^j growth rate of population; that is, $N_t^j = (1+n^j)N_{t-1}^j$. In their first period of life, the young inelastically supply labor to earn the competitive wage and then allocate their wage income between consumption and saving, which is equal to investment to acquire capital in the next period. In their second period of life, the young become the old and retire from work, consuming all saving and its returns from the previous period. There is no international borrowing and lending since allowing it would result in indeterminacy of production and trade in equilibrium. Let c_{yt} and c_{ot} be the consumption in period *t* of each young and old individual. Then, the budget constraints in accordance with the behaviors of country *j*'s representative individual in overlapping generations are

$$p_{yt}^j c_{yt}^j + s_t^j = w_t^j \qquad (3.1.1)$$

$$p_{yt+1}^j c_{ot+1}^j = R_{t+1}^j s_t^j \qquad (3.1.2)$$

where s_t^j is individual saving, and $R_{t+1}^j = 1 + r_{t+1}^j - \delta$ is the return on saving held from period *t* to *t+1*. Since it is assumed that the depreciation rate is zero, then $R_{t+1}^j = 1 + r_{t+1}^j$. p_{yt}^j and p_{yt+1}^j are country *j*'s prices of final good in each period.

The utility function of an individual born at *t*, $U(c_{yt}, c_{ot+1})$, follows Cobb Douglas utility function, which is monotonically increasing, twice continuously differentiable and strictly quasi-concave, $u'(\cdot) > 0, u''(\cdot) < 0$, satisfying the Inada conditions. That is, by monotonic transformation in the logarithmic form,

$$U(c_{yt}, c_{ot+1}) = \beta^j \log c_{yt}^j + (1-\beta^j)\log c_{ot+1}^j, \qquad (3.1.3)$$

where β^j stands for how individual deciding to allocate their consumption between young and old.

Consumer's problem is that the individual born in period t chooses (c_{yt}, c_{ot+1}, s_t) to maximize utility (3.1.3) subject to the budget constraints (3.1.1) and (3.1.2), which can be rearranged to obtain the intertemporal budget constraint as

$$p_{yt}^j c_{yt}^j + \frac{p_{yt+1}^j c_{ot+1}^j}{(1+r_{t+1}^j)} = w_t^j . \qquad (3.1.4)$$

Then, the first order condition is

$$\frac{\beta^j}{c_{yt}^j} = \lambda p_{yt}^j , \qquad (3.1.5)$$

$$\frac{1-\beta^j}{c_{0t+1}^j} = \frac{\lambda p_{yt+1}^j}{(1+r_{t+1}^j)} . \qquad (3.1.6)$$

Combining together both (3.1.5) and (3.1.6) gives us the standard intertemporal Euler equation,

$$\frac{U'_{c_{yt}}(\cdot)}{U'_{c_{ot+1}}(\cdot)} = \frac{c_{ot+1}^j}{c_{yt}^j} = \frac{1-\beta^j}{\beta^j}\frac{p_{yt}^j}{p_{yt+1}^j}(1+r_{t+1}^j) , \qquad (3.1.7)$$

which is the marginal rate of substitution between consumption of the young and old.

Finally, the optimization obtains the following results,

$$c_{yt}^j = \beta^j \frac{w_t^j}{p_{yt}^j} , \qquad (3.1.8)$$

$$c_{ot+1}^j = (1-\beta^j)(1+r_{t+1}^j)\frac{w_t^j}{p_{yt+1}^j} , \qquad (3.1.9)$$

$$s_t^j = (1-\beta^j) w_t^j . \qquad (3.1.10)$$

As shown by the above three equations, the optimal consumption is a fraction of real wage, as well as the optimal saving. Furthermore, the optimal consumption in both periods is an increasing function in wage, as well as the optimal saving, $\frac{\partial s_t}{\partial w_t} > 0$, under the assumption that consumption in both periods is a normal good. According to transformed logarithmic utility function, $\frac{\partial s_t}{\partial r_{t+1}} = 0$, which implies that the elasticity of substitution between consumption in the two periods is equal to one (Bianconi, 1995).

In equilibrium, total saving in the present period, $s_t^j N_t^j$, is equal total capital in the next period,

$$K_{t+1}^j = s_t^j N_t^j . \qquad (3.1.11)$$

Substituting equation (3.1.10) into (3.1.11), and $N_{t+1}^j / N_t^j = (1+n^j)$, then the equilibrium condition becomes

$$(1+n^j)k_{t+1}^j = (1-\beta^j)w_t^j ,$$

$$k_{t+1}^j = \frac{(1-\beta^j)}{(1+n^j)} w_t^j , \qquad (3.1.12)$$

which implies that the next period capital-labor ratio in each economy depends on a wage rate, w_t^j, a saving proportion, $(1-\beta^j)$, and a population growth rate, n^j, or dependency ratio, $\frac{1}{(1+n^j)}$.

3.1.2 Each Small Economy’s Production

The production functions of both intermediate and final goods are assumed to be Cobb-Douglas production technologies, which satisfy standard neoclassical production functions; homogeneous of degree one in all inputs, twice continuously differentiable, and with positive and diminishing marginal products of each input. The production technologies of the two intermediate goods are identical across countries, as well as that of the final good, but different across commodities. To specify, technology of good 2 requires more capital intensive than that of good 1 for all relevant factor-prices ratios; that is assumed to be no factor intensity reversals. Assume further that there is no depreciation rate of capital for simplicity, and the market is perfectly competitive in both intermediate and final goods.

a) Intermediate goods

According to the assumptions of intermediate good technology mentioned above, the intermediate goods 1 and 2 produced in country *j* at time *t*, given by x_{1t}^{j}, x_{2t}^{j} respectively, are expressed in per worker terms as follows,

$$x_{1t}^{j} = (k_{1t}^{j})^{\alpha_1}(l_{1t}^{j})^{1-\alpha_1}, \quad 0<\alpha_1<1, \qquad (3.1.13)$$

$$x_{2t}^{j} = (k_{2t}^{j})^{\alpha_2}(l_{2t}^{j})^{1-\alpha_2}, \quad 0<\alpha_2<1, \qquad (3.1.14)$$

where $x_{it}^{j} = \dfrac{X_{it}^{j}}{N_t^{j}}$ is per worker intermediate good *i* produced in country *j* at time *t*,

$k_{it}^{j} = \dfrac{K_{it}^{j}}{N_t^{j}}$ is per worker capital used for producing good *i* in country *j* at time *t*,

$l_{it}^{j} = \dfrac{L_{it}^{j}}{N_t^{j}}$ is per worker labor used for producing good *i* in country *j* at time *t*,

all for *i* = 1, 2.

Since intermediate good 1 production is relatively less capital intensive than intermediate good 2 production, then $\alpha_1 < \alpha_2$.

By this notation, total labor supply at time t can be written as $L_t^{j} = N_t^{j}\bar{l}$, where $\bar{l}$ represents the fixed amount of labor inelastically supplied by each young (Sayan, 2004). Full employment of both factors requires that factors needed to produce intermediate goods 1 and 2 are equal to total factor supplies in each period. That is,

for labor,

$$L_{1t}^{j} + L_{2t}^{j} = L_t^{j},$$

$$l_{1t}^{j} + l_{2t}^{j} = \bar{l}_t^{\,j}, \qquad (3.1.15)$$

which normalizing $\bar{l}^{\,j}$ to be 1, and

for capital,

$$K_{1t}^{j} + K_{2t}^{j} = K_t^{j},$$

$$k_{1t}^{j} + k_{2t}^{j} = k_t^{j}. \qquad (3.1.16)$$

Competitive intermediate goods firms decide how to allocate the total capital and labor available in that country across the two goods, taking world prices of intermediate goods, p_{1t} and p_{2t}, and domestic factor prices, w_t^j and r_t^j, as given. Then, each intermediate good firm solves the following profit maximization problems:

$$\max \ \pi_{1t}^j = p_{1t}(k_{1t}^j)^{\alpha_1}(l_{1t}^j)^{1-\alpha_1} - r_t^j k_{1t}^j - w_t^j l_{1t}^j, \qquad (3.1.17)$$

$$\max \ \pi_{2t}^j = p_{2t}(k_{2t}^j)^{\alpha_2}(l_{2t}^j)^{1-\alpha_2} - r_t^j k_{2t}^j - w_t^j l_{2t}^j. \qquad (3.1.18)$$

The first order conditions of profit maximizing are

$$w_t^j = (1-\alpha_1)p_{1t}(k_{1t}^j)^{\alpha_1}(l_{1t}^j)^{-\alpha_1} = (1-\alpha_2)p_{2t}(k_{2t}^j)^{\alpha_2}(l_{2t}^j)^{-\alpha_2}, \qquad (3.1.19)$$

$$r_t^j = \alpha_1 p_{1t}(k_{1t}^j)^{\alpha_1-1}(l_{1t}^j)^{1-\alpha_1} = \alpha_2 p_{2t}(k_{2t}^j)^{\alpha_2-1}(l_{2t}^j)^{1-\alpha_2}, \qquad (3.1.20)$$

Equation (3.1.19) and (3.1.20) represents that each factor price equals its marginal product in both two intermediate sectors. As a result of solving profit maximization problems, the demands for factors are obtained as

$$l_{1t}^j = \left(\frac{\alpha_1}{1-\alpha_1}\frac{w_t^j}{r_t^j}\right)^{-\alpha_1} x_{1t}^j \ , \quad l_{2t}^j = \left(\frac{\alpha_2}{1-\alpha_2}\frac{w_t^j}{r_t^j}\right)^{-\alpha_2} x_{2t}^j, \qquad (3.1.21)$$

$$k_{1t}^j = \left(\frac{\alpha_1}{1-\alpha_1}\frac{w_t^j}{r_t^j}\right)^{1-\alpha_1} x_{1t}^j \ , \quad k_{2t}^j = \left(\frac{\alpha_2}{1-\alpha_2}\frac{w_t^j}{r_t^j}\right)^{1-\alpha_2} x_{2t}^j. \qquad (3.1.22)$$

Define $a_{il} = \left(\frac{\alpha_i}{1-\alpha_i}\frac{w_t^j}{r_t^j}\right)^{-\alpha_i}$ as the labor used for one unit of production in sector *i*, and $a_{ik} = \left(\frac{\alpha_i}{1-\alpha_i}\frac{w_t^j}{r_t^j}\right)^{1-\alpha_i}$ as the capital used for one unit of production in sector *i*, for *i*=1,2.

Thus, factor market clearing conditions require that factor supplies are equal to factor demands, according to equation (3.1.15) and (3.1.16),

$$\bar{l}_t^j = l_{1t}^j + l_{2t}^j = \left(\frac{\alpha_1}{1-\alpha_1}\frac{w_t^j}{r_t^j}\right)^{-\alpha_1} x_{1t}^j + \left(\frac{\alpha_2}{1-\alpha_2}\frac{w_t^j}{r_t^j}\right)^{-\alpha_2} x_{2t}^j = 1, \qquad (3.1.23)$$

$$k_t^j = k_{1t}^j + k_{2t}^j = \left(\frac{\alpha_1}{1-\alpha_1}\frac{w_t^j}{r_t^j}\right)^{1-\alpha_1} x_{1t}^j + \left(\frac{\alpha_2}{1-\alpha_2}\frac{w_t^j}{r_t^j}\right)^{1-\alpha_2} x_{2t}^j. \qquad (3.1.24)$$

Cost function is simply the sum of total wage and total rental given to factor supplies needed in each sector;

$$c_{1t}(w,r,x_1) \;=\; w_t^j l_{1t}^j + r_t^j k_{1t}^j \;=\; \alpha_1^{-\alpha_1}(1-\alpha_1)^{-(1-\alpha_1)} w_t^{1-\alpha_1} r_t^{\alpha_1} x_{1t}^j\,, \quad (3.1.25)$$

$$c_{2t}(w,r,x_2) \;=\; w_t^j l_{2t}^j + r_t^j k_{2t}^j \;=\; \alpha_2^{-\alpha_2}(1-\alpha_2)^{-(1-\alpha_2)} w_t^{1-\alpha_2} r_t^{\alpha_2} x_{2t}^j\,. \quad (3.1.26)$$

Free entry ensures zero profit for the intermediate goods producers. Then, in Cobb Douglas technology with constant returns to scale, zero-profit conditions are given by the equality between price and unit cost as expressed follows,

$$p_{1t} = c_{1x_1}(w,r,x_1) \;=\; \alpha_1^{-\alpha_1}(1-\alpha_1)^{-(1-\alpha_1)} w_t^{1-\alpha_1} r_t^{\alpha_1} \quad (3.1.27)$$

$$p_{2t} = c_{2x_2}(w,r,x_2) \;=\; \alpha_2^{-\alpha_2}(1-\alpha_2)^{-(1-\alpha_2)} w_t^{1-\alpha_2} r_t^{\alpha_2} \quad (3.1.28)$$

Solving for w_t^j and r_t^j from equation (3.1.27) and (3.1.28) then obtain

$$w_t^j \;=\; (1-\alpha_1)A^{\alpha_1}\left(\frac{p_{1t}^{\alpha_2}}{p_{2t}^{\alpha_1}}\right)^{\frac{1}{\alpha_2-\alpha_1}} \;=\; (1-\alpha_2)B^{\alpha_2}\left(\frac{p_{1t}^{\alpha_2}}{p_{2t}^{\alpha_1}}\right)^{\frac{1}{\alpha_2-\alpha_1}}, \quad (3.1.29)$$

$$r_t^j \;=\; \alpha_1 A^{\alpha_1-1}\left(\frac{p_{1t}^{\alpha_2-1}}{p_{2t}^{\alpha_1-1}}\right)^{\frac{1}{\alpha_2-\alpha_1}} \;=\; \alpha_2 B^{\alpha_2-1}\left(\frac{p_{1t}^{\alpha_2-1}}{p_{2t}^{\alpha_1-1}}\right)^{\frac{1}{\alpha_2-\alpha_1}}, \quad (3.1.30)$$

where $$A=\left[\left(\frac{\alpha_2}{\alpha_1}\right)^{\alpha_2}\left(\frac{1-\alpha_2}{1-\alpha_1}\right)^{1-\alpha_2}\right]^{\frac{1}{\alpha_1-\alpha_2}},\quad B=\left[\left(\frac{\alpha_2}{\alpha_1}\right)^{\alpha_1}\left(\frac{1-\alpha_2}{1-\alpha_1}\right)^{1-\alpha_1}\right]^{\frac{1}{\alpha_1-\alpha_2}},$$

and since $\alpha_1 < \alpha_2$, then $A < B$. As can be seen, wage and rental rates are determined by prices of intermediate goods, and equalized in both two intermediate sectors due to no factor intensity reversal.

Thus, the diversification range of capital-labor ratio is

$$A\left(\frac{p_{1t}}{p_{2t}}\right)^{\frac{1}{\alpha_2-\alpha_1}} \;\le\; k_t^j \le B\left(\frac{p_{1t}}{p_{2t}}\right)^{\frac{1}{\alpha_2-\alpha_1}}. \quad (3.1.31)$$

The country's endowment is assumed to be within the diversification range so that both intermediate goods are produced. Since wage and rental rates are the functions of prices at the world market, together with the country diversifying its production of intermediate goods, factor price equalization condition is ensured to be held.

Moreover, this diversification range changes over time since the capital-labor ratio and the equilibrium prices of the integrated economy change over time (Bajona and Kehoe, 2006).

From factor market clearing conditions, since w^j and r^j are known as a function of intermediate good prices, use factor market clearing condition to solve for x_{1t}^j and x_{2t}^j by using Cremer's rule (see Appendix A for solving details). Then, supplies of intermediate good 1 and 2 as a function of intermediate good prices and capital-labor ratio in that economy, $x_{it}^j = x_i(p_{1t}, p_{2t}, k_t^j)$ for $i = 1, 2$, are given by

$$x_{1t}^j = \frac{A^{\alpha_1}}{B-A}\left[B - k_t^j\left(\frac{p_{1t}}{p_{2t}}\right)^{\frac{1}{\alpha_1-\alpha_2}}\right]\left(\frac{p_{1t}}{p_{2t}}\right)^{\frac{\alpha_1}{\alpha_2-\alpha_1}}, \qquad (3.1.32)$$

$$x_{2t}^j = \frac{B^{\alpha_2}}{B-A}\left[k_t^j\left(\frac{p_{1t}}{p_{2t}}\right)^{\frac{1}{\alpha_1-\alpha_2}} - A\right]\left(\frac{p_{1t}}{p_{2t}}\right)^{\frac{\alpha_2}{\alpha_2-\alpha_1}}. \qquad (3.1.33)$$

Consequently, the demands for factors of each good as a function of intermediate good prices and capital-labor ratio in that economy in period t, $l_{it}^j = l_i(p_{1t}, p_{2t}, k_t^j)$ and $k_{it}^j = k_i(p_{1t}, p_{2t}, k_t^j)$, for $i = 1, 2$, are expressed as

$$l_{1t}^j = \frac{1}{B-A}\left[B - k_t^j\left(\frac{p_{1t}}{p_{2t}}\right)^{\frac{1}{\alpha_1-\alpha_2}}\right], \qquad (3.1.34)$$

$$l_{2t}^j = \frac{1}{B-A}\left[-A + k_t^j\left(\frac{p_{1t}}{p_{2t}}\right)^{\frac{1}{\alpha_1-\alpha_2}}\right], \qquad (3.1.35)$$

$$k_{1t}^j = \frac{A}{B-A}\left[B\left(\frac{p_{1t}}{p_{2t}}\right)^{\frac{1}{\alpha_2-\alpha_1}} - k_t^j\right], \qquad (3.1.36)$$

$$k_{2t}^j = \frac{B}{B-A}\left[-A\left(\frac{p_{1t}}{p_{2t}}\right)^{\frac{1}{\alpha_2-\alpha_1}} + k_t^j\right]. \qquad (3.1.37)$$

That is, the allocation of capital and labor between two intermediate sectors depends on the domestic capital-labor ratio, k_t^j. Countries that have a higher capital-labor ratio devote a larger fraction of capital and labor to the capital-intensive sector 2 whereas countries that have a lower capital-labor ratio allocate a larger fraction of capital and labor to the labor-intensive sector 1.

b) Final Good

The technology of final good also follows Cobb Douglas production function, $y_t^j = (z_{1t}^j)^{\alpha_3}(z_{2t}^j)^{1-\alpha_3}$, where y_t^j is final good per worker produced by country *j* at time *t*, z_{it}^j is per worker intermediate good 1 and 2 needed to produce final good *y* by country *j* at time *t*. Competitive final good firms choose how to allocate intermediate good 1 and 2, produced domestically or purchased from abroad, taking the current world prices of intermediate goods, p_{1t}, p_{2t}, and final good, p_{yt}. Thus, firm problem of profit maximization is

$$\max \ \pi_y = p_y(z_{1t}^j)^{\alpha_3}(z_{2t}^j)^{1-\alpha_3} - p_{1t}z_{1t}^j - p_{2t}z_{2t}^j. \qquad (3.1.38)$$

Then, first order conditions for profit maximizing problem are

$$p_{1t} = \alpha_3 p_y (z_{1t}^j)^{\alpha_3 - 1}(z_{2t}^j)^{1-\alpha_3}, \qquad (3.1.39)$$

$$p_{2t} = (1-\alpha_3) p_y (z_{1t}^j)^{\alpha_3}(z_{2t}^j)^{-\alpha_3}. \qquad (3.1.40)$$

Solving equation (3.1.39) and (3.1.40) yields demands for intermediate goods as expressed follows,

$$z_{1t}^j = \left(\frac{\alpha_3}{1-\alpha_3}\frac{p_{2t}}{p_{1t}}\right)^{1-\alpha_3} y_t^j, \qquad (3.1.41)$$

$$z_{2t}^j = \left(\frac{\alpha_3}{1-\alpha_3}\frac{p_{2t}}{p_{1t}}\right)^{-\alpha_3} y_t^j. \qquad (3.1.42)$$

Thus, demands for intermediate goods are given by the equilibrium prices of the integrated economy and the final good needed in that country.

c) International Trade

Since all goods are tradable between the economies, the quantities of intermediate goods and final good can differ from the quantities produced in the small economy. It is assumed as in standard dynamic Heckscher-Ohlin models that trade is balanced in each period for each economy, due to an absence of international borrowing and lending. That is, trade balance must satisfy the following condition;

$$p_{1t}(z_{1t}^{j} - x_{1t}^{j}) + p_{2t}(z_{2t}^{j} - x_{2t}^{j}) + p_{yt}(y_{dt}^{j} - y_{st}^{j}) = 0 \text{, or}$$

$$p_{1t}z_{1t}^{j} + p_{2t}z_{2t}^{j} + p_{yt}y_{dt}^{j} = p_{1t}x_{1t}^{j} + p_{2t}x_{2t}^{j} + p_{yt}y_{st}^{j} \qquad (3.1.43)$$

Capital abundant country, for example, will export an intermediate good 1 and import an intermediate good 2. Although factor prices are equalized in each period, trade in intermediate goods still occurs in the long run dynamic Heckscher-Ohlin model according to Chen (1992). Moreover, as long as each country's capital-labor ratio is not equal to the world average level of capital-labor ratio, there exists an international trade in every steady state (Bajona and Kehoe, 2006).

From equation (3.1.41) and (3.1.42), the ratio of demanded intermediate goods is

$$\frac{z_{2t}^{j}}{z_{1t}^{j}} = \frac{1-\alpha_3}{\alpha_3}\frac{p_{1t}}{p_{2t}}. \qquad (3.1.44)$$

Since every economy faces the same price ratios, $\frac{p_{1t}}{p_{2t}}$, of intermediate goods at the world market, then the ratio of each economy's intermediate good demand is the same as the ratio of the world average demand of intermediate goods, as well as the ratio of the world average supply of intermediate goods. That is,

$$\frac{z_{2t}^{j}}{z_{1t}^{j}} = \frac{z_{2t}}{z_{1t}} = \frac{x_{2t}}{x_{1t}}, \qquad (3.1.45)$$

where $z_{it} = \frac{\sum_j Z_t^j}{\sum_j N_t^j} = \sum_j \eta_t^j z_{it}^j$, the world average demand of intermediate good *i* at time *t*, and $x_{it} = \frac{\sum_j X_t^j}{\sum_j N_t^j} = \sum_j \eta_t^j x_{it}^j$, the world average supply of intermediate good *i* at time *t*. It is essential to note that each economy is a small country whose population is a subset of world population. Thus, $N_t^j,\ j = 1,...,J$ is the number of individuals born in country *j* each period. Then, $\sum_j N_t^j = N_t^w$ is the total individuals born at time *t* in the world, and $\eta_t^j \equiv \frac{N_t^j}{N_t^w}$ is the fraction of world labor available in country *j* in each period.

Thus, solving the small economy problem requires the integrated economy (or world average) approach. In addition, to easily solve for an equilibrium prices of the world economy, it needs each country's endowment being in the diversification cone to ensure the factor price equalization (Bajona and Kehoe, 2006).

Similarly, final good price clearing also need final good demands and final good supplies from all over the world. This clearing explanation will be addressed in the next section.

3.2 The Integrated Economy (World Average)

As stated by Ventura (1997) and Bajona and Kehoe (2006), 'the equilibrium values of the variables for one country cannot be solved an optimal growth problem for that country in isolation'. Since both intermediate goods firms and final good firms take the world prices of intermediate goods as given, all demands and supplies also depend on the world prices and eventually the world's capital-labor ratio. These results will obtain from an integrated economy solving approach as can be seen in this section.

To clarify, the integrated world is considered as a closed economy with initial factor endowments equal to the world endowments of the factors of production (See Dixit and Norman 1980, and Freenstra 2004). Since each economy is assumed to have identical production technology and no factor intensity reversal warrants factor price equalization, summing of demands for total labor and capital used in each country finally equals total factors used in world demand when considering as closed economy. This is applied to the supply side as well. Therefore, the equilibrium of total demands and supplies form each economy results exactly the same as the equilibrium of world demand and endowment as a closed economy with a representative firm. If the total amount of capital is divided by the world labor at time *t* – called the world average, it will be equal to summing of each economy's capital-labor ratio weighted by each country's share of world labor at time *t*. Thus, the integrated economy can be viewed as world average that all endowments, demands, and supplies are represented in per capita born at time *t*. Define $k_t^w = k_t$ is the world endowment of capital per young capita (worker), $\sum_j K_t^j \Big/ \sum_j N_t^j = K_t^w \big/ N_t^w$, as well as $l_t^w = l_t$ is the world endowment of labor per young capita, L_t^w / N_t^w. Since the integrated economy approach is the solution to the closed economy planner's problem, the equilibrium prices of the traded goods, p_{1t}, p_{2t}, and p_{yt}, will be solved endogenously in the integrated economy.

3.2.1 Production and Intermediate good prices in the integrated economy

The production technology of intermediate goods 1 and 2 in the integrated economy is the same as that in each economy, but with the world endowments.

$$x_{1t} = k_{1t}^{\alpha_1} l_{1t}^{1-\alpha_1}, \qquad 0 < \alpha_1 < 1, \tag{3.2.1}$$

$$x_{2t} = k_{2t}^{\alpha_2} l_{2t}^{1-\alpha_2}, \qquad 0 < \alpha_2 < 1, \tag{3.2.2}$$

where $x_{it} = \dfrac{X_{it}^w}{N_t^w}$, $k_{it} = \dfrac{K_{it}^w}{N_t^w}$, $l_{it} = \dfrac{L_{it}^w}{N_t^w}$, for $i = 1, 2$ and $\alpha_1 < \alpha_2$,

$X_{it}^{\ w}$ is the total intermediate good *i* in the world market at time *t*, $\sum_j X_{it}^j$,

$K_{it}^{\ w}$ is the capital used for producing intermediate good *i* in the integrated economy,

$L_{it}^{\ w}$ is the labor used for producing intermediate good *i* in the integrated economy.

First order conditions of world producer's profit maximization problems are

$$w_t = (1-\alpha_1) p_{1t} k_{1t}^{\alpha_1} l_{1t}^{-\alpha_1} = (1-\alpha_2) p_{2t} k_{2t}^{\alpha_2} l_{2t}^{-\alpha_2}, \qquad (3.2.3)$$

$$r_t = \alpha_1 p_{1t} k_{1t}^{\alpha_1 -1} l_{1t}^{1-\alpha_1} = \alpha_2 p_{2t} k_{2t}^{\alpha_2 -1} l_{2t}^{1-\alpha_2}, \qquad (3.2.4)$$

where w_t and r_t are the competitive wage and interest in the integrated economy.

Thus, integrated factor market clearing conditions are defined as the world factor endowments are equal to the integrated average demands of factors; that is,

$$\bar{l}_t = l_{1t} + l_{2t} = \left(\frac{\alpha_1}{1-\alpha_1} \frac{w}{r} \right)^{-\alpha_1} x_{1t} + \left(\frac{\alpha_2}{1-\alpha_2} \frac{w}{r} \right)^{-\alpha_2} x_{2t} = 1, \quad (3.2.5)$$

$$k_t = k_{1t} + k_{2t} = \left(\frac{\alpha_1}{1-\alpha_1} \frac{w}{r} \right)^{1-\alpha_1} x_{1t} + \left(\frac{\alpha_2}{1-\alpha_2} \frac{w}{r} \right)^{1-\alpha_2} x_{2t}. \qquad (3.2.6)$$

Zero-profit conditions are followed by world economy prices equal to unit cost,

$$p_{1t} = \alpha_1^{-\alpha_1} (1-\alpha_1)^{-(1-\alpha_1)} w_t^{1-\alpha_1} r_t^{\alpha_1} \qquad (3.2.7)$$

$$p_{2t} = \alpha_2^{-\alpha_2} (1-\alpha_2)^{-(1-\alpha_2)} w_t^{1-\alpha_2} r_t^{\alpha_2} \qquad (3.2.8)$$

Solve for w_t and r_t from zero-profit conditions, similar to small country results, to get

$$w_t = \left(\frac{\alpha_2^{\alpha_1\alpha_2} (1-\alpha_2)^{\alpha_1(1-\alpha_2)}}{\alpha_1^{\alpha_1\alpha_2} (1-\alpha_1)^{(1-\alpha_1)\alpha_2}} \frac{p_{2t}^{\alpha_1}}{p_{1t}^{\alpha_2}} \right)^{\frac{1}{\alpha_1-\alpha_2}}, \qquad (3.2.9)$$

$$r_t = \left(\frac{\alpha_1^{\alpha_1(1-\alpha_2)} (1-\alpha_1)^{(1-\alpha_1)(1-\alpha_2)}}{\alpha_2^{(1-\alpha_1)\alpha_2} (1-\alpha_2)^{(1-\alpha_1)(1-\alpha_2)}} \frac{p_{2t}^{\alpha_1-1}}{p_{1t}^{\alpha_2-1}} \right)^{\frac{1}{\alpha_1-\alpha_2}}, \qquad (3.2.10)$$

Final good in the integrated world which is considered as closed economy has the production function as

$$y_t = x_{1t}^{\alpha_3} x_{2t}^{1-\alpha_3}. \qquad (3.2.11)$$

Then, the final good firm problem is to maximize $\pi_y = x_{1t}^{\alpha_3} x_{2t}^{1-\alpha_3} - p_{1t}x_{1t} - p_{2t}x_{2t}$, and the first order conditions are

$$p_{1t} = \alpha_3 x_{1t}^{\alpha_3 -1} x_{2t}^{1-\alpha_3}, \qquad (3.2.12)$$

$$p_{2t} = (1-\alpha_3) x_{1t}^{\alpha_3} x_{2t}^{-\alpha_3}. \qquad (3.2.13)$$

The price ratio as a function of the ratio of intermediate good can be expressed as

$$\frac{p_{1t}}{p_{2t}} = \frac{\alpha_3}{1-\alpha_3}\frac{x_{2t}}{x_{1t}} \quad . \qquad (3.2.14)$$

Zero-profit condition for final good is according to

$$1 = \alpha_3^{-\alpha_3}(1-\alpha_3)^{-(1-\alpha_3)} p_{1t}^{\alpha_3} p_{2t}^{1-\alpha_3} . \qquad (3.2.15)$$

Solve for the integrated equilibrium prices p_{1t} and p_{2t} from equation (3.2.14) and (3.2.15) by substituting $p_{1t} = \frac{\alpha_3}{1-\alpha_3}\frac{x_{2t}}{x_{1t}} p_{2t}$ into equation (3.2.15). The integrated equilibrium prices in terms of the ratio of intermediate good are expressed as

$$p_{1t} = \alpha_3^{\alpha_3}(1-\alpha_3)^{1-\alpha_3}\left(\frac{1-\alpha_3}{\alpha_3}\frac{x_{2t}}{x_{1t}}\right)^{1-\alpha_3} = \alpha_3\left(\frac{x_{2t}}{x_{1t}}\right)^{1-\alpha_3} , \qquad (3.2.16)$$

$$p_{2t} = \alpha_3^{\alpha_3}(1-\alpha_3)^{1-\alpha_3}\left(\frac{1-\alpha_3}{\alpha_3}\frac{x_{1t}}{x_{2t}}\right)^{\alpha_3} = (1-\alpha_3)\left(\frac{x_{2t}}{x_{1t}}\right)^{-\alpha_3} \qquad (3.2.17)$$

If $\frac{x_{2t}}{x_{1t}}$ can be expressed in terms of k_t, then the equilibrium price can be stated as the function of k_t (See further details in Appendix B). That is,

$$\frac{x_{2t}}{x_{1t}} = \frac{X_{2t}}{X_{1t}} = \frac{\kappa_{2t}^{\alpha_2} l_{2t}}{\kappa_{1t}^{\alpha_1} l_{1t}} = \left[\frac{\alpha_2(1-\gamma)}{\gamma(1-\alpha_2)}\right]^{\alpha_2}\left[\frac{\gamma(1-\alpha_1)}{\alpha_1(1-\gamma)}\right]^{\alpha_1}\frac{1-\gamma}{\alpha_3(1-\alpha_1)}\frac{(1-\alpha_3)(1-\alpha_2)}{1-\gamma}k_t^{\alpha_2-\alpha_1}$$

$$\frac{x_{2t}}{x_{1t}} = \frac{\alpha_2^{\alpha_2}(1-\alpha_2)^{1-\alpha_2}}{\alpha_1^{\alpha_1}(1-\alpha_1)^{1-\alpha_1}}\frac{(1-\alpha_3)}{\alpha_3}\left(\frac{1-\gamma}{\gamma}\right)^{\alpha_2-\alpha_1} k_t^{\alpha_2-\alpha_1} \qquad (3.2.18)$$

From equation (3.2.18), the ratio of intermediate good can be expressed as a function of capital-labor ratio. Then, we can find $p_{1t} = p(k_t)$ and $p_{2t} = p(k_t)$.

$$p_{1t} = \alpha_3\left[\frac{\alpha_2^{\alpha_2}(1-\alpha_2)^{1-\alpha_2}}{\alpha_1^{\alpha_1}(1-\alpha_1)^{1-\alpha_1}}\frac{(1-\alpha_3)}{\alpha_3}\left(\frac{1-\gamma}{\gamma}\right)^{\alpha_2-\alpha_1}\right]^{1-\alpha_3} k_t^{(1-\alpha_3)(\alpha_2-\alpha_1)} \qquad (3.2.19)$$

$$p_{2t} = (1\text{-}\alpha_3)\left[\frac{\alpha_2^{\alpha_2}(1-\alpha_2)^{1-\alpha_2}}{\alpha_1^{\alpha_1}(1-\alpha_1)^{1-\alpha_1}}\frac{(1-\alpha_3)}{\alpha_3}\left(\frac{1-\gamma}{\gamma}\right)^{\alpha_2-\alpha_1}\right]^{-\alpha_3} k_t^{-\alpha_3(\alpha_2-\alpha_1)} \qquad (3.2.20)$$

Since the equilibrium prices can be stated in terms of capital-labor ratio, the competitive wage and interest can also be expressed in terms of capital-labor ratio as follows. (See solving details and further explanation in Appendix C.)

$$w_t = \alpha_3^{\alpha_3}(1-\alpha_3)^{1-\alpha_3}\left[\alpha_1^{\alpha_1}(1-\alpha_1)^{1-\alpha_1}\right]^{\alpha_3}\left[\alpha_2^{\alpha_2}(1-\alpha_2)^{1-\alpha_2}\right]^{(1-\alpha_3)}\left(\frac{1-\gamma}{\gamma}\right)^{\gamma} k_t^{\gamma},$$

$$w_t = \left(\frac{1-\gamma}{\gamma}\right)^{\gamma} a\left(k_t^{w}\right)^{\gamma}, \qquad (3.2.21)$$

$$r_t = \alpha_3^{\alpha_3}(1-\alpha_3)^{1-\alpha_3}\left[\alpha_1^{\alpha_1}(1-\alpha_1)^{1-\alpha_1}\right]^{\alpha_3}\left[\alpha_2^{\alpha_2}(1-\alpha_2)^{1-\alpha_2}\right]^{(1-\alpha_3)}\left(\frac{1-\gamma}{\gamma}\right)^{\gamma-1} k_t^{\gamma-1},$$

$$r_t = \left(\frac{1-\gamma}{\gamma}\right)^{\gamma-1} a\left(k_t^{w}\right)^{\gamma-1}, \qquad (3.2.22)$$

where $a = \alpha_3^{\alpha_3}(1-\alpha_3)^{1-\alpha_3}\left[\alpha_1^{\alpha_1}(1-\alpha_1)^{1-\alpha_1}\right]^{\alpha_3}\left[\alpha_2^{\alpha_2}(1-\alpha_2)^{1-\alpha_2}\right]^{(1-\alpha_3)}$.

3.2.2 Final good clearing in the integrated economy

Each individual's optimal consumptions are stated in the section 3.1, equation (3.1.8) to (3.1.10). To find the final good integrated demand in period t, the summation of both young and old consumptions and saving is needed. Total optimal consumption of young in each country is $C_{yt}^j = c_{yt}^j N_t^j$, and total optimal consumption of old in each country at period *t* follows that $C_{0t}^j = c_{0t}^j N_{t-1}^j$. Similarly, total optimal saving by young in each country at period t, $S_t^j = s_t^j N_t^j$, is also included as an expansion of capital in the next period, $K_{t+1}^j = s_t^j N_t^j$, in the country's demand, Y_{dt}^j. Since the final good is used for consumption and capital expansion, then the integrated demand consisting of all small countries' demands is

$$\sum_{j=1}^{J} Y_{dt}^j = \sum_{j=1}^{J} C_{yt}^j + \sum_{j=1}^{J} C_{ot}^j + \sum_{j=1}^{J} K_{t+1}^j$$

$$\sum_{j=1}^{J} Y_{dt}^j = \sum_{j=1}^{J} c_{yt}^j N_t^j + \sum_{j=1}^{J} c_{ot}^j N_{t-1}^j + \sum_{j=1}^{J} s_t^j N_t^j \qquad (3.2.23)$$

Since $N_t = (1+n)N_{t-1}$, rearranging equation (3.2.23) and dividing by total worker in period *t*, $N_t^w = \sum_{j=1}^{J} N_t^j$ yields

$$y_{dt}^w = \frac{\sum_j Y_{dt}^j}{N_t^w} = \frac{\sum_j c_{yt}^j N_t^j}{\sum_j N_t^j} + \left(\sum_j \frac{c_{ot}^j N_t^j}{(1+n^j)}\right)\frac{1}{\sum_j N_t^j} + \frac{\sum_j s_t^j N_t^j}{\sum_j N_t^j}$$

$$y_{dt}^w = \left(\frac{c_{yt}^1 N_t^1}{N_t^w} + \ldots + \frac{c_{yt}^J N_t^J}{N_t^w}\right) + \left(\frac{c_{ot}^1 N_t^1}{(1+n^1)N_t^w} + \ldots + \frac{c_{ot}^J N_t^J}{(1+n^J)N_t^w}\right) + \left(\frac{s_t^1 N_t^1}{N_t^w} + \ldots + \frac{s_t^J N_t^J}{N_t^w}\right)$$

As previously identified in section 3.1, $\eta_t^j = N_t^j / N_t^w$. Thus, the world average demand for final good follows

$$y_{dt}^w = \sum_{j=1}^{J} c_{yt}^j \eta_t^j + \sum_{j=1}^{J} \frac{c_{ot}^j \eta_t^j}{(1+n^j)} + \sum_{j=1}^{J} s_t^j \eta_t^j \qquad (3.2.24)$$

For simplicity, define consumption and saving at world average as $c_{yt} = \sum_j c_{yt}^j \eta_t^j$ for young, $c_{ot} = \sum_j c_{ot}^j \eta_{t-1}^j$ for old, and $s_t = \sum_j s_t^j \eta_t^j$ for saving, where $\eta_{t-1}^j = \eta_t^j / (1+n^j)$.

To clear the competitive world market, total demand for final good must be equal total supply of final good, $y_{st}^w = y_{dt}^w$, where $y_{st}^w = \sum_j y_{st}^j$. As a result, the final good price must also be equated at the integrated economy. It seems that every country take final good price at the world market as given since each economy is a small one. Additionally, there is only one final good market. Thus, the equilibrium clearing final good price can be also normalized to be 1, $p_{yt}^j = p_{yt} = 1$, in order to simplify further analysis.

Then, the optimization of individuals in overlapping-generation in the integrated economy is characterized by

$$c_{yt} = \sum_j \eta_t^j \beta^j w_t = \beta w_t \qquad (3.2.25)$$

$$c_{ot+1} = \sum_j \eta_{t+1}^j (1-\beta^j)(1+r_{t+1}) w_t = (1-\beta)(1+r_{t+1}) w_t \qquad (3.2.26)$$

$$s_t = \sum_j \eta_t^j (1-\beta^j) w_t = (1-\beta) w_t \qquad (3.2.27)$$

Let $\varphi^j = (1-\beta^j)$, then average propensity to save is $\varphi = \sum_j \eta_t^j \varphi^j$, and thus equation (3.2.27) can be expressed as

$$s_t = \varphi w_t \tag{3.2.28}$$

In equilibrium of the world economy, total saving in the present period, $s_t N_t^w$, is equal to total capital in the next period,

$$K_{t+1}^w = s_t N_t^w,$$

or

$$(1+n)k_{t+1}^w = \varphi w_t \tag{3.2.29}$$

The next period capital-labor ratio in integrated economy depends on a wage rate, w_t, world average saving proportion, φ, and world average population growth rate, n, or dependency ratio, $\frac{1}{(1+n)}$.

3.3 Resource allocation in small open economy and Results on trade aspects

According to the integrated economy equilibrium, all clearing prices are in expression of world capital-labor ratio, as can be seen from equation (3.2.19) to (3.2.22). In order to know the resource allocation, all these prices have to be substituted into demands for factors and supplies of intermediate goods. Moreover, since the model is based on Heckscher-Ohlin and assumes factor price equalization, it should be investigated Stloper-Samuelson and Rybczynski theorem as well.

3.3.1 Characteristics of diversification cone and resource allocation

Recall the integrated economy equilibrium prices characterized as equation (3.2.19) and (3.2.20). Then, the price ratio is followed by

$$\begin{aligned}\frac{p_{2t}}{p_{1t}} &= \frac{(1\text{-}\alpha_3)}{\alpha_3}\left[\frac{\alpha_2^{\alpha_2}(1-\alpha_2)^{1-\alpha_2}}{\alpha_1^{\alpha_1}(1-\alpha_1)^{1-\alpha_1}}\frac{(1-\alpha_3)}{\alpha_3}\left(\frac{1-\gamma}{\gamma}\right)^{\alpha_2-\alpha_1}\right]^{-\alpha_3-1+\alpha_3} k_t^{-\alpha_3(\alpha_2-\alpha_1)-(1-\alpha_3)(\alpha_2-\alpha_1)} \\ &= \left[\frac{\alpha_1^{\alpha_1}(1-\alpha_1)^{1-\alpha_1}}{\alpha_2^{\alpha_2}(1-\alpha_2)^{1-\alpha_2}}\left(\frac{1-\gamma}{\gamma}\right)^{\alpha_1-\alpha_2}\right]k_t^{\alpha_1-\alpha_2}\end{aligned} \tag{3.3.1}$$

Consider the cone of diversification,

$$A\left(\frac{p_{2t}}{p_{1t}}\right)^{\frac{1}{\alpha_1-\alpha_2}} = \left[\left(\frac{\alpha_2}{\alpha_1}\right)^{\alpha_2}\left(\frac{1-\alpha_2}{1-\alpha_1}\right)^{1-\alpha_2}\right]^{\frac{1}{\alpha_1-\alpha_2}}\left[\frac{\alpha_1^{\alpha_1}(1-\alpha_1)^{1-\alpha_1}}{\alpha_2^{\alpha_2}(1-\alpha_2)^{1-\alpha_2}}\left(\frac{1-\gamma}{\gamma}\right)^{\alpha_1-\alpha_2}\right]^{\frac{1}{\alpha_1-\alpha_2}} k_t^w$$

$$= \left[\alpha_1^{\alpha_1-\alpha_2}(1-\alpha_1)^{(1-\alpha_1)-(1-\alpha_2)}\right]^{\frac{1}{\alpha_1-\alpha_2}}\left(\frac{1-\gamma}{\gamma}\right)k_t^w$$

$$= \frac{\alpha_1}{(1-\alpha_1)}\frac{(1-\gamma)}{\gamma}k_t^w \qquad (3.3.2)$$

$$B\left(\frac{p_{2t}}{p_{1t}}\right)^{\frac{1}{\alpha_1-\alpha_2}} = \left[\left(\frac{\alpha_2}{\alpha_1}\right)^{\alpha_1}\left(\frac{1-\alpha_2}{1-\alpha_1}\right)^{1-\alpha_1}\right]^{\frac{1}{\alpha_1-\alpha_2}}\left[\frac{\alpha_1^{\alpha_1}(1-\alpha_1)^{1-\alpha_1}}{\alpha_2^{\alpha_2}(1-\alpha_2)^{1-\alpha_2}}\left(\frac{1-\gamma}{\gamma}\right)^{\alpha_1-\alpha_2}\right]^{\frac{1}{\alpha_1-\alpha_2}} k_t^w$$

$$= \left[\alpha_2^{\alpha_1-\alpha_2}(1-\alpha_2)^{(1-\alpha_1)-(1-\alpha_2)}\right]^{\frac{1}{\alpha_1-\alpha_2}}\left(\frac{1-\gamma}{\gamma}\right)k_t^w \qquad (3.3.3)$$

$$= \frac{\alpha_2}{(1-\alpha_2)}\frac{(1-\gamma)}{\gamma}k_t^w$$

Since $\alpha_1 < \alpha_2$, then $\frac{\alpha_1}{(1-\alpha_1)}\frac{(1-\gamma)}{\gamma}k_t^w < \frac{\alpha_2}{(1-\alpha_2)}\frac{(1-\gamma)}{\gamma}k_t^w$

In section 3.1 equation (3.1.31), the cone of diversification of small economy is

$$A\left(\frac{p_{2t}}{p_{1t}}\right)^{\frac{1}{\alpha_1-\alpha_2}} < k_t^j < B\left(\frac{p_{2t}}{p_{1t}}\right)^{\frac{1}{\alpha_1-\alpha_2}}$$

Substituting price ratio, then

$$\frac{\alpha_1}{(1-\alpha_1)}\frac{(1-\gamma)}{\gamma}k_t^w < k_t^j < \frac{\alpha_2}{(1-\alpha_2)}\frac{(1-\gamma)}{\gamma}k_t^w,$$

$$\frac{\alpha_1}{(1-\alpha_1)}\frac{(1-\gamma)}{\gamma} < \frac{k_t^j}{k_t^w} < \frac{\alpha_2}{(1-\alpha_2)}\frac{(1-\gamma)}{\gamma}. \qquad (3.3.4)$$

Since $\gamma = \alpha_1\alpha_3 + \alpha_2(1-\alpha_3)$ is weighted average between α_1 and α_2, then $\alpha_1 < \gamma$ and $\alpha_2 > \gamma$, which leads to $\frac{\alpha_1}{(1-\alpha_1)}\frac{(1-\gamma)}{\gamma} < 1$ and $\frac{\alpha_2}{(1-\alpha_2)}\frac{(1-\gamma)}{\gamma} > 1$. In order to ensure factor price equalization, each country's relative capital-labor ratio to the world's must be situated between $\frac{\alpha_1}{(1-\alpha_1)}\frac{(1-\gamma)}{\gamma}$ and $\frac{\alpha_2}{(1-\alpha_2)}\frac{(1-\gamma)}{\gamma}$.

As stated by Chatterjee and Shukayev (2006), 'the small country will maintain a constant ratio between the domestic aggregate capital-labor ratio and the world when it starts within the cone. If two small countries start within the diversification cone, but with different capital-labor ratios relative to the world economy, they will maintain those relative positions'. When the equilibrium prices are substituted into the allocations of labor and capital in that particular country, those allocations can be characterized by the country-to-world relative capital-labor ratio.

From equation (3.1.34), substituting equilibrium price, then

$$l_{1t}^{j} = \frac{1}{B-A}\left[B - k_t^{j}\left(\frac{\alpha_1^{\alpha_1}(1-\alpha_1)^{1-\alpha_1}}{\alpha_2^{\alpha_2}(1-\alpha_2)^{1-\alpha_2}}\left(\frac{1-\gamma}{\gamma}\right)^{\alpha_1-\alpha_2}(k_t^{w})^{\alpha_1-\alpha_2}\right)^{\frac{1}{\alpha_2-\alpha_1}}\right]$$

$$l_{1t}^{j} = \frac{B}{B-A}\left[1-\left(\frac{1-\alpha_2}{\alpha_2}\right)\left(\frac{\gamma}{1-\gamma}\right)\frac{k_t^{j}}{k_t^{w}}\right] \qquad (3.3.5)$$

For any country j, i.e. country N and country S, if $\frac{k_t^N}{k_t^w} > \frac{k_t^S}{k_t^w}$, then $l_{1t}^N < l_{1t}^S$. Countries with more capital abundance supplies their labor to labor intensive good 1 less than countries with less capital abundance.

Similarly, from equation (3.1.34),

$$l_{2t}^{j} = \frac{1}{B-A}\left[k_t^{j}\left(\frac{\alpha_1^{\alpha_1}(1-\alpha_1)^{1-\alpha_1}}{\alpha_2^{\alpha_2}(1-\alpha_2)^{1-\alpha_2}}\left(\frac{1-\gamma}{\gamma}\right)^{\alpha_1-\alpha_2}(k_t^{w})^{\alpha_1-\alpha_2}\right)^{\frac{1}{\alpha_2-\alpha_1}} - A\right]$$

$$l_{1t}^{j} = \frac{A}{B-A}\left[\left(\frac{1-\alpha_2}{\alpha_2}\right)\left(\frac{\gamma}{1-\gamma}\right)\frac{k_t^{j}}{k_t^{w}}-1\right] \qquad (3.3.6)$$

Since $\frac{k_t^N}{k_t^w} > \frac{k_t^S}{k_t^w}$, then $l_{2t}^N > l_{2t}^S$. Countries with more capital abundance supplies their labor to capital intensive good 2 much more than countries with less capital abundance.

From equation (3.1.36) and (3.1.37), substituting the equilibrium prices ratio, then

$$k_{1t}^{j} = \frac{A}{B-A}\left[\frac{\alpha_2}{(1-\alpha_2)}\frac{(1-\gamma)}{\gamma}k_t^{w} - k_t^{j}\right] \qquad (3.3.7)$$

Since $k_t^N > k_t^S$, then $k_{1t}^N < k_{1t}^S$. Countries with more capital abundance supplies their capital to labor intensive good 1 less than countries with less capital abundance.

And, finally

$$k_{2t}^{j} = \frac{B}{B-A}\left[k_t^{j} - \frac{\alpha_1}{(1-\alpha_1)}\frac{(1-\gamma)}{\gamma}k_t^{w}\right] \qquad (3.3.8)$$

Since $k_t^N > k_t^S$, then $k_{2t}^N > k_{2t}^S$. Countries with more capital abundance supplies their capital to capital intensive good 2 much more than countries with less capital abundance.

Thus, ratio of country *j*’s capital used in each industry to world capital are

$$\frac{k_{1t}^{j}}{k_t^{w}} = \frac{A}{B-A}\left[\frac{\alpha_2}{(1-\alpha_2)}\frac{(1-\gamma)}{\gamma} - \frac{k_t^{j}}{k_t^{w}}\right], \qquad (3.3.9)$$

$$\frac{k_{2t}^{j}}{k_t^{w}} = \frac{B}{B-A}\left[\frac{k_t^{j}}{k_t^{w}} - \frac{\alpha_1}{(1-\alpha_1)}\frac{(1-\gamma)}{\gamma}\right]. \qquad (3.3.10)$$

These support the above allocation that more capital abundant countries would relatively require more capital to produce capital intensive good 2 while less capital abundant countries would relatively require more capital to produce labor intensive good 1.

For intermediate goods, each country produces the abundant factor intensive good in more amount than the other one. That is,

$$x_{1t}^{j} = \frac{A^{\alpha_1}}{B-A}\left[B - k_t^{j}\left(\frac{p_{1t}}{p_{2t}}\right)^{\frac{1}{\alpha_1-\alpha_2}}\right]\left(\frac{p_{1t}}{p_{2t}}\right)^{\frac{\alpha_1}{\alpha_2-\alpha_1}}$$

$$x_{1t}^{j} = l_{1t}^{j}A^{\alpha_1}\left(\frac{p_{1t}}{p_{2t}}\right)^{\frac{\alpha_1}{\alpha_2-\alpha_1}} \qquad (3.3.11)$$

Since $l_{1t}^N < l_{1t}^S$, then $x_{1t}^N < x_{1t}^S$. Country S, labor abundant economy, produces more labor intensive intermediate good 1 than country N.

$$x_{2t}^j = \frac{B^{\alpha_2}}{B-A}\left[k_t^j\left(\frac{p_{1t}}{p_{2t}}\right)^{\frac{1}{\alpha_1-\alpha_2}} - A\right]\left(\frac{p_{1t}}{p_{2t}}\right)^{\frac{\alpha_2}{\alpha_2-\alpha_1}}$$

$$x_{2t}^j = l_{2t}^j B^{\alpha_2}\left(\frac{p_{1t}}{p_{2t}}\right)^{\frac{\alpha_2}{\alpha_2-\alpha_1}} \qquad (3.3.12)$$

Since $l_{2t}^N > l_{2t}^S$, then $x_{2t}^N > x_{2t}^S$. Country N, capital abundant economy, produces more capital intensive intermediate good 2 than country S.

To be concluded, equation (3.3.5) through (3.3.12) confirm the pattern of trade according to the Heckscher-Ohlin theorem with the verified assumption of factor price equalization.

3.3.2 Stolper-Samuelson and Rybczynski effects

As a result of trade in two intermediate goods based on Heckscher-Ohlin theorem, there exist other two effects called Stolper-Samuelson and Rybcynski. In Stolper-Samuelson, provided that both goods are produced, an increase in relative price of a good will increase the real return to the factor used intensively in that good, and reduce the real return to the other factor (Feenstra, 2004). In Rybcynski proposition, it states that, when both goods are produced, an increase in factor resource will increase the output of good used that factor relatively intensively while it will reduce the production of good used that factor relatively less intensively.

In this study, it assumes constant returns to scale on production and no factor intensity reversal. Stolper-Samuelson is satisfied according to Galor (1992) that when good 1 is less capital intensive comparable to good 2, then

$$\frac{d\omega_t}{dp_t} > 0,\ \frac{dr_t}{dp_t} < 0,\ \text{and}\ \frac{dw_t}{dp_t}\frac{p_t}{w_t} > 1,$$

where ω_t is a wage-rental ratio, and p_t is a price of an intermediate good 1, or relative price of good 1 over good 2. Thus, from equation (A.2), (3.1.29), and (3.1.30) with numerical proof, the above result is valid.

For Rybcynski theorem, from equation (3.3.11) and (3.3.12) together with equation (3.3.5) and (3.3.6), it can be concluded that an increase in capital-labor ratio of country *j* reduces the labors needed in production of intermediate good 1, and thus results in the reduction in the output of intermediate good 1 that uses less capital. On the other hand, an increase in capital-labor ratio of country *j* raises the labors needed in production of intermediate good 2, and thus results in the increasing in the output of intermediate good 2 that uses capital intensively.

3.4 Dynamic Equilibrium and Analysis

Due to the assumption that initial endowment of each country is not much different so that it is in the diversification cone, when the trade occurs, the prices of factor are also equalized. Thus, with the integrated economy approach, each country's equilibrium path of the capital stock and the steady state value depends on both its initial endowment of capital and the world's equilibrium path of capital stock and its steady state. However, it is necessary to state the autarky equilibrium in overlapping-generations so that it can be compared to the free trade equilibrium. Thus, except for the integrated economy approach at the beginning, this section will concern in three states of equilibrium; autarky equilibrium, integrated economy equilibrium, and free trade equilibrium**.**

3.4.1 Autarky equilibrium

Autarkic equilibrium is analyzed in the closed economic environment. Thus, the production of final good and other prices are solved in the same way as shown in Appendix C. Since the optimal consumption of young and the saving are the proportion of wage income while the optimal consumption of old depends on both wage income and next period rental rate, substituting the wage income and the rental rate expressed in terms of capital-labor ratio, results in the following expressions;

$$c_{yt}^j = \beta^j \left(\frac{1-\gamma}{\gamma}\right)^{\gamma} a\left(k_t^j\right)^{\gamma} \tag{3.4.1}$$

$$c_{ot+1}^j = \left[1+\left(\frac{1-\gamma}{\gamma}\right)^{\gamma-1} a\left(k_{t+1}^j\right)^{\gamma-1}\right]\varphi^j \left(\frac{1-\gamma}{\gamma}\right)^{\gamma} a\left(k_t^j\right)^{\gamma} \tag{3.4.2}$$

$$s_t^j = \varphi^j \left(\frac{1-\gamma}{\gamma}\right)^{\gamma} a\left(k_t^j\right)^{\gamma} \tag{3.4.3}$$

Then, the dynamic of economy is followed by

$$(1+n)k_{t+1}^j = \varphi^j w_t^j$$

$$(1+n^j)k_{t+1}^j = \varphi^j \left(\frac{1-\gamma}{\gamma}\right)^{\gamma} a\left(k_t^j\right)^{\gamma}$$

$$k_{t+1}^j = \frac{\varphi^j}{(1+n^j)}\left(\frac{1-\gamma}{\gamma}\right)^{\gamma} a\left(k_t^j\right)^{\gamma} \tag{3.4.4}$$

Equation (3.4.4) is the law of motion for capital per worker. Since the utility function is logarithmic and the production function is Cobb-Douglas, these ensure a unique positive steady state.

At steady state, $k_{t+1}^j = k_t^j = \hat{k}^j$. Then, $\hat{k}^j = \frac{\varphi}{(1+n)}\left(\frac{1-\gamma}{\gamma}\right)^{\gamma} a\left(\hat{k}^j\right)^{\gamma}$ can be solved for a unique steady state as

$$\hat{k}^j = \left[\frac{\varphi^j}{(1+n^j)}\left(\frac{1-\gamma}{\gamma}\right)^{\gamma} a\right]^{\frac{1}{1-\gamma}} \tag{3.4.5}$$

3.4.2 Equilibrium and Dynamics in the Integrated Economy (World Average)

Similar to the autarkic equilibrium, the integrated economy is solved as the closed economy. Thus, substituting the equilibrium wage rate and rental rate that resulting in equation (3.2.21) and (3.2.22) into c_{yt}, c_{ot+1}, and s_t of the world average overlapping generations gives the following optimalities.

$$c_{yt} = \beta\left(\frac{1-\gamma}{\gamma}\right)^{\gamma} a\left(k_t^w\right)^{\gamma} \tag{3.4.6}$$

$$c_{ot+1} = \left[1 + \left(\frac{1-\gamma}{\gamma}\right)^{\gamma-1} a\left(k_{t+1}^{w}\right)^{\gamma-1}\right]\varphi\left(\frac{1-\gamma}{\gamma}\right)^{\gamma} a\left(k_{t}^{w}\right)^{\gamma} \qquad (3.4.7)$$

$$s_t = \varphi\left(\frac{1-\gamma}{\gamma}\right)^{\gamma} a\left(k_{t}^{w}\right)^{\gamma} \qquad (3.4.8)$$

Then, the dynamic of integrated economy is followed by

$$(1+n)k_{t+1}^{w} = \varphi w_t$$

$$(1+n)k_{t+1}^{w} = \varphi\left(\frac{1-\gamma}{\gamma}\right)^{\gamma} a\left(k_{t}^{w}\right)^{\gamma}$$

$$k_{t+1}^{w} = \frac{\varphi}{(1+n)}\left(\frac{1-\gamma}{\gamma}\right)^{\gamma} a\left(k_{t}^{w}\right)^{\gamma} \qquad (3.4.9)$$

Equation (3.4.9) is the law of motion for world capital per worker. At a steady state, $k_{t+1} = k_t = \hat{k}$. Then, $\hat{k} = \frac{\varphi}{(1+n)}\left(\frac{1-\gamma}{\gamma}\right)^{\gamma} a\left(\hat{k}\right)^{\gamma}$ can be solved for a unique steady state

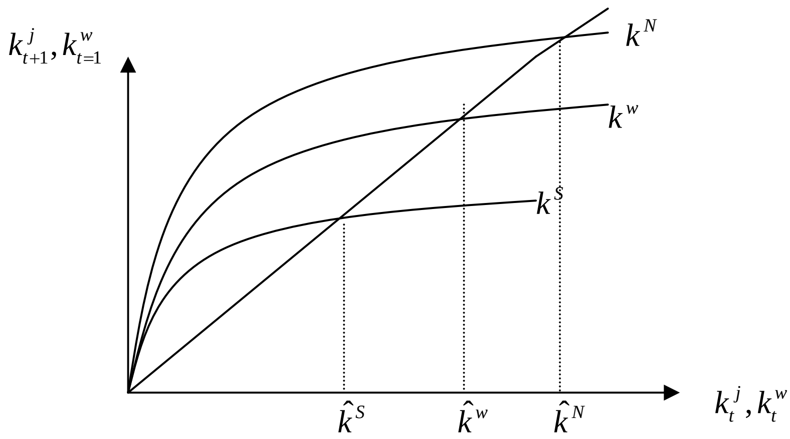
as

$$\hat{k} = \left[\frac{\varphi}{(1+n)}\left(\frac{1-\gamma}{\gamma}\right)^{\gamma} a\right]^{\frac{1}{1-\gamma}} \qquad (3.4.10)$$

Thus, figure 3.4.1 shows the dynamics of capital per worker in country *j*, when *j* = N, S, and in the integrated economy when those two countries have different saving rates and dependency ratios.

Figure 3.4.1

The dynamics of capital per worker in country N, S, and the integrated economy, labeled k^N, k^S ,and k^w, respectively.



According to the law of motion in equation (3.4.4), the dynamics of each country depend crucially on the saving rate and growth rate, as well as its initial conditions. A particular country that has high saving rate and low growth rate will move to a higher steady state than one that has low saving rate and high growth rate. Compared to the world dynamics that represent the average saving rate and growth rate, the autarkic path of a specific country can lie above or below the world average path.

3.4.3 Equilibrium and Dynamics in Small Economy under Free Trade

In the interdependent small economy, the equilibrium prices, as well as consumption, production and the dynamics of capital stock coincide with those of the integrated world average. However, it would be advantage to review the characteristics of small economy under free trade, and then disaggregate them from the world average economy.

a) Dynamics of small open economy and comparisons to the integrated economy

Since Competitive intermediate goods firms take as given the world prices of intermediate goods, p_{1t} and p_{2t} , to allocate the total capital and labor available in that country across the two goods, after trading, the returns to factors are the same across countries and given by the world capital-labor ratio. Thus, consumption of young and old, as well as saving are also given by world capital-labor ratio, characterized as the below formulas.

$$c_{yt}^{j} = \beta^{j}\left(\frac{1-\gamma}{\gamma}\right)^{\gamma} a\left(k_t^w\right)^{\gamma} \qquad (3.4.11)$$

$$c_{ot+1}^{j} = \left[1+\left(\frac{1-\gamma}{\gamma}\right)^{\gamma-1} a\left(k_{t+1}^w\right)^{\gamma-1}\right]\varphi^{j}\left(\frac{1-\gamma}{\gamma}\right)^{\gamma} a\left(k_t^w\right)^{\gamma} \qquad (3.4.12)$$

$$s_t^{j} = \varphi^{j}\left(\frac{1-\gamma}{\gamma}\right)^{\gamma} a\left(k_t^w\right)^{\gamma} \qquad (3.4.13)$$

Then, the law of motion under free trade is characterized by

$$(1+n)k_{t+1}^{j} = \varphi^{j} w_t\,(k_t^{w})$$

$$k_{t+1}^{j} = \frac{\varphi^{j}}{(1+n^{j})}\left(\frac{1-\gamma}{\gamma}\right)^{\gamma} a\left(k_t^w\right)^{\gamma} \qquad (3.4.14)$$

The capital stock per worker of small economy will move along with the capital stock per worker of the integrated economy. Since the world capital stock is followed by the law of motion, $k_{t+1}^{w} = \frac{\varphi}{(1+n)}\left(\frac{1-\gamma}{\gamma}\right)^{\gamma} a\left(k_t^w\right)^{\gamma}$, then the relative capital stock per worker of country j to that of integrated economy is presented as

$$\frac{k_{t+1}^{j}}{k_{t+1}^{w}} = \frac{\varphi^{j}/(1+n^{j})}{\varphi/(1+n)} \qquad (3.4.15)$$

From equation (3.4.15), the relative capital-labor ratio of a particular country to the world capital-labor ratio is also the relative specific rate of saving and dependency ratio of that country to the world average's. To disaggregate the country form the integrated world, the similar method in Bajona and Kehoe (2006) has been applied. Then, the difference between the level of income per worker in a given country in period *t+1*, $y_{t+1}^{j} = w_{t+1} + r_{t+1}k_{t+1}^{j}$, to the world's average , $y_{t+1}^{w} = w_{t+1} + r_{t+1}k_{t+1}^{w}$, is followed

$$y^j_{t+1} - y^w_{t+1} = r_{t+1}(k^j_{t+1} - k^w_{t+1}) \,. \qquad (3.4.16)$$

Since the difference between capital-labor ratio in a given country to the world's average is

$$\begin{aligned} k^j_{t+1} - k^w_{t+1} &= \frac{\varphi^j}{(1+n^j)} w(k_t) - \frac{\varphi}{(1+n)} w(k_t) \\ &= \left[\frac{\varphi^j}{(1+n^j)} - \frac{\varphi}{(1+n)} \right] w(k_t) \end{aligned} , \qquad (3.4.17)$$

then, equation (3.4.16) becomes

$$y^j_{t+1} - y^w_{t+1} = r(k_{t+1}) w(k_t) \left[\frac{\varphi^j}{(1+n^j)} - \frac{\varphi}{(1+n)} \right].$$

To compare the difference with the world average's income level, divide the above equation with y^w_{t+1}; that is,

$$\frac{y^j_{t+1} - y^w_{t+1}}{y^w_{t+1}} = \frac{r(k_{t+1}) w(k_t)}{y^w_{t+1}} \left[\frac{\varphi^j}{(1+n^j)} - \frac{\varphi}{(1+n)} \right].$$

Since $\frac{r(k^w_{t+1})}{y^w_{t+1}} = \frac{(1-\gamma)^{\gamma-1} \gamma^{1-\gamma} a(k^w_{t+1})^{\gamma-1}}{(1-\gamma)^{\gamma-1} \gamma^{-\gamma} a(k^w_{t+1})^{\gamma}} = \frac{\gamma}{k^w_{t+1}}$, then

$$\frac{y^j_{t+1} - y^w_{t+1}}{y^w_{t+1}} = \frac{\gamma w(k^w_t)}{k^w_{t+1}} \left[\frac{\varphi^j}{(1+n^j)} - \frac{\varphi}{(1+n)} \right]. \qquad (3.4.18)$$

Thus, the difference between the level of income per worker in a given country in period *t+1* to the world's average depends on the country's relative of a dependency ratio, $\frac{1}{(1+n^j)}$, and saving rate, φ^j, to the world's.

b) Convergence under free trade

Since the dynamics of each country depend crucially on the saving rate and growth rate, as well as its initial conditions. From equation (3.4.15), it can be concluded that an increase in saving rate will lead to an increase in next period relative capital-labor ratio in that country. Similarly, an increase in dependency ratio – in other words, smaller population growth rate – will lead to an increase in next period relative capital-labor ratio in that country. Thus, if saving rate and dependency ratio of any particular country increase, then that country would face a fast growing economy and speedy expansion of capital stock compared to the world's capital stock. In this case, country that is previously labor abundant, but recently increases its saving rate and dependency ratio can move faster and the difference of capital-labor ratio to the world's capital-labor ratio would lessen, according to equation (3.4.17).

In addition, from equation (3.4.18), the difference between the level of income per worker in a given country in period *t+1* to the average world's income level will reduce if the less capital abundant country increases its saving rate and dependency ratio. Thus, the convergence can be occurred.

c) Rybcynski revisited

From sub-section b), a rise in saving rate will lead to an increase in next period relative capital-labor ratio in that country. As a result, the production of intermediate good 2 that relatively more capital intensive will increase while the production of intermediate good 1 that is relatively less capital intensive will decrease. Hence, country that accumulates more capital will finally become capital intensive economy and export more capital intensive goods as can be seen from Korea and Singapore.

In conclusion, the small economy model can describe the convergence in an open economy through trade and capital accumulation by comparable to the world resources and dynamics. Countries whose saving rate is increase overtime, such as Korea and Singapore, can catch up with the developed countries whose saving rate are almost constant during the time. However, if the rate does not increase high enough, there will be a club convergence, resulting in different steady states of income levels. In the real world that there is a rising trend in the dependency ratio, especially in developed countries such as Japan, this trend makes developed countries remain in high capital-labor ratio compared to the world average according to the model. Countries that face both increases in the saving rate and the dependency ratio contribute to very high economic growth. China is also in groups which the saving rate and the dependency ratio is increasing overtime. It still needs to see what would happen in the near future.