APPENDIX D

SOLVING LINEAR RATIONAL EXPECTATION AND THE KALMAN FILTER

D.1 Writing Equations as Linear Rational Expectation System.

Linear rational expectations system (LRE System) as:

$$0 = Ax_t + Bx_{t-1} + Cy_t + Dz_t \tag{D1.1}$$

$$0 = FE_t\left(x_{t+1}\right) + Gx_t + Hx_{t-1} + JE_t\left(y_{t+1}\right) + Ky_t + LE_t\left(z_{t+1}\right) + Mz_t \tag{D1.2}$$

$$E_t\left(z_{t+1}\right) = Nz_t + E_t\left(\xi_{t+1}\right), E_t\left(\xi_{t+1}\right) = 0 \tag{D1.3}$$

for $t = 0,1,2,...$

We can write all equations in terms of the linear rational expectations system as following:

NON-EXPECTATIONAL EQUATIONS (6.1):

$$0 = -\psi_t - \left[\left(1-\alpha\right)s_t + q_t\right] \tag{D1.4}$$

$$0 = -\left(s_t - s_{t-1}\right) + \pi_{F,t} - \pi_{H,t} + \varepsilon_t^s \tag{D1.5}$$

$$0 = -\pi_t + \left(1-\alpha\right)\pi_{H,t} + \alpha\pi_{F,t} \tag{D1.6}$$

$$0 = -mc_t + \frac{\sigma}{1-h}\left(c_t - hc_{t-1}\right) + \varphi y_t + \alpha s_t - \left(1+\varphi\right)a_t \tag{D1.7}$$

$$0 = -y_t + \left(2-\alpha\right)\alpha\eta s_t + \left(1-\alpha\right)c_t + \alpha\eta\psi_t + \alpha y_t^* \tag{D1.8}$$

$$0 = -r_t + \rho_r r_{t-1} + \left(1-\rho_r\right)\left(\phi_1\pi_t + \phi_2 y_t\right) + \varepsilon^{r_t} \tag{D1.10}$$

$$0 = -y_t^* + \lambda_1 y_{t-1}^* + \varepsilon_t^{y^*} \tag{D1.11}$$

$$0 = -r_t^* + \rho_{r*} r_{t-1}^* + \varepsilon_t^{r*} \tag{D1.12}$$

EXPECTATIONAL EQUATIONS (6.2):

$$0 = E_t\left\{q_{t+1} - q_t - r_t^* + r_t - \pi_{t+1} + \varepsilon_t^q\right\} \tag{D1.13}$$

$$0 = E_t\left\{-\pi_{F,t} + \beta\left(1-\beta\theta_F\right)\pi_{F,t+1} + \theta_F\pi_{F,t-1} + \lambda_F\psi_t + \varepsilon_t^{\pi_F}\right\} \tag{D1.14}$$

$$0 = E_t\left\{-\pi_{H,t} + \beta\left(1-\beta\theta_H\right)\pi_{H,t+1} + \theta_H\pi_{H,t-1} + \lambda_H mc_t + \varepsilon_t^{\pi_H}\right\} \tag{D1.15}$$

$$0 = E_t\left\{c_{t+1} - hc_t - \frac{(1-h)}{\sigma}r_t + \frac{(1-h)}{\sigma}\pi_{t+1} - y_t^* + hy_{t-1}^* + \frac{1-h}{\sigma}q_t\right\} \tag{D1.16}$$

EXOGENOUS EQUATIONS (6.3):

$$a_{t+1} = \rho_a a_t + \nu_{t+1}^a \quad \text{with} \quad E_t\left(\nu_{t+1}^a\right) = 0 \tag{D1.17}$$

$$\varepsilon_{t+1}^s = 0\varepsilon_t^s + \nu_{t+1}^s \quad \text{with} \quad E_t\left(\nu_{t+1}^s\right) = 0 \tag{D1.18}$$

$$\varepsilon_{t+1}^q = 0\varepsilon_t^q + \nu_{t+1}^q \quad \text{with} \quad E_t\left(\nu_{t+1}^q\right) = 0 \tag{D1.19}$$

$$\varepsilon_{t+1}^{\pi_H} = 0\varepsilon_t^{\pi_H} + \nu_{t+1}^{\pi_H} \quad \text{with} \quad E_t\left(\nu_{t+1}^{\pi_H}\right) = 0 \tag{D1.20}$$

$$\varepsilon_{t+1}^{\pi_F} = 0\varepsilon_t^{\pi_F} + \nu_{t+1}^{\pi_F} \quad \text{with} \quad E_t\left(\nu_{t+1}^{\pi_F}\right) = 0 \tag{D1.21}$$

$$\varepsilon_{t+1}^r = 0\varepsilon_t^r + \nu_{t+1}^r \quad \text{with} \quad E_t\left(\nu_{t+1}^r\right) = 0 \tag{D1.22}$$

$$\varepsilon_{t+1}^{y^*} = 0\varepsilon_t^{y^*} + \nu_{t+1}^{y^*} \quad \text{with} \quad E_t\left(\nu_{t+1}^{y^*}\right) = 0 \tag{D1.23}$$

$$\varepsilon_{t+1}^{r^*} = 0\varepsilon_t^{r^*} + \nu_{t+1}^{r^*} \quad \text{with} \quad E_t\left(\nu_{t+1}^{r^*}\right) = 0 \tag{D1.24}$$

for all t and every ν_{t+1} with $E_t\left(\nu_{t+1}\right) = 0$

The vector x_t is the endogenous state vector, y_t is the endogenous vector of unobservable variables (control variable) and z_t is the exogenous stochastic process.

It can descript simply that in equation (D1.1) there are non-expectational equations, in our model, equation (5.1), (5.2), (5.4), (5.7), (5.11), (5.12), (5.13) and (5.14). In equation (D1.2) there are expectational equations, in our model, equation (5.3), (5.5), (5.6), (5.9), (5.10). The equation (D1.33) is for exogenous equations connected to the shocks and innovation in the model with respect to the restriction of $E_t\left(\xi_{t+1}\right) = 0$.

The vector x_t is the endogenous state vector, y_t is the endogenous vector of unobservable variables (control variable) and z_t is the exogenous stochastic process. The matrices of system $A_{8\times10}$, $B_{8\times10}$, $C_{8\times2}$, $D_{8\times8}$, $F_{4\times10}$, $G_{4\times10}$, $H_{4\times10}$, $J_{4\times2}$, $K_{4\times2}$, $L_{4\times8}$ $M_{4\times8}$ and $N_{8\times8}$.

$$A=\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & -(1-\alpha) & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & (2-\alpha)\alpha\eta & 1-\alpha & 0 & \alpha & 0 \\ (1-\rho_r)\phi_2 & 0 & -1 & (1-\rho_r)\phi_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ \varphi & 0 & 0 & 0 & 0 & \alpha & \frac{\sigma}{1-h} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & \alpha & 0 & 0 & 0 & 0 & 1-\alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

$$B=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_r & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-\sigma h}{1-h} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_{r^*} & 0 & 0 \end{bmatrix}$$

$$C=\begin{bmatrix} -1 & 0 \\ \alpha\eta & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$D=\begin{bmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&1&0&0\\0&1&0&0&0&0&0&0\\-(1+\varphi)&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix}$$

$$F=\begin{bmatrix}0&1&0&-1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&\beta(1-\theta_H)\\0&0&0&0&\beta(1-\theta_F)&0&0&0&0&0\\0&0&0&\frac{1-h}{\sigma}&0&0&1&0&0&0\end{bmatrix}$$

$$G=\begin{bmatrix}0&-1&1&0&0&0&0&-1&0&0\\0&0&0&0&0&0&0&0&0&-1\\0&0&0&0&-1&0&0&0&0&0\\0&\frac{1-h}{\sigma}&\frac{-(1-h)}{\sigma}&0&0&0&-h&0&-1&0\end{bmatrix}$$

$$H=\begin{bmatrix}0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&\theta_H\\0&0&0&0&\theta_F&0&0&0&0&0\\0&0&0&0&0&0&0&0&h&0\end{bmatrix}$$

$$K=\begin{bmatrix}0&0\\0&\frac{(1-\beta\theta_H)(1-\theta_H)}{\theta_H}\\\frac{(1-\beta\theta_F)(1-\theta_F)}{\theta_F}&0\\0&0\end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N = \begin{bmatrix} \rho_a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix J and L are matrices of zeros.

What one is looking for is the recursive equilibrium law of motion as Uhlig (1995)

$$x_t = Px_{t-1} + Qz_t$$
$$y_t = Rx_{t-1} + Sz_t$$
$$z_t = Nz_{t-1} + \varepsilon_t$$

In our model, we can set it as following

$$x_t = \left\{ y_t, q_t, r_t, \pi_t, \pi_{F,t}, s_t, c_t, r_t^*, y_t^*, \pi_{H,t} \right\}$$

$$y_t = \left\{ \psi_t, mc_t \right\}$$

$$z_t = \left\{ a_t, \varepsilon_t^s, \varepsilon_t^q, \varepsilon_t^{\pi_H}, \varepsilon_t^{\pi_F}, \varepsilon_t^r, \varepsilon_t^{y^*}, \varepsilon_t^{r^*} \right\}$$

Next, how we can get the structural parameter (P, Q, R, S) is a routine of DYNARE for solving a matrix quadratic equation.

D.2 An Alternative Expression for The Likelihood Function

Let's suppose the full set of observations in a (Tx1) vector.

$$\boldsymbol{y} = \left(y1, y2, ..., yT\right)'$$

This vector could be viewed as a single realization from a T-dimensional Gaussian distribution. The mean of this (Tx1) vector is

$$\begin{bmatrix} E(Y_1) \\ E(Y_2) \\ \vdots \\ E(Y_T) \end{bmatrix} = \begin{bmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{bmatrix},$$

or it could be written as

$$E(\boldsymbol{Y}) = \boldsymbol{\mu},$$

where μ denotes the (Tx1). The variance-covariance matrix of Y is given by

$$E\left[(\boldsymbol{Y} - \boldsymbol{u})(\boldsymbol{Y} - \boldsymbol{u})'\right] = \boldsymbol{\Omega}.$$

Viewing the observed sample of y as a single draw from a $N(\mu, \Omega)$ distribution, the sample likelihood could be written down immediately from the formula for the multivariate Gaussian density:

$$f_Y(\boldsymbol{y}; \boldsymbol{\theta}) = (2\pi)^{-T/2} \left|\boldsymbol{\Omega}^{-1}\right|^{1/2} \exp\left[-\frac{1}{2}(\boldsymbol{y} - \boldsymbol{\mu})' \boldsymbol{\Omega}^{-1} (\boldsymbol{y} - \boldsymbol{\mu})\right],$$

with log likelihood Hamilton (1994)

$$L(\boldsymbol{\theta}) = (-T/2)\log(2\pi) + \frac{1}{2}\log\left|\boldsymbol{\Omega}^{-1}\right| - \frac{1}{2}(\boldsymbol{y} - \boldsymbol{\mu})' \boldsymbol{\Omega}^{-1} (\boldsymbol{y} - \boldsymbol{\mu});$$

or in terms of innovations representation:

$$L(\boldsymbol{\theta}) = (-T/2)\log(2\pi) + \frac{1}{2}\log\left|\boldsymbol{\Omega}^{-1}\right| - \frac{1}{2}\boldsymbol{v}'\boldsymbol{\Omega}^{-1}\boldsymbol{v}$$

D.3 From the Kalman Filter to the Formulation of Likelihood Function

The state-space can be represented as

$$
\begin{aligned}
s_{t+1} &= \Gamma_1 s_t + \Gamma_2 w_{t+1} && \text{(State Eq.)} \\
Y_t &= \Lambda s_t + \nu_t && \text{(Measurement Eq).} \\
\nu_t &= D\nu_{t-1} + \eta_t
\end{aligned}
$$

where D is a matrix whose eigenvalues are strictly below unity in modulus and $En_t n_t' = R$ and $Ew_{t+1}n_s' = 0$ (No serially correlated shocks) for all t and s.

It can be useful to transform the observed vector as a quasi-differenced process:

$$\bar{Y}_t \equiv Y_{t+1} - DY_t$$

Using measurement equation at $t+1$ and the definition of state equation, it follows that

$$\bar{Y}_t \equiv \left(\Lambda\Gamma_1 - D\Lambda\right) s_t + \Lambda\Gamma_2 w_{t+1} + \eta_{t+1}$$

Thus, $\left(s_t, \bar{Y}_t\right)$ is governed by the state space system

$$
\begin{aligned}
s_{t+1} &= \Gamma_1 s_t + \Gamma_2 w_{t+1} \\
\bar{Y}_t &= \bar{\Lambda} s_t + \Lambda\Gamma_2 w_{t+1} + \eta_{t+1}
\end{aligned},
$$

where $\bar{\Lambda} = \left(\Lambda\Gamma_1 - D\Lambda\right)$

By applying the Kalman filter, we can obtain a gain sequence K_t with which to construct the associated innovations representation (proof in Hansen and Sargent, 2005: chapter 9)

$$
\begin{aligned}
\hat{s}_{t+1} &= A^0 \hat{s}_t + K_t u_t \\
\bar{Y}_t &= \bar{\Lambda}\hat{s}_t + u_t
\end{aligned}
$$

D.4 Recursive Formulation of Likelihood Function

The likelihood function $\left\{y_s\right\}_{s=0}^{T}$ is defined as the density $f\left(y_T, y_{T-1}, \dots, y_0\right)$. It is convenient to factor the likelihood function

$$f\left(y_T, y_{T-1}, \dots, y_0\right) = f_T\left(y_T \left| y_{T-1}, \dots, y_0\right.\right) f_{T-1}\left(y_{T-1} \left| y_{T-2}, \dots, y_0\right.\right) \dots f_1\left(y_1 \left| y_0\right.\right) f_0\left(y_0\right)$$

The Gaussian likelihood function for an $n \times 1$ random vector y with mean μ and covariance matrix V is

$$N(\mu, V) = 2\pi^{-n/2} |V|^{-1/2} \exp\left(-\frac{1}{2}(y-\mu)' V^{-1} (y-\mu) \right)$$

Therefore, the above series of u_t and the metrix Ω_t are use to construct the logarithm of the likelihood function $Y \sim N(\nu, \Omega)$. This is given by (Hansen and Sargent: 2005)

$$\log L\left(Y^T \,|\Theta\right) = \frac{1}{2}\sum_{t-1}^{T}\left[N\log 2\pi + \log\left|\Omega_{t|t-1}\right| + \sum_{t-1}^{T}\nu_t'\left|\Omega_{t|t-1}^{-1}\right|\nu_t \right]$$

where:

$$\Theta = \{\Gamma_1, \Gamma_2, \Lambda, \Xi, \Upsilon\}$$
$$\Omega_{t|t-1} = \Lambda' \sum\nolimits_{t|t-1} \Lambda + \Upsilon$$
$$\sum\nolimits_{t|t-1} = \Gamma_1 \sum\nolimits_{t-1|t-1} \Gamma_1' + \Gamma_2 \Xi \Gamma_2'$$

For the sequence of draws holds:

$$\{\theta^j\}_1^N \sim p\left(\theta \middle| Y^T\right)$$

and then it is used the law of large numbers:

$$E_\theta\left(g(\theta)\middle|Y^T\right) = \frac{1}{N}\sum_{j=1}^{N} g\left(\theta^j\right)$$

where g(.) is a suitable function

The sequence of posterior draws $\{\theta^j\}_1^N$ used in the law of the large numbers is obtained using Markov Chain which is generated by the Monte Carlo method (MCMC).