ภาคผนวก

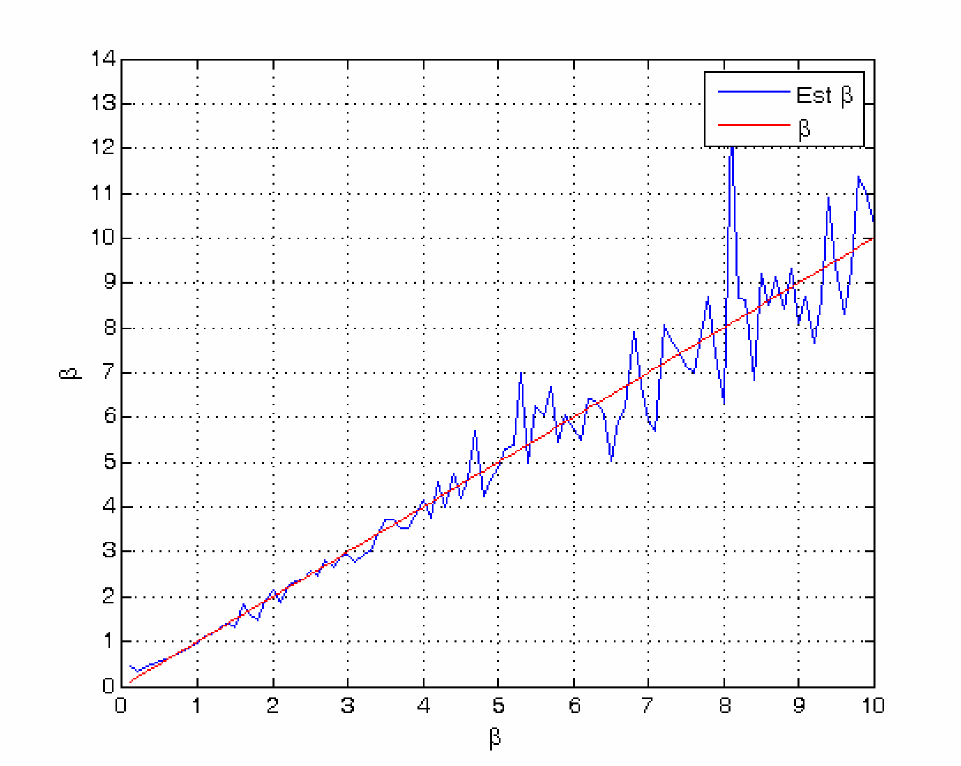
ภาคผนวก ก.

การประมาณค่าพารามิเตอร์ของแบบจำลองทางสถิติแบบเกาส์เซียนทั่วไป
และแบบแอลฟ่าสเตเบิลที่สมมาตร

ก.1 ผลการประมาณพารามิเตอร์เบต้า (β) ของการแจกแจงแบบเกาส์เซียนทั่วไป

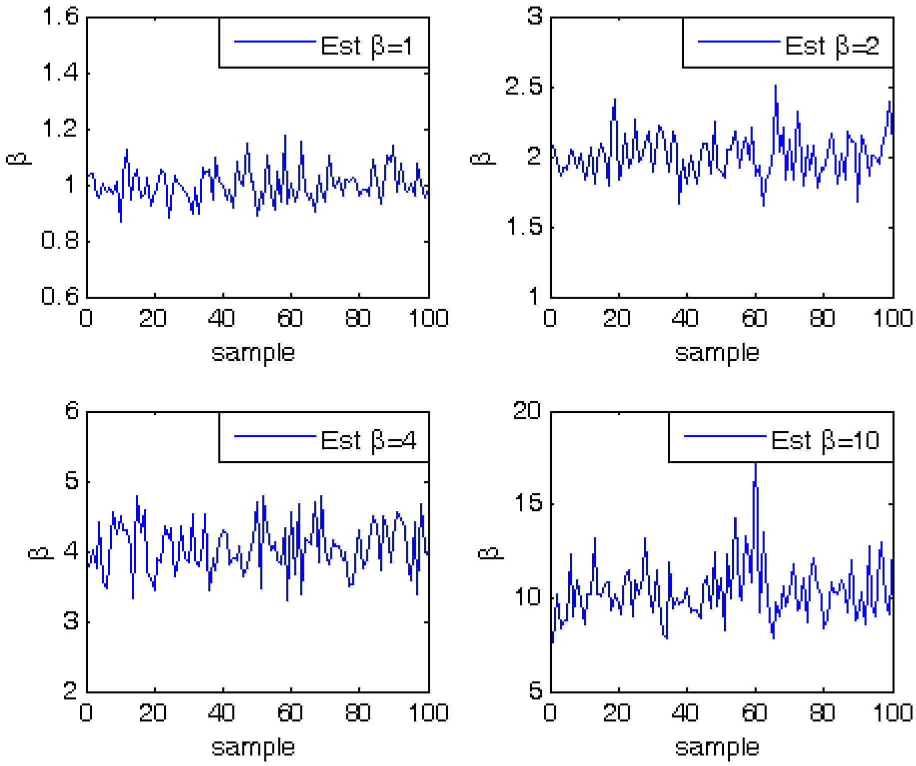
งานวิจัยนี้ประมาณค่าพารามิเตอร์เบต้าด้วยวิธี method of moment estimator (MME) ดังสมการที่ (2.5)-(2.7) ในบทที่ 2

การทดลองนี้ได้สร้างสัญญาณแบบเกาส์เซียนทั่วไปที่มีค่า β ตั้ง 0.1 จนถึง 10 แต่ละค่า β มีจำนวนตัวอย่างเท่ากับ 1000 แล้วคำนวณค่า β ที่ค่าต่าง ๆ จะได้ผลการทดลองดังรูปที่ ก.1



รูปที่ ก.1 ผลการประมาณค่าเบต้าด้วยวิธี MME สำหรับค่าเบต้าในช่วง (0.1-10.0)

หลังจากนั้น สร้างสัญญาณแบบเกาส์เซียนทั่วไปที่ค่า β = 1, 2, 4 และ 10 โดยที่แต่ละค่าของ β จะสร้างสัญญาณแบบเกาส์เซียนทั่วไปจำนวน 100 ชุด และทำการประมาณค่า โดยมีผลการประมาณค่าดังในรูปที่ ก.2



รูปที่ ก.2 ผลการประมาณค่า β = 1, 2, 4 และ 10 จำนวน 100 ครั้ง

ค่าความถูกต้องในการประมาณค่าถูกแสดงอยู่ในรูปของค่าเฉลี่ยและส่วนเบี่ยงเบนมาตราฐาน (μ,σ) จากกระประมาณค่า β จำนวน 1,000 ครั้ง เมื่อจำนวนตัวอย่างเท่ากับ 5,000, 1,000 และ400 ดังตาราง

ตารางที่ ก. 1 ผลการประมาณค่าเบต้า เมื่อ β = 1, 2, 4 และ 10 จำนวน 1000 ครั้ง

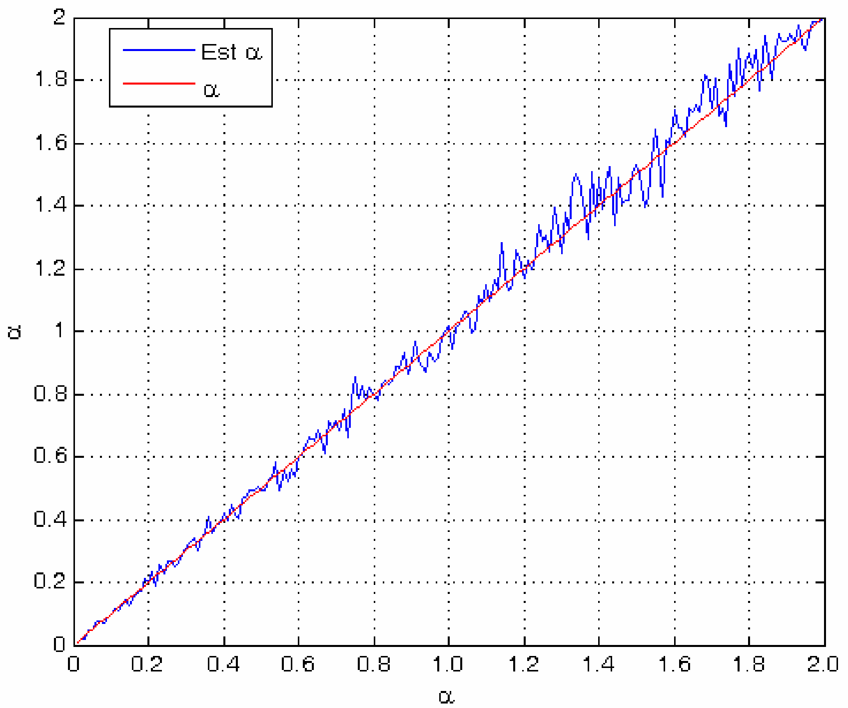
จำนวนตัวอย่าง	β = 1	β = 2	β = 4	β = 10
5,000	(1.0012,0.0273)	(2.0018,0.0647)	(4.0076,0.1651)	(10.0207,0.5804)
1,000	(1.0021,0.0629)	(2.0114,0.1432)	(4.0438,0.3980)	(10.0823,1.4310)
400	(1.0166,0.0942)	(2.0335,0.2399)	(4.0529,0.6254)	(10.2477,2.6314)

สรุปได้ว่าการประมาณค่าแบบ MME จะให้ผลมีความถูกต้องสำหรับค่า β ในช่วง 0.5-4.5 และเมื่อค่า β มีค่าสูงเกิน 4.5 พบว่ามีความคลาดเคลื่อนสูงขึ้นเรื่อย ๆ ดังในรูปที่ ก.1, ก.2 ซึ่งจะมีผลในขั้นตอนการเปรียบเทียบฟังก์ชันการแจกแจงความน่าจะเป็นด้วย KLD ดังนั้นในงานวิจัยนี้จึงใช้ค่า β ในช่วง 0.5-4.5

ก.2 ผลการประมาณพารามิเตอร์แอลฟ่า (α) ของการแจกแจงแบบแอลฟ่าสเตเบิลที่สมมาตร

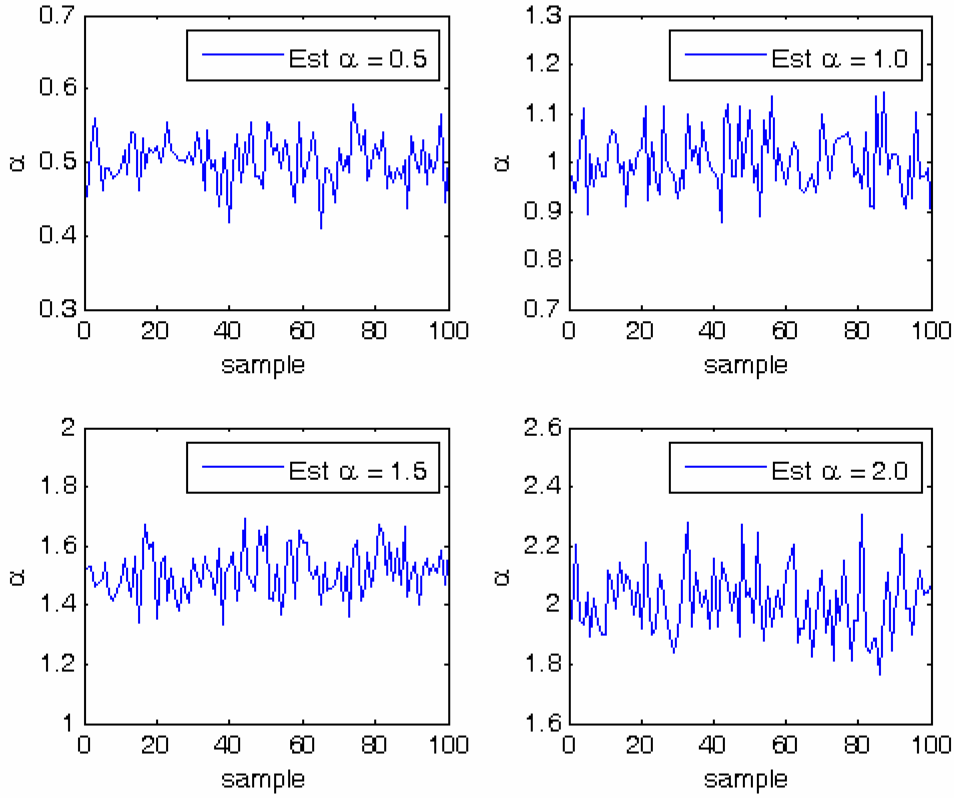
งานวิจัยนี้ประมาณค่าพารามิเตอร์แอลฟ่าด้วยวิธี method of sample characteristic function ดังสมการที่ (2.10)-(2.14) ในบทที่ 2

การทดลองนี้ได้สร้างสัญญาณแบบแอลฟ่าสเตเบิลที่สมมาตรที่มีค่า α ตั้ง 0.1 จนถึง 2 แต่ละค่า α มีจำนวนตัวอย่างเท่ากับ 1,000 แล้วประมาณค่า α ที่ค่าต่าง ๆ จะได้ผลการประมาณค่าดังรูปที่ ก.3



รูปที่ ก.3 แสดงผลการประมาณค่าแอลฟ่าด้วยวิธี method of sample characteristic function สำหรับค่าแอลฟ่าในช่วง (0.1-2.0)

หลังจากนั้น สร้างสัญญาณแบบแอลฟ่าสเตเบิลที่สมมาตรที่ค่า α = 0.5, 1.0, 1.5 และ 2.0 โดยที่แต่ละค่าของ α จะสร้างสัญญาณแบบแอลฟ่าสเตเบิลที่สมมาตรจำนวน 100 ชุด และทำการประมาณค่า โดยมีผลการประมาณค่าดังในรูปที่ ก.4



รูปที่ ก.4 แสดงการประมาณค่า α = 0.5, 1.0, 1.5 และ 2.0 จำนวน 100 ครั้ง

ค่าความถูกต้องในการประมาณค่าถูกแสดงอยู่ในรูปของค่าเฉลี่ยและส่วนเบี่ยงเบนมาตราฐาน (μ,σ) จากกระประมาณค่า α จำนวน 1,000 ครั้ง เมื่อจำนวนตัวอย่างมีค่า 5,000, 1,000 และ 400 ดังตาราง โดยแบ่งเป็นสองกลุ่มกลุ่มแรกมีค่า γ = 10 และกลุ่มที่สอง γ = 50 ดังตาราง

ตารางที่ ก.2 แสดงผลการประมาณค่าแอลฟ่าสำหรับ γ=10 จำนวน 1000 ครั้ง

จำนวนตัวอย่าง	α = 0.5	α = 1.0	α = 1.5	α = 2.0
5,000	(0.4971,0.0281)	(1.0003,0.0340)	(1.5033,0.0926)	(2.0000,0.0001)
1,000	(0.5025,0.0631)	(1.0053,0.0847)	(1.5152,0.1738)	(2.0000,0.0002)
400	(0.4911,0.0939)	(1.0301,0.1334)	(1.5254,0.3014)	(2.0000,0.0002)

ตารางที่ ก.3 แสดงผลการประมาณค่าแอลฟ่าสำหรับ γ =50 จำนวน 1,000 ครั้ง

จำนวนตัวอย่าง	α = 0.5	α = 1.0	α = 1.5	α = 2.0
5,000	(0.4935,0.1066)	(0.9995,0.0276)	(1.5021,0.0374)	(2.0000,0.0003)
1,000	(0.4573,0.2417)	(0.9928,0.5603)	(1.5117,0.1071)	(2.0000,0.0006)
400	(0.4776,0.3596)	(1.0015,0.1115)	(1.5566,0.1919)	(2.0000,0.0006)

สรุปได้ว่าการประมาณค่าแอลฟ่าด้วยวิธี method of sample characteristic function จะให้ผลมีความถูกต้องสำหรับค่า γ ในช่วง (1.0-50) สำหรับเมื่อค่า γ > 50 สัญญาณมีส่วนเบี่ยงเบนมาตราฐานที่สูงทำให้มีความคลาดเคลื่อนในการประมาณค่า α ซึ่งจะมีผลทำให้การประมาณค่า α ในช่วง (0.1-1.0) มีผลคลาดเคลื่อนดังตารางที่ ก.3 ดังนั้นในงานวิจัยนี้จึงใช้ค่า α ในช่วง (1.0-2.0)

ภาคผนวก ข.

การเปรียบเทียบฟังก์ชันการแจกแจงความน่าแบบเกาส์เซียนหลายตัวแปร ด้วย Kullback-Leibler Divergence

ข.1 ฟังก์ชันการแจกแจงความน่าจะเป็นแบบเกาส์เซียนหลายตัวแปร

$$f(\mathbf{X}) = \frac{1}{(2\pi)^{N/2} |\Sigma|^{1/2}} \exp\left(-\frac{1}{2} (\mathbf{X} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{X} - \boldsymbol{\mu}) \right) \quad (ข.1)$$

สมการของ Kullback-Leibler Divergence

$$D\left(f_1(\mathbf{X}) \,\|\, f_2(\mathbf{X}) \right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_1(\mathbf{X}) \ln \frac{f_1(\mathbf{X})}{f_2(\mathbf{X})} dx_1 dx_2 \cdots dx_N \quad (ข.2)$$

$$= \mathbb{E}_{f_1} \left\{ \ln \frac{f_1(x_1, ..., x_N)}{f_2(x_1, ..., x_N)} \right\} \quad (ข.3)$$

$$= \mathbb{E}_{f_1} \left\{ \ln \frac{(2\pi)^{N/2} |\Sigma_2|^{1/2}}{(2\pi)^{N/2} |\Sigma_1|^{1/2}} \frac{e^{-\frac{1}{2}(\mathbf{X}-\boldsymbol{\mu}_1)^T \Sigma_1^{-1} (\mathbf{X}-\boldsymbol{\mu}_1)}}{e^{-\frac{1}{2}(\mathbf{X}-\boldsymbol{\mu}_2)^T \Sigma_2^{-1} (\mathbf{X}-\boldsymbol{\mu}_2)}} \right\} \quad (ข.4)$$

$$= \mathbb{E}_{f_1} \left\{ \ln \frac{|\Sigma_2|^{1/2}}{|\Sigma_1|^{1/2}} + \ln \frac{e^{-\frac{1}{2}(\mathbf{X}-\boldsymbol{\mu}_1)^T \Sigma_1^{-1} (\mathbf{X}-\boldsymbol{\mu}_1)}}{e^{-\frac{1}{2}(\mathbf{X}-\boldsymbol{\mu}_2)^T \Sigma_2^{-1} (\mathbf{X}-\boldsymbol{\mu}_2)}} \right\} \quad (ข.5)$$

$$= \ln \frac{|\Sigma_2|^{1/2}}{|\Sigma_1|^{1/2}} + \mathbb{E}_{f_1} \left\{ \ln e^{-\frac{1}{2}(\mathbf{X}-\boldsymbol{\mu}_1)^T \Sigma_1^{-1} (\mathbf{X}-\boldsymbol{\mu}_1)} - \ln e^{-\frac{1}{2}(\mathbf{X}-\boldsymbol{\mu}_2)^T \Sigma_2^{-1} (\mathbf{X}-\boldsymbol{\mu}_2)} \right\} \quad (ข.6)$$

$$= \frac{1}{2} \ln \frac{|\Sigma_2|}{|\Sigma_1|} + \mathbb{E}_{f_1} \left\{ -\frac{1}{2} (\mathbf{X} - \boldsymbol{\mu}_1)^T \Sigma_1^{-1} (\mathbf{X} - \boldsymbol{\mu}_1) + \frac{1}{2} (\mathbf{X} - \boldsymbol{\mu}_2)^T \Sigma_2^{-1} (\mathbf{X} - \boldsymbol{\mu}_2) \right\} \quad (ข.7)$$

$$= \frac{1}{2} \ln \frac{|\Sigma_2|}{|\Sigma_1|} - \frac{1}{2} \mathbb{E}_{f_1} \left\{ (\mathbf{X} - \boldsymbol{\mu}_1)^T \Sigma_1^{-1} (\mathbf{X} - \boldsymbol{\mu}_1) \right\} + \frac{1}{2} \mathbb{E}_{f_1} \left\{ (\mathbf{X} - \boldsymbol{\mu}_2)^T \Sigma_2^{-1} (\mathbf{X} - \boldsymbol{\mu}_2) \right\} \quad (ข.8)$$

พิจารณาพจน์ $\mathbb{E}_{f_1}\left\{\left(\mathbf{X}-\boldsymbol{\mu}_1\right)^T \Sigma_1^{-1}\left(\mathbf{X}-\boldsymbol{\mu}_1\right)\right\}$

กำหนดให้ $K_1^T K_1 = \Sigma_1^{-1}$

$$\mathbb{E}_{f_1}\left\{\left(\mathbf{X}-\boldsymbol{\mu}_1\right)^T \Sigma_1^{-1}\left(\mathbf{X}-\boldsymbol{\mu}_1\right)\right\} = \mathbb{E}_{f_1}\left\{\left(\mathbf{X}-\boldsymbol{\mu}_1\right)^T K_1^T K_1\left(\mathbf{X}-\boldsymbol{\mu}_1\right)\right\} \quad \text{(ข.9)}$$

$$= \mathbb{E}_{f_1}\left\{\left(\mathbf{X}-\boldsymbol{\mu}_1\right)^T K_1^T K_1\left(\mathbf{X}-\boldsymbol{\mu}_1\right)\right\} \quad \text{(ข.10)}$$

$$= tr\left(\mathbb{E}_{f_1}\left\{K_1\left(\mathbf{X}-\boldsymbol{\mu}_1\right)\left(\mathbf{X}-\boldsymbol{\mu}_1\right)^T K_1^T\right\}\right) \quad \text{(ข.11)}$$

$$= tr\left(K_1\mathbb{E}_{f_1}\left\{\left(\mathbf{X}-\boldsymbol{\mu}_1\right)\left(\mathbf{X}-\boldsymbol{\mu}_1\right)^T\right\} K_1^T\right) \quad \text{(ข.12)}$$

$$= tr\left(K_1\Sigma_1 K_1^T\right) \quad \text{(ข.13)}$$

จาก $tr\left(AB\right) = tr\left(BA\right)$ (ข.14)

$$= tr\left(\Sigma_1 K_1^T K_1\right) \quad \text{(ข.15)}$$

$$= tr\left(\Sigma_1\Sigma_1^{-1}\right) \quad \text{(ข.16)}$$

$$= tr\left(I\right) \quad \text{(ข.17)}$$

$$= N \quad \text{(ข.18)}$$

พิจารณาพจน์ $\mathbb{E}_{f_1}\left\{\left(\mathbf{X}-\boldsymbol{\mu}_2\right)^T \Sigma_2^{-1}\left(\mathbf{X}-\boldsymbol{\mu}_2\right)\right\}$

$$\mathbb{E}_{f_1}\left\{\left(\mathbf{X}-\boldsymbol{\mu}_2\right)^T \Sigma_2^{-1}\left(\mathbf{X}-\boldsymbol{\mu}_2\right)\right\} = \mathbb{E}_{f_1}\left\{\left(\mathbf{X}-\boldsymbol{\mu}_2\right)^T \Sigma_2^{-1}\left(\mathbf{X}-\boldsymbol{\mu}_2\right)\right\} \quad \text{(ข.19)}$$

$$= \mathbb{E}_{f_1}\left\{\mathbf{X}^T\Sigma_2^{-1}\mathbf{X} - \mathbf{X}^T\Sigma_2^{-1}\boldsymbol{\mu}_2 - \boldsymbol{\mu}_2^T\Sigma_2^{-1}\mathbf{X} + \boldsymbol{\mu}_2^T\Sigma_2^{-1}\boldsymbol{\mu}_2\right\} \quad \text{(ข.20)}$$

$$= \mathbb{E}_{f_1}\left\{\mathbf{X}^T\Sigma_2^{-1}\mathbf{X}\right\} - \mathbb{E}_{f_1}\left\{\mathbf{X}^T\Sigma_2^{-1}\boldsymbol{\mu}_2\right\} - \mathbb{E}_{f_1}\left\{\boldsymbol{\mu}_2^T\Sigma_2^{-1}\mathbf{X}\right\} + \mathbb{E}_{f_1}\left\{\boldsymbol{\mu}_2^T\Sigma_2^{-1}\boldsymbol{\mu}_2\right\} \quad \text{(ข.21)}$$

$$= \mathbb{E}_{f_1}\left\{\mathbf{X}^T\Sigma_2^{-1}\mathbf{X}\right\} - \boldsymbol{\mu}_1^T\Sigma_2^{-1}\boldsymbol{\mu}_2 - \boldsymbol{\mu}_2^T\Sigma_2^{-1}\mu_1 + \boldsymbol{\mu}_2^T\Sigma_2^{-1}\mu_2 \quad \text{(ข.22)}$$

พิจารณา $\mathbb{E}_{f_1}\left\{\mathbf{X}^T\Sigma_2^{-1}\mathbf{X}\right\}$

$$\mathbb{E}_{f_1}\left\{\mathbf{X}^T\Sigma_2^{-1}\mathbf{X}\right\}=\mathbb{E}_{f_1}\left\{\left(\mathbf{X}-\boldsymbol{\mu}_1\right)^T\Sigma_2^{-1}\left(\mathbf{X}-\boldsymbol{\mu}_1\right)\right\}+\boldsymbol{\mu}_1^T\Sigma_2^{-1}\boldsymbol{\mu}_1 \quad (ข.23)$$

สำหรับพจน์ $\mathbb{E}_{f_1}\left\{\left(\mathbf{X}-\boldsymbol{\mu}_1\right)^T\Sigma_2^{-1}\left(\mathbf{X}-\boldsymbol{\mu}_1\right)\right\}$ กำหนดให้ $\mathbf{Z}=K\left(\mathbf{X}-\boldsymbol{\mu}_1\right)$

$$\mathbb{E}_{f_1}\left\{\left(\mathbf{X}-\boldsymbol{\mu}_1\right)^T\Sigma_2^{-1}\left(\mathbf{X}-\boldsymbol{\mu}_1\right)\right\}=\mathbb{E}_{f_1}\left\{\mathbf{Z}^T\mathbf{Z}\right\} \quad (ข.24)$$

$$=tr\left\{\mathbb{E}\left\{\mathbf{Z}\mathbf{Z}^T\right\}\right\} \quad (ข.25)$$

$$=tr\left\{\mathbb{E}\left\{K_2\left(\mathbf{X}-\boldsymbol{\mu}_1\right)\left(\mathbf{X}-\boldsymbol{\mu}_1\right)^T K_2^T\right\}\right\} \quad (ข.26)$$

$$=tr\left\{K_2\mathbb{E}\left\{\left(\mathbf{X}-\boldsymbol{\mu}_1\right)\left(\mathbf{X}-\boldsymbol{\mu}_1\right)^T\right\}K_2^T\right\} \quad (ข.27)$$

$$=tr\left\{K_2\Sigma_1K_2^T\right\} \quad (ข.28)$$

$$=tr\left\{K_2^TK_2\Sigma_1\right\} \quad (ข.29)$$

$$=tr\left\{\Sigma_2^{-1}\Sigma_1\right\} \quad (ข.30)$$

แทนสมการที่ (ข.22) และ (ข.29) ใน (ข.21)

$$\mathbb{E}_{f_1}\left\{\left(\mathbf{X}-\boldsymbol{\mu}_2\right)^T\Sigma_2^{-1}\left(\mathbf{X}-\boldsymbol{\mu}_2\right)\right\}$$

$$=tr\left\{\Sigma_2^{-1}\Sigma_1\right\}+\boldsymbol{\mu}_1^T\Sigma_2^{-1}\boldsymbol{\mu}_1-\boldsymbol{\mu}_1^T\Sigma_2^{-1}\boldsymbol{\mu}_2-\boldsymbol{\mu}_2^T\Sigma_2^{-1}\boldsymbol{\mu}_1+\boldsymbol{\mu}_2^T\Sigma_2^{-1}\boldsymbol{\mu}_2 \quad (ข.31)$$

$$=tr\left\{\Sigma_2^{-1}\Sigma_1\right\}+\left(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2\right)^T\Sigma_2^{-1}\left(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2\right) \quad (ข.32)$$

แทนสมการที่ (ข.17) และ (ข.31) ใน (ข.8)

$$D\left(f_1\,\|\,f_2\right)=\frac{1}{2}\ln\frac{\left|\Sigma_2\right|}{\left|\Sigma_1\right|}-\frac{1}{2}N+\frac{1}{2}\left(tr\left\{\Sigma_2^{-1}\Sigma_1\right\}+\left(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2\right)^T\Sigma_2^{-1}\left(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2\right)\right) \quad (ข.33)$$

$$=\frac{1}{2}\left(\ln\frac{\left|\Sigma_2\right|}{\left|\Sigma_1\right|}-N+tr\left\{\Sigma_2^{-1}\Sigma_1\right\}+\left(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2\right)^T\Sigma_2^{-1}\left(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2\right)\right) \quad (ข.34)$$

ถ้า $N=1$ จะเป็นการแจกแจงแบบเกาส์เซียนตัวแปรเดียว ซึ่งมี KLD เป็น

$$D\left(f_1(x)\,\|\,f_2(x)\right)=\frac{1}{2}\left(\ln\frac{\sigma_2^2}{\sigma_1^2}-1+\frac{\sigma_1^2}{\sigma_2^2}+\frac{\left(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2\right)^2}{\sigma_2^2}\right) \quad (ข.35)$$

ภาคผนวก ค.

การเปรียบเทียบฟังก์ชันการแจกแจงความน่าแบบเกาส์เซียนทั่วไปหลายตัวแปร ด้วย Kullback-Leibler Divergence

ค.1 ฟังก์ชันการแจกแจงความน่าจะเป็นแบบเกาส์เซียนทั่วไปหลายตัวแปร

$$f(\mathbf{X};\sigma,\beta,\Sigma) = C\exp\left(-\frac{1}{\lambda}\left(\left(\mathbf{X}-\mu\right)^T \Sigma^{-1}\left(\mathbf{X}-\mu\right)\right)^{\beta/2}\right) \qquad (ค.1)$$

เมื่อ $C = \left(\dfrac{\beta}{2\lambda^{\frac{1}{\beta}}\Gamma\left(1/\beta\right)}\right)^N \dfrac{1}{|\Sigma|^{1/2}}$ เป็นค่าที่ทำให้ปริมาตรของการแจกแจงเป็นหนึ่งและ

$$\lambda = \left(\frac{\Gamma\left(1/\beta\right)}{\Gamma\left(3/\beta\right)}\right)^{\beta/2}$$

สมการของ Kullback-Leibler Divergence

$$D\left(f_1(\mathbf{X}) \,\|\, f_2(\mathbf{X})\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_1(\mathbf{X})\ln\frac{f_1(\mathbf{X})}{f_2(\mathbf{X})}dx_1 dx_2\cdots dx_N$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_1(\mathbf{X})\left(\ln\frac{C_1 e^{-\frac{1}{\lambda_1}\left(\left(\mathbf{X}-\mu_1\right)^T\Sigma_1^{-1}\left(\mathbf{X}-\mu_1\right)\right)^{\beta_1/2}}}{C_2 e^{-\frac{1}{\lambda_2}\left(\left(\mathbf{X}-\mu_2\right)^T\Sigma_2^{-1}\left(\mathbf{X}-\mu_2\right)\right)^{\beta_2/2}}}\right)dx_1\cdots dx_N \qquad (ค.2)$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_1(\mathbf{X})\left(\ln\frac{C_1}{C_2} + \ln\frac{e^{-\frac{1}{\lambda_1}\left(\left(\mathbf{X}-\mu_1\right)^T\Sigma_1^{-1}\left(\mathbf{X}-\mu_1\right)\right)^{\beta_1/2}}}{e^{-\frac{1}{\lambda_2}\left(\left(\mathbf{X}-\mu_2\right)^T\Sigma_2^{-1}\left(\mathbf{X}-\mu_2\right)\right)^{\beta_2/2}}}\right)dx_1\cdots dx_N \qquad (ค.3)$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_1(\mathbf{X}) \left(\ln \frac{C_1}{C_2} + \ln e^{-\frac{1}{\lambda_1}\left((\mathbf{X}-\boldsymbol{\mu}_1)^T \Sigma_1^{-1} (\mathbf{X}-\boldsymbol{\mu}_1)\right)^{\beta_1/2}} - \ln e^{-\frac{1}{\lambda_2}\left((\mathbf{X}-\boldsymbol{\mu}_2)^T \Sigma_2^{-1} (\mathbf{X}-\boldsymbol{\mu}_2)\right)^{\beta_2/2}} \right) dx_1 \cdots dx_N$$

(ค.4)

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_1(\mathbf{X}) \left(\ln \frac{C_1}{C_2} - \frac{1}{\lambda_1}\left((\mathbf{X}-\boldsymbol{\mu}_1)^T \Sigma_1^{-1} (\mathbf{X}-\boldsymbol{\mu}_1)\right)^{\beta_1/2} + \frac{1}{\lambda_2}\left((\mathbf{X}-\boldsymbol{\mu}_2)^T \Sigma_2^{-1} (\mathbf{X}-\boldsymbol{\mu}_2)\right)^{\beta_2/2} \right) dx_1 \cdots dx_N \quad \text{(ค.5)}$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_1(\mathbf{X}) \ln \frac{C_1}{C_2} dx_1 \cdots dx_N \\ &- \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_1(\mathbf{X}) \frac{1}{\lambda_1}\left((\mathbf{X}-\boldsymbol{\mu}_1)^T \Sigma_1^{-1} (\mathbf{X}-\boldsymbol{\mu}_1)\right)^{\beta_1/2} dx_1 \cdots dx_N \\ &+ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_1(\mathbf{X}) \frac{1}{\lambda_2}\left((\mathbf{X}-\boldsymbol{\mu}_2)^T \Sigma_2^{-1} (\mathbf{X}-\boldsymbol{\mu}_2)\right)^{\beta_2/2} dx_1 \cdots dx_N \end{aligned} \quad \text{(ค.6)}$$

$$\begin{aligned} D(f_1 \parallel f_2) = \ln \frac{C_1}{C_2} &- \mathbb{E}_{f_1}\left\{ \frac{1}{\lambda_1}\left((\mathbf{X}-\boldsymbol{\mu}_1)^T \Sigma_1^{-1} (\mathbf{X}-\boldsymbol{\mu}_1)\right)^{\beta_1/2} \right\} \\ &+ \mathbb{E}_{f_1}\left\{ \frac{1}{\lambda_2}\left((\mathbf{X}-\boldsymbol{\mu}_2)^T \Sigma_2^{-1} (\mathbf{X}-\boldsymbol{\mu}_2)\right)^{\beta_2/2} \right\} \end{aligned} \quad \text{(ค.7)}$$

พิจารณา

$$\mathbb{E}_{f_1}\left\{ \frac{1}{\lambda_1}\left((\mathbf{X}-\boldsymbol{\mu}_1)^T \Sigma_1^{-1} (\mathbf{X}-\boldsymbol{\mu}_1)\right)^{\beta_1/2} \right\}$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{\lambda_1}\left((\mathbf{X}-\boldsymbol{\mu}_1)^T \Sigma_1^{-1} (\mathbf{X}-\boldsymbol{\mu}_1)\right)^{\beta_1/2} \left(C_1 e^{-\frac{1}{\lambda_1}\left((\mathbf{X}-\boldsymbol{\mu}_1)^T \Sigma_1^{-1} (\mathbf{X}-\boldsymbol{\mu}_1)\right)^{\beta_1/2}} \right) dx_1 \cdots dx_N \quad \text{(ค.8)}$$

ใช้กฎการเปลี่ยนตัวแปรหลายตัวแปรของการอินทิเกรตการแจกแจง MGGD หลายตัวแปร

$$\int_{-\infty}^{\infty} f_{\mathbf{Y}}(y) dy = \int_{-\infty}^{\infty} f_{\mathbf{X}}\left(\phi^{-1}(y)\right) \left| \frac{d\phi^{-1}}{dy} \right| dx \quad \text{(ค.9)}$$

กำหนดให้ $\phi(x) = \mathbf{Y} = K_1(\mathbf{X} - \boldsymbol{\mu}_1)$ เมื่อ $K_1 = E_1 \Lambda_1^{-\frac{1}{2}}$

$$\phi^{-1}(\mathbf{Y}) = K_1^{-1}\mathbf{Y} + \mu_1 \quad \text{(ค.10)}$$

Differentiate ทั้งสองข้าง และกำหนดให้ $|K| = \det(K)$ จะได้

$$\left|\frac{d\phi^{-1}(y)}{dy}\right| = \frac{1}{|K_1|} \quad (\text{ค}.11)$$

แทนสมการ (ค.11) ในสมการ (ค.9)

$$f_Y(y) = \frac{1}{|K_1|} \cdot f_X\left(\phi^{-1}(y)\right) \quad (\text{ค}.12)$$

กำหนดให้ $\Sigma_1^{-1} = K_1^T K_1$ แทนค่า $\phi^{-1}(y) = K_1^{-1}Y + \mu_1$ ในสมการ (ค.12) จะได้

$$\mathbb{E}_{f_1}\left\{\frac{1}{\lambda_1}\left(\left(\mathbf{X}-\boldsymbol{\mu}_1\right)^T \Sigma_1^{-1}\left(\mathbf{X}-\boldsymbol{\mu}_1\right)\right)^{\beta_1/2}\right\}$$
$$= \frac{1}{|K_1|} \cdot \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{\lambda_1}\left(\mathbf{Y}^T\mathbf{Y}\right)^{\beta_1/2}\left(C_1 e^{-\frac{1}{\lambda_1}\left(\mathbf{Y}^T\mathbf{Y}\right)^{\beta_1/2}}\right) dy_1 \cdots dy_N \quad (\text{ค}.13)$$

จาก

$$\mathbf{Y}^T\mathbf{Y} = \sum_{i=1}^{N} y_i^2 \quad (\text{ค}.14)$$

แทน $\mathbf{Y}^T\mathbf{Y} = \sum_{i=1}^{N} y_i^2$ ในสมการ (ค.17)

$$\mathbb{E}_{f_1}\left\{\frac{1}{\lambda_1}\left(\left(\mathbf{X}-\boldsymbol{\mu}_1\right)^T \Sigma_1^{-1}\left(\mathbf{X}-\boldsymbol{\mu}_1\right)\right)^{\beta_1/2}\right\}$$
$$= \frac{1}{|K_1|} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{\lambda_1}\left(\sum_{i=1}^{N} y_i^2\right)^{\beta_1/2}\left(C_1 e^{-\frac{1}{\lambda_1}\left(\sum_{i=1}^{N} y_i^2\right)^{\beta_1/2}}\right) dy_1 \cdots dy_N \quad (\text{ค}.15)$$

$$= \frac{C_1}{|K_1|} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{\lambda_1}\left(\sum_{i=1}^{N} y_i^2\right)^{\beta_1/2}\left(e^{-\frac{1}{\lambda_1}\left(\sum_{i=1}^{N} y_i^2\right)^{\beta_1/2}}\right) dy_1 \cdots dy_N \quad (\text{ค}.16)$$

ประมาณค่า $\left(\sum_{i=1}^{N} y_i^2\right)^{\frac{\beta_1}{2}}$

$$\left(\sum_{i=1}^{N} y_i^2\right)^{\frac{\beta_1}{2}} \approx \sum_{i=1}^{N} |y_i|^{\beta_1} \qquad (\text{ค}.17)$$

แทนสมการ (ค.17) ในสมการ (ค.16) จะได้

$$\begin{aligned}&\mathbb{E}_{f_1}\left\{\frac{1}{\lambda_1}\left(\left(\mathbf{X}-\boldsymbol{\mu}_1\right)^T \Sigma_1^{-1}\left(\mathbf{X}-\boldsymbol{\mu}_1\right)\right)^{\beta_1/2}\right\} \\ &\approx \frac{C_1}{|K_1|}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{1}{\lambda_1}\left(\sum_{i=1}^{N}|y_i|^{\beta_1}\right)\left(e^{-\frac{1}{\lambda_1}\sum_{i=1}^{N}|y_i|^{\beta_1}}\right)dy_1\cdots dy_N\end{aligned} \qquad (\text{ค}.18)$$

แทนค่า $\sum_{i=1}^{N}|y_i|^{\beta_1} = |y_1|^{\beta_1}+|y_2|^{\beta_1}+\cdots+|y_N|^{\beta_1}$ ในสมการ (ค.18)

$$\begin{aligned}&\mathbb{E}_{f_1}\left\{\frac{1}{\lambda_1}\left(\left(\mathbf{X}-\boldsymbol{\mu}_1\right)^T \Sigma_1^{-1}\left(\mathbf{X}-\boldsymbol{\mu}_1\right)\right)^{\beta_1/2}\right\} \\ &\approx \frac{C_1}{|K_1|}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\frac{|y_1|^{\beta_1}}{\lambda_1}+\frac{|y_2|^{\beta_1}}{\lambda_1}+\cdots+\frac{|y_N|^{\beta_1}}{\lambda_1}\right)\left(e^{-\left(\frac{|y_1|^{\beta_1}}{\lambda_1}+\frac{|y_2|^{\beta_1}}{\lambda_1}+\cdots+\frac{|y_N|^{\beta_1}}{\lambda_1}\right)}\right)dy_1\cdots dy_N\end{aligned} \qquad (\text{ค}.19)$$

$$\begin{aligned}&= \frac{C_1}{|K_1|}\left[\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{|y_1|^{\beta_1}}{\lambda_1}\left(e^{-\left(\frac{|y_1|^{\beta_1}}{\lambda_1}+\frac{|y_2|^{\beta_1}}{\lambda_1}+\cdots+\frac{|y_N|^{\beta_1}}{\lambda_1}\right)}\right)dy_1\cdots dy_N\right. \\ &\quad +\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{|y_2|^{\beta_1}}{\lambda_1}\left(e^{-\left(\frac{|y_1|^{\beta_1}}{\lambda_1}+\frac{|y_2|^{\beta_1}}{\lambda_1}+\cdots+\frac{|y_N|^{\beta_1}}{\lambda_1}\right)}\right)dy_1\cdots dy_N \\ &\quad \left.+\cdots+\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{|y_N|^{\beta_1}}{\lambda_1}\left(e^{-\left(\frac{|y_1|^{\beta_1}}{\lambda_1}+\frac{|y_2|^{\beta_1}}{\lambda_1}+\cdots+\frac{|y_N|^{\beta_1}}{\lambda_1}\right)}\right)dy_1\cdots dy_N\right]\end{aligned} \qquad (\text{ค}.20)$$

$$=\frac{C_1}{|K_1|}\left[\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{|y_1|^{\beta_1}}{\lambda_1}\left(e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}\cdots e^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}\right)dy_1\cdots dy_N\right.$$
$$+\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{|y_2|^{\beta_1}}{\lambda_1}\left(e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}\cdots e^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}\right)dy_1\cdots dy_N \quad (ค.21)$$
$$\left.+\cdots+\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{|y_N|^{\beta_1}}{\lambda_1}\left(e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}\cdots e^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}\right)dy_1\cdots dy_N\right]$$

$$=\frac{C_1}{|K_1|}\left[\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}\cdots e^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}dy_N\cdots dy_2\right)\left(\frac{|y_1|^{\beta_1}}{\lambda_1}e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}dy_1\right)\right.$$
$$+\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}e^{-\frac{|y_3|^{\beta_1}}{\lambda_1}}\cdots e^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}dy_N\cdots dy_3dy_1\right)\left(\frac{|y_2|^{\beta_1}}{\lambda_1}e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}dy_2\right) \quad (ค.22)$$
$$\left.+\cdots+\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}\cdots e^{-\frac{|y_{N-1}|^{\beta_1}}{\lambda_1}}dy_{N-1}\cdots dy_1\right)\left(\frac{|y_N|^{\beta_1}}{\lambda_1}e^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}dy_N\right)\right]$$

สำหรับ GGD หนึ่งมิติ: $f(y;\lambda,\beta)=ce^{-\frac{|y|^{\beta}}{\lambda}}$; $c=\frac{\beta}{2\lambda^{\frac{1}{\beta}}\Gamma(1/\beta)}$

$$c\int_{-\infty}^{\infty}e^{-\frac{|y|^{\beta}}{\lambda}}dx=1 \quad (ค.23)$$

สำหรับฟังก์ชันแกมม่า (Gamma function)

$$\int_{-\infty}^{\infty}|y|^{n}ce^{-\frac{|y|^{\beta}}{\lambda}}dy=\lambda^{\frac{n}{\beta}}\frac{\Gamma\left(\frac{n+1}{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)} \quad (ค.24)$$

แทน $n=\beta$ แทนค่าในสมการ (ค.29)

$$\lambda\frac{\Gamma\left(\frac{\beta+1}{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)}=\lambda\frac{\Gamma\left(\frac{1}{\beta}+1\right)}{\Gamma\left(\frac{1}{\beta}\right)}=\frac{\lambda}{\beta}\frac{\Gamma\left(\frac{1}{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)}=\frac{\lambda}{\beta} \quad (ค.25)$$

ประยุกต์สมการ (ค.23)- (ค.25) ในสมการ (ค.22)

$$
\begin{aligned}
&\mathbb{E}_{f_1}\left\{\frac{1}{\lambda_1}\left(\left(\mathbf{X}-\boldsymbol{\mu}_1\right)^T \Sigma_1^{-1}\left(\mathbf{X}-\boldsymbol{\mu}_1\right)\right)^{\beta_1/2}\right\} \\
&=\frac{C_1}{|K_1|c^N}\left[\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(c_2 e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}\cdots c_N e^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}dy_2\cdots dy_N\right)\left(\frac{|y_1|^{\beta_1}}{\lambda_1}c_1 e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}dy_1\right)\right. \\
&\quad+\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(c_1 e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}c_3 e^{-\frac{|y_3|^{\beta_1}}{\lambda_1}}\cdots c_N e^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}dy_1 dy_3\cdots dy_N\right)\left(\frac{|y_2|^{\beta_1}}{\lambda_1}c_2 e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}dy_2\right) \qquad (ค.26)\\
&\quad\left.+\cdots+\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(c_1 e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}\cdots c_{N-1} e^{-\frac{|y_{N-1}|^{\beta_1}}{\lambda_1}}dy_1\cdots dy_{N-1}\right)\left(\frac{|y_N|^{\beta_1}}{\lambda_1}c_N e^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}dy_N\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{C_1}{|K_1|c^N}\left[\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(c_2 e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}\cdots c_N e^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}dy_2\cdots dy_N\right)}_{1}\left(\int_{-\infty}^{\infty}\frac{|y_1|^{\beta_1}}{\lambda_1}c_1 e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}dy_1\right)\right. \\
&\quad\underbrace{+\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(c_1 e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}c_3 e^{-\frac{|y_3|^{\beta_1}}{\lambda_1}}\cdots c_N e^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}dy_1 dy_3\cdots dy_N\right)}_{1}\left(\int_{-\infty}^{\infty}\frac{|y_2|^{\beta_1}}{\lambda_1}c_2 e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}dy_2\right) \qquad (ค.27)\\
&\quad\left.+\cdots+\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(c_1 e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}\cdots c_{N-1} e^{-\frac{|y_{N-1}|^{\beta_1}}{\lambda_1}}dy_1\cdots dy_{N-1}\right)}_{1}\left(\int_{-\infty}^{\infty}\frac{|y_N|^{\beta_1}}{\lambda_1}c_3 e^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}dy_N\right)\right]
\end{aligned}
$$

$$
=\frac{C_1}{|K_1|c_N}\left[\left(\frac{\Gamma\left(\frac{\beta_1+1}{\beta_1}\right)}{\Gamma\left(\frac{1}{\beta_1}\right)}\right)+\left(\frac{\Gamma\left(\frac{\beta_1+1}{\beta_1}\right)}{\Gamma\left(\frac{1}{\beta_1}\right)}\right)+\cdots+\left(\frac{\Gamma\left(\frac{\beta_1+1}{\beta_1}\right)}{\Gamma\left(\frac{1}{\beta_1}\right)}\right)\right] \qquad (ค.28)
$$

แทนค่าคงที่ $C_1=\left(\frac{\beta_1}{2\lambda_1^{\frac{1}{\beta_1}}\Gamma\left(1/\beta_1\right)}\right)^N\frac{1}{|\Sigma_1|^{1/2}}$ และ $c=\frac{\beta}{2\lambda^{\frac{1}{\beta}}\Gamma(1/\beta)}$ ในสมการ (ค.28)

$$= \underbrace{\left(\frac{C_1}{|K_1| c_N} \right)}_{1} N \left(\frac{\frac{1}{\beta_1} \Gamma\left(\frac{1}{\beta_1}\right)}{\Gamma\left(\frac{1}{\beta_1}\right)} \right) \quad \text{(ค.29)}$$

$$= \frac{N}{\beta_1} \quad \text{(ค.30)}$$

ดังนั้นพจน์

$$\mathbb{E}_{f_1}\left\{ \frac{1}{\lambda_1} \left(\left(\mathbf{X}-\boldsymbol{\mu}_1\right)^T \Sigma_1^{-1} \left(\mathbf{X}-\boldsymbol{\mu}_1\right) \right)^{\beta_1/2} \right\} \approx \frac{N}{\beta_1} \quad \text{(ค.31)}$$

พิจารณาพจน์ $\mathbb{E}_{f_1}\left\{ \frac{1}{\lambda_2} \left(\left(\mathbf{X}-\boldsymbol{\mu}_2\right)^T \Sigma_2^{-1} \left(\mathbf{X}-\boldsymbol{\mu}_2\right) \right)^{\beta_2/2} \right\}$ ในสมการ (ค.7)

$$\mathbb{E}_{f_1}\left\{ \frac{1}{\lambda_2} \left(\left(\mathbf{X}-\boldsymbol{\mu}_2\right)^T \Sigma_2^{-1} \left(\mathbf{X}-\boldsymbol{\mu}_2\right) \right)^{\beta_2/2} \right\}$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{\lambda_2} \left(\left(\mathbf{X}-\boldsymbol{\mu}_2\right)^T \Sigma_2^{-1} \left(\mathbf{X}-\boldsymbol{\mu}_2\right) \right)^{\beta_2/2} \left(C_1 e^{-\frac{1}{\lambda_1}\left(\left(\mathbf{X}-\boldsymbol{\mu}_1\right)^T \Sigma_1^{-1} \left(\mathbf{X}-\boldsymbol{\mu}_1\right) \right)^{\beta_1/2}} \right) dx_1 \cdots dx_N \quad \text{(ค.32)}$$

ใช้กฎการเปลี่ยนตัวแปรการอินทิเกรตในสมการ (ค.9)- (ค.12) และแทนค่า $\phi^{-1}(y) = K_1^{-1}\mathbf{Y} + \boldsymbol{\mu}_1$ ในสมการ (ค.32)

$$\mathbb{E}_{f_1}\left\{ \frac{1}{\lambda_2} \left(\left(\mathbf{X}-\boldsymbol{\mu}_2\right)^T \Sigma_2^{-1} \left(\mathbf{X}-\boldsymbol{\mu}_2\right) \right)^{\beta_2/2} \right\}$$

$$= \frac{1}{|K_1|} \cdot \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{\lambda_2} \left(\left(K_1^{-1}\mathbf{Y}+\boldsymbol{\mu}_0\right)^T \Sigma_2^{-1} \left(K_1^{-1}\mathbf{Y}+\boldsymbol{\mu}_0\right) \right)^{\beta_2/2} \left(C_1 e^{-\frac{1}{\lambda_1}\left(\mathbf{Y}^T\mathbf{Y}\right)^{\frac{\beta_1}{2}}} \right) dy_1 \cdots dy_N \quad \text{(ค.33)}$$

จะได้ $\boldsymbol{\mu}_0 = \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2$

$$= \frac{1}{|K_1|} \cdot \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{\lambda_2} \left(\left(\mathbf{Y}^T K_1^{-T} + \boldsymbol{\mu}_0^T\right) \Sigma_2^{-1} \left(K_1^{-1}\mathbf{Y}+\boldsymbol{\mu}_0\right) \right)^{\beta_2/2} \left(C_1 e^{-\frac{1}{\lambda_1}\left(\mathbf{Y}^T\mathbf{Y}\right)^{\frac{\beta_1}{2}}} \right) dy_1 \cdots dy_N \quad \text{(ค.34)}$$

$$= \frac{1}{|K_1|} \cdot \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{\lambda_2} \left(\left(\mathbf{Y}^T K_1^{-T} + \boldsymbol{\mu}_0^T\right) K_2^T K_2 \left(K_1^{-1}\mathbf{Y}+\boldsymbol{\mu}_0\right) \right)^{\beta_2/2} \left(C_1 e^{-\frac{1}{\lambda_1}\left(\mathbf{Y}^T\mathbf{Y}\right)^{\frac{\beta_1}{2}}} \right) dy_1 \cdots dy_N \quad \text{(ค.35)}$$

$$=\frac{1}{|K_1|}\cdot\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{1}{\lambda_2}\left(\left(\mathbf{Y}^T K_1^{-T}K_2^T+\boldsymbol{\mu}_0^T K_2^T\right)\left(K_2K_1^{-1}\mathbf{Y}+K_2\boldsymbol{\mu}_0\right)\right)^{\beta_2/2}\left(C_1e^{-\frac{1}{\lambda_1}\left(\mathbf{Y}^T\mathbf{Y}\right)^{\frac{\beta_1}{2}}}\right)dy_1\cdots dy_N$$

(ค.36)

$$=\frac{1}{|K_1|}\cdot\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{1}{\lambda_2}\left(\left(K_2K_1^{-1}\mathbf{Y}+K_2\boldsymbol{\mu}_0\right)^T\left(K_2K_1^{-1}\mathbf{Y}+K_2\boldsymbol{\mu}_0\right)\right)^{\beta_2/2}\left(C_1e^{-\frac{1}{\lambda_1}\left(\mathbf{Y}^T\mathbf{Y}\right)^{\frac{\beta_1}{2}}}\right)dy_1\cdots dy_N$$

(ค.37)

แทนค่า

$$\left(\left(K_2K_1^{-1}\mathbf{Y}+K_2\boldsymbol{\mu}_0\right)^T\left(K_2K_1^{-1}\mathbf{Y}+K_2\boldsymbol{\mu}_0\right)\right)^{\beta_2/2}=\left(\sum_{i=1}^{N}\left(K_{2_i}K_{1_i}^{-1}y_i+K_{2_i}\boldsymbol{\mu}_{0_i}\right)^2\right)^{\beta_2/2} \quad \text{(ค.38)}$$

ในสมการ (ค.37)

$$=\frac{1}{|K_1|}\cdot\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{1}{\lambda_2}\left(\sum_{i=1}^{N}\left(K_{2_i}K_{1_i}^{-1}y_i+K_{2_i}\boldsymbol{\mu}_{0_i}\right)^2\right)^{\beta_2/2}\left(C_1e^{-\frac{1}{\lambda_1}\left(\mathbf{Y}^T\mathbf{Y}\right)^{\frac{\beta_1}{2}}}\right)dy_1\cdots dy_N \quad \text{(ค.39)}$$

ประยุกต์สมการ (ค.14) ในสมการ (ค.39)

$$\approx\frac{1}{|K_1|}\cdot\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{1}{\lambda_2}\sum_{i=1}^{N}\left|K_{2_i}K_{1_i}^{-1}y_i+K_{2_i}\boldsymbol{\mu}_{0_i}\right|^{\beta_2}\left(C_1e^{-\frac{1}{\lambda_1}\sum_{i=1}^{N}|y_i|^{\beta_1}}\right)dy_1\cdots dy_N \quad \text{(ค.40)}$$

ประมาณค่า

$$\sum_{i=1}^{N}\left|K_{2_i}K_{1_i}^{-1}y_i+K_{2_i}\boldsymbol{\mu}_{0_i}\right|^{\beta_2}\approx\sum_{i=1}^{N}\left|\left(K_{2_i}K_{1_i}^{-1}y_1\right)^{\beta_2}+\left(K_{2_i}\boldsymbol{\mu}_{0_i}\right)^{\beta_2}\right| \quad \text{(ค.41)}$$

เมื่อ *i* คือ marginal density และสำหรับ matrix K_{1_i} เป็นลำดับในแนวทแยงมุม (diagonal)

$$\approx\frac{1}{|K_1|}\cdot\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{1}{\lambda_2}\sum_{i=1}^{N}\left|\left(K_{2_i}K_{1_i}^{-1}y_i\right)^{\beta_2}+\left(K_{2_i}\boldsymbol{\mu}_{0_i}\right)^{\beta_2}\right|\left(C_1e^{-\frac{1}{\lambda_1}\sum_{i=1}^{N}|y_i|^{\beta_1}}\right)dy_1\cdots dy_N \quad \text{(ค.42)}$$

$$=\frac{1}{|K_1|}\cdot\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{1}{\lambda_2}\sum_{i=1}^{N}\left|\left(K_{2_i}K_{1_i}^{-1}y_i\right)\right|^{\beta_2}+\sum_{i=1}^{N}\left(K_{2_i}\boldsymbol{\mu}_{0_i}\right)^{\beta_2}\left(C_1e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}\cdots e^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}\right)dy_1\cdots dy_N$$

(ค.43)

$$=\frac{1}{|K_1|}\cdot\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{1}{\lambda_2}\left(\left(\left(K_{2_1}K_{1_1}^{-1}y_1\right)^{\beta_2}+\left(K_{2_2}K_{1_2}^{-1}y_2\right)^{\beta_2}+\cdots+\left(K_{2_N}K_{1_N}^{-1}y_N\right)^{\beta_2}\right)+\sum_{i=1}^{N}\left(K_{2_i}\boldsymbol{\mu}_{0_i}\right)^{\beta_2}\right)$$
$$\left(C_1e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}\cdots e^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}\right)dy_1\cdots dy_N \qquad (ค.44)$$

$$=\frac{1}{|K_1|}\cdot\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{1}{\lambda_2}\left(\left(K_{2_1}K_{1_1}^{-1}y_1\right)^{\beta_2}\left(C_1e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}\cdots e^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}\right)dy_1\cdots dy_N\right.$$
$$+\left(K_{2_2}K_{1_2}^{-1}y_2\right)^{\beta_2}\left(C_1e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}\cdots e^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}\right)dy_1\cdots dy_N$$
$$+\cdots+\left(K_{2_N}K_{1_N}^{-1}y_N\right)^{\beta_2}\left(C_1e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}\cdots e^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}\right)dy_1\cdots dy_N \qquad (ค.45)$$
$$\left.+\sum_{i=1}^{N}\left(K_{2_i}\boldsymbol{\mu}_{0_i}\right)^{\beta_2}\left(C_1e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}\cdots e^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}\right)dy_1\cdots dy_N\right)$$

$$=\frac{1}{|K_1|}\cdot\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{1}{\lambda_2}\left(\left(K_{2_1}K_{1_1}^{-1}\right)^{\beta_2}\left(y_1\right)^{\beta_2}\left(C_1e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}\cdots e^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}\right)dy_1\cdots dy_N\right.$$
$$+\left(K_{2_2}K_{1_2}^{-1}\right)^{\beta_2}\left(y_2\right)^{\beta_2}\left(C_1e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}\cdots e^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}\right)dy_1\cdots dy_N$$
$$+\cdots+\left(K_{2_N}K_{1_N}^{-1}\right)^{\beta_2}\left(y_N\right)^{\beta_2}\left(C_1e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}\cdots e^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}\right)dy_1\cdots dy_N \qquad (ค.46)$$
$$\left.+\sum_{i=1}^{N}\left(K_{2_i}\boldsymbol{\mu}_{0_i}\right)^{\beta_2}\left(C_1e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}\cdots e^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}\right)dy_1\cdots dy_N\right)$$

ประยุกต์สมการ (ค.28)- (ค.31) ในสมการ (ค.46) และ

$$c^N=c_1\cdot c_2\cdots c_N \qquad (ค.47)$$

เนื่องจากค่า c มีค่าขึ้นอยู่กับค่า β เพียงค่าเดียว

$$
\begin{aligned}
&=\frac{C_1}{\lambda_2|K_1|c^N}\cdot\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\left(K_{2_1}K_{1_1}^{-1}\right)^{\beta_2}\left(y_1\right)^{\beta_2}\left(c_1e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}c_2e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}\cdots c_Ne^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}\right)dy_1\cdots dy_N\right.\\
&+\left(K_{2_2}K_{1_2}^{-1}\right)^{\beta_2}\left(y_2\right)^{\beta_2}\left(c_1e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}c_2e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}\cdots c_Ne^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}\right)dy_1\cdots dy_N\\
&+\cdots+\left(K_{2_N}K_{1_N}^{-1}\right)^{\beta_2}\left(y_N\right)^{\beta_2}\left(c_1e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}c_2e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}\cdots c_Ne^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}\right)dy_1\cdots dy_N\\
&\left.+\sum_{i=1}^{N}\left(K_{2_i}\boldsymbol{\mu}_{0_i}\right)^{\beta_2}\left(c_1e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}c_2e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}\cdots c_Ne^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}\right)dy_1\cdots dy_N\right)
\end{aligned}
\qquad (\text{ค}.48)
$$

$$
\begin{aligned}
&=\frac{C_1}{\lambda_2|K_1|c^N}\left(\left(K_{2_1}K_{1_1}^{-1}\right)^{\beta_2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(y_1\right)^{\beta_2}c_1e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}\left(c_2e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}\cdots c_Ne^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}\right)dy_1\cdots dy_N\right.\\
&+\left(K_{2_2}K_{1_2}^{-1}\right)^{\beta_2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(y_2\right)^{\beta_2}c_2e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}\left(c_1e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}c_3e^{-\frac{|y_3|^{\beta_1}}{\lambda_1}}\cdots c_Ne^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}\right)dy_1\cdots dy_N\\
&+\cdots+\left(K_{2_N}K_{1_N}^{-1}\right)^{\beta_2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(y_N\right)^{\beta_2}c_Ne^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}\left(c_1e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}\cdots c_{N-1}e^{-\frac{|y_{N-1}|^{\beta_1}}{\lambda_1}}\right)dy_1\cdots dy_N\\
&\left.+\sum_{i=1}^{N}\left(K_{2_i}\boldsymbol{\mu}_{0_i}\right)^{\beta_2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(c_1e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}\cdots c_Ne^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}\right)dy_1\cdots dy_N\right)
\end{aligned}
\qquad (\text{ค}.49)
$$

$$
\begin{aligned}
&=\frac{C_1}{\lambda_2|K_1|c^N}\left(\left(K_{2_1}K_{1_1}^{-1}\right)^{\beta_2}\left(\int_{-\infty}^{\infty}\left(y_1\right)^{\beta_2}c_1e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}dy_1\right)\underbrace{\left(\int_{-\infty}^{\infty}c_2e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}\cdots c_Ne^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}dy_2\cdots dy_N\right)}_{1}\right.\\
&+\left(K_{2_2}K_{1_2}^{-1}\right)^{\beta_2}\left(\int_{-\infty}^{\infty}\left(y_2\right)^{\beta_2}c_2e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}}dy_2\right)\underbrace{\left(\int_{-\infty}^{\infty}c_1e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}c_3e^{-\frac{|y_3|^{\beta_1}}{\lambda_1}}\cdots c_Ne^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}dy_1dy_3\cdots dy_N\right)}_{1}\\
&\left.+\cdots+\left(K_{2_N}K_{1_N}^{-1}\right)^{\beta_2}\left(\int_{-\infty}^{\infty}\left(y_N\right)^{\beta_2}c_Ne^{-\frac{|y_N|^{\beta_1}}{\lambda_1}}dy_N\right)\underbrace{\left(\int_{-\infty}^{\infty}c_1e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}}\cdots c_{N-1}e^{-\frac{|y_{N-1}|^{\beta_1}}{\lambda_1}}dy_2\cdots dy_N\right)}_{1}+\sum_{i=1}^{N}\left(K_{2_i}\boldsymbol{\mu}_{0_i}\right)^{\beta_2}\right)
\end{aligned}
$$

(ค.50)

$$= \frac{C_1}{\lambda_2 |K_1| c^N} \left(\left(K_{2_1} K_{1_1}^{-1} \right)^{\beta_2} \left(\int_{-\infty}^{\infty} (y_1)^{\beta_2} c_1 e^{-\frac{|y_1|^{\beta_1}}{\lambda_1}} dy_1 \right) + \left(K_{2_2} K_{1_2}^{-1} \right)^{\beta_2} \left(\int_{-\infty}^{\infty} (y_2)^{\beta_2} c_2 e^{-\frac{|y_2|^{\beta_1}}{\lambda_1}} dy_2 \right) + \cdots + \left(K_{2_N} K_{1_N}^{-1} \right)^{\beta_2} \left(\int_{-\infty}^{\infty} (y_N)^{\beta_2} c_N e^{-\frac{|y_N|^{\beta_1}}{\lambda_1}} dy_N \right) + \sum_{i=1}^{N} \left(K_{2_i} \boldsymbol{\mu}_{0_i} \right)^{\beta_2} \right) \quad (ค.51)$$

ประยุกต์สมการ (ค.24) $\int_{-\infty}^{\infty} |y|^n c e^{-\frac{|y|^{\beta}}{\lambda}} dy = \lambda^{\frac{n}{\beta}} \frac{\Gamma\left(\frac{n+1}{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)}$ ในสมการ (ค.51)

$$= \frac{C_1}{\lambda_2 |K_1| c^N} \left(\left(K_{2_1} K_{1_1}^{-1} \right)^{\beta_2} \lambda_1^{\frac{\beta_2}{\beta_1}} \frac{\Gamma\left(\frac{\beta_2+1}{\beta_1}\right)}{\Gamma\left(\frac{1}{\beta_1}\right)} + \left(K_{2_2} K_{1_2}^{-1} \right)^{\beta_2} \lambda_1^{\frac{\beta_2}{\beta_1}} \frac{\Gamma\left(\frac{\beta_2+1}{\beta_1}\right)}{\Gamma\left(\frac{1}{\beta_1}\right)} + \cdots + \left(K_{2_N} K_{1_N}^{-1} \right)^{\beta_2} \lambda_1^{\frac{\beta_2}{\beta_1}} \frac{\Gamma\left(\frac{\beta_2+1}{\beta_1}\right)}{\Gamma\left(\frac{1}{\beta_1}\right)} + \sum_{i=1}^{N} \left(K_{2_i} \boldsymbol{\mu}_{0_i} \right)^{\beta_2} \right) \quad (ค.52)$$

$$= \frac{C_1}{\lambda_2 |K_1| c^N} \left(\lambda_1^{\frac{\beta_2}{\beta_1}} \frac{\Gamma\left(\frac{\beta_2+1}{\beta_1}\right)}{\Gamma\left(\frac{1}{\beta_1}\right)} \left(\left(K_{2_1} K_{1_1}^{-1} \right)^{\beta_2} + \left(K_{2_2} K_{1_2}^{-1} \right)^{\beta_2} + \cdots + \left(K_{2_N} K_{1_N}^{-1} \right)^{\beta_2} \right) + \sum_{i=1}^{N} \left(K_{2_i} \boldsymbol{\mu}_{0_i} \right)^{\beta_2} \right) \quad (ค.53)$$

จะได้

$$tr\left(\left(K_2 K_1^{-1} \right)^{\beta_2} \right) = \left(\left(K_{2_1} K_{1_1}^{-1} \right)^{\beta_2} + \left(K_{2_2} K_{1_2}^{-1} \right)^{\beta_2} + \cdots + \left(K_{2_N} K_{1_N}^{-1} \right)^{\beta_2} \right) \quad (ค.54)$$

$$\approx \frac{C_1}{\lambda_2 |K_1| c^N} \left(\lambda_1^{\frac{\beta_2}{\beta_1}} tr\left(\left(K_2 K_1^{-1} \right)^{\beta_2} \right) \frac{\Gamma\left(\frac{\beta_2+1}{\beta_1}\right)}{\Gamma\left(\frac{1}{\beta_1}\right)} + \sum_{i=1}^{N} \left(K_{2_i} \boldsymbol{\mu}_{0_i} \right)^{\beta_2} \right) \quad (ค.55)$$

แทนค่า C_1 , c^N ➔ $\frac{C_1}{|K_1| c^N} = 1$

$$\mathbb{E}_{f_1}\left\{\frac{1}{\lambda_2}\left(\left(\mathbf{X}-\boldsymbol{\mu}_2\right)^T\Sigma_2^{-1}\left(\mathbf{X}-\boldsymbol{\mu}_2\right)\right)^{\beta_2/2}\right\}$$

$$=\frac{\lambda_1^{\frac{\beta_2}{\beta_1}}}{\lambda_2}tr\left(\left(K_2K_1^{-1}\right)^{\beta_2}\right)\frac{\Gamma\left(\frac{\beta_2+1}{\beta_1}\right)}{\Gamma\left(\frac{1}{\beta_1}\right)}+\frac{1}{\lambda_2}\sum_{i=1}^{N}\left(K_{2_i}\boldsymbol{\mu}_{0_i}\right)^{\beta_2} \tag{ค.56}$$

ดังนั้นแทนสมการ (ค.31) และ (ค.56) ในสมการ (ค.7) จะได้

$$D\left(f_1\,||\,f_2\right)\approx\ln\frac{C_1}{C_2}-\frac{N}{\beta_1}+\frac{\lambda_1^{\frac{\beta_2}{\beta_1}}}{\lambda_2}tr\left(\left(K_2K_1^{-1}\right)^{\beta_2}\right)\frac{\Gamma\left(\frac{\beta_2+1}{\beta_1}\right)}{\Gamma\left(\frac{1}{\beta_1}\right)}+\frac{1}{\lambda_2}\sum_{i=1}^{N}\left(K_{2_i}\boldsymbol{\mu}_{0_i}\right)^{\beta_2} \tag{ค.57}$$

$$D\left(f_1\,||\,f_2\right)\approx N\ln\frac{\beta_1\lambda_2^{\frac{1}{\beta_2}}\Gamma\left(\frac{1}{\beta_2}\right)}{\beta_2\lambda_1^{\frac{1}{\beta_1}}\Gamma\left(\frac{1}{\beta_1}\right)}+\frac{1}{2}\ln\frac{|\Sigma_2|}{|\Sigma_1|}-\frac{N}{\beta_1}+\frac{\lambda_1^{\frac{\beta_2}{\beta_1}}}{\lambda_2}tr\left(\left(K_2K_1^{-1}\right)^{\beta_2}\right)\frac{\Gamma\left(\frac{\beta_2+1}{\beta_1}\right)}{\Gamma\left(\frac{1}{\beta_1}\right)} \tag{ค.58}$$
$$+\frac{1}{\lambda_2}\sum_{i=1}^{N}\left(K_{2_i}\left(\boldsymbol{\mu}_{1_i}-\boldsymbol{\mu}_{2_i}\right)\right)^{\beta_2}$$

เมื่อ $N=1$ จะเป็นการแจกแจงเกาส์เซียนทั่วไปตัวแปรเดียว (GGD) ซึ่งมีสมการ KLD ดังนี้

$$D\left(f_1\,||\,f_2\right)=\ln\left(\frac{\beta_1\lambda_2^{\frac{1}{\beta_2}}\sigma_2\Gamma\left(1/\beta_2\right)}{\beta_2\lambda_1^{\frac{1}{\beta_1}}\sigma_1\Gamma\left(1/\beta_1\right)}\right)-\frac{1}{\beta_1}+\left(\frac{\lambda_1^{\frac{1}{\beta_1}}\sigma_1}{\lambda_2^{\frac{1}{\beta_2}}\sigma_2}\right)^{\beta_2}\frac{\Gamma\left(\frac{\beta_2+1}{\beta_1}\right)}{\Gamma\left(\frac{1}{\beta_1}\right)} \tag{ค.59}$$
$$+\frac{1}{\lambda_2}\left(\frac{\boldsymbol{\mu}_1-\boldsymbol{\mu}_2}{\sigma_2}\right)^{\beta_2}$$

จากสมการ (ค.58) เมื่อ $\beta_1=\beta_2=2$ จะได้ KLD ของการแจกแจงแบบ MGD

$$D\left(f_1\,||\,f_2\right)=\frac{1}{2}\left(\ln\frac{|\Sigma_2|}{|\Sigma_1|}-N+tr\left\{\Sigma_2^{-1}\Sigma_1\right\}+\left(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2\right)^T\Sigma_2^{-1}\left(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2\right)\right) \tag{ค.60}$$

ภาคผนวก ง.

การเปรียบเทียบฟังก์ชันลักษณะเฉพาะของการแจกแจงความน่าจะเป็นแบบแอลฟ่าสเตเบิลที่สมมาตรหลายตัวแปรด้วย Kullback-Leibler Divergence

ง.1 ฟังก์ชันลักษณะเฉพาะของแอลฟ่าสเตเบิลที่สมมาตรหลายตัวแปร

$$\varphi(t) = \exp\left(-\frac{1}{2}\left(t^T \Sigma t\right)^{\alpha/2}\right) \tag{ง.1}$$

และฟังก์ชันการแจกแจงลักษณะเฉพาะของแอลฟ่าสเตเบิล

$$\varphi'(t) = A\varphi(t) \tag{ง.2}$$

เมื่อ $A = \dfrac{\alpha^N |\Sigma|}{\left(2\Gamma\left(\frac{1}{\alpha}\right)\right)^N}$ เป็นค่าที่ทำให้ปริมาตรของการแจกแจงเป็นหนึ่งและ

สมการของ Kullback-Leibler Divergence ของสมการลัษณะเฉพาะ

$$D\left(\varphi_1'(t) \,||\, \varphi_2'(t)\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \varphi_1'(t) \ln \frac{\varphi_1'(t)}{\varphi_2'(t)} dt_1 dt_2 \cdots dt_N$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \varphi_1' \ln \frac{A_1 e^{-\left(\frac{1}{2}\left(t^T\Sigma_1 t\right)^{\alpha_1/2}\right)}}{A_2 e^{-\left(\frac{1}{2}\left(t^T\Sigma_2 t\right)^{\alpha_2/2}\right)}} dt_1 \cdots dt_N \tag{ง.3}$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \varphi_1' \left(\ln \frac{A_1}{A_2} + \ln \frac{e^{-\left(\frac{1}{2}\left(t^T\Sigma_1 t\right)^{\alpha_1/2}\right)}}{e^{-\left(\frac{1}{2}\left(t^T\Sigma_2 t\right)^{\alpha_2/2}\right)}} \right) dt_1 \cdots dt_N \tag{ง.4}$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \varphi_1' \left(\ln \frac{A_1}{A_2} + \ln e^{-\left(\frac{1}{2}\left(t^T\Sigma_1 t\right)^{\alpha_1/2}\right)} - \ln e^{-\left(\frac{1}{2}\left(t^T\Sigma_2 t\right)^{\alpha_2/2}\right)} \right) dt_1 \cdots dt_N \tag{ง.5}$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi_1' \left(\ln \frac{A_1}{A_2} - \frac{1}{2}\left(t^T \Sigma_1 t\right)^{\alpha_1/2} + \frac{1}{2}\left(t^T \Sigma_2 t\right)^{\alpha_2/2} \right) dt_1 \cdots dt_N \tag{ง.6}$$

$$= \ln \frac{A_1}{A_2} - \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi_1' \frac{1}{2}\left(t^T \Sigma_1 t\right)^{\alpha_1/2} dt_1 \cdots dt_N + \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi_1' \frac{1}{2}\left(t^T \Sigma_2 t\right)^{\alpha_2/2} dt_1 \cdots dt_N \tag{ง.7}$$

พิจารณาพจน์ $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi_1' \frac{1}{2}\left(t^T \Sigma_1 t\right)^{\alpha_1/2} dt_1 \cdots dt_N$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi_1' \frac{1}{2}\left(t^T \Sigma_1 t\right)^{\alpha_1/2} dt_1 \cdots dt_N = \frac{A_1}{2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\left(\frac{1}{2}\left(t^T \Sigma_1 t\right)^{\alpha_1/2}\right)} \left(t^T \Sigma_1 t\right)^{\alpha_1/2} dt_1 \cdots dt_N \tag{ง.8}$$

ใช้กฎการเปลี่ยนตัวแปรในการอินทิเกรตหลายตัวแปร

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi_\omega(\omega) d\omega_1 \cdots d\omega_N = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi_t\left(\phi^{-1}\left(\omega\right)\right) \left|\frac{d\phi^{-1}}{d\omega}\right| dt_1 \cdots dt_N \tag{ง.9}$$

กำหนดให้ $\phi(t) = \omega = H_1^T t$

$$\phi^{-1}(\omega) = H_1^{-T} \omega \tag{ง.10}$$

Differentiate ทั้งสองค่า และกำหนดให้ $|H| = \det(H)$

$$\left|\frac{d\phi^{-1}(\omega)}{d\omega}\right| = \frac{1}{\left|H_1^T\right|} \tag{ง.11}$$

แทนสมการ (ง.9) ในสมการ (ง.7) จะได้

$$\varphi_\omega(\omega) = \frac{1}{\left|H_1^T\right|} \cdot \varphi_t\left(\phi^{-1}\left(\omega\right)\right) \tag{ง.12}$$

การกระจาย Covariance matrix

$$\Sigma_1 = E_1 \Lambda_1 E_1^T = E_1 \Lambda_1^{\frac{1}{2}} \Lambda_1^{\frac{1}{2}} E_1^T = E_1 \Lambda_1^{\frac{1}{2}} \left(E_1 \Lambda_1^{\frac{1}{2}} \right)^T = H_1 H_1^T \tag{ง.13}$$

แทนค่า $\phi^{-1}(\omega) = H_1^{-T} \omega$ ในสมการ (ง.12)

$$\frac{A_1}{2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\left(\frac{1}{2}\left(t^T\Sigma_1 t\right)^{\alpha_1/2}\right)}\left(t^T\Sigma_1 t\right)^{\alpha_1/2} dt_1\cdots dt_N$$

$$=\frac{A_1}{2\left|H_1^T\right|}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\omega^T\omega\right)^{\alpha_1/2} e^{-\left(\frac{1}{2}\left(\omega^T\omega\right)^{\alpha_1/2}\right)} d\omega_1\cdots d\omega_N \qquad (ง.14)$$

จาก $\omega^T\omega=\sum_{i=1}^{N}\omega_i^2$ แทนค่าในสมการ (ง.14)

$$=\frac{A_1}{2\left|H_1^T\right|}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\sum_{i=1}^{N}\omega_i^2\right)^{\alpha_1/2} e^{-\left(\frac{1}{2}\left(\sum_{i=1}^{N}\omega_i^2\right)^{\alpha_1/2}\right)} d\omega_1\cdots d\omega_N \qquad (ง.15)$$

ประมาณค่า $\left(\sum_{i=1}^{N}\omega_i^2\right)^{\alpha_1/2}$

$$\left(\sum_{i=1}^{N}\omega_i^2\right)^{\alpha_1/2} \approx \sum_{i=1}^{N}\left|\omega_i\right|^{\alpha_1} \qquad (ง.16)$$

แทนสมการ (ง.16) ในสมการ (ง.15) จะได้

$$\approx\frac{A_1}{2\left|H_1^T\right|}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sum_{i=1}^{N}\left|\omega_i\right|^{\alpha_1} e^{-\left(\frac{1}{2}\sum_{i=1}^{N}\left|\omega_i\right|^{\alpha_1}\right)} d\omega_1\cdots d\omega_N \qquad (ง.17)$$

แทนค่า $\sum_{i=1}^{N}\left|\omega_i\right|^{\alpha_1}=\left|\omega_1\right|^{\alpha_1}+\left|\omega_2\right|^{\alpha_1}+\cdots+\left|\omega_N\right|^{\alpha_1}$ ในสมการ (ง.17)

$$=\frac{A_1}{2\left|H_1^T\right|}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\left|\omega_1\right|^{\alpha_1}+\left|\omega_2\right|^{\alpha_1}+\cdots+\left|\omega_N\right|^{\alpha_1}\right) e^{-\frac{1}{2}\left(\left|\omega_1\right|^{\alpha_1}+\left|\omega_2\right|^{\alpha_1}+\cdots+\left|\omega_N\right|^{\alpha_1}\right)} d\omega_1\cdots d\omega_N \qquad (ง.18)$$

$$\begin{aligned}=\frac{A_1}{2\left|H_1^T\right|}\Bigg[&\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|\omega_1\right|^{\alpha_1} e^{-\frac{1}{2}\left(\left|\omega_1\right|^{\alpha_1}+\left|\omega_2\right|^{\alpha_1}+\cdots+\left|\omega_N\right|^{\alpha_1}\right)} d\omega_1\cdots d\omega_N\\ &+\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|\omega_2\right|^{\alpha_1} e^{-\frac{1}{2}\left(\left|\omega_1\right|^{\alpha_1}+\left|\omega_2\right|^{\alpha_1}+\cdots+\left|\omega_N\right|^{\alpha_1}\right)} d\omega_1\cdots d\omega_N\\ &+\cdots+\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|\omega_N\right|^{\alpha_1} e^{-\frac{1}{2}\left(\left|\omega_1\right|^{\alpha_1}+\left|\omega_2\right|^{\alpha_1}+\cdots+\left|\omega_N\right|^{\alpha_1}\right)} d\omega_1\cdots d\omega_N\Bigg]\end{aligned} \qquad (ง.19)$$

$$=\frac{A_1}{2\left|H_1^T\right|}\left[\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|\omega_1\right|^{\alpha_1}\left(e^{-\frac{1}{2}\left|\omega_1\right|^{\alpha_1}}e^{-\frac{1}{2}\left|\omega_2\right|^{\alpha_1}}\cdots e^{-\frac{1}{2}\left|\omega_N\right|^{\alpha_1}}\right)d\omega_1\cdots d\omega_N\right.$$
$$+\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|\omega_2\right|^{\alpha_1}\left(e^{-\frac{1}{2}\left|\omega_1\right|^{\alpha_1}}e^{-\frac{1}{2}\left|\omega_2\right|^{\alpha_1}}\cdots e^{-\frac{1}{2}\left|\omega_N\right|^{\alpha_1}}\right)d\omega_1\cdots d\omega_N \qquad (ง.20)$$
$$\left.+\cdots+\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|\omega_N\right|^{\alpha_1}\left(e^{-\frac{1}{2}\left|\omega_1\right|^{\alpha_1}}e^{-\frac{1}{2}\left|\omega_2\right|^{\alpha_1}}\cdots e^{-\frac{1}{2}\left|\omega_N\right|^{\alpha_1}}\right)d\omega_1\cdots d\omega_N\right]$$

$$=\frac{A_1}{2\left|H_1^T\right|}\left[\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(e^{-\frac{1}{2}\left|\omega_2\right|^{\alpha_1}}\cdots e^{-\frac{1}{2}\left|\omega_N\right|^{\alpha_1}}d\omega_2\cdots d\omega_N\right)\left(\left|\omega_1\right|^{\alpha_1}e^{-\frac{1}{2}\left|\omega_1\right|^{\alpha_1}}d\omega_1\right)\right.$$
$$+\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(e^{-\frac{1}{2}\left|\omega_1\right|^{\alpha_1}}e^{-\frac{1}{2}\left|\omega_3\right|^{\alpha_1}}\cdots e^{-\frac{1}{2}\left|\omega_N\right|^{\alpha_1}}d\omega_1 d\omega_3\cdots d\omega_N\right)\left(\left|\omega_2\right|^{\alpha_1}e^{-\frac{1}{2}\left|\omega_1\right|^{\alpha_1}}d\omega_2\right) \qquad (ง.21)$$
$$\left.+\cdots+\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(e^{-\frac{1}{2}\left|\omega_1\right|^{\alpha_1}}\cdots e^{-\frac{1}{2}\left|\omega_{N-1}\right|^{\alpha_1}}d\omega_1\cdots d\omega_{N-1}\right)\left(\left|\omega_N\right|^{\alpha_1}e^{-\frac{1}{2}\left|\omega_N\right|^{\alpha_1}}d\omega_N\right)\right]$$

คูณ $\frac{a^N}{a^N}$ ในสมการ (ง.21)

$$=\frac{A_1}{2\left|H_1^T\right|a^N}\left[\underbrace{\left(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}ae^{-\frac{1}{2}\left|\omega_2\right|^{\alpha_1}}\cdots ae^{-\frac{1}{2}\left|\omega_N\right|^{\alpha_1}}d\omega_2\cdots d\omega_N\right)}_{1}\left(\int_{-\infty}^{\infty}\left|\omega_1\right|^{\alpha_1}ae^{-\frac{1}{2}\left|\omega_1\right|^{\alpha_1}}d\omega_1\right)\right.$$
$$+\underbrace{\left(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}ae^{-\frac{1}{2}\left|\omega_1\right|^{\alpha_1}}ae^{-\frac{1}{2}\left|\omega_3\right|^{\alpha_1}}\cdots ae^{-\frac{1}{2}\left|\omega_N\right|^{\alpha_1}}d\omega_1 d\omega_3\cdots d\omega_N\right)}_{1}\left(\int_{-\infty}^{\infty}\left|\omega_2\right|^{\alpha_1}ae^{-\frac{1}{2}\left|\omega_1\right|^{\alpha_1}}d\omega_2\right) \qquad (ง.22)$$
$$\left.+\cdots+\underbrace{\left(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}ae^{-\frac{1}{2}\left|\omega_1\right|^{\alpha_1}}\cdots ae^{-\frac{1}{2}\left|\omega_{N-1}\right|^{\alpha_1}}d\omega_1\cdots d\omega_{N-1}\right)}_{1}\left(\int_{-\infty}^{\infty}\left|\omega_N\right|^{\alpha_1}ae^{-\frac{1}{2}\left|\omega_N\right|^{\alpha_1}}d\omega_N\right)\right]$$

$$=\frac{A_1}{2\left|H_1^T\right|a^N}\left[\left(\int_{-\infty}^{\infty}\left|\omega_1\right|^{\alpha_1}ae^{-\frac{1}{2}\left|\omega_1\right|^{\alpha_1}}d\omega_1\right)+\left(\int_{-\infty}^{\infty}\left|\omega_2\right|^{\alpha_1}ae^{-\frac{1}{2}\left|\omega_1\right|^{\alpha_1}}d\omega_2\right)\right.$$
$$\left.+\cdots+\left(\int_{-\infty}^{\infty}\left|\omega_N\right|^{\alpha_1}ae^{-\frac{1}{2}\left|\omega_N\right|^{\alpha_1}}d\omega_N\right)\right] \qquad (ง.23)$$

จาก SAS หนึ่งตัวแปร

$$\varphi'=ae^{-\gamma^{\alpha}\left|t\right|^{\alpha}}\ ,\ a=\frac{\alpha\gamma}{2\Gamma\left(\frac{1}{\alpha}\right)} \qquad (ง.24)$$

สำหรับฟังก์ชันแกมม่า $\int_{-\infty}^{\infty} |\omega|^n ae^{-\gamma^\alpha |\omega|^\alpha} d\omega = \left(\frac{1}{\gamma}\right)^n \frac{\Gamma\left(\frac{n+1}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}\right)}$ (ง.25)

$n = \alpha, \gamma = \left(\frac{1}{2}\right)^{\frac{1}{\alpha}}$ จะได้ $\left(\frac{1}{\gamma}\right)^n \frac{\Gamma\left(\frac{n+1}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}\right)} = \frac{2}{\alpha}$ (ง.26)

แทนสมการ (ง.26) ในสมการ (ง.23) จะได้

$$= \frac{A_1}{2\left|H_1^T\right| a^N}\left[\left(\int_{-\infty}^{\infty} |\omega_1|^{\alpha_1} ae^{-\frac{1}{2}|\omega_1|^{\alpha_1}} d\omega_1\right) + \left(\int_{-\infty}^{\infty} |\omega_2|^{\alpha_1} ae^{-\frac{1}{2}|\omega_1|^{\alpha_1}} d\omega_2\right) + \cdots + \left(\int_{-\infty}^{\infty} |\omega_N|^{\alpha_1} ae^{-\frac{1}{2}|\omega_N|^{\alpha_1}} d\omega_N\right)\right] \quad (ง.27)$$

$$= \frac{A_1}{2\left|H_1^T\right| a^N}\left[\frac{2}{\alpha_1} + \frac{2}{\alpha_1} + \cdots + \frac{2}{\alpha_1}\right] \quad (ง.28)$$

$$= \frac{A_1}{\left|H_1^T\right| a^N} \frac{N}{\alpha_1} \quad (ง.29)$$

แทนค่า $A_1 = a_1^N \left|\Sigma_1\right|^{\frac{1}{2}}$ ในสมการ (ง.29)

$$= \frac{a^N \left|\Sigma_1\right|^{\frac{1}{2}}}{\left|H_1^T\right| a^N} \frac{N}{\alpha_1} \quad (ง.30)$$

จากสมการที่ (ง.13) $\left|\Sigma_1^{\frac{1}{2}}\right| = \left|H_1^T\right|$

$$= \frac{N}{\alpha_1} \quad (ง.31)$$

ดังนั้นจะได้

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi_1' \frac{1}{2}\left(t^T \Sigma_1 t\right)^{\alpha_1/2} dt_1 \cdots dt_N = \frac{N}{\alpha_1} \quad (ง.32)$$

พิจารณาพจน์ $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi_1' \frac{1}{2}\left(t^T \Sigma_2 t\right)^{\alpha_2/2} dt_1 \cdots dt_N$ ในสมการ (ง.7)

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\varphi_1'\frac{1}{2}\left(t^T\Sigma_2 t\right)^{\alpha_2/2} dt_1\cdots dt_N = \frac{A_1}{2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(t^T\Sigma_2 t\right)^{\alpha_2/2} e^{-\frac{1}{2}\left(t^T\Sigma_1 t\right)^{\alpha_1/2}} dt_1\cdots dt_N \qquad (ง.33)$$

ใช้กฎการแทนตัวแปรในการอินทิเกรตหลายตัวแปรในสมการที่ (ง.9)-(ง.12) และแทนค่า $\phi^{-1}(\omega)=H_1^{-T}\omega$ ในสมการ (ง.33) จะได้

$$=\frac{A_1}{2\left|H_1^T\right|}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\omega^T H_1^{-1}\Sigma_2 H_1^{-T}\omega\right)^{\alpha_2/2} e^{-\frac{1}{2}\left(\omega^T\omega\right)^{\alpha_1/2}} dt_1\cdots dt_N \qquad (ง.34)$$

การแยก Covariance matrix

$$\Sigma_2 = E_2\Lambda_2 E_2^T = E_2\Lambda_2^{\frac{1}{2}}\Lambda_2^{\frac{1}{2}}E_2^T = E_2\Lambda_2^{\frac{1}{2}}\left(E_2\Lambda_2^{\frac{1}{2}}\right)^T = H_2 H_2^T \qquad (ง.35)$$

แทนสมการ (ง.35) ในสมการ (ง.34) จะได้

$$=\frac{A_1}{2\left|H_1^T\right|}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\omega^T H_1^{-1}H_2 H_2^T H_1^{-T}\omega\right)^{\alpha_2/2} e^{-\frac{1}{2}\left(\omega^T\omega\right)^{\alpha_1/2}} dt_1\cdots dt_N \qquad (ง.36)$$

กำหนดให้ $H_0^T = H_1^{-1}H_2$ และแทนค่าในสมการ (ง.36) จะได้

$$=\frac{A_1}{2\left|H_1^T\right|}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\omega^T H_0^T H_0\omega\right)^{\alpha_2/2} e^{-\frac{1}{2}\left(\omega^T\omega\right)^{\alpha_1/2}} dt_1\cdots dt_N \qquad (ง.37)$$

จาก $\omega^T\omega=\sum_{i=1}^{N}\omega_i^2$ แทนค่าในสมการ (ง.37)

$$=\frac{A_1}{2\left|H_1^T\right|}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\sum_{i=1}^{N}\left(\omega_i H_{0_i}\right)^2\right)^{\alpha_2/2} e^{-\frac{1}{2}\left(\sum_{i=1}^{N}\omega_i^2\right)^{\alpha_1/2}} dt_1\cdots dt_N \qquad (ง.38)$$

ประยุกต์สมการ (ง.16) ใน สมการ (ง.38) เพื่อประมาณค่า

$$\left(\sum_{i=1}^{N}\left(\omega_i H_{0_i}\right)^2\right)^{\alpha_2/2} \approx \sum_{i=1}^{N}\left|\omega_i H_{0_i}\right|^{\alpha_2} \qquad (ง.39)$$

$$\approx\frac{A_1}{2\left|H_1^T\right|}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\sum_{i=1}^{N}\left|\omega_i H_{0_i}\right|^{\alpha_2}\right) e^{-\frac{1}{2}\sum_{i=1}^{N}\left|\omega_i\right|^{\alpha_1}} dt_1\cdots dt_N \qquad (ง.40)$$

$$= \frac{A_1}{2\left|H_1^T\right|}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\left|\omega_1 H_{0_1}\right|^{\alpha_2}+\left|\omega_2 H_{0_2}\right|^{\alpha_2}+\cdots+\left|\omega_N H_{0_N}\right|^{\alpha_2}\right)e^{-\frac{1}{2}\sum_{i=1}^{N}\left|\omega_i\right|^{\alpha_1}}dt_1\cdots dt_N \qquad (ง.41)$$

$$= \frac{A_1}{2\left|H_1^T\right|}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\left|\omega_1 H_{0_1}\right|^{\alpha_2}+\left|\omega_2 H_{0_2}\right|^{\alpha_2}+\cdots+\left|\omega_N H_{0_N}\right|^{\alpha_2}\right)e^{-\frac{1}{2}\left(\left|\omega_1\right|^{\alpha_1}+\left|\omega_2\right|^{\alpha_1}+\cdots\left|\omega_N\right|^{\alpha_1}\right)}dt_1\cdots dt_N \qquad (ง.42)$$

$$\begin{aligned}= \frac{A_1}{2\left|H_1^T\right|}\Bigg[&\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|\omega_1 H_{0_1}\right|^{\alpha_2} e^{-\frac{1}{2}\left(\left|\omega_1\right|^{\alpha_1}+\left|\omega_2\right|^{\alpha_1}+\cdots+\left|\omega_N\right|^{\alpha_1}\right)}d\omega_1\cdots d\omega_N \\ &+\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|\omega_2 H_{0_2}\right|^{\alpha_2} e^{-\frac{1}{2}\left(\left|\omega_1\right|^{\alpha_1}+\left|\omega_2\right|^{\alpha_1}+\cdots+\left|\omega_N\right|^{\alpha_1}\right)}d\omega_1\cdots d\omega_N \qquad (ง.43)\\ &+\cdots+\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|\omega_N H_{0_N}\right|^{\alpha_2} e^{-\frac{1}{2}\left(\left|\omega_1\right|^{\alpha_1}+\left|\omega_2\right|^{\alpha_1}+\cdots+\left|\omega_N\right|^{\alpha_1}\right)}d\omega_1\cdots d\omega_N\Bigg]\end{aligned}$$

$$\begin{aligned}= \frac{A_1}{2\left|H_1^T\right|}\Bigg[&\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|\omega_1 H_{0_1}\right|^{\alpha_2}\left(e^{-\frac{1}{2}\left|\omega_1\right|^{\alpha_1}}e^{-\frac{1}{2}\left|\omega_2\right|^{\alpha_1}}\cdots e^{-\frac{1}{2}\left|\omega_N\right|^{\alpha_1}}\right)d\omega_1\cdots d\omega_N \\ &+\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|\omega_2 H_{0_2}\right|^{\alpha_2}\left(e^{-\frac{1}{2}\left|\omega_1\right|^{\alpha_1}}e^{-\frac{1}{2}\left|\omega_2\right|^{\alpha_1}}\cdots e^{-\frac{1}{2}\left|\omega_N\right|^{\alpha_1}}\right)d\omega_1\cdots d\omega_N \qquad (ง.44)\\ &+\cdots+\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left|\omega_N H_{0_N}\right|^{\alpha_2}\left(e^{-\frac{1}{2}\left|\omega_1\right|^{\alpha_1}}e^{-\frac{1}{2}\left|\omega_2\right|^{\alpha_1}}\cdots e^{-\frac{1}{2}\left|\omega_N\right|^{\alpha_1}}\right)d\omega_1\cdots d\omega_N\Bigg]\end{aligned}$$

$$\begin{aligned}= \frac{A_1}{2\left|H_1^T\right|}\Bigg[&\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(e^{-\frac{1}{2}\left|\omega_2\right|^{\alpha_1}}\cdots e^{-\frac{1}{2}\left|\omega_N\right|^{\alpha_1}}d\omega_2\cdots d\omega_N\right)\left(\left|\omega_1 H_{0_1}\right|^{\alpha_2}e^{-\frac{1}{2}\left|\omega_1\right|^{\alpha_1}}d\omega_1\right) \\ &+\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(e^{-\frac{1}{2}\left|\omega_1\right|^{\alpha_1}}e^{-\frac{1}{2}\left|\omega_3\right|^{\alpha_1}}\cdots e^{-\frac{1}{2}\left|\omega_N\right|^{\alpha_1}}d\omega_1 d\omega_3\cdots d\omega_N\right)\left(\left|\omega_2 H_{0_2}\right|^{\alpha_2}e^{-\frac{1}{2}\left|\omega_1\right|^{\alpha_1}}d\omega_2\right) \qquad (ง.45)\\ &+\cdots+\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(e^{-\frac{1}{2}\left|\omega_1\right|^{\alpha_1}}\cdots e^{-\frac{1}{2}\left|\omega_{N-1}\right|^{\alpha_1}}d\omega_1\cdots d\omega_{N-1}\right)\left(\left|\omega_N H_{0_N}\right|^{\alpha_2}e^{-\frac{1}{2}\left|\omega_N\right|^{\alpha_1}}d\omega_N\right)\Bigg]\end{aligned}$$

คูณ $\frac{a^N}{a^N}$ ในสมการ (ง.45) จะได้

$$= \frac{A_1}{2\left|H_1^T\right|a^N}\Bigg[\underbrace{\left(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} ae^{-\frac{1}{2}|\omega_2|^{\alpha_1}}\cdots ae^{-\frac{1}{2}|\omega_N|^{\alpha_1}} d\omega_2\cdots d\omega_N\right)}_{1}\left(\int_{-\infty}^{\infty}\left|\omega_1 H_{0_1}\right|^{\alpha_2} ae^{-\frac{1}{2}|\omega_1|^{\alpha_1}} d\omega_1\right)$$

$$+\underbrace{\left(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} ae^{-\frac{1}{2}|\omega_1|^{\alpha_1}} ae^{-\frac{1}{2}|\omega_3|^{\alpha_1}}\cdots ae^{-\frac{1}{2}|\omega_N|^{\alpha_1}} d\omega_1 d\omega_3\cdots d\omega_N\right)}_{1}\left(\int_{-\infty}^{\infty}\left|\omega_2 H_{0_2}\right|^{\alpha_2} ae^{-\frac{1}{2}|\omega_1|^{\alpha_1}} d\omega_2\right) \quad (ง.46)$$

$$+\cdots+\underbrace{\left(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} ae^{-\frac{1}{2}|\omega_1|^{\alpha_1}}\cdots ae^{-\frac{1}{2}|\omega_{N-1}|^{\alpha_1}} d\omega_1\cdots d\omega_{N-1}\right)}_{1}\left(\int_{-\infty}^{\infty}\left|\omega_N H_{0_N}\right|^{\alpha_2} ae^{-\frac{1}{2}|\omega_N|^{\alpha_1}} d\omega_N\right)\Bigg]$$

$$= \frac{A_1}{2\left|H_1^T\right|a^N}\Bigg[\left(\int_{-\infty}^{\infty}\left|\omega_1 H_{0_1}\right|^{\alpha_2} ae^{-\frac{1}{2}|\omega_1|^{\alpha_1}} d\omega_1\right)+\left(\int_{-\infty}^{\infty}\left|\omega_2 H_{0_2}\right|^{\alpha_2} ae^{-\frac{1}{2}|\omega_1|^{\alpha_1}} d\omega_2\right)$$

$$+\cdots+\left(\int_{-\infty}^{\infty}\left|\omega_N H_{0_N}\right|^{\alpha_2} ae^{-\frac{1}{2}|\omega_N|^{\alpha_1}} d\omega_N\right)\Bigg] \quad (ง.47)$$

$$= \frac{A_1}{2\left|H_1^T\right|a^N}\Bigg[\left(H_{0_1}^{\alpha_2}\int_{-\infty}^{\infty}\left|\omega_1\right|^{\alpha_2} ae^{-\frac{1}{2}|\omega_1|^{\alpha_1}} d\omega_1\right)+\left(H_{0_2}^{\alpha_2}\int_{-\infty}^{\infty}\left|\omega_2\right|^{\alpha_2} ae^{-\frac{1}{2}|\omega_1|^{\alpha_1}} d\omega_2\right)$$

$$+\cdots+\left(H_{0_N}^{\alpha_2}\int_{-\infty}^{\infty}\left|\omega_N\right|^{\alpha_2} ae^{-\frac{1}{2}|\omega_N|^{\alpha_1}} d\omega_N\right)\Bigg] \quad (ง.48)$$

จากสมการ (ง.24) ฟังก์ชันแกมม่า

$$\int_{-\infty}^{\infty}|\omega|^n ae^{-\gamma^\alpha|\omega|^\alpha} d\omega = \left(\frac{1}{\gamma}\right)^n \frac{\Gamma\left(\frac{n+1}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}\right)}$$

$$n=\alpha_2, \gamma_1 = \left(\frac{1}{2}\right)^{\frac{1}{\alpha_1}} \quad ; \quad (2)^{\frac{\alpha_2}{\alpha_1}}\frac{\Gamma\left(\frac{\alpha_2+1}{\alpha_1}\right)}{\Gamma\left(\frac{1}{\alpha_1}\right)} \quad (ง.49)$$

ประยุกต์สมการ (ง.49) ใน (ง.48) จะได้

$$=\frac{A_1}{2\left|H_1^T\right|a^N}\left[\left(H_{0_1}^{\alpha_2}(2)^{\frac{\alpha_2}{\alpha_1}}\frac{\Gamma\left(\frac{\alpha_2+1}{\alpha_1}\right)}{\Gamma\left(\frac{1}{\alpha_1}\right)}\right)+\left(H_{0_2}^{\alpha_2}(2)^{\frac{\alpha_2}{\alpha_1}}\frac{\Gamma\left(\frac{\alpha_2+1}{\alpha_1}\right)}{\Gamma\left(\frac{1}{\alpha_1}\right)}\right)+\cdots+\left(H_{0_N}^{\alpha_2}(2)^{\frac{\alpha_2}{\alpha_1}}\frac{\Gamma\left(\frac{\alpha_2+1}{\alpha_1}\right)}{\Gamma\left(\frac{1}{\alpha_1}\right)}\right)\right] \quad \text{(ง.50)}$$

$$=\frac{A_1}{2\left|H_1^T\right|a^N}(2)^{\frac{\alpha_2}{\alpha_1}}\frac{\Gamma\left(\frac{\alpha_2+1}{\alpha_1}\right)}{\Gamma\left(\frac{1}{\alpha_1}\right)}\left(\left(H_{0_1}^{\alpha_2}\right)+\left(H_{0_2}^{\alpha_2}\right)+\cdots+\left(H_{0_N}^{\alpha_2}\right)\right) \quad \text{(ง.51)}$$

แทนค่า $tr\left(H_0^{\alpha_2}\right)=\left(\left(H_{0_1}^{\alpha_2}\right)+\left(H_{0_2}^{\alpha_2}\right)+\cdots+\left(H_{0_N}^{\alpha_2}\right)\right)$ (ง.52)

ในสมการ (ง.51)

$$\approx\frac{A_1}{2\left|H_1^T\right|a^N}(2)^{\frac{\alpha_2}{\alpha_1}}\frac{\Gamma\left(\frac{\alpha_2+1}{\alpha_1}\right)}{\Gamma\left(\frac{1}{\alpha_1}\right)}tr\left(H_0^{\alpha_2}\right) \quad \text{(ง.53)}$$

แทนค่า $a, A => \frac{A_1}{\left|H_1^T\right|a^N}=1$ ในสมการ (ง.53) จะได้

$$=(2)^{\frac{\alpha_2}{\alpha_1}-1}\frac{\Gamma\left(\frac{\alpha_2+1}{\alpha_1}\right)}{\Gamma\left(\frac{1}{\alpha_1}\right)}tr\left(H_0^{\alpha_2}\right) \quad \text{(ง.54)}$$

แทนค่า $H_0=H_2^TH_1^{-T}$ ในสมการ (ง.54)

$$=(2)^{\frac{\alpha_2}{\alpha_1}-1}tr\left(\left(H_2^TH_1^{-T}\right)^{\alpha_2}\right)\frac{\Gamma\left(\frac{\alpha_2+1}{\alpha_1}\right)}{\Gamma\left(\frac{1}{\alpha_1}\right)} \quad \text{(ง.55)}$$

ดังนั้น

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\varphi_1'\frac{1}{2}\left(t^T\Sigma_2 t\right)^{\alpha_2/2}dt_1\cdots dt_N \approx (2)^{\frac{\alpha_2}{\alpha_1}-1} tr\left(\left(H_2^T H_1^{-T}\right)^{\alpha_2}\right)\frac{\Gamma\left(\frac{\alpha_2+1}{\alpha_1}\right)}{\Gamma\left(\frac{1}{\alpha_1}\right)} \quad (ง.56)$$

แทนสมการ (ง.32) และ (ง.56) ในสมการ (ง.7) จะได้

$$D\left(\varphi_1' \| \varphi_2'\right) \approx \ln\frac{A_1}{A_2} - \frac{N}{\alpha_1} + (2)^{\frac{\alpha_2}{\alpha_1}-1} tr\left(\left(H_2^T H_1^{-T}\right)^{\alpha_2}\right)\frac{\Gamma\left(\frac{\alpha_2+1}{\alpha_1}\right)}{\Gamma\left(\frac{1}{\alpha_1}\right)} \quad (ง.57)$$

$$D\left(\varphi_1' \| \varphi_2'\right) \approx N\ln\frac{\alpha_1\Gamma\left(\frac{1}{\alpha_2}\right)}{\alpha_2\Gamma\left(\frac{1}{\alpha_1}\right)} + \ln\frac{|\Sigma_1|}{|\Sigma_2|} + -\frac{N}{\alpha_1} + (2)^{\frac{\alpha_2}{\alpha_1}-1} tr\left(\left(H_2^T H_1^{-T}\right)^{\alpha_2}\right)\frac{\Gamma\left(\frac{\alpha_2+1}{\alpha_1}\right)}{\Gamma\left(\frac{1}{\alpha_1}\right)} \quad (ง.58)$$

เมื่อ $N=1$ จะเป็นการแจกแจงแบบแอลฟ่าสเตเบิลตัวแปรเดียว และมีสมการ KLD ดังนี้

$$D\left(\varphi_1' \| \varphi_2'\right) = \ln\left(\frac{a_1}{a_2}\right) - \frac{1}{\alpha_1} + \left(\frac{\gamma_2}{\gamma_1}\right)^{\alpha_2}\frac{\Gamma\left(\frac{\alpha_2+1}{\alpha_1}\right)}{\Gamma\left(\frac{1}{\alpha_1}\right)} \quad (ง.59)$$

Visual Inspection in Textured Materials Using Generalized Gaussian Density and Kullback-Leibler Distance

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Abstract

This paper presents a new statistical method for inspection of textured materials. The algorithm consists of feature extraction (FE) with similarity measure (SM) and thresholding (TS). In the FE step, each pixel of the test image is modeled using generalized Gaussian Density (GGD) with parameters estimated from its neighboring pixels in $(r \times c)$ window. Then Kullback-Leibler distance (KLD) gives the similarity between the obtained GGD of each pixel and the GGD of the template obtained from the non-defective image. The decision making step or the thresholding (TS) step uses maximum-value distance obtained from the template image. We evaluate the performance of the proposed method with experiments on fabric and metal images.

Keywords: defect detection, statistical modeling, generalized Gaussian density, Kullback-Leibler distance.

1. Introduction

Surface inspection is one of the crucial steps in manufacturing. Visual inspection of industrial products requires low-cost, high-speed, and high-quality detection of defects. Some of the most challenging industrial inspection problems deal with the textured materials such as textile web, paper, wood, and metal [1]. At present human operators still perform most of these visual inspection tasks. But automatic visual inspection machines seem to provide a better solution since human's fatigue and carelessness often cause errors in the inspection. These errors will cause severe damages and losses as unqualified products are shipped to customers or must be discarded.

The inspection problem encountered in textured materials becomes texture analysis problems at microscopic levels. Image texture analysis also plays an important role in many image processing tasks, medical image analysis, document processing, remote sensing, query by content in large image databases and visual inspection on surface [1]-[7]. Several researchers work on defect detection using statistical models such as gray-level texture feature extracted from co-occurrence matrix [2], mean and standard deviations, autocorrelation of subimages [3]. The texture demonstrates a high order of periodicity and, hence, Fourier-domain features have been used for the detection of fabric defects [4]. Then detection of local defects requires multiresolution decomposition of fabric images across several scales. A feature vector composed of significant features at each scales is used for the identification of defects. Such a multiresolution analysis of fabric using discrete wavelet transform (DWT) has been detailed in [5]. Jasper *et al*. [6] use texture-adapted wavelet bases whose response is close to zero for normal fabric texture and significantly different for fabric defect, thereby enabling detection. Escofet *et al*. [7] use multiscale Gabor filters for textile web defect detection. Other researchers [8]-[10] propose the use of wavelet transform as features extraction and then model the wavelet coefficients with generalized Gaussian density (GGD). Then they use Kullback-Leibler distance (KLD) as a way to measure similarity between the GGDs and then apply this technique to image retrieval. All of these methods are computational intensive and they require optimization of several parameters.

This paper proposes a defect inspection algorithm based on statistical models of textured images. The research effort intends to develop an automatic visual inspection system that can reduce human errors and increase productivity. The paper explores an idea of defect detection in test images using generalized Gaussian density (GGD) modelling of the gray-scale value of each pixel of the test image without any pre-processing such as Gabor filtering or wavelet transformation. The KLD measures the similarity between the GGD models of each pixel in the test images and the template of non-defective image. The obtained KLD map will be used as features and the defect detection step will use variable threshold to obtain the final defect map. The proposed method requires less computational time and is simpler for implementation.

The organization of this paper is as follows. Section 2 describes the proposed visual inspection method. Section 3 shows experimental results on fabric and metallic defect samples. Then Section 4 gives conclusions of the paper.

2. Defect Inspection based on GGD models of pixels

We use the generalized Gaussian density (GGD) to model the probability density function (pdf) of each pixel of an image. The extracted densities are obtained from the pixels' neighbors that we call template (or the non-defective example of the textured image) and subimages (the overlapping, windowed samples from the image under inspection). So we obtain the parameters for GGD model for the template image and another GGD model for the pixel of the test subimage. Then the KLD measures the distance between these two GGDs and gives a distance map of the test image. The low-valued distance pixel indicates small difference of textures (non-defective pixel) of the pixel while the high-valued distance pixel indicates large difference of textures or high possibility of the pixel being defective. Finally, we use threshold that depends on the maximum-valued distance of template to detect defective pixels. The detailed algorithm is shown in Fig. 1.

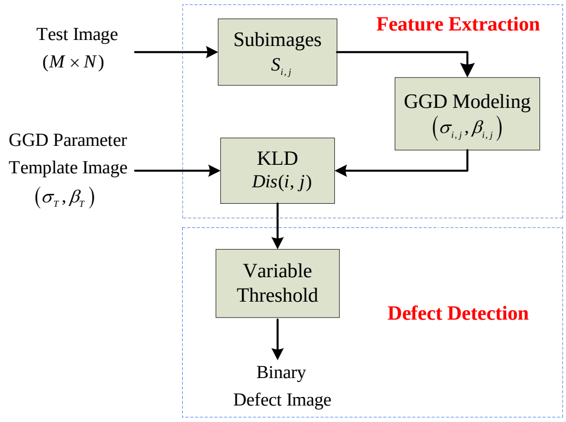


Fig. 1 The proposed visual inspection system based on GGD and KLD with variable threshold.

2.1 GGD Modeling of Gray Scale Value

Generalized Gaussian density (GGD) has been use as a model of wavelet coefficients [11] and used in several applications that include image retrieval [10] and texture analysis [12]. Our method employs the GGD as a probability density function of each gray-scale pixel by directly fitting the histograms of the neighboring pixels. The GGD has the form [10]-[12]

$$p(x;\sigma,\beta) = Ke^{-(|x|/\sigma)^{\beta}} \tag{1}$$

where β is inversely proportional to the decreasing rate of the histogram peak, σ models the width of the histogram (or standard deviation), and $K = \dfrac{\beta}{2\sigma\Gamma(1/\beta)}$ is a normalization constant. Usually σ is referred to as the scale parameter and β as the shape parameter. The GGD model contains the Gaussian density when $\beta = 2$ and the Laplacian density when $\beta = 1$.

Here we use a window of $r \times c$ pixels moving on a test image of size $M \times N$ pixels to get the subimages $S_{i,j}$. The window is moved on row and column to crop the test image. We model each pixel (i, j) of the test image (the image under inspection) with a GGD model with parameter $\sigma_{i,j}$ and $\beta_{i,j}$. The two parameters $\sigma_{i,j}$ and $\beta_{i,j}$ are estimated using all the $(r \times c)$ pixels in the subimages $S_{i,j}$ as follows

$$\sigma = m_1 \frac{\Gamma(1/\beta)}{\Gamma(2/\beta)} \text{ and } \beta = F^{-1}\left(\frac{m_1^2}{m_2}\right) \tag{2}$$

where $\Gamma(\cdot)$ is the Gamma function $\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt$ and

$$F(x) = \frac{\Gamma^2(2/x)}{\Gamma(3/x)\Gamma(1/x)} \tag{3}$$

The parameters (the moments) m_1 and m_2 in (2) can be estimated from neighboring pixels in the $r \times c$ window:

$$m_1 = \frac{1}{rc}\sum_r\sum_c \left|S_{i,j}(x,y)\right| \tag{4}$$

$$m_2 = \frac{1}{rc}\sum_r\sum_c \left|S_{i,j}(x,y) - m_1\right| \tag{5}$$

2.2 Kullback-Leibler Distance

The Kullback-Leibler distance is possibly the most frequently used information-theoretic distance measurement [13]. If $p(x,\theta_q)$ and $p(x,\theta_T)$ are two probability densities where θ_q is a set of model parameter (test image) and θ_T is a set of model parameter of template (non-defective image) the Kullback-Leibler distance is defined by

$$D\left(p(x;\theta_q) \,||\, p(x;\theta_T)\right) = \int p(x;\theta_q)\log\frac{p(x;\theta_q)}{p(x;\theta_T)}dx. \tag{6}$$

The closed form for the KLD between two GGDs has the form [4]

$$D\left(p_{\sigma_1,\beta_1} \,||\, p_{\sigma_2,\beta_2}\right) = \log\left(\frac{\beta_1\sigma_2\Gamma(1/\beta_2)}{\beta_2\sigma_1\Gamma(1/\beta_1)}\right) + \left(\frac{\sigma_1}{\sigma_2}\right)^{\beta_2}\frac{\Gamma\left(\frac{\beta_2+1}{\beta_1}\right)}{\Gamma\left(\frac{1}{\beta_1}\right)} - \frac{1}{\beta_1} \tag{7}$$

where $\sigma_1 = \sigma_{i,j}$, $\beta_1 = \beta_{i,j}$ are parameters of the GGD model of the pixel (i, j) and $\sigma_2 = \sigma_T$, $\beta_2 = \beta_T$ are parameters of the GGD model of the template. Note that in general $D(p_1 \,||\, p_2) \neq D(p_2 \,||\, p_1)$ where p_1 and p_2 are two pdf's and thus KLD is not symmetric. So it is not truly a distance function but it is always non-negative [13]. We can modify our distance measure as $D(p_1, p_2) = D(p_1 \,||\, p_2) + D(p_2 \,||\, p_1)$ or other variants to make it symmetric. In this paper we use the original form of KLD in Eq (6).

2.3 Defect Detection

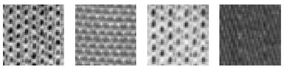
The distance image $Dis(i, j)$ is generated when we compute the distance of GGD models. Next the feature under test is transformed to binary image as follows:

$$B(i,j) = \begin{cases} 0 & Dis(i,j) > \gamma \\ 1 & Otherwise \end{cases} \tag{8}$$

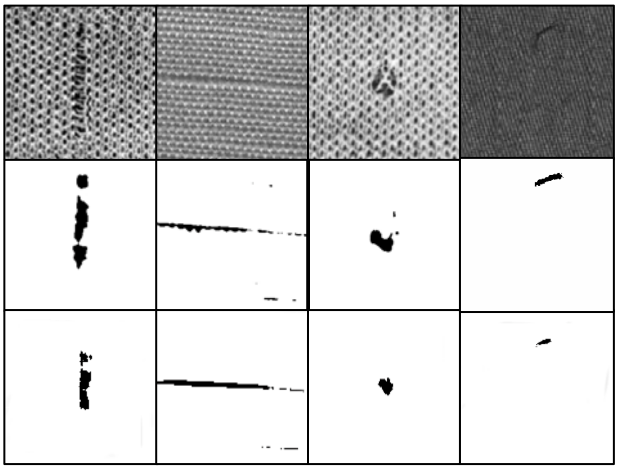
where γ is the threshold obtained from the maximum-valued distance of the template image. Thresholding produces a binary image of possible defects in the image under inspection. The defective pixel has value 0 (black) and a normal pixel has value 1 (white). The threshold γ determines the sensitivity of this algorithm. The threshold depends on texture of the material and the environment in capturing the images.

3. Experimental Results

We tested our algorithm on gray-scale images that included images of fabric and metallic textures materials as shown in Fig. 2 and Fig. 3. The defective samples included defects commonly found in industry [14]. We chose the template image from non-defective parts of the surface materials as shown in Fig. 2a and Fig. 3a. We estimated the parameters of the GGD models of the template material images (σ_T, β_T) to calculate the KLD of template and used the maximum-valued distance as the threshold for defect detection step.



(a)



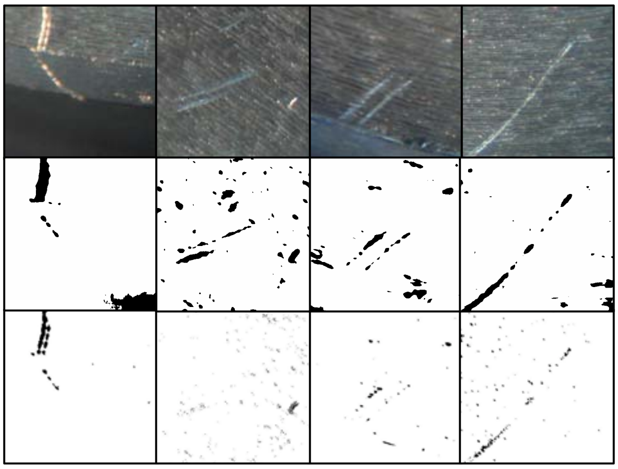
(b) (c) (d) (e)

Fig. 2 Defect detection of fabric textures. (a) templates. (b)-(e) test images (top row: (b) tear defect. (c) coarse pick defect. (d) hole defect. (e) big-knot defect.) and detection results: (middle row: our method; bottom row: Gabor wavelet filtering.)

We used a window size $r \times c = 12 \times 12$ for fabric images in Fig. 2 (top row of Fig. 2b-e). The fabric defect detection results using the proposed algorithm are shown in the middle row of Fig. 2b-e where the thresholds are 0.3, 0.1, 0.2 and 0.1. The results using Gabor filtering are shown in bottom row of Fig. 2b-e for comparison. The proposed method can detect more defects than Gabor filer. Fig. 3 shows the results of defect detection for images of metallic surfaces (top row of Fig. 3b-e). In this case we used a window size $r \times c = 10 \times 10$. The results of our proposed algorithm are shown in the middle row of Fig. 3b-e (with thresholds 0.7, 0.2, 0.3 and 0.3). The bottom row of Fig. 3b-e shows the results using Gabor filtering. Our method can detect more parts of the defects.

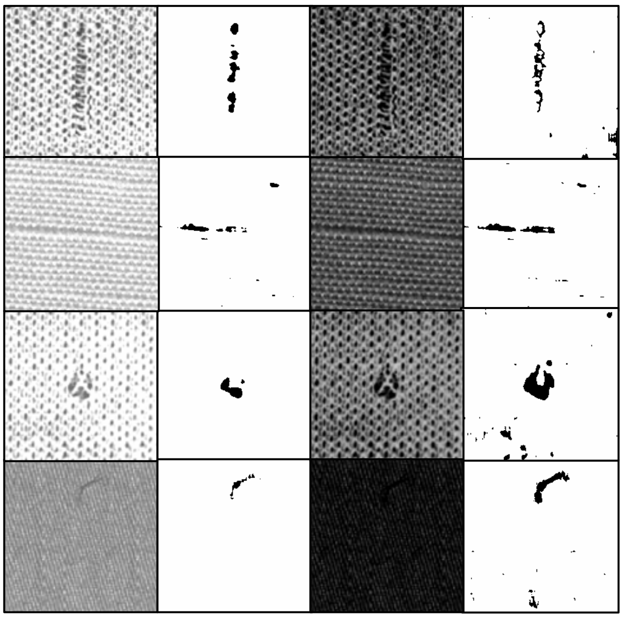
Next we tested the robustness of our algorithm on the same set of images using simulated illumination (by addition or subtraction of pixel values). Fig 4a shows the samples of images with addition of 30% constant illumination and Fig 4b with subtraction of 30% constant illumination. So the images in Fig 4a become brighter and Fig 4b become darker. We use the window size, threshold and template as same as Fig. 3 and Fig. 4. The defect detection results are in Fig 4b and Fig 4d.

(a)



(b) (c) (d) (e)

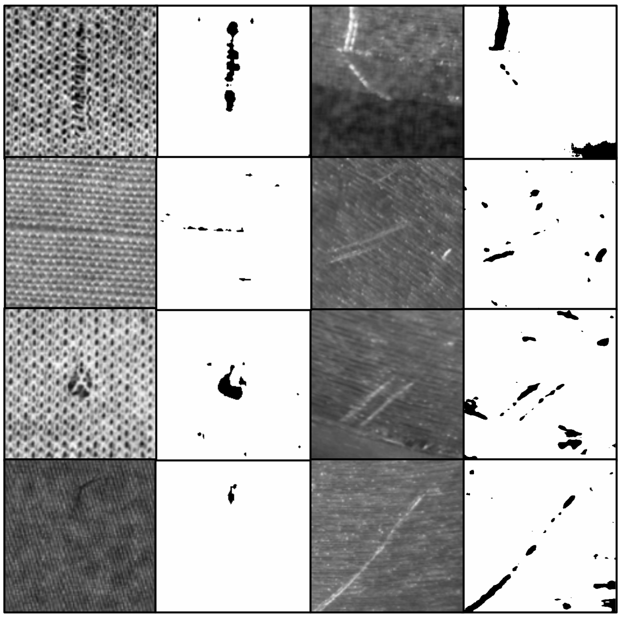
Fig. 3 Defect detection of metallic surfaces.
(a) templates. (b)-(e) test images and detection results:
(middle row: our method; bottom row: Gabor filtering.)



(a) (b) (c) (d)

Fig. 4 Test of uniformly-illuminated images. (a) modified fabric surfaces with addition of 30% brightness. (b) defect detection results of (a). (c) modified fabric surfaces with subtraction of 30% darkness. (d) defect detection results of (c).

Fig. 5 shows another test case where the pixels values that are added to the original images have Gaussian distribution with zero-mean and variance 80 (based on the pixel value 255) and then low-pass filtered with a window size 20×20 that the value of window is 1/400. The defect detection results are good.



(a) (b) (c) (d)

Fig. 5 Test of Gaussian illuminated images. (a) modified fabric surfaces. (b) defect detection results of (a). (c) modified metallic surfaces. (d) defect detection results of (c).

4. Conclusions

This paper presents an algorithm based on generalized Gaussian density (GGD) modeling of pixel values of textures and Kullback-Leibler distance (KLD) for fabric and metal defect detection. Experiments show that this simple algorithm can perform well comparing to the Gabor filtering presently used in texture analysis and defect detection. The advantages of our method are as follows: The number of required parameters for GGD to be estimated is small in contrast to Gabor wavelet filtering [14], [15]. The implementation would be simpler and the required processing time would be smaller. The algorithm is robust for the variation of uniform and non-uniform illumination across the image using the same threshold. Adaptive threshold may improve this robustness.

However, the method has several limitations. The textured images must always be periodic or look alike. The algorithm is solely based on the pdf of the pixels and hence it would fail for two different textures (or defects) that have similar pdf's. We assume that the case in which defects have similar pdf's to the templates are rare. Finally the size of the window affects detection performance. Too big window size would ignore small defects and too small window size would give noisy detection. Future research will also focus on optimal window sizing that depends on characteristics of textures and defects. Then adaptive threshold will add the performance of algorithm.

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