CHAPTER 2 THEORIES

Calculation of dimension is the origin of fractal dimension estimation. Dimensions of geometric structures are simple to understand and the dimensions are not fractional number.

2.1 Calculation of Dimension

For the line, square, and cube there is a simple power law that relates the reduction factor, the number of pieces and the dimension. This relationship is given by the equation (Peitgen, 2004)

$$N = \frac{1}{R^D} \tag{2.1}$$

When taking logarithms on both sides,

$$D = \frac{\log N}{\log\left(1/R\right)} \tag{2.2}$$

where D is the dimension
R is the reduction factor
N is the number of pieces (Figure2.1).

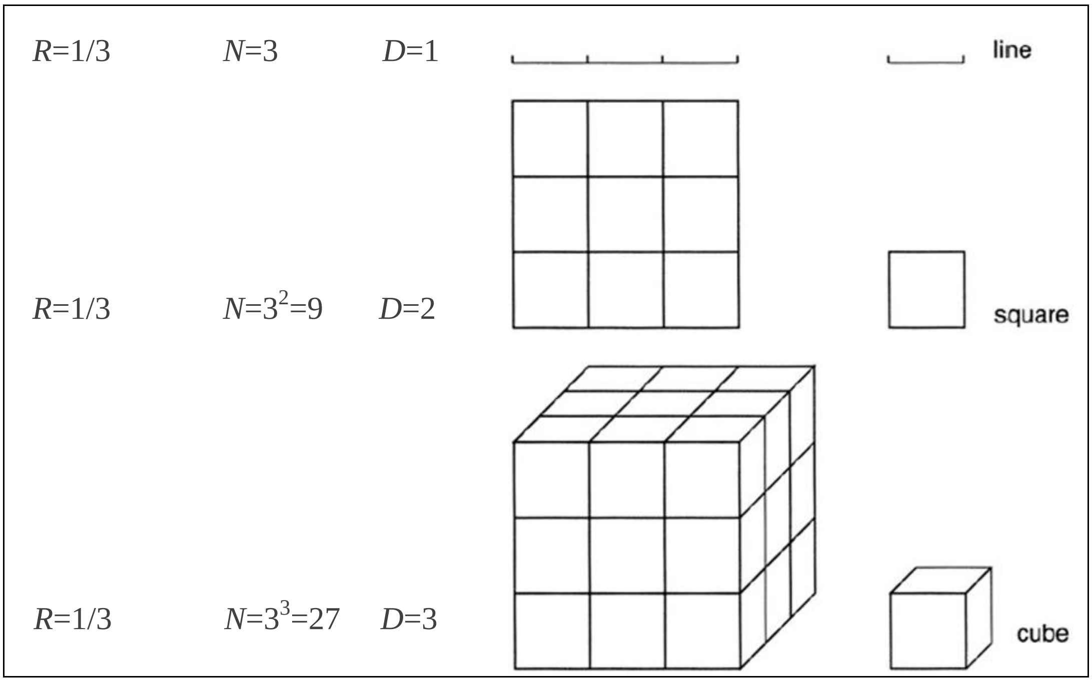

Figure 2.1 Structures of line, square and cube (Peitgen, 2004).

The reduction factors and the corresponding number of pieces are shows in Table 2.1.

Table 2.1 The reduction factors and the associated number of pieces for line, square and cube (Peitgen, 2004).

Object	Reduction Factor (R)	Number of Pieces (N)	Dimension (D)
Line	1/3	3	1
Line	1/6	6	1
Square	1/3	$9=3^2$	2
Square	1/6	$36=6^2$	2
Cube	1/3	$27=3^3$	3
Cube	1/6	$216=6^3$	3

From Table 2.1, for line, square and cube the dimensions are integer. If line, square, cube object are incompleted, the line, square and cube will have dimensions between zero and one, one and two and two and three, respectively.

An example of fractal object is Koch curve (Figure 2.2). The relationship of the reduction factor and the number of pieces for Koch curve is not so obvious. But being guided by the relation for the line square, and cube. The dimension of Koch curve is non-integer and is called fractal dimension.

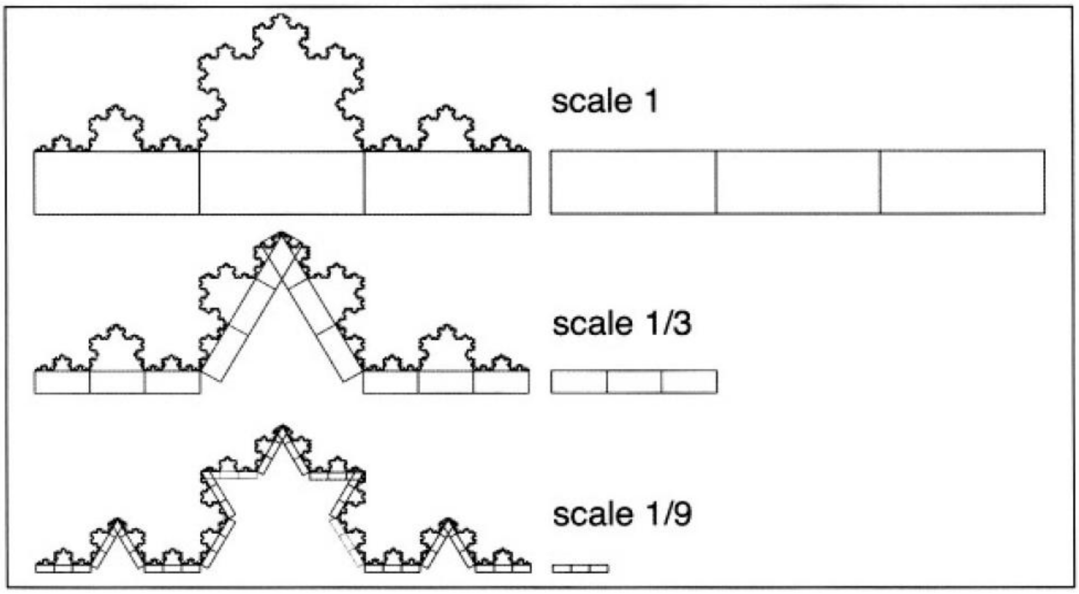

Figure 2.2 Measuring the length of the Koch curve with different scale (reduction factor) settings (Peitgen, 2004).

According to the calculation of fractal dimension of Koch Curve by the power law, it is very difficult to apply. Therefore, approximations of fractal dimension by other methods are applied.

2.2 Approximation of Fractal Dimension

There are many methods to estimate fractal dimension. However, only 6 methods are presented in this section.

2.2.1 Box-Counting Method

The box-counting method is intuitive and easy to apply. It estimates fractal dimension by counting the number of boxes that cover the fractal curve (Breslin, 1999).

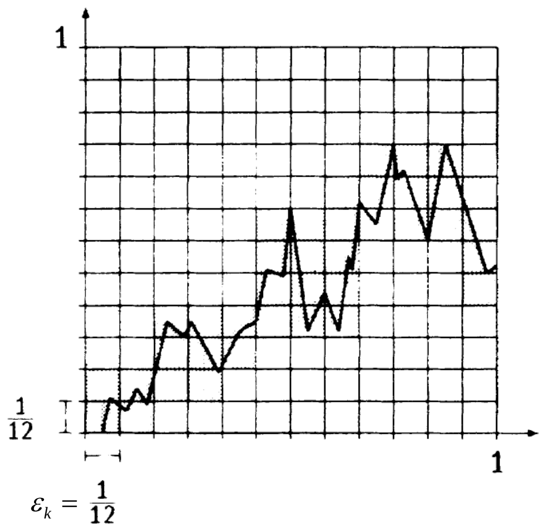

Figure 2.3 Determination of ε_k (Dubuc at al.1989).

The set of time series curve (Figure 2.3) can be covered by a grid with pixel length, ε_k, and the number of pixel, $N(\varepsilon_k)$, the fractal dimension (D_B) can then be written as

$$D_B = \lim_{\varepsilon_k \to 0} \frac{\log N(\varepsilon_k)}{\log(1/\varepsilon_k)} \tag{2.3}$$

where ε_k is the size of the pixels
$N(\varepsilon_k)$ is the number of pixels that covering the curve
K is the number of time unit.

The fractal dimension is given by the slope of $\log N(\varepsilon_k)$ versus $\log(1/\varepsilon_k)$.

2.2.2 Rescale Range Analysis

Rescale range (R/S) analysis (Komn, 1994) is the ratio between the range and standard deviation of time series data, $\xi(t)$, containing N data points for the chosen time period τ as shown in Figure 2.4.

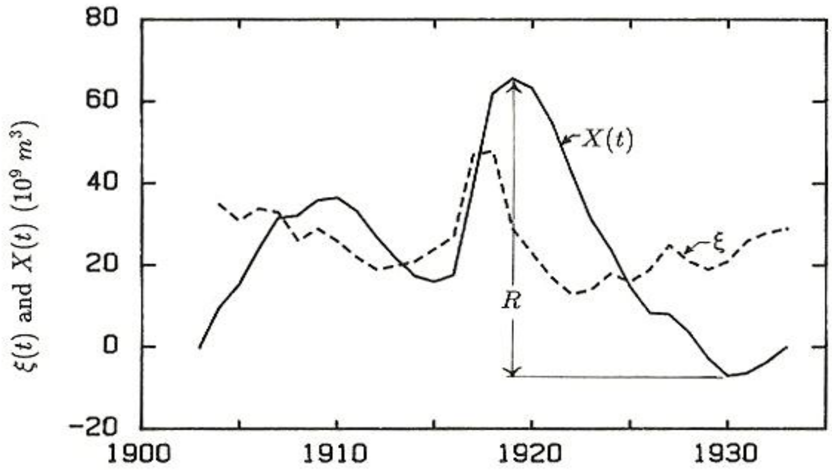

Figure 2.4 Plot of a time series (Feder, 1988).

For the variable ξ, the average $\langle \xi \rangle_{\tau}$ is given by the equation

$$\langle \xi \rangle_{\tau} = \frac{1}{\tau} \sum_{t=1}^{\tau} \xi(t) \tag{2.4}$$

the length of time increment, (τ), and the range, $R(\tau)$, is the difference between the maximum and minimum of data as follows

$$R(\tau) = \max_{1 \le t \le \tau} \xi(t) - \min_{1 \le t \le \tau} \xi(t) \tag{2.5}$$

The standard deviation, $S(\tau)$, is given by the equation

$$S(\tau) = \left(\frac{1}{\tau} \sum_{t=1}^{\tau} \left\{ \xi(t) - \langle \xi \rangle_{\tau} \right\}^{2} \right)^{1/2} \tag{2.6}$$

The ratio between the range and standard deviation, *R/S*, will generally increase with τ and can be described as

$$R/S = (\tau/2)^{H} \tag{2.7}$$

where *H* is the Hurst exponent which can be written as

$$H = \frac{\log(R/S)}{\log(\tau/2)} \tag{2.8}$$

The values of Hurst exponent are shown in Table 2.2 for particle movement in Brownian motion.

Table 2.2 Properties of the value of Hurst exponent (Chang et al., 2012).

Hurst Exponent	Character of Particle Movement
$(0 < H < ½)$	Persistence
$H = 1/2$	Independence
$(½ < H < 1)$	Anti-persistence

$$D_{R/S} = 2 - H \qquad (2.9)$$

The fractal dimension for the rescale range analysis, $D_{R/S}$, can be estimated by the slope of $\log(R/S)$ versus $\log \tau$.

2.2.3 Higuchi Method

Coyt et al., (2013) present a method for estimate fractal dimension of time series data demonstrated by measure the length of the curve as in Figure 2.5.

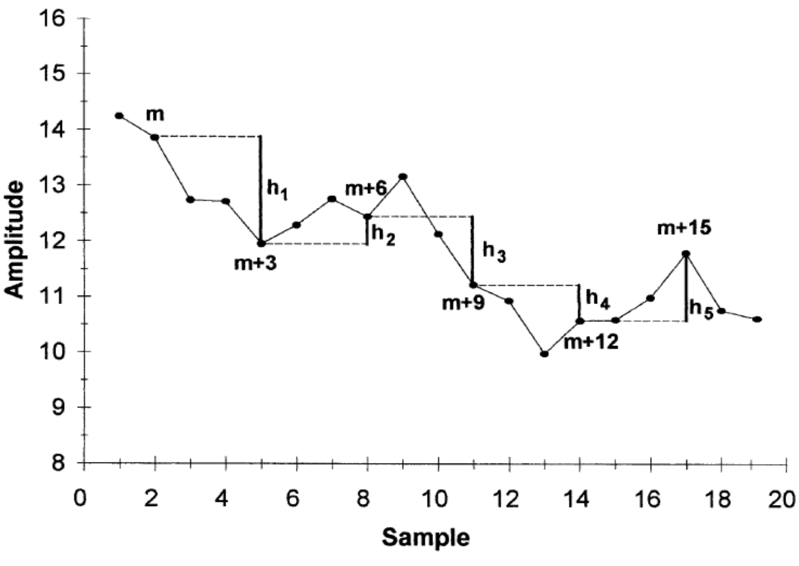

Figure 2.5 Measure the length of the curve by Higuchi method (Pracha, 2011).

Higuchi method is a method for estimate the fractal dimension in a short time data. This method calculates fractal dimension based on the given finite time series $X = \{X(1), X(2), ..., X(N)\}$. A new time series, X_m^k, is given by the equation

$$X_m^k = \left[X(m), X(m+k), X(m+2k), ..., X\left(m + \left[\frac{N-m}{k} \right] k \right) \right], \; m = 1, 2, ..., k \qquad (2.10)$$

where m is the integer number of the initial time

k is the integer number of the interval time.

The length of each curves, $L_m(k)$, is calculated as

$$L_m(k) = \frac{1}{k} \left[\left(\sum_{i=1}^{\left[\frac{N-m}{k}\right]} \left| X(m+ik) - X(m+(i-1)k) \right| \right) \frac{N-1}{\left[\frac{N-m}{k} \right] k} \right] \qquad (2.11)$$

Where N is the total number of samples

$\frac{N-1}{\left[\frac{N-m}{k} \right] k}$ is the normalization factor.

In the fractal dimension by Higuchi method, D_{HM}, is given by the slope of $\log L(k)$ versus $\log(k)$.

2.2.4 Scaling Properties of Variance

A generalized scaling property of variance (GSPV) (Nimkerdphol et al, 2008) is a method to directly estimate fractal dimension for time series data by using the Hurst exponent. This method calculates the variance (σ_α) of the amplitude increments of the time series data on different scale range as follows

$$\sigma_\alpha(t) = \left\langle \left| f(t+\tau) - f(t) \right|^\alpha \right\rangle \sim |\tau|^{\alpha H_\alpha} \qquad (2.12)$$

where α is the order of the number of the time series signal

τ is a range of time increment.

Thus, Hurst exponent can be used to estimate fractal dimension as

$$H_\alpha = \frac{1}{\alpha} \frac{\Delta \log \left\langle \left| f(t+\tau) - f(t) \right|^\alpha \right\rangle}{\Delta \log(|\tau|)} \qquad (2.13)$$

The fractal dimension for scaling properties of variance, D_α, is related to Hurst exponent. Thus, the fractal dimension for this method can be estimated by

$$D_\alpha = 1/H_\alpha \qquad (2.14)$$

2.2.5 Horizontal Structuring Element Method

Horizontal structuring element method (HSEM) (Dubuc et al., 1989) is a method to find fractal dimension by consider the total area that covers the horizontal line segment, containing the square element with the ε length.

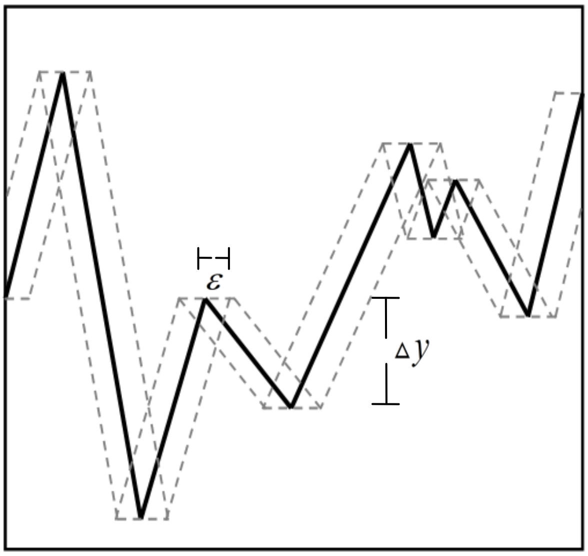

Figure 2.6 Determination of ε (Dubuc at al., 2006).

The fractal dimension by the HSEM, D_{HSEM}, is given by the equation

$$D_{HSEM} = \lim_{\varepsilon \to 0} \frac{\log \frac{1}{\varepsilon^2} U(\varepsilon)}{\log(1/\varepsilon)} \tag{2.15}$$

where ε is the length of the horizontal segment (Figure2.6)

$U(\varepsilon)$ is the total area (union) that covering the horizontal segment.

The fractal dimension is given by the slope of $\log \frac{1}{\varepsilon^2} U(\varepsilon)$ versus $\log(\frac{1}{\varepsilon})$.

2.2.6 Variation Method (VM)

Variation method (Dubuc et al., 1989) is a method of estimating fractal dimension that incorporates upper and lower boundaries of the horizontal segment cover into the estimation.

A covering of the variation method is constructed by calculating the 'ε-oscillation' at points along the curve. For the continuous curve $y = f(x)$, the ε-oscillation at a point $v_i(x,\varepsilon)$ is simply

$$v_i(x,\varepsilon) = \max f(x_i) - \min f(x_i), \quad |x_i - x| < \varepsilon \tag{2.16}$$

This corresponds to the height of the cover (Figure 2.7). That is, ε gives the scale at which the oscillation is measured, as ε decreases, so does the cover. The area of the cover, found by integrating v, is known as the ε-variation of f, and is denoted as $V(\varepsilon)$,

$$V(\varepsilon) = \int_0^1 v_i(x,\varepsilon)dx \tag{2.17}$$

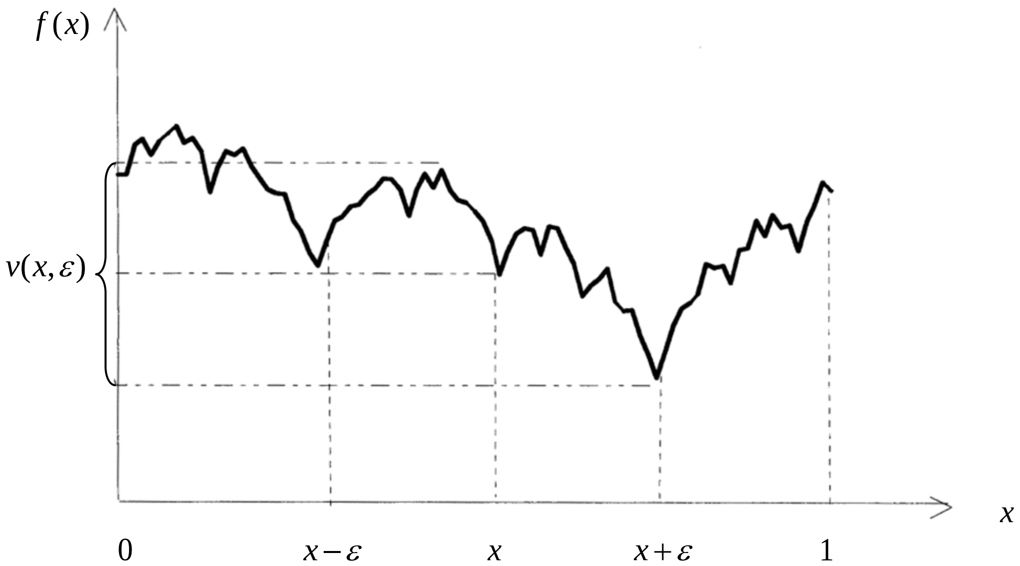

Figure 2.7 ε-oscillation of $f(x)$ used in the variation method (Charkaluk at al., 1998).

This is the area covered by upper and lower boundaries along the curve $f(x)$ with a horizontal distance of ε. To find the fractal dimension, the rate at which the area, $V(\varepsilon)$, tends to 0 as ε tends to 0 is calculated. The fractal dimension is the slope of a log-log plot of ($V(\varepsilon)/\varepsilon^2$) vs. ($1/\varepsilon$). That is, the fractal dimension for the variation method, D_v, is defined according to the equation

$$N = \frac{1}{R^{D_v}}$$

$$\log N = \log\left(\frac{1}{R^{D_v}}\right)$$

$$\log N = \log 1 - \log R^{D_v}$$

$$\log N = -\log R^{D_v}$$

$$\log N = -D_v \log R$$

$$D_v = \frac{\log N}{-\log R}$$

$$D_v = \frac{\log N}{\log\left(\frac{1}{R}\right)}$$

From $N = \frac{1}{\varepsilon^2}V(\varepsilon)$ and $R = \varepsilon$ therefore,

$$D_V = \lim_{\varepsilon \to 0} \frac{\log\left(\frac{1}{\varepsilon^2}V(\varepsilon)\right)}{\log(1/\varepsilon)} \tag{2.18}$$

2.3 Comparison of Methods for Estimating Fractal Dimension

From Section 2.2, six methods for estimating fractal dimension are explained. Comparisons of these methods are presented in Table 2.3 based on fractional Brownian motion (FBM) curve. Fractal dimension of FBM curve are 1.4 by theoretical analysis.

Table 2.3 Comparison of the methods for estimating fractal dimension.

Methods	Speed of Calculation	Estimation Errors	Remark	Reference
1. Box-Counting	Fast	Small	For FBM curve, the dimension is 1.338.	Dubuc et al., 1989
2. R/S	Fast	Small	Estimation error for Box-Counting and R/S are close.	Elena, 1996
3. Higuchi	Medium	Large	For FBM curve, the dimension is 1.5075.	Coyt et al., 2013
4. SPV	Medium	Large	SPV and Higuchi are close.	Nimkerdphol et al., 2008
5. HSEM	Slow	Medium	For FBM curve, the dimension is 1.482.	Dubuc et al., 1989
6. Variation	Slow	Smallest	For FBM curve, the dimension is 1.408.	Dubuc et al., 1989

From Table 2.3, variation method has the smallest error but with slow calculation speed due to complex algorithm. In this study, this method is developed further and is used to calculate fractal dimension of summer monsoon wind data.