THESIS APPROVAL

GRADUATE SCHOOL, KASETSART UNIVERSITY

Master of Science (Mathematics)

DEGREE

Mathematics — **FIELD**

Mathematics — **DEPARTMENT**

TITLE: The $Q_{\alpha}-$Convolution of Arithmetic Functions

NAME: Miss Sunanta Srisopha

THIS THESIS HAS BEEN ACCEPTED BY

THESIS ADVISOR

(Assistant Professor Pattira Ruengsinsub, Ph.D.)

DEPARTMENT HEAD

(Mr. Suriya Na Nhongkai, Dr.rer.nat)

APPROVED BY THE GRADUATE SCHOOL ON

DEAN

(Associate Professor Gunjana Theeragool, D.Agr.)

THESIS

THE Q_α– CONVOLUTION OF ARITHMETIC FUNCTIONS

SUNANTA SRISOPHA

A Thesis Submitted in Partial Fulfillment of
the Requirements for the Degree of
Master of Science (Mathematics)
Graduate School, Kasetsart University
2012

Sunanta Srisopha 2012: The $\mathrm{Q}_\alpha-$ Convolution of Arithmetic Functions. Master of Science (Mathematics), Major Field: Mathematics, Department of Mathematics. Thesis Advisor: Assistant Professor Pattira Ruengsinsub, Ph.D. 54 pages.

Let $\mathcal{A}$ denote the set of arithmetic functions. Let $\alpha \in \mathcal{A}$ be such that $\alpha(n) \neq 0$ for all $n \in \mathbb{N}$, for $f, g \in \mathcal{A}$, we define the $\mathrm{Q}_\alpha-$ convolution as

$$(f \diamond g)(n) = \sum_{xy=n} \frac{\alpha(n)}{\alpha(x)\alpha(y)} f(x)g(y).$$

In this thesis, we establish some properties of the $\mathrm{Q}_\alpha-$ convolution $\diamond$, connections between the Dirichlet convolution $*$ and $\mathrm{Q}_\alpha-$ convolution $\diamond$, characterizations of completely multiplicative functions under the $\mathrm{Q}_\alpha-$ convolution and the algebraic independence of arithmetic funtions under the $\mathrm{Q}_\alpha-$ convolution.

Let g^{*k} denote the convolution power $g * \cdots * g$ with k factor $g \in \mathcal{A}$. Consider the polynomial convolution equation of the form

$$Tg = a_d * g^{*d} + a_{d-1} * g^{*(d-1)} + \cdots + a_1 * g + a_0 = 0 \tag{1}$$

with fixed coefficients $a_d, a_{d-1}, \ldots, a_1, a_0 \in \mathcal{A}$ and $a_d \neq 0$.

In 2007, H. Glöckner, L. G. Lucht and $\tilde{\mathrm{S}}$. Porubský gave a condition which is necessary for existence of solutions $g \in \mathcal{A}$ to equation (1) as follows: if z_0 is a simple zero of the polynomial

$$f(z) = a_d(1)z^d + a_{d-1}(1)z^{d-1} + \cdots + a_1(1)z + a_0(1),$$

then there exists a uniquely determined solution $g \in \mathcal{A}$ to the polynomial convolution equation $Tg = 0$ satisfying $g(1) = z_0$. We investigate the solvability of polynomial convolution equation $Tg = 0$ where $f(z)$ has no simple zero and of polynomial $\mathrm{Q}_\alpha-$ convolution equation

$$T_\alpha g = a_d \diamond g^{\diamond d} + a_{d-1} \diamond g^{\diamond(d-1)} + \cdots + a_1 \diamond g + a_0 = 0.$$

________________ ________________ —/—/—

Student's signature Thesis Advisor's signature

ACKNOWLEDGEMENTS

I would like to express my gratitudes to all those who help, encourage and support me throughout my study and research undertaking here at Kasetsart University. In particular, I would like to thank Assistant Professor Dr. Pattira Ruengsinsub, my thesis adviser, whose guidance and patience have yielded and resulted in the research work that has been completed as this thesis.

I would like to thank Professor Dr. Vichian Laohakosol, my graduate teacher for his advice, lots of good ideas and valuable suggestions for my research work.

Finally, I would like to thank my family for their support, understanding, patience, love and care which have been given to me during all my student years.

Sunanta Srisopha
June 2012

TABLE OF CONTENTS

Page

LIST OF SYMBOLS

LIST OF SYMBOLS (Continued)

THE $\mathrm{Q}_\alpha-$ CONVOLUATION OF ARITHMETIC FUNCTIONS

INTRODUCTION

An *arithmetic function* is a complex-valued function whose domain is the set of positive integers, $\mathbb{N}$, and whose range is a subset of the set of complex numbers, $\mathbb{C}$. A nonzero arithmetic function f is said to be *multiplicative* if $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$ and is called *completely multiplicative* if this equality holds for all $m, n \in \mathbb{N}$.

Let $\mathcal{A}$ be the set of arithmetic functions equipped with addition, usual multiplication and Dirichlet convolution (or Dirichlet product) defined over $\mathbb{N}$, respectively, by

$$(f+g)(n) = f(n) + g(n),\ fg(n) = f(n)g(n),\ (f*g)(n) = \sum_{xy=n} f(x)g(y).$$

We write $g^d = g \cdot g \cdots g$ and $g^{*d} = g * g * \cdots * g$ (d times). The usual multiplication identity of $\mathcal{A}$ is the unit function u defined by $u(n) = 1$ for all $n \in \mathbb{N}$. The Dirichlet convolution identity $I \in \mathcal{A}$ is defined by $I(1) = 1$ and $I(n) = 0$ for $n > 1$. It is well known that $(\mathcal{A}, +, *)$ is an integral domain and $(\mathcal{A}, +, *, \mathbb{C})$ is a $\mathbb{C}-$algebra.

Let $\mathcal{D}$ be the set of formal Dirichlet series $D(f, s) = \displaystyle\sum_{n=1}^{\infty} f(n)n^{-s}$; $f(n) \in \mathbb{C}$. It is well known that the $\mathbb{C}-$algebra $(\mathcal{D}, +, \cdot, \mathbb{C})$ is isomorphic to $(\mathcal{A}, +, *, \mathbb{C})$ under the mapping $f \mapsto D(f, s)$. The Riemann zeta function ζ defined by $\zeta(s) = D(u, s) = \displaystyle\sum_{n=1}^{\infty} n^{-s}$ plays a crucial role in the $\mathbb{C}-$algebra $(\mathcal{D}, +, \cdot, \mathbb{C})$.

In 1968, D. Rearick proved basic properties of arithmetic functions with respect to the Dirichlet convolution; see also (Apostol, 1971; Haukkanen, 2001). In 1966, J. Lambek established characterizations of completely multiplicative functions in terms of the Dirichlet convolution; see also (Langford, 1973; Laohakosol and Pabhapote, 2004). In 1986, H. N. Shapiro and G. H. Sparer investigated the algebraic independence in $\mathcal{A}$.

The binomial convolution of arithmetic functions f and g is defined as

$$(f \circ g)(n) = \sum_{d|n} \left(\prod_p \binom{\nu_p(n)}{\nu_p(d)} \right) f(d)g(n/d),$$

where $\binom{a}{b}$ is the binomial coefficient, and $\nu_p(n)$ is the highest power of p dividing n. We can also denote $f \circ g$ as

$$(f \circ g)(n) = \sum_{xy=n} \frac{\xi(n)}{\xi(x)\xi(y)} f(x)g(y),$$

where $\xi(n) = \prod_p \nu_p(n)!$ for all $n \in \mathbb{N}$. This convolution and its basic properties was first introduced in 1996 by P. Haukkanen; see also (Haukkanen, 2001). In 2009, L. Tóth and P. Haukkanen proved that $(\mathcal{A}, +, \circ, \mathbb{C})$ is a $\mathbb{C}-$algebra with the binomial convolution identity I, and characterized completely multiplicative functions via distributivity under the binomial convolution.

In this thesis we further extend the binomial convolution to a new convolution denoted by $\diamond$. Let $\alpha \in \mathcal{A}$ be such that $\alpha(n) \neq 0$ for all $n \in \mathbb{N}$. For $f, g \in \mathcal{A}$, define

$$(f \diamond g)(n) = \sum_{xy=n} \frac{\alpha(n)}{\alpha(x)\alpha(y)} f(x)g(y),$$

and call it the $\mathrm{Q}_\alpha-$ convolution of f and g. It is easy to see that $(\mathcal{A}, +, \diamond)$ is a ring with identity αI which is defined by $\alpha I(1) = \alpha(1)$ and $\alpha I(n) = 0$ for $n > 1$. We note that

- if α is a completely multiplicative function, then $f \diamond g = f * g$;

- if $\alpha = \xi$, then $f \diamond g = f \circ g$.

In this thesis, we establish

- basic properties of the $\mathrm{Q}_\alpha-$ convolution,

- connections between the Dirichlet convolution and $\mathrm{Q}_\alpha-$ convolution,

- characterizations of completely multiplicative function and

- the algebraic independence of certain arithmetic functions under the $\mathrm{Q}_\alpha-$ convolution.

Furthermore we investigate the solubility of some arithmetic convolution equations. In 2007, H. Glöckner, L. G. Lucht and S̃. Porubský solved the polynomial equation

$$Tg = a_d * g^{*d} + a_{d-1} * g^{*(d-1)} + \cdots + a_1 * g + a_0 = 0 \tag{1}$$

with fixed coefficients $a_d, a_{d-1}, \ldots, a_1, a_0 \in \mathcal{A}$ and $a_d \neq 0$. They showed that if z_0 is a simple zero of the characteristic polynomial

$$f(z) = a_d(1)z^d + a_{d-1}(1)z^{d-1} + \cdots + a_1(1)z + a_0(1),$$

then there exists a uniquely determined solution $g \in \mathcal{A}$ of the polynomial convolution equation $Tg = 0$ satisfying $g(1) = z_0$. In this thesis, we investigate

- the solubility of polynomial convolution equations $Tg = 0$ when $f(z)$ has no simple zero,
- the solubility of the polynomial binomial convolution equation

  $$T_\xi g = a_d \circ g^{\circ d} + a_{d-1} \circ g^{\circ(d-1)} + \cdots + a_1 \circ g + a_0 = 0$$

  and
- the solubility of the polynomial $\mathrm{Q}_\alpha-$ convolution equation

  $$T_\alpha g = a_d \diamond g^{\diamond d} + a_{d-1} \diamond g^{\diamond(d-1)} + \cdots + a_1 \diamond g + a_0 = 0.$$

OBJECTIVES

1. To introduce the $\mathrm{Q}_\alpha-$ extending the binomial convolution.

2. To establish basic properties of arithmetic function under the $\mathrm{Q}_\alpha-$ convolution and compare our results with those of the Dirichlet convolution.

3. To investigate the algebraic independence of certain arithmetic functions under the $\mathrm{Q}_\alpha-$ convolution.

4. To investigate the solubility of certain polynomial convolution equations consisting of

4.1 polynomial convolution equations whose characteristic polynomial has no simple zero,

4.2 polynomial binomial convolution equation,

2.3 polynomial $\mathrm{Q}_\alpha-$ convolution equation.

LITERATURE REVIEW

In 1971, T. M. Apostol show a characterizations of completely multiplicative functions with respect to the convolution $*$ as follows.

Let f^{-1*} denote the inverse of f under the Dirichlet convolution.

Theorem 1. *Assume that f is multiplicative. Then for every squarefree integer n,*

$$f^{-1*}(n) = \mu(n)f(n).$$

when μ is the Möbius function defined by

$$\mu(n) = \begin{cases} 1, & n = 1; \\ (-1)^r, & n = p_1p_2\cdots p_r,\ p_i \, \textit{distinct primes} \\ 0, & \textit{otherwise} \end{cases}$$

Moreover, if p is any prime then $f^{-1*}(p^2) = f(p)^2 - f(p^2)$.

Theorem 2. *Assume that f is multiplicative. Then f is completely multiplicative if and only if $f^{-1*}(p^a) = 0$ for all primes p and all integers $a \geq 2$.*

Theorem 3. *Assume that f is multiplicative. Then f is completely multiplicative if and only if $f(g * h) = fg * fh$, for all arithmetic functions g and h.*

Theorem 4. *Assume that f is multiplicative. Then f is completely multiplicative if and only if $(fg)^{-1*} = fg^{-1*}$, for every arithmetic function g with $g(1) \neq 0$.*

This result is similar to Lembek's Theorem (see Lembek, 1966).

Theorem 5. *The multiplicative function f satisfies $f(g * h) = fg * fh$, for all arithmetic functions g and h, if and only if f is completely multiplicative.*

In 1973, E. Langford established characterizations of completely multiplicative functions in terms of the Dirichlet convolution.

Definition 1. For $g, h \in \mathcal{A}$, the product $k = g * h$ is said to be *discriminative* if the relation

$$k(n) = g(1)h(n) + g(n)h(1)$$

holds only when n is prime and is said to be *partially discriminative* if for every prime power p^i $(i \in \mathbb{N})$ the relation

$$k(p^i) = g(1)h(p^i) + g(p^i)h(1)$$

implies that $i = 1$.

Theorem 6. *Suppose that $f(1) \neq 0$. Then f is completely multiplicative if and only if it distributes over some discrimitive product $k = g * h$.*

Theorem 7. *Suppose that f is multiplicative. Then f is completely multiplicative if and only if it distributes over some partially discrimitive product $k = g * h$.*

In 2010, V. Laohakosol and N. Pabhapote present some properties which related to completely multiplicative functions.

Let $\mathcal{M}$ and $\mathcal{C}$ be the set of all multiplicative functions and completely multiplicative functions, respectively.

Theorem 8. *Let $f, g \in \mathcal{M}$. Then $f * g \in \mathcal{C} \Leftrightarrow$ either*

$$\begin{aligned} g(p^a) =& f^{-1*}(p^a) + (f(p) + g(p))f^{-1*}(p^{a-1}) \\ &+ \cdots + (f(p) + g(p))^{a-1}f^{-1*}(p) + (f(p) + g(p))^a \end{aligned}$$

or

$$\begin{aligned} f(p^a) =& g^{-1*}(p^a) + (f(p) + g(p))g^{-1*}(p^{a-1}) \\ &+ \cdots + (f(p) + g(p))^{a-1}g^{-1*}(p) + (f(p) + g(p))^a \end{aligned}$$

for all primes p and all $a \in \mathbb{N}$.

Corollary 1. *Let* $f \in \mathcal{C}$ *and* $g \in \mathcal{M}$*. Then*

$$f * g \in C \Leftrightarrow g(p^a) = g(p)(g(p) + f(p))^{a-1}$$

for all primes p *and all* $a \in \mathbb{N}$*.*

Corollary 2. *Let* $f, g \in \mathcal{C}$*. Then*

$$f * g \in \mathcal{C} \Leftrightarrow f(p)g(p) = 0$$

for all primes p*.*

Definition 2. Let $r \in \mathbb{R}$ and $n = \prod_{p \ prime} p^{\nu_p(n)}$ be the prime factorization of n. The generalized Möbius function is defined by

$$\mu_r(n) = \prod_{p|n} \binom{r}{\nu_p(n)} (-1)^{\nu_p(n)}.$$

Note that

1. $\mu_1 = \mu$, the Möbius function,

2. $\mu_0 = I$, the Dirichlet convolution identity,

3. $\mu_{-1} = u$, the unit function,

4. $\mu_{s+t} = \mu_s * \mu_t$; $s, t \in \mathbb{R}$.

Theorem 9. *Let* f *be a nonzero multiplicative function and* r *a nonzero real number. Then* f *is completely multiplicative if and only if* $(\mu_r f)^{-1*} = \mu_{-r} f$*.*

Rearick's Logarithm (see Rearick, 1968).

Let P stand for the set of all real valued functions f such that $f(1) > 0$.

Definition 3. For $f \in P$, let

$$\begin{aligned} Lf(1) &= \log f(1), \\ Lf(n) &= \sum_{d|n} f(d) f^{-1*}(n/d) \log d, \qquad \text{if } n > 1. \end{aligned}$$

Theorem 10. *For all* $f, g \in P$*,* $L(f * g) = Lf + Lg$*.*

The algebraic independence in $\mathcal{A}$ (see Shapiro and Sparer, 1986).

In 1986, H. N. Shapiro and G. H. Sparer investigated the algebraic independence in $\mathcal{A}$.

Definition 4. Let $\mathcal{E}$ be a subring of $\mathcal{A}$. For $k > 1$ we say that $f_1, f_2, \ldots, f_k \in \mathcal{A}$. are *algebraically dependent* over $\mathcal{E}$. If there exist $P \in \mathcal{E}[f_1, f_2, \ldots, f_k] \setminus \{0\}$ such that

$$P(f_1, \ldots, f_k) = \sum_{(i)} a_i * f_1^{*i_1} * \cdots * f_k^{*i_k} = 0$$

and is said to be *algebraically independent* over $\mathcal{E}$ otherwise.

We say that f_1 is *algebraic* over $\mathcal{E}[f_2, \ldots, f_k]$ if $f_1, f_2, \ldots, f_k$ are *algebraically dependent* over $\mathcal{E}$.

The algebraic independence of arithmetic functions under the Dirichlet convolution can be considered relative to a given subring $\mathcal{R}$ of $\mathcal{A}$. In particular, $\mathcal{A}$ contains the complex numbers via the identification of a $c \in \mathbb{C}$ with the function $cI(n)$ of $\mathcal{A}$.

Definition 5. A derivation d over $\mathcal{A}$ is a map of $\mathcal{A}$ into itself satisfying

$$d(f * g) = df * g + f * dg, \quad d(c_1 f + c_2 g) = c_1 df + c_2 dg,$$

where $f, g \in \mathcal{A}$, $c_1, c_2 \in \mathbb{C}$.

Two typical examples of derivation are

(i) the p-basic derivation, p prime, defined by

$$d_p(f) = f(np)\nu_p(np) \qquad (\forall n \in \mathbb{N})$$

(ii) the log-derivation, defined by

$$d_L(n) = f(n)log(n) \qquad (\forall n \in \mathbb{N}).$$

Lemma 1. *Let $\mathcal{E}$ be a subring of $\mathcal{A}$, and f a given function of $\mathcal{A}$ such that there exists a derivation d over $\mathcal{A}$ which annihilates all of $\mathcal{E}$ and $d(f) \neq 0$. Then f is not algebraic over $\mathcal{E}$.*

Definition 6. Given $f_1, f_2, \ldots, f_k$ in $\mathcal{A}$ and derivations $d_1, d_2, \ldots, d_k$ over $\mathcal{A}$, the Jacobian of the f_i relative to the d_i is given by the $k \times k$ determinant

$$J(f_1, \ldots, f_k/d_1, \ldots, d_k) = det(d_i(f_j)),$$

for $i, j \in \{1, 2, \ldots, k\}$. Clearly a Jacobian is an element of $\mathcal{A}$. In the case where each d is a basic derivation d_{p_i}, corresponding to some prime p_i, we shall use the notation $J(f_1, \ldots, f_k/p_1, \ldots, p_k)$ for the corresponding Jacobian.

Theorem 11. *Let $f_1, \ldots, f_k$ be given functions of $\mathcal{A}$ and $d_1, \ldots, d_k$ derivations over $\mathcal{A}$ which annihilate all elements of the subring $\mathcal{E}$. Then if $J(f_1, \ldots, f_k/d_1, \ldots, d_k) \neq 0$, the $f_1, \ldots, f_k$ are algebraically independent over $\mathcal{E}$.*

Theorem 12. *Let $\mathcal{E}$ be a subring $\mathcal{A}$ such that, for some set of r distinct primes $p_1, \ldots, p_r$, the corresponding basic derivations d_{p_i} all annihilate $\mathcal{E}$. Then, for $2s + 1 \leq r$, the functions $I_j(n) = n^j, -s \leq j \leq s$, are algebraically independent over $\mathcal{E}$.*

Corollary 3. *Let $\mathcal{E}$ be a subring $\mathcal{A}$ such that for infinitely many primes p the basic derivations d_p annihilate all of $\mathcal{E}$. Then the functions $I_j(n), j = 0, \pm 1, \pm 2, \ldots$ are algebraically independent over $\mathcal{E}$.*

Theorem 13. *Let $\mathcal{E}$ be a subring of $\mathcal{A}$ such that given any finite subset $\mathcal{E}^* \subset \mathcal{E}$ there are infinitely many primes p such the derivations d_p annihilate all of $\mathcal{E}^*$. Then given any sequence of complex numbers $r_i, i = 1, 2, \ldots$ with distinct real parts, and any sequence of integers s_j (not necessarily distinct), the functions*

$$f_{ij}(n) = n^{r_i}(log\ n)^{s_j}$$

are algebraically independent over $\mathcal{E}$.

Corollary 4. *Let r_i, $i = 1, 2, \ldots, L$ be complex numbers with distinct real parts, and m_i any non-negative integers. Then, the functions*

$$\zeta^{(m_1)}(s - r_1), \ldots, \zeta^{(m_L)}(s - r_L)$$

are algebraically independent over $\mathbb{C}$.

Corollary 5. *The zeta function does not satisfy any nontrivial algebraic differential difference equation over* $\mathbb{C}$.

The binomial convolution of arithmetic functions.

The works of L. Tóth and P. Haukkanen was presented in 2009. They introduced a new convolution as follows. Let $n = \prod_p p^{\nu_p(n)}$ denote the canonical factorization of $n \in \mathbb{N}$. The binomial convolution of arithmetic functions f and g is defined as

$$(f \circ g)(n) = \sum_{d|n} \left(\prod_p \binom{\nu_p(n)}{\nu_p(d)} \right) f(d)g(n/d),$$

where $\binom{a}{b}$ is the binomial coefficient, $\nu_p(n)$ is the highest power of p dividing n and I is the binomial convolution identity. Note that $f^{\circ k} = f \circ f \circ \cdots \circ f$ (k times).

Theorem 14. *The algebras* $(\mathcal{A}, +, \circ, \mathbb{C})$ *and* $(\mathcal{A}, +, *, \mathbb{C})$ *are isomorphic under the mapping* $f \mapsto \dfrac{f}{\xi}$, *where* $\xi(n) = \prod_p \nu_p(n)!$.

Denote $f^{-1\circ}$ be the inverse of f under the binomial convolution.

Theorem 15. *for any* $f \in \mathcal{A}$ *with* $f(1) \neq 0$,

$$f^{-1\circ} = \xi \left(\frac{f}{\xi} \right)^{-1*} \tag{2}$$

and

$$f^{-1*} = \frac{(\xi f)^{-1\circ}}{\xi}. \tag{3}$$

Theorem 16. *If* f *is multiplicative and* $f(p^a) = 0$ *for all prime powers* p^a *with* $a \geq 2$, *then for every* $n \geq 1$,

$$f^{-1\circ}(n) = (-1)^{\Omega(n)} \xi(n) \prod_p f(p)^{\nu_p(n)} = \lambda(n)\xi(n) \prod_p f(p)^{\nu_p(n)} \tag{4}$$

and

$$f^{-1*}(n) = (-1)^{\Omega(n)} \prod_p f(p)^{\nu_p(n)} = \lambda(n) \prod_p f(p)^{\nu_p(n)} \tag{5}$$

where $\Omega(n) = \sum_p \nu_p(n)$ *and* $\lambda(n) = (-1)^{\Omega(n)}$.

Theorem 17. *Let f be a multiplicative function. Then f is completely multiplicative if and only if $f(g \circ h) = fg \circ fh$, for all $g, h \in \mathcal{A}$.*

Definition 7. An arithmetic function f is said to be semimultiplicative if

$$f(m)f(n) = f((m,n))f([m,n])$$

for $m, n \in \mathbb{N}$ where (m,n) and $[m,n]$ stand for the g.c.d. and l.c.m. of m and n. Notice that for semimultiplicative functions f and g we have that fg is also a semimultiplicative function.

Proposition 1. *An arithmetic function F (not identically zero) is semimultiplicative if and only if there exists a nonzero constant C_F, a positive integer a_F and a multiplicative function F' such that*

$$F(n) = c_F F'(n/a_F), \qquad for\ all\ n \in \mathbb{N}$$

(see section 4 of Tóth and Haukkanen, 2009)*, where a_F is the smallest positive integer k such that $F(k) \neq 0$ and $c_F = F(a_F)$. Note that an arithmetic function F' possesses the property that $F'(x) = 0$ if $x \notin \mathbb{N}$.*

Proposition 2. *Semimultiplicative functions form a commutative semigroup with identity under the Dirichlet convolution and*

$$a_{F*G} = a_F a_G, \quad c_{F*G} = c_F c_G, \quad (F*G)' = F' * G',$$

$$a_{F \circ G} = a_F a_G, \quad c_{F \circ G} = c_F c_G \frac{\xi(a_F a_G)}{\xi(a_F)\xi(a_G)},$$

$$(F \circ G)' = \frac{\xi_{a_F a_G}}{\xi(a_F a_G)\xi}\left[\left(\frac{\xi(a_F)\xi}{\xi_{a_F}}F'\right) \circ \left(\frac{\xi(a_G)\xi}{\xi_{a_G}}G'\right)\right],$$

where $\xi_a(n) = \xi(an)$ for all $a, n \in \mathbb{N}$.

Proposition 3. *If F is semimultiplicative (not identically zero) and f is multiplicative with $f(a_F) \neq 0$, then*

$$a_{fF} = a_F, \quad c_{fF} = f(a_F)c_F, \quad (fF)' = \frac{f_{a_F}}{f(a_F)}F',$$

where $f_a(n) = f(an)$ for all $a, n \in \mathbb{N}$.

Definition 8. For an arithmetic function f, exponential Dirichlet series is defined as

$$\tilde{D}(f,s) = D\left(\frac{f}{\alpha}, s\right) = \sum_{n=1}^{\infty} \frac{f(n)}{\xi(n)n^s}.$$

Theorem 18. *The product of exponential Dirichlet series is the exponential Dirichlet series of the binomial convolution of the corresponding arithmetic functions, i.e.,*

$$\tilde{D}(f,s)\tilde{D}(g,s) = \tilde{D}(f \circ g, s).$$

remark. The algebra $(\tilde{D}, +, ., \mathbb{C})$ of exponential Dirichlet series is isomorphic to the algebra $(\mathcal{A}, +, \circ, \mathbb{C})$.

The polynomial convolution equation.

In 2007, H. Glöckner, L. G. Lucht and Š. Porubský investigated the solubility of polynomial convolution equation of the form

$$a_d * g^{*d} + a_{d-1} * g^{*(d-1)} + \cdots + a_1 * g + a_0 = 0$$

with fixed coefficients $a_d, a_{d-1}, \ldots, a_1, a_0 \in \mathcal{A}$.

Theorem 19. *For $d \in \mathbb{N}$, let $T : \mathcal{A} \to \mathcal{A}$ be defined by*

$$Tg = a_d * g^{*d} + a_{d-1} * g^{*(d-1)} + \cdots + a_1 * g + a_0 \tag{6}$$

for $g \in \mathcal{A}$ with $a_d, a_{d-1}, \ldots, a_1, a_0 \in \mathcal{A}$ and $a_d \neq 0$. If z_0 is a simple zero of the polynomial

$$f(z) = a_d(1)z^d + a_{d-1}(1)z^{d-1} + \cdots + a_1(1)z + a_0(1), \tag{7}$$

then there exists a uniquely determined solution $g \in \mathcal{A}$ to the convolution equation $Tg = 0$ satisfying $g(1) = z_0$. If $f(z)$ has no simple zero, then $Tg = 0$ need not possess any solution. In any case $Tg = 0$ has at most d solutions.

MATERIALS AND METHODS

Let $\mathcal{M}$, $\mathcal{C}$ and $\mathcal{S}$ denote the set of all multiplicative, completely multiplicative functions and semi-multiplicative functions, respectively.

Definition 9. Let $\alpha \in \mathcal{A}$ be such that $\alpha(n) \neq 0$ for all $n \in \mathbb{N}$. The $\mathrm{Q}_\alpha-$ convolution of arithmetic functions f and g is defined as

$$(f \diamond g)(n) = \sum_{xy=n} \frac{\alpha(n)}{\alpha(x)\alpha(y)} f(x)g(y).$$

We denote $f^{\diamond k}$ as $f \diamond f \diamond \cdots \diamond f$ (k times).

The $\mathrm{Q}_\alpha-$ convolution can be expressed in term of the Dirichlet convolution as follows:

$$f \diamond g = \alpha\left(\frac{f}{\alpha} * \frac{g}{\alpha}\right),$$

that is

$$\alpha f \diamond \alpha g = \alpha\left(\frac{\alpha f}{\alpha} * \frac{\alpha g}{\alpha}\right).$$

So

$$\alpha(f * g) = \alpha f \diamond \alpha g \tag{8}$$

or equivalently

$$f * g = \frac{\alpha f \diamond \alpha g}{\alpha}. \tag{9}$$

In 2009, L. Tóth and P. Haukkanen showed that $(\mathcal{A}, +, *, \mathbb{C})$ is isomorphic to $(\mathcal{A}, +, \circ, \mathbb{C})$. They also compared properties of arithmetic functions under the Dirichlet convolution with binomial convolution. We now show that the algebra $(\mathcal{A}, +, *, \mathbb{C})$ is isomorphic to $(\mathcal{A}, +, \diamond, \mathbb{C})$ and showed that most basic properties under Dirichlet convolution are analogous to those under the $\mathrm{Q}_\alpha-$ convolution.

Theorem 20. *The algebras $(\mathcal{A}, +, *, \mathbb{C})$ and $(\mathcal{A}, +, \diamond, \mathbb{C})$ are isomorphic under the mapping $f \mapsto \dfrac{f}{\alpha}$.*

Proof. First, we will show that $(\mathcal{A},+,\diamond,\mathbb{C})$ is a $\mathbb{C}-$algebra. It is easy to see that $(\mathcal{A},+)$ is a vector space over $\mathbb{C}$. Let $f_1, f_2, f_3 \in \mathcal{A}$ and $c \in \mathbb{C}$, then

1. $(f_1 \diamond f_2)(n) = \sum_{xy=n} \frac{\alpha(n)}{\alpha(x)\alpha(y)} f_1(x) f_2(y) \quad \in \mathbb{C}$.

Thus $f_1 \diamond f_2 \in \mathcal{A}$.

2. $$(f_1 \diamond (f_2+f_3))(n) = \sum_{xy=n} \frac{\alpha(n)}{\alpha(x)\alpha(y)} f_1(x)(f_2+f_3)(y)$$

$$= \sum_{xy=n} \frac{\alpha(n)}{\alpha(x)\alpha(y)} (f_1(x)f_2(y) + f_1(x)f_3(y))$$

$$= (f_1 \diamond f_2)(n) + (f_1 \diamond f_3)(n).$$

Thus $f_1 \diamond (f_2+f_3) = f_1 \diamond f_2 + f_1 \diamond f_3$.

3. Using expression similar to 2 we obtain $(f_1+f_2) \diamond f_3 = f_1 \diamond f_3 + f_2 \diamond f_3$.

From 1-3, we obtain $(\mathcal{A},+,\diamond)$ is a ring.

4. $$c(f_1 \diamond f_2)(n) = c \sum_{xy=n} \frac{\alpha(n)}{\alpha(x)\alpha(y)} f_1(x)f_2(y) = \sum_{xy=n} \frac{\alpha(n)}{\alpha(x)\alpha(y)} cf_1(x)f_2(y)$$
$$= (cf_1 \diamond f_2)(n).$$

Thus $c(f_1 \diamond f_2) = cf_1 \diamond f_2$ and similarly $c(f_1 \diamond f_2) = f_1 \diamond cf_2$.

From 1-4, we obtain $(\mathcal{A},+,\diamond,\mathbb{C})$ is a $\mathbb{C}-$algebra. We next show that the mapping $f \mapsto \frac{f}{\alpha}$ is a bijection on $\mathcal{A}$. We let $H : \mathcal{A} \to \mathcal{A}$ be defined by $H(f) = \frac{f}{\alpha}$.

5. Let $f, g \in \mathcal{A}$. Then $f = g \Leftrightarrow \frac{f}{\alpha} = \frac{g}{\alpha} \Leftrightarrow H(f) = H(g)$.

Thus H is a 1-1 function.

6. Since $f = \frac{\alpha f}{\alpha} = H(\alpha f)$ and $\alpha f \in \mathcal{A}$, then

Thus H is an onto function.

From 5-6, we obtain the mapping $f \mapsto \frac{f}{\alpha}$ is a bijection on $\mathcal{A}$.

7. $$(f \diamond g)(n) = \sum_{xy=n} \frac{\alpha(n)}{\alpha(x)\alpha(y)} f(x)f(y) = \alpha(n)\left(\frac{f}{\alpha} * \frac{g}{\alpha}\right)(n)$$

or $f \diamond g = \alpha(\frac{f}{\alpha} * \frac{g}{\alpha})$, that is $\frac{f \diamond g}{\alpha} = \frac{f}{\alpha} * \frac{g}{\alpha}$. So $H(f \diamond g) = H(f) * H(g)$.

$$H(f+g) = \frac{f+g}{\alpha} = \frac{f}{\alpha} + \frac{g}{\alpha} = H(f) + H(g),$$

$$H(cf) = \frac{cf}{\alpha} = c\frac{f}{\alpha} = cH(f).$$

These show that H is an homomorphism. We conclude that $(\mathcal{A},+,*,\mathbb{C})$ is isomorphic to $(\mathcal{A},+,\diamond,\mathbb{C})$. $\square$

Remark. It is well known that $(\mathcal{A},+,*)$ is an integral domain. Consequently, $(\mathcal{A},+,\diamond)$ is also an integral domain.

We denote by $f^{-1\diamond}$ the inverse of f under the $\mathrm{Q}_\alpha-$ convolution $\diamond$. Then for $f \in \mathcal{A}$, f^{-1*} and $f^{-1\diamond}$ exist if and only if $f(1) \neq 0$.

Theorem 21. *For any $f \in \mathcal{A}$ with $f(1) \neq 0$,*

$$f^{-1*} = \frac{(\alpha f)^{-1\diamond}}{\alpha} \tag{10}$$

$$f^{-1\diamond} = \alpha\left(\frac{f}{\alpha}\right)^{-1*} \tag{11}$$

Proof. Since $f * f^{-1*} = I$ and from (8) we have $\alpha f \diamond \alpha f^{-1*} = \alpha I$.
Thus $\alpha f^{-1*} = (\alpha f)^{-1\diamond}$ i.e., $f^{-1*} = \dfrac{(\alpha f)^{-1\diamond}}{\alpha}$. On the other hand, we have $f \diamond f^{-1\diamond} = \alpha I$ that is $\alpha\left(\dfrac{f}{\alpha} * \dfrac{f^{-1\diamond}}{\alpha}\right) = \alpha I$, so $\left(\dfrac{f}{\alpha} * \dfrac{f^{-1\diamond}}{\alpha}\right) = I$.
Therefore $\dfrac{f^{-1\diamond}}{\alpha} = \left(\dfrac{f}{\alpha}\right)^{-1*}$ or that $f^{-1\diamond} = \alpha\left(\dfrac{f}{\alpha}\right)^{-1*}$. $\square$

Example 1. $\alpha^{-1\diamond} = \alpha\left(\dfrac{\alpha}{\alpha}\right)^{-1*} = \alpha u^{-1*} = \alpha\mu$, so $u^{-1*} = \dfrac{\alpha^{-1\diamond}}{\alpha} = \dfrac{\alpha\mu}{\alpha} = \mu$.

Theorem 22. *If $f, \alpha \in \mathcal{M}$ and $f(p^a) = 0$ for all primes power p^a with $a \geq 2$ then for every $n \in \mathbb{N}$,*

$$f^{-1\diamond}(n) = (-1)^{\Omega(n)}\alpha(n)\prod_p \frac{f}{\alpha}(p)^{\nu_p(n)}.$$

Proof. Since $\dfrac{f}{\alpha} \in \mathcal{M}$, $\dfrac{f}{\alpha}(p^a) = \dfrac{f(p^a)}{\alpha(p^a)} = 0$, for all $a \geq 2$ and from Theorem 21 we have $f^{-1\diamond} = \alpha\left(\dfrac{f}{\alpha}\right)^{-1*}$, hence by Theorem 16 we obtain
$f^{-1\diamond}(n) = \alpha\left(\dfrac{f}{\alpha}\right)^{-1*}(n) = (-1)^{\Omega(n)}\alpha(n)\displaystyle\prod_p \frac{f}{\alpha}(p)^{\nu_p(n)}$. $\square$

Example 2. For $\alpha \in \mathcal{M}$ we have

$$\mu^{-1\diamond}(n) = \begin{cases} 1, & n = 1; \\ \dfrac{\alpha(n)}{\prod\limits_{p} \alpha(p)^{\nu_p(n)}}, & n > 1. \end{cases}$$

To prove this identity, we first find the $Q_\alpha-$ convolution inverse of Möbius function, $\mu^{-1\diamond}$. Since

$$\alpha\left(\frac{\mu}{\alpha} * \frac{\mu^{-1\diamond}}{\alpha}\right) = \mu \diamond \mu^{-1\diamond} = \alpha I,$$

then

$$\frac{\mu}{\alpha} * \frac{\mu^{-1\diamond}}{\alpha} = I.$$

It follows that $\frac{\mu}{\alpha}(1)\frac{\mu^{-1\diamond}}{\alpha}(1) = I(1) = 1$. For $n > 1$,

$$\alpha(n)\left(\frac{\mu}{\alpha} * \frac{\mu^{-1\diamond}}{\alpha}\right)(n) = \alpha I(n).$$

That is

$$\frac{\mu}{\alpha}(1)\frac{\mu^{-1\diamond}}{\alpha}(n) + \sum_{\substack{xy=n \\ y<n}} \frac{\mu}{\alpha}(x)\frac{\mu^{-1\diamond}}{\alpha}(y) = 0.$$

Thus,

$$\mu^{-1\diamond}(n) = -\alpha(n)\sum_{\substack{xy=n \\ y<n}} \frac{\mu}{\alpha}(x)\frac{\mu^{-1\diamond}}{\alpha}(y).$$

Therefore

$$\mu^{-1\diamond}(n) = \begin{cases} 1, & n = 1; \\ -\alpha(n)\sum\limits_{\substack{xy=n \\ y<n}} \dfrac{\mu}{\alpha}(x)\dfrac{\mu^{-1\diamond}}{\alpha}(y), & n > 1, \end{cases}$$

Let $n \geq 2$ and assume that $\mu^{-1\diamond}(m) = \dfrac{\alpha(m)}{\prod_p \alpha(p)^{\nu_p(m)}}$ for $m < n$, then we obtain

$$\begin{aligned}
\mu^{-1\diamond}(n) &= -\alpha(n) \sum_{\substack{xy=n \\ y<n}} \frac{\mu}{\alpha}(x) \frac{\mu^{-1\diamond}}{\alpha}(y) \\
&= -\alpha(n) \sum_{\substack{xy=n \\ y<n}} \frac{\mu(x)\alpha(y)}{\alpha(x)\alpha(y) \prod_p \alpha(p)^{\nu_p(y)}} \qquad \text{(by inductive assumption)} \\
&= -\alpha(n) \sum_{\substack{xy=n \\ y<n}} \frac{\mu(x)}{\prod_p \alpha(p)^{\nu_p(x)} \alpha(p)^{\nu_p(y)}} \qquad (x = p_1 \cdots p_r,\ p_i \text{ distinct}) \\
&= -\frac{\alpha(n)}{\prod_p \alpha(p)^{\nu_p(n)}} \sum_{\substack{xy=n \\ y<n}} \mu(x)u(y) \\
&= -\frac{\alpha(n)}{\prod_p \alpha(p)^{\nu_p(n)}} \left(\sum_{xy=n} \mu(x)u(y) - \mu(1)u(n) \right) \\
&= -\frac{\alpha(n)}{\prod_p \alpha(p)^{\nu_p(n)}} (-1) \\
&= \frac{\alpha(n)}{\prod_p \alpha(p)^{\nu_p(n)}}.
\end{aligned}$$

We give here another proof. From Theorem 22 we have

$$\begin{aligned}
\mu^{-1\diamond}(n) &= (-1)^{\Omega(n)} \alpha(n) \prod_p \frac{\mu}{\alpha}(p)^{\nu_p(n)} \\
&= (-1)^{\Omega(n)} \alpha(n) \prod_p \frac{(-1)^{\nu_p(n)}}{\alpha(p)^{\nu_p(n)}} \\
&= (-1)^{\Omega(n)} \alpha(n) \frac{(-1)^{\Omega(n)}}{\prod_p \alpha(p)^{\nu_p(n)}} \\
&= \frac{\alpha(n)}{\prod_p \alpha(p)^{\nu_p(n)}}.
\end{aligned}$$

Various characterizations of completely multiplicative functions have been discovered by J. Lambek (Lambek, 1966), E. Langford (Langford, 1973), P. Haukkanen (Haukkanen, 2001), N. Pabhapote and V. Laohakosol (Pabhapote and Laohakosol, 2004).

In 2009, L. Tóth and P. Haukkanen (see section 3 of Tóth and Haukkanen, 2009) proved a characterization of completely multiplicative functions using the notion of distributivity with respect to the binomial convolution. We now extend some of these characterizations of completely multiplicative function through the use of the $Q_\alpha-$ convolution.

Theorem 23. *Let $f \in \mathcal{M}$. Then $f \in \mathcal{C}$ if and only if $f(g \diamond h) = fg \diamond fh$ for all $g, h \in \mathcal{A}$.*

Proof. Assume that $f \in \mathcal{C}$. Let $g, h \in \mathcal{A}$. Then

$$f(g \diamond h) = f\alpha(\frac{g}{\alpha} * \frac{h}{\alpha}) = \alpha(\frac{fg}{\alpha} * \frac{fh}{\alpha}) = fg \diamond fh.$$

Conversely, assume that $f(g \diamond h) = fg \diamond fh$ for all $g, h \in \mathcal{A}$. Then

$$\alpha f(g * h) = f(\alpha g \diamond \alpha h) = \alpha fg \diamond \alpha fh = \alpha(\frac{\alpha fg}{\alpha} * \frac{\alpha fh}{\alpha}) = \alpha(fg * fh)$$

so, $f(g * h) = fg * fh$ and so by Theorem 3, f is a completely multiplicative. $\square$

Location and Duration of Research

Location, Department of Mathematics, Kasetsart University.

Duration of Research, July 2011- Fabuary 2012.

RESULTS AND DISCUSSION

1. The properties of the $\mathrm{Q}_\alpha-$ convolution.

In 1973, E. Langford characterized completely multiplicative functions using a distributivity property with respect to the Dirichlet convolution. We extend some of these characterizations of completely multiplicative functions through the $\mathrm{Q}_\alpha-$ convolution.

Definition 10. For $g, h \in \mathcal{A}$, the product $k = g \diamond h$ is said to be Q_α*−discriminative*, if the relation

$$\alpha(1)k(n) = g(1)h(n) + g(n)h(1) \tag{12}$$

holds only when n is prime, is said to be $Q_\alpha -$ *partially discriminative*, if for every prime power p^i $(i \in \mathbb{N})$ the relation

$$\alpha(1)k(p^i) = g(1)h(p^i) + g(p^i)h(1) \tag{13}$$

implies that $i = 1$ and is said to be $Q_\alpha - semi - discriminative$, if the relation

$$\alpha(1)k(n) = g(1)h(n) + g(n)h(1) \tag{14}$$

holds only when $n = 1$ or n is prime.

Theorem 24. *Suppose that $f(1) \neq 0$. Then $f \in \mathcal{C}$ if and only if it distributes over a $Q_\alpha - discriminative$ product.*

Proof. Let $f \in \mathcal{C}$, then by Theorem 23 we obtain $f(g \diamond h) = fg \diamond fh$ for all $g, h \in \mathcal{A}$. Now we prove the conversely. Assume that f distributes over a $\mathrm{Q}_\alpha-$ discriminative product $k = g \diamond h$ where $g, h \in \mathcal{A}$. First we show that f(1)=1. If $k(1) = 0$ then $0 = k(1) = \dfrac{g(1)h(1)}{\alpha(1)}$, so we get $\dfrac{g(1)h(1)}{\alpha(1)} + \dfrac{g(1)h(1)}{\alpha(1)} = 0 = k(1)$. That is, $\alpha(1)k(1) = g(1)h(1) + g(1)h(1)$. This equation express (12) holds for n is not prime which is a contradiction, hence $k(1) \neq 0$.
Since $f(1)k(1) = fk(1) = \alpha(1)\dfrac{fg}{\alpha}(1)\dfrac{fh}{\alpha}(1) = f(1)^2\alpha(1)\dfrac{g}{\alpha}(1)\dfrac{h}{\alpha}(1) = f(1)^2k(1)$, it follows that $f(1)^2k(1) - f(1)k(1) = 0$. But $f(1)k(1) \neq 0$, thus we obtain $f(1) = 1$. Next, we show that for all prime $p_1, \ldots, p_m$ (not necessary distinct),

$$f(p_1 \cdots p_m) = f(p_1) \cdots f(p_m) \tag{15}$$

We now show it by induction on m. Equation (15) is trivial if $m = 1$. Let $m \geq 2$ and assume that $f(p_{i_1} \cdots p_{i_j}) = f(p_{i_1}) \cdots f(p_{i_j})$ for $1 \leq j < m$, $p_{i_1}, \ldots, p_{i_j} \in \{p_1, \ldots, p_m\}$. Then by distributive property $f(g \diamond h) = fg \diamond fh$ we get that

$$\begin{aligned}
\alpha(p_{i_1} \cdots p_{i_m}) f(p_{i_1} \cdots p_{i_m}) \sum_{j=0}^{m} \frac{g}{\alpha}(p_{i_1} \cdots p_{i_m}) \frac{h}{\alpha}(p_{i_1} \cdots p_{i_m}) \\
= \alpha(p_{i_1} \cdots p_{i_m}) \sum_{j=0}^{m} \frac{fg}{\alpha}(p_{i_1} \cdots p_{i_m}) \frac{fh}{\alpha}(p_{i_1} \cdots p_{i_m}),
\end{aligned}$$

$$\begin{aligned}
f(p_{i_1} \cdots p_{i_m})\Big[\frac{g}{\alpha}(1)\frac{h}{\alpha}(p_{i_1} \cdots p_{i_m}) + \frac{g}{\alpha}(p_{i_1} \cdots p_{i_m})\frac{h}{\alpha}(1) \\
+ \sum_{j=1}^{m-1} \frac{g}{\alpha}(p_{i_1} \cdots p_{i_m})\frac{h}{\alpha}(p_{i_1} \cdots p_{i_m})\Big] \\
= \frac{fg}{\alpha}(1)\frac{fh}{\alpha}(p_{i_1} \cdots p_{i_m}) + \frac{fg}{\alpha}(p_{i_1} \cdots p_{i_m})\frac{fh}{\alpha}(1) \\
+ \sum_{j=1}^{m-1} \frac{fg}{\alpha}(p_{i_1} \cdots p_{i_m})\frac{fh}{\alpha}(p_{i_1} \cdots p_{i_m}),
\end{aligned}$$

$$\begin{aligned}
f(p_{i_1} \cdots p_{i_m}) \sum_{j=1}^{m-1} \frac{g}{\alpha}(p_{i_1} \cdots p_{i_m})\frac{h}{\alpha}(p_{i_1} \cdots p_{i_m}) \\
= \sum_{j=1}^{m-1} \frac{fg}{\alpha}(p_{i_1} \cdots p_{i_m})\frac{fh}{\alpha}(p_{i_1} \cdots p_{i_m}).
\end{aligned}$$

Since,

$$\sum_{\substack{xy=n \\ x,y<n}} \frac{g}{\alpha}(x)\frac{h}{\alpha}(y) = \sum_{xy=n} \frac{g}{\alpha}(x)\frac{h}{\alpha}(y) - \frac{g}{\alpha}(1)\frac{h}{\alpha}(n) - \frac{g}{\alpha}(n)\frac{h}{\alpha}(1)$$

and $k = g \diamond h$ is $\mathrm{Q}_\alpha-$discriminative product, it follows that $\sum_{\substack{xy=n \\ x,y<n}} \frac{g}{\alpha}(x)\frac{h}{\alpha}(y) \neq 0$. Thus $f(p_1 \cdots p_m) = f(p_1) \cdots f(p_m)$. The proof is complete. □

Theorem 25. *Let $f \in \mathcal{M}$. Then $f \in \mathcal{C}$ if and only if it distributes over a $\mathrm{Q}_\alpha-$partially discriminative product $k = g \diamond h$.*

Proof. If $f \in \mathcal{C}$ then Theorem 23 show that $f(g \diamond h) = fg \diamond fh$ for all $g, h \in \mathcal{A}$. Conversely, assume that f distributes over a $\mathrm{Q}_\alpha-$partially discriminative product $k = g \diamond h$. We will show that for all primes p, $f(p^m) = f(p)^m$; for all $m \in \mathbb{N}$.

If $m=1$, obvious. Let $m\geq 2$ and assume that $f(p^r)=f(p)^r$ for $1\leq r<m$. Since $f(g\diamond h)=fg\diamond fh$, we obtain

$$\alpha(p^m)f(p^m)\sum_{i=0}^{m}\frac{g}{\alpha}(p^i)\frac{h}{\alpha}(p^{m-i})=\alpha(p^m)\sum_{i=0}^{m}\frac{fg}{\alpha}(p^i)\frac{fh}{\alpha}(p^{m-i}),$$

$$\begin{aligned} f(p^m)&\left(\frac{g}{\alpha}(1)\frac{h}{\alpha}(p^m)+\frac{g}{\alpha}(p^m)\frac{h}{\alpha}(1)+\sum_{i=1}^{m-1}\frac{g}{\alpha}(p^i)\frac{h}{\alpha}(p^{m-i})\right)\\ &=\frac{fg}{\alpha}(1)\frac{fh}{\alpha}(p^m)+\frac{fg}{\alpha}(p^m)\frac{fh}{\alpha}(1)+\sum_{i=1}^{m-1}\frac{fg}{\alpha}(p^i)\frac{fh}{\alpha}(p^{m-i}),\\ f(p^m)&\sum_{i=1}^{m-1}\frac{g}{\alpha}(p^i)\frac{h}{\alpha}(p^{m-i})=\sum_{i=1}^{m-1}\frac{fg}{\alpha}(p^i)\frac{fh}{\alpha}(p^{m-i}). \end{aligned}$$

By induction hypothesis, $f(p^m)\sum_{i=1}^{m-1}\frac{g}{\alpha}(p^i)\frac{h}{\alpha}(p^{m-i})=f(p)^m\sum_{i=1}^{m-1}\frac{g}{\alpha}(p^i)\frac{h}{\alpha}(p^{m-i})$. It follows from $g\diamond h$ is $\mathrm{Q}_\alpha-$partially discriminative product that

$$\sum_{i=1}^{m-1}\frac{g}{\alpha}(p^i)\frac{h}{\alpha}(p^{m-i})=\sum_{i=0}^{m}\frac{g}{\alpha}(p^i)\frac{h}{\alpha}(p^{m-i})-\frac{g}{\alpha}(1)\frac{h}{\alpha}(p^m)-\frac{g}{\alpha}(p^m)\frac{h}{\alpha}(1)\neq 0.$$

Therefore $f(p^m)=f(p)^m$. We conclude that $f\in\mathcal{C}$. □

Theorem 26. *Suppose that $f(1)=1$. Then $f\in\mathcal{C}$ if and only if it distributes over a $Q_\alpha-semi-discriminative$ product.*

Proof. If $f\in\mathcal{C}$ then Theorem 23 show that $f(g\diamond h)=fg\diamond fh$ for all $g,h\in\mathcal{A}$. The conversely follows through the same proof as in the last half of Theorem 24. □

Definition 11. Let $r\in\mathbb{N},\ r>2$ and let $g_1,g_2,\ldots,g_r\in\mathcal{A}\setminus\{0\}$. We say that the product $k=g_1\diamond g_2\diamond\cdots\diamond g_r$ is

- $Q_\alpha-r\ fold\ discriminative$, or "$\diamond-$ r. d." for notation, if the relation

$$\alpha(1)^{r-1}k(n)=\sum_{j=1}^{r}g_1(1)\cdots g_{j-1}(1)g_j(n)g_{j+1}(1)\cdots g_r(1) \tag{16}$$

  holds only when n is prime;

- $Q_\alpha - r$ *fold partially discriminative*, or "$\diamond-$ r. p. d." for notation, if the relation

$$\alpha(1)^{r-1}k(p^i) = \sum_{j=1}^{r} g_1(1)\cdots g_{j-1}(1)g_j(p^i)g_{j+1}(1)\cdots g_r(1) \tag{17}$$

implies that $i = 1$;

- $Q_\alpha - r$ *fold semi − discriminative*, or "$\diamond-$ r. s. d." for notation, if the relation

$$\alpha(1)^{r-1}k(n) = \sum_{j=1}^{r} g_1(1)\cdots g_{j-1}(1)g_j(n)g_{j+1}(1)\cdots g_r(1) \tag{18}$$

holds only when $n = 1$ or n is prime.

Theorem 27. *Suppose that $f(1) \neq 0$. Then $f \in \mathcal{C}$ if and only if it distributes over a $\diamond-$ r. d. product.*

Proof. Let $f \in \mathcal{C}$, then by Theorem 23 we obtain $f(g\diamond h) = fg\diamond fh$ for all $g, h \in \mathcal{A}$. Conversely, assume that f distributes over a $\diamond-$ r. d. product $k = g_1 \diamond g_2 \diamond \cdots \diamond g_r$. First we show that f(1)=1. So we get
$\dfrac{1}{\alpha(1)^{r-1}} \displaystyle\sum_{j=1}^{r} g_1(1)\cdots g_{j-1}(1)g(1)g_{j+1}(1)\cdots g_r(1) = 0 = k(1)$, i.e.
$\alpha(1)^{r-1}k(1) = \displaystyle\sum_{j=1}^{r} g_1(1)\cdots g_{j-1}(1)g_j(1)g_{j+1}(1)\cdots g_r(1)$, so the equation (16) holds for $n = 1$ which is a contradiction. Thus $k(1) \neq 0$. Since

$$fk = f(g_1 \diamond g_2 \diamond \cdots \diamond g_r) = fg_1 \diamond fg_2 \diamond \cdots \diamond fg_r,$$

we get

$$\begin{aligned} f(1)k(1) &= \alpha(1)\frac{fg_1}{\alpha}(1)\frac{fg_2}{\alpha}(1)\cdots\frac{fg_r}{\alpha}(1) \\ &= f(1)^r\left(\frac{g_1(1)g_2(1)\cdots g_r(1)}{\alpha(1)^{r-1}}\right) \\ &= f(1)^r k(1). \end{aligned}$$

It follows from $f(1)k(1) \neq 0$ that $f(1)^{r-1} = 1$. Next we show that for all prime $p_1, \ldots, p_m$ (not necessary distinct),

$$f(p_1\cdots p_m) = f(p_1)\cdots f(p_m). \tag{19}$$

We proceed by induction on m. This is trivial if $m=1$, so assume that $n = p_1 \cdots p_m$; $m \geq 2$ and that (19) is true for all integers which the number of prime factor (not necessary distinct) is less then m. Since

$$f(g_1 \diamond g_2 \diamond \cdots \diamond g_r) = fg_1 \diamond fg_2 \diamond \cdots \diamond fg_r.$$

Then

$$\begin{aligned} f(p_1 \cdots p_m)\alpha(p_1 \cdots p_m) & \sum_{d_1 \cdots d_r = p_1 \cdots p_m} \frac{g_1}{\alpha}(d_1) \cdots \frac{g_r}{\alpha}(d_r) \\ = \alpha(p_1 \cdots p_m) & \sum_{d_1 \cdots d_r = p_1 \cdots p_m} \frac{fg_1}{\alpha}(d_1) \cdots \frac{fg_r}{\alpha}(d_r). \end{aligned}$$

Using the induction hypothesis, we get

$$\begin{aligned} & f(p_1 \cdots p_m) \sum_{\substack{d_1 \cdots d_r = p_1 \cdots p_m \\ d_j \neq p_1 \cdots p_m \text{for all } j \in \{1,\ldots,r\}}} \frac{g_1}{\alpha}(d_1) \cdots \frac{g_1}{\alpha}(d_r) \\ & + f(p_1 \cdots p_m) \sum_{\substack{d_1 \cdots d_r = p_1 \cdots p_m \\ d_j = p_1 \cdots p_m \text{for some } j \in \{1,\ldots,r\}}} \frac{g_1}{\alpha}(d_1) \cdots \frac{g_1}{\alpha}(d_r) \\ = & f(p_1) \cdots f(p_m) \sum_{\substack{d_1 \cdots d_r = p_1 \cdots p_m \\ d_j \neq p_1 \cdots p_m \text{for all } j \in \{1,\ldots,r\}}} \frac{g_1}{\alpha}(d_1) \cdots \frac{g_1}{\alpha}(d_r) \\ & + f(p_1 \cdots p_m) f(1)^{r-1} \sum_{\substack{d_1 \cdots d_r = p_1 \cdots p_m \\ d_j = p_1 \cdots p_m \text{for some } j \in \{1,\ldots,r\}}} \frac{g_1}{\alpha}(d_1) \cdots \frac{g_1}{\alpha}(d_r). \end{aligned}$$

But $f(1)^{r-1} = 1$. Hence

$$[f(p_1 \cdots p_m) - f(p_1) \cdots f(p_m)] \sum_{\substack{d_1 \cdots d_r = p_1 \cdots p_m \\ d_j \neq p_1 \cdots p_m \text{for all } j \in \{1,\ldots,r\}}} \frac{g_1}{\alpha}(d_1) \cdots \frac{g_1}{\alpha}(d_r) = 0.$$

Since

$$\begin{aligned} & \sum_{\substack{d_1 \cdots d_r = p_1 \cdots p_m \\ d_j \neq p_1 \cdots p_m \text{for all } j \in \{1,\ldots,r\}}} \frac{g_1}{\alpha}(d_1) \cdots \frac{g_1}{\alpha}(d_r) \\ & = k(p_1 \cdots p_m) - \sum_{\substack{d_1 \cdots d_r = p_1 \cdots p_m \\ d_j = p_1 \cdots p_m \text{for some } j \in \{1,\ldots,r\}}} \frac{g_1}{\alpha}(d_1) \cdots \frac{g_1}{\alpha}(d_r) \end{aligned}$$

and k is $\diamond-$ r. d. product, it follows that
$\displaystyle\sum_{\substack{d_1 \cdots d_r = p_1 \cdots p_m \\ d_j \neq p_1 \cdots p_m \text{for all } j \in \{1,\ldots,r\}}} \frac{g_1}{\alpha}(d_1) \cdots \frac{g_1}{\alpha}(d_r) \neq 0$. Thus $f(p_1 \cdots p_m) = f(p_1) \cdots f(p_m)$.

This show that $f \in \mathcal{C}$. □

Theorem 28. *Suppose that $f(1) \in \mathcal{M}$. Then $f \in \mathcal{C}$ if and only if it distributes over a $\diamond-$ r. p. d. product.*

Proof. If $f \in \mathcal{C}$ then Theorem 23 show that $f(g \diamond h) = fg \diamond fh$ for all $g, h \in \mathcal{A}$. Conversely, assume that f distributes over a $\diamond-$ r. p. d. product $k = g_1 \diamond g_2 \diamond \cdots \diamond g_r$. Since $f \in \mathcal{M}$. Thus it suffices to show that for all primes p, $f(p^m) = f(p)^m$; for all $m \in \mathbb{N}$. The case $m = 1$ being trivial, so assume that $m \geq 2$ and $f(p^t) = f(p)^t$ holds for $t < m$. Using distributive property and induction hypothesis, we get

$$f(g_1 \diamond g_2 \diamond \cdots \diamond g_r)(p^m) = (fg_1 \diamond fg_2 \diamond \cdots \diamond fg_r)(p^m)$$

$$\begin{aligned}
&f(p^m) \sum_{\substack{d_1 \cdots d_r = p^m \\ d_j \neq p^m \text{for all } j \in \{1,\ldots,r\}}} \frac{g_1}{\alpha}(d_1) \cdots \frac{g_1}{\alpha}(d_r) \\
&\quad + f(p^m) \sum_{\substack{d_1 \cdots d_r = p^m \\ d_j = p^m \text{for some } j \in \{1,\ldots,r\}}} \frac{g_1}{\alpha}(d_1) \cdots \frac{g_1}{\alpha}(d_r) \\
&= f(p)^m \sum_{\substack{d_1 \cdots d_r = p^m \\ d_j \neq p^m \text{for all } j \in \{1,\ldots,r\}}} \frac{g_1}{\alpha}(d_1) \cdots \frac{g_1}{\alpha}(d_r) \\
&\quad + f(p^m) f(1)^{r-1} \sum_{\substack{d_1 \cdots d_r = p^m \\ d_j = p^m \text{for some } j \in \{1,\ldots,r\}}} \frac{g_1}{\alpha}(d_1) \cdots \frac{g_1}{\alpha}(d_r).
\end{aligned}$$

But $f(1)^{r-1} = 1$. Hence

$$[f(p^m) - f(p)^m] \sum_{\substack{d_1 \cdots d_r = p^m \\ d_j \neq p^m \text{for all } j \in \{1,\ldots,r\}}} \frac{g_1}{\alpha}(d_1) \cdots \frac{g_1}{\alpha}(d_r) = 0.$$

Since

$$\begin{aligned}
&\sum_{\substack{d_1 \cdots d_r = p^m \\ d_j \neq p^m \text{for all } j \in \{1,\ldots,r\}}} \frac{g_1}{\alpha}(d_1) \cdots \frac{g_1}{\alpha}(d_r) \\
&= k(p^m) - \sum_{\substack{d_1 \cdots d_r = p^m \\ d_j = p^m \text{for some } j \in \{1,\ldots,r\}}} \frac{g_1}{\alpha}(d_1) \cdots \frac{g_1}{\alpha}(d_r)
\end{aligned}$$

and k is $\diamond-$ r. p. d. product, it follows that

$$\sum_{\substack{d_1 \cdots d_r = p^m \\ d_j \neq p^m \text{for all } j \in \{1,\ldots,r\}}} \frac{g_1}{\alpha}(d_1) \cdots \frac{g_1}{\alpha}(d_r) \neq 0. \text{ Thus } f(p^m) = f(p)^m.$$ □

Theorem 29. *Suppose that $f(1) = 1$. Then $f \in \mathcal{C}$ if and only if it distributes over a $\diamond-$ r. s. d. product.*

Proof. If $f \in \mathcal{C}$ then Theorem 23 show that $f(g \diamond h) = fg \diamond fh$ for all $g, h \in \mathcal{A}$. The conversely follows through the same proof as in the last half of Theorem 27. $\square$

Theorem 30. *Let $f \in \mathcal{M}$. Then $f \in \mathcal{C}$ if and only if $(fg)^{-1\diamond} = fg^{-1\diamond}$ for all $g \in \mathcal{A}$ with $g(1) \neq 0$.*

Proof. Since $f \in \mathcal{C}$ and we have $f\alpha I = f(g \diamond g^{-1\diamond})$, so by Theorem 23,
$\alpha I = f\alpha I = fg \diamond fg^{-1\diamond}$. That is $(fg)^{-1\diamond} = fg^{-1\diamond}$.
Conversely, assume that $(fg)^{-1\diamond} = fg^{-1\diamond}$ for all $g \in \mathcal{A}$ with $g(1) \neq 0$, then $(f\alpha)^{-1\diamond} = f\alpha^{-1\diamond}$, i.e., $\alpha \left(\frac{f\alpha}{\alpha}\right)^{-1*} = f\alpha\mu$. So $f^{-1*} = f\mu$. By Theorem 4 we get that $f \in \mathcal{C}$. $\square$

Theorem 31. *Let $f \in \mathcal{M}$. Then $f \in \mathcal{C}$ if and only if $(\alpha f)^{-1\diamond} = \mu\alpha f$.*

Proof. If $f \in \mathcal{C}$ then by Theorem 30, $(\alpha f)^{-1\diamond} = \alpha^{-1\diamond} f = \mu\alpha f$.
Conversely, assume that $(\alpha f)^{-1\diamond} = \mu\alpha f$. Then we get

$$\begin{aligned}
\alpha f \diamond \mu\alpha f &= \alpha I \\
\alpha\left(\frac{\alpha f}{\alpha} * \frac{\mu\alpha f}{\alpha}\right) &= \alpha I, \\
f * \mu f &= I, \\
f^{-1*} &= \mu f.
\end{aligned}$$

By Theorem 4 we conclude that $f \in \mathcal{C}$. $\square$

Recall that $\mu_r(n) = \prod_{p|n} \binom{r}{\nu_p(n)} (-1)^{\nu_p(n)};\ r \in \mathbb{R}$.

Theorem 32. *Let $f \in \mathcal{M}$ and $r \in \mathbb{R} - \{0\}$. Then $f \in \mathcal{C}$ if and only if $(\mu_r\alpha f)^{-1\diamond} = \mu_{-r}\alpha f$.*

Proof. Let $f \in \mathcal{C}$, then

$$\begin{aligned}
\alpha I =& f\alpha I \\
=& f\alpha\mu_0 \\
=& f\alpha(\mu_r * \mu_{-r}) \\
=& f\alpha\left(\frac{\alpha\mu_r \diamond \alpha\mu_{-r}}{\alpha}\right) \\
=& f\alpha\mu_r \diamond f\alpha\mu_{-r},
\end{aligned}$$

i.e., $(\mu_r\alpha f)^{-1\diamond} = \mu_{-r}\alpha f$.
Conversely, let $(\mu_r\alpha f)^{-1\diamond} = \mu_{-r}\alpha f$, we have $\alpha\left(\frac{\mu_r\alpha f}{\alpha}\right)^{-1*} = \mu_{-r}\alpha f$.
So $(\mu_r f)^{-1*} = \mu_{-r}f$. Theorem 9 implies that $f \in \mathcal{C}$. □

Remark. Let $f \in \mathcal{M}$. Then $f \in \mathcal{C}$ if and only if $(\mu\alpha f)^{-1\diamond} = \mu_{-1}\alpha f$.

Theorem 33. *Let $f \in \mathcal{M}$. Then $f \in \mathcal{C}$ if and only if $(f\alpha)^{-1\diamond}(p^a) = 0$ for all prime p and all $a \geq 2$.*

Proof. If $f \in \mathcal{C}$ then by Theorem 30, $(f\alpha)^{-1\diamond} = f\alpha^{-1\diamond}$.
Hence $(f\alpha)^{-1\diamond}(p^a) = f\alpha^{-1\diamond}(p^a) = f\alpha\mu(p^a) = 0$.
Conversely, assume that for all primes p and all $a \geq 2, (f\alpha)^{-1\diamond}(p^a) = 0$.
From Theorem 21 we have $(\alpha f)^{-1\diamond} = \alpha f^{-1*}$, so that $0 = (\alpha f)^{-1\diamond}(p^a) = \alpha f^{-1*}(p^a)$
i.e.,$f^{-1*}(p^a) = 0$. Theorem 2 implies that $f \in \mathcal{C}$. □

Proposition 4. *For $f \in \mathcal{M}$, $f^{-1\diamond}(p) = -\alpha(1)^2 f(p)$ for all primes p.*

Proof. From $(f \diamond f^{-1\diamond})(p) = (\alpha I)(p)$, we get

$$\begin{aligned}
\alpha(p)\left(\frac{f}{\alpha}(1)\frac{f^{-1\diamond}}{\alpha}(p) + \frac{f}{\alpha}(p)\frac{f^{-1\diamond}}{\alpha}(1)\right) &= 0 \\
\frac{f}{\alpha}(1)\frac{f^{-1\diamond}}{\alpha}(p) + \frac{f}{\alpha}(p)\frac{\alpha(1)^2}{\alpha(1)} &= 0 \\
f^{-1\diamond}(p) &= -\alpha(1)^2 f(p).
\end{aligned}$$

□

Remark. If $\alpha(1) = 1$ then $f^{-1\diamond}(p) = -f(p)$ for all $f \in \mathcal{M}$.

Theorem 34. *Let $f, g \in \mathcal{M}$. Then $f \diamond g \in \mathcal{C} \Leftrightarrow$ either*

$$g(p^a) = \frac{f^{-1\diamond}(p^a)}{\alpha(1)} + \alpha(p^a)\left[\frac{f^{-1\diamond}(p^{a-1})(f(p)+g(p))}{\alpha(1)\alpha(p^{a-1})\alpha(p)} + \cdots + \frac{f^{-1\diamond}(p)(f(p)+g(p))^{a-1}}{\alpha(1)^{a-1}\alpha(p)\alpha(p^{a-1})}\right] + \frac{(f(p)+g(p))^a}{\alpha(1)^{a-1}} \tag{20}$$

or

$$f(p^a) = \frac{g^{-1\diamond}(p^a)}{\alpha(1)} + \alpha(p^a)\left[\frac{g^{-1\diamond}(p^{a-1})(f(p)+g(p))}{\alpha(1)\alpha(p^{a-1})\alpha(p)} + \cdots + \frac{g^{-1\diamond}(p)(f(p)+g(p))^{a-1}}{\alpha(1)^{a-1}\alpha(p)\alpha(p^{a-1})}\right] + \frac{(f(p)+g(p))^a}{\alpha(1)^{a-1}} \tag{21}$$

for all prime p and all $a \in \mathbb{N}$.

Proof. Assume that $f \diamond g \in \mathcal{C}$. Let p be a prime and $a \in \mathbb{N}$, then $g(p^a) = (f \diamond g) \diamond f^{-1\diamond}(p^a)$.

$$\begin{aligned} g(p^a) &= \alpha(p^a)\left[\sum_{i=0}^{a} \frac{f^{-1\diamond}}{\alpha}(p^{a-1})\frac{f \diamond g}{\alpha}(p^i)\right] \\ &= \alpha(p^a)\left[\frac{f^{-1\diamond}(p^a)(f \diamond g)(1)}{\alpha(p^a)\alpha(1)} + \cdots + \frac{f^{-1\diamond}(1)(f \diamond g)(p^a)}{\alpha(1)\alpha(p^a)}\right] \end{aligned} \tag{22}$$

Since $f \diamond g \in \mathcal{C}$, thus
$(f \diamond g)(p^i) = (f \diamond g)(p)^i = \left[\alpha(p^a)\left(\dfrac{f(p)+g(p)}{\alpha(1)\alpha(p)}\right)\right]^i = \dfrac{(f(p)+g(p))^i}{\alpha(1)^i}$ for $i \geq 1$.
We get

$$g(p^a) = \frac{f^{-1\diamond}(p^a)}{\alpha(1)} + \alpha(p^a)\left[\frac{f^{-1\diamond}(p^{a-1})(f(p)+g(p))}{\alpha(1)\alpha(p^{a-1})\alpha(p)} + \cdots + \frac{f^{-1\diamond}(p)(f(p)+g(p))^{a-1}}{\alpha(1)^{a-1}\alpha(p)\alpha(p^{a-1})}\right] + \frac{(f(p)+g(p))^a}{\alpha(1)^{a-1}}.$$

Similarly, we can prove the equation (21) by interchanging f and g.

Conversely, Assume that the equation (20) holds.
We obtain $(f \diamond g)(p^i) = \dfrac{(f(p)+g(p))^i}{\alpha(1)^i}$ by comparing (20) and (22) for successive values of $i \in \mathbb{N}$.
But $(f \diamond g)(p)^i = \left[\alpha(p)\left(\dfrac{f(p)+g(p)}{\alpha(1)\alpha(p)}\right)\right]^i = \dfrac{(f(p)+g(p))^i}{\alpha(1)^i}$ $(\forall i \in \mathbb{N})$.
Hence $(f \diamond g)^(p^i) = (f \diamond g)(p)^i$ $(\forall i \in \mathbb{N})$, that is $f \diamond g \in \mathcal{C}$. The same result holds similarly for the equation (21). $\square$

In 2009, Tóth and Haukkanen (see section 4 of Tóth and Haukkanen, 2009) proved some properties of semi-multiplicative functions with respect to the binomial convolution. Similar results can also be derived for the $\mathrm{Q}_\alpha-$ convolution.

Proposition 5. *If $f \in \mathcal{M}$ then $f \in \mathcal{S}$.*

Proof. Let $m, n \in \mathbb{N}$, $(m, n) = d$ and we get that $m = dk$ and $n = dl$ with $(d, k) = 1, (d, l) = 1$ and $(k, l) = 1$. It follows that

$$\begin{aligned} f(mn) &= f(dk)f(dl) = f(d)f(k)f(d)F(l) = f(d)f(kdl) \\ &= f((m,n))f(mn/d) = f((m,n))f([m,n]). \end{aligned}$$

Thus $f \in \mathcal{S}$. □

Theorem 35. *Let $\alpha \in \mathcal{M}$, then $\mathcal{S}$ is a commutative semigroup with identity under the $Q_\alpha-$ convolution.*

Proof. It is known that semimultiplicative functions form a commutative semigroup with identity under the Dirichlet convolution (see section 4 of Tóth and Haukkanen, 2009). Since $f \diamond g = \alpha(\frac{f}{\alpha} * \frac{g}{\alpha})$, hence it is suffices to show that for all $f \in \mathcal{S}$, $\frac{f}{\alpha} \in \mathcal{S}$ and $\alpha I \in \mathcal{S}$. Let $f \in \mathcal{S}$. Then for $m, n \in \mathbb{N}$,

$$\begin{aligned} \frac{f}{\alpha}(m)\frac{f}{\alpha}(n) &= \frac{f(m)f(n)}{\alpha(m)\alpha(n)} \\ &= \frac{f((m,n))f([m,n])}{\alpha((m,n))\alpha([m,n])} \\ &= \frac{f}{\alpha}((m,n))\frac{f}{\alpha}([m,n]). \end{aligned}$$

Thus $\frac{f}{\alpha} \in \mathcal{S}$. Therefore for all $f, g \in \mathcal{S}$, $f \diamond g = \alpha(\frac{f}{\alpha} * \frac{g}{\alpha}) \in \mathcal{S}$.
Since $\alpha, I \in \mathcal{M}$ thus by proposition 5, we get $\alpha, I \in \mathcal{S}$, so that $\alpha I \in \mathcal{S}$.
We conclude that $\mathcal{S}$ is a commutive semigroup with identity under the $\mathrm{Q}_\alpha-$ convolution. □

Adopting notations of Proposition 1, we get:

Theorem 36. *Let $\alpha \in \mathcal{M}$ and $f, g \in \mathcal{S}$, then*

(i) $a_{f \diamond g} = a_f a_g$.

(ii) $c_{f \diamond g} = \dfrac{\alpha(a_f a_g)}{\alpha(a_f)\alpha(a_g)} c_f c_g$.

(iii) $(f \diamond g)' = \dfrac{\alpha_{a_f a_g} \alpha(a_f)\alpha(a_g)}{\alpha\alpha(a_f a_g)} \left(\dfrac{\alpha f'}{\alpha_{a_f}} \diamond \dfrac{\alpha g'}{\alpha_{a_g}} \right)$.

Proof. (i) Since $f \diamond g = \alpha(\dfrac{f}{\alpha} * \dfrac{g}{\alpha})$, thus

$$\begin{aligned}
a_{f \diamond g} &= a_{\alpha(\frac{f}{\alpha} * \frac{g}{\alpha})} \\
&= a_{\frac{f}{\alpha} * \frac{g}{\alpha}} \qquad \text{(by Proposition 3)} \\
&= a_{\frac{f}{\alpha}} a_{\frac{g}{\alpha}} \qquad \text{(by Proposition 2)} \\
&= a_f a_g \qquad \text{(by Proposition 3 and } \frac{u}{\alpha} \in \mathcal{M}).
\end{aligned}$$

(ii) $c_{f \diamond g} = (f \diamond g)(a_{f \diamond g}) = (f \diamond g)(a_f a_g) = \alpha(a_f a_g)(\frac{f}{\alpha} * \frac{g}{\alpha})(a_f a_g)$.

Consider

$$\begin{aligned}
\left(\frac{f}{\alpha} * \frac{g}{\alpha} \right)(a_f a_g) &= \sum_{rs = a_f a_g} \frac{f}{\alpha}(r) \frac{g}{\alpha}(s) \\
&= \sum_{\substack{rs = a_f a_g \\ r < a_f \\ s > a_g}} \frac{f}{\alpha}(r) \frac{g}{\alpha}(s) + \frac{f}{\alpha}(a_f) \frac{g}{\alpha}(a_g) + \sum_{\substack{rs = a_f a_g \\ r > a_f \\ s < a_g}} \frac{f}{\alpha}(r) \frac{g}{\alpha}(s) \\
&= \frac{f}{\alpha}(a_f) \frac{g}{\alpha}(a_g) \\
&= \frac{c_f c_g}{\alpha(a_f)\alpha(a_g)} \quad ,
\end{aligned}$$

hence $\quad c_{f \diamond g} = \dfrac{\alpha(a_f a_g)}{\alpha(a_f)\alpha(a_g)} c_f c_g$.

(iii) Since $f \diamond g = \alpha(\frac{f}{\alpha} * \frac{g}{\alpha})$, thus

$$\begin{aligned}
(f \diamond g)' &= \left(\alpha\left(\frac{f}{\alpha} * \frac{g}{\alpha}\right)\right)' \\
&= \frac{\alpha_{a_f a_g}}{\alpha(a_f a_g)}\left(\frac{f}{\alpha} * \frac{g}{\alpha}\right)' && \text{(by proposition 3)} \\
&= \frac{\alpha_{a_f a_g}}{\alpha(a_f a_g)}\left(\left(\frac{f}{\alpha}\right)' * \left(\frac{g}{\alpha}\right)'\right) && \text{(by proposition 2)} \\
&= \frac{\alpha_{a_f a_g}}{\alpha(a_f a_g)}\left(\frac{\alpha(a_f)}{\alpha_{a_f}} f' * \frac{\alpha(a_g)}{\alpha_{a_g}} g'\right) && \text{(by proposition 3 and } \frac{u}{\alpha} \in \mathcal{M}) \\
&= \frac{\alpha_{a_f a_g}}{\alpha(a_f a_g)\alpha}\left[\left(\frac{\alpha(a_f)\alpha}{\alpha_{a_f}} f'\right) \diamond \left(\frac{\alpha(a_g)\alpha}{\alpha_{a_g}} g'\right)\right] \\
&= \frac{\alpha_{a_f a_g}\alpha(a_f)\alpha(a_g)}{\alpha(a_f a_g)\alpha}\left(\frac{\alpha f'}{\alpha_{a_f}} \diamond \frac{\alpha g'}{\alpha_{a_g}}\right).
\end{aligned}$$

□

Since Dirichlet series is one of the most fundamental concepts in analytic number theory which has been investigated widely in the literatures (see McCarthy, 1985; Shapiro, 1983), generalizations of this concept is naturally of interest. In 2009, Tóth and Huakkanen proposed a generalization of Dirichlet series by exponential Dirichlet series. A similar result in term of $\alpha-$ Dirichlet series is presented as follows:

Definition 12. For arithmetic function f we define the α-Dirichlet series by

$$\hat{D}(f, s) = D(\frac{f}{\alpha}, s) = \sum_{n=1}^{\infty} \frac{f(n)}{\alpha(n) n^s}, \quad f(n) \in \mathbb{C}.$$

Notice that

(1) if $\alpha = \xi$ then $\hat{D}(f, s) = \tilde{D}(f, s)$ where $\tilde{D}(f, s)$ is the exponential Dirichlet series (see section 5 of Tóth and Huakkanen, 2009);

(2) $\hat{D}(\alpha, s) = \zeta(s)$ where $\zeta(s)$ is the arithmetic zeta function (see also section 5 of Tóth and Huakkanen, 2009).

Let $\hat{\mathcal{D}}$ denote the set of all $\alpha-$Dirichlet series. We will show that $\hat{\mathcal{D}}$ is a $\mathbb{C}-$algebra under the usual addition and multiplication of series and isomorphic to $(\mathcal{A}, +, \diamond, \mathbb{C})$.

Proposition 6. $(\hat{\mathcal{D}}, +, \cdot, \mathbb{C})$ *is a* $\mathbb{C}-$ *algebra.*

Proof. It is easy to see that $\hat{\mathcal{D}}$ is a vector space over field $\mathbb{C}$.
Let $F_1(s), F_2(s), F_3(s) \in \hat{\mathcal{D}}$. Then $F_k(s) = \hat{D}(f_k, s) = \sum_{n=1}^{\infty} \frac{f_k(n)}{\alpha(n)n^s}, \quad f_k(n) \in \mathcal{A}$, $k = 1, 2, 3$

1. $$F_1(s)F_2(s) = \sum_{n=1}^{\infty} \frac{f_1(n)}{\alpha(n)n^s} \sum_{n=1}^{\infty} \frac{f_2(n)}{\alpha(n)n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \left(\sum_{xy=n} \frac{f_1}{\alpha}(x) \frac{f_2}{\alpha}(y) \right)$$
$$= \sum_{n=1}^{\infty} \frac{1}{n^s} \left(\frac{f_1}{\alpha} * \frac{f_2}{\alpha} \right)(n) = \sum_{n=1}^{\infty} \frac{(f_1 \diamond f_2)(n)}{\alpha(n)n^s}.$$

Thus $F_1(s)F_2(s) \in \hat{\mathcal{D}}$.

2. $$F_1(s)(F_2(s) + F_3(s)) = \sum_{n=1}^{\infty} \frac{f_1(n)}{\alpha(n)n^s} \left(\sum_{n=1}^{\infty} \frac{f_2(n)}{\alpha(n)n^s} + \sum_{n=1}^{\infty} \frac{f_3(n)}{\alpha(n)n^s} \right)$$
$$= \sum_{n=1}^{\infty} \frac{1}{n^s} \left(\sum_{xy=n} \frac{f_1}{\alpha}(x) \frac{f_2}{\alpha}(y) + \sum_{xy=n} \frac{f_1}{\alpha}(x) \frac{f_3}{\alpha}(y) \right)$$
$$= \sum_{n=1}^{\infty} \frac{1}{n^s} \left(\frac{f_1}{\alpha} * \frac{f_2}{\alpha} + \frac{f_1}{\alpha} * \frac{f_3}{\alpha} \right)(n)$$
$$= (F_1)(s)(F_2)(s) + (F_1)(s)(F_3)(s).$$

Thus $F_1(s)(F_2(s) + F_3(s)) = F_1(s)F_2(s) + F_1(s)F_3(s)$.

3. Using expression similar to 2.

Thus $(F_1(s) + F_2(s))F_3(s) = F_1(s)F_3(s) + F_2(s)F_3(s)$.

4. $$c(F_1(s)F_2(s)) = c \sum_{n=1}^{\infty} \frac{(f_1 \diamond f_2)(n)}{\alpha(n)n^s}.$$
$$(cF_1)(s)F_2(s) = \sum_{n=1}^{\infty} \frac{(cf_1 \diamond f_2)(n)}{\alpha(n)n^s} = c \sum_{n=1}^{\infty} \frac{(f_1 \diamond f_2)(n)}{\alpha(n)n^s}$$
$$F_1(s)(cF_2)(s) = \sum_{n=1}^{\infty} \frac{(f_1 \diamond cf_2)(n)}{\alpha(n)n^s} = c \sum_{n=1}^{\infty} \frac{(f_1 \diamond f_2)(n)}{\alpha(n)n^s}.$$

Thus $c(F_1(s)F_2(s)) = (cF_1(s))F_2(s) = F_1(s)(cF_2(s))$.

From 1-4 we conclude that $(\hat{\mathcal{D}}, +, \cdot, \mathbb{C})$ is a $\mathbb{C}$-algebra. □

Theorem 37. *The algebras* $(\mathcal{A}, +, \diamond, \mathbb{C})$ *and* $(\hat{\mathcal{D}}, +, \cdot, \mathbb{C})$ *are isomorphic.*

Proof. It is easy to see that the mapping $f \mapsto \hat{D}(f, s)$ is a bijection on $\hat{\mathcal{D}}$.
Moreover $f + g \mapsto \hat{D}(f + g, s) = \hat{D}(f, s) + \hat{D}(g, s)$ and $cf \mapsto \hat{D}(cf, s) = c\hat{D}(f, s)$

and we can express that

$$\begin{aligned}\hat{D}(f \diamond g, s) &= \hat{D}\left(\alpha\left(\frac{f}{\alpha} * \frac{g}{\alpha}\right), s\right) \\ &= D\left(\frac{f}{\alpha} * \frac{g}{\alpha}, s\right) \\ &= D\left(\frac{f}{\alpha}, s\right) D\left(\frac{g}{\alpha}, s\right) \\ &= \hat{D}(f, s)\hat{D}(g, s).\end{aligned}$$

This prove is complete. □

We proceed to derive further properties of $\mathrm{Q}_\alpha-$ convolution. Properties involving the Rearick's logarithm and $p-$basic derivation with respect to the Dirichlet convolution are well known in number theory. These properties have been published in the literature (see Laohakol *et al*, 2002; Raerick, 1968; Shapiro, 1983). We now show that these properties have analogues with respect to the $\mathrm{Q}_\alpha-$ convolution.

Proposition 7. *For $f, g \in \mathcal{A}$, $n \in \mathbb{N}$, let $d_L f : \mathcal{A} \to \mathcal{A}$ be defined by*

$$d_L f = f(n) log\ n,$$

let $\tilde{d}_p f : \mathcal{A} \to \mathcal{A}$ be defined by

$$\tilde{d}_p f = \alpha(n)\frac{f}{\alpha}(np)\nu_p(np),$$

or equivalently

$$\tilde{d}_p f = \alpha(n)\tilde{d}_p\left(\frac{f}{\alpha}\right)(n).$$

Then d_L and $\tilde{d}_p$ are derivations on $(\mathcal{A}, +, \diamond)$.

Proof. We will show that

(i) $d_L(c_1 f + c_2 g) = c_1 d_L f + c_2 d_L g,\ d_L(f \diamond g) = f \diamond d_L g + g \diamond d_L f.$

(ii) $\tilde{d}_p(c_1 f + c_2 g) = c_1 \tilde{d} f + c_2 \tilde{d} g,\ \tilde{d}_p(f \diamond g) = f \diamond \tilde{d}_p g + g \diamond \tilde{d}_p f.$

Let $n \in \mathbb{N}$. Then

$$\begin{aligned}
d_L(c_1f+c_2g)(n) &= (c_1f+c_2g)(n)log\ n \\
&= [(c_1f)(n)+(c_2g)(n)]\ log\ n \\
&= c_1f(n)log\ n + c_2g(n)log\ n \\
&= c_1d_Lf(n)+c_2d_Lg(n). \\
d_L(f\diamond g) &= (f\diamond g)(n)log\ n \\
&= \left(\sum_{xy=n}\frac{\alpha(n)}{\alpha(x)\alpha(y)}f(x)g(y)\right)log\ n \\
&= \alpha(n)\left(\sum_{xy=n}\frac{f}{\alpha}(x)\frac{g}{\alpha}(y)logx+\sum_{xy=n}\frac{f}{\alpha}(x)\frac{g}{\alpha}(y)logy\right) \\
&= \alpha(n)\left(\frac{d_Lf}{\alpha}*\frac{g}{\alpha}+\frac{d_Lg}{\alpha}*\frac{f}{\alpha}\right)(n) \\
&= (f\diamond d_Lg)(n)+(g\diamond d_Lf)(n). \\
\tilde{d}_p(c_1f+c_2g)(n) &= \alpha(n)\left(\frac{c_1f+c_2g}{\alpha}\right)(n)\nu_p(np) \\
&= \alpha(n)\left(\frac{c_1f}{\alpha}\right)(n)\nu_p(np)+\alpha(n)\left(\frac{c_2g}{\alpha}\right)(n)\nu_p(np) \\
&= c_1\alpha(n)\left(\frac{f}{\alpha}\right)(n)\nu_p(np)+c_2\alpha(n)\left(\frac{g}{\alpha}\right)(n)\nu_p(np) \\
&= c_1\tilde{d}f+c_2\tilde{d}g. \\
\tilde{d}_p(f\diamond g) &= \alpha(n)\left(\frac{f\diamond g}{\alpha}\right)(np)\nu_p(np) \\
&= \alpha(n)\left[\frac{\alpha(np)\left(\frac{f}{\alpha}*\frac{g}{\alpha}\right)(np)}{\alpha(np)}\right]\nu_p(np) \\
&= \alpha(n)\left(\frac{f}{\alpha}*\frac{g}{\alpha}\right)(np)\nu_p(np) \\
&= \alpha(n)d_p\left(\frac{f}{\alpha}*\frac{g}{\alpha}\right)(n) \\
&= \alpha(n)\left(\frac{f}{\alpha}*d_p\frac{g}{\alpha}+\frac{g}{\alpha}*d_p\frac{f}{\alpha}\right)(n) \\
&= \alpha(n)\left[\frac{f}{\alpha}*\frac{\alpha}{\alpha}\left(d_p\frac{g}{\alpha}\right)\right](n)+\alpha(n)\left[\frac{g}{\alpha}*\frac{\alpha}{\alpha}\left(d_p\frac{f}{\alpha}\right)\right](n) \\
&= \alpha(n)\left[\frac{f}{\alpha}*\frac{\tilde{d}_pg}{\alpha}\right](n)+\alpha(n)\left[\frac{g}{\alpha}*\frac{\tilde{d}_pf}{\alpha}\right](n) \\
&= f\diamond\tilde{d}_pg+g\diamond\tilde{d}_pf.
\end{aligned}$$

Hence d_L and $\tilde{d}_p$ are derivations on $(\mathcal{A}, +, \diamond)$. $\square$

Definition 13. For $f \in P$, let

$$L_\alpha f(1) = \log f(1) \quad \text{and}$$
$$L_\alpha f(n) = \alpha(n) \sum_{d|n} \frac{f}{\alpha}(d) \frac{f^{-1\diamond}}{\alpha}(n/d) \log d, \qquad \text{if } n > 1.$$

(we recall that P is the set of all real valued functions f such that $f(1) > 0$.)

Proposition 8. *For any $f, g \in \mathcal{A}$ with $f(1) \neq 0, g(1) \neq 0$,*

$$(f \diamond g)^{-1\diamond} = f^{-1\diamond} \diamond g^{-1\diamond}.$$

Proof. $(f \diamond g) \diamond (f^{-1\diamond} \diamond g^{-1\diamond}) = f \diamond g \diamond g^{-1\diamond} \diamond f^{-1\diamond} = f \diamond f^{-1\diamond} = \alpha I$.
That is $(f \diamond g)^{-1\diamond} = f^{-1\diamond} \diamond g^{-1\diamond}$. $\square$

Theorem 38. *For all $f, g \in P$,*

$$L_\alpha(f \diamond g)(1) = \frac{1}{\alpha(1)} \left(L_\alpha f(1) + L_\alpha g(1)\right) \quad \text{and}$$

$$L_\alpha(f \diamond g)(n) = L_\alpha f(n) + L_\alpha g(n) \quad \text{when } n > 1.$$

Proof. For $n = 1$, we get that

$$\begin{aligned}
L_\alpha(f \diamond g)(1) &= log((f \diamond g)(1)) \\
&= log\left(\alpha(1)\left(\frac{f}{\alpha} * \frac{g}{\alpha}\right)(1)\right) \\
&= \frac{1}{\alpha(1)} log f(1) g(1) \\
&= \frac{1}{\alpha(1)} (log f(1) + log g(1)) \\
&= \frac{1}{\alpha(1)} (L_\alpha f(1) + L_\alpha g(1)).
\end{aligned}$$

For $n > 1$, $L_\alpha f(n)$ can be expressed using $d_L f$ as follow:

$$\begin{aligned}
L_\alpha f(n) &= \alpha(n) \sum_{d|n} \frac{f}{\alpha}(d) \frac{f^{-1\diamond}}{\alpha}(n/d) \log d \\
&= \alpha(n) \left(\frac{f^{-1\diamond}}{\alpha} * \frac{d_L f}{\alpha} \right)(n) \\
&= (f^{-1\diamond} \diamond d_L f)(n), \\
L_\alpha(f \diamond g)(n) &= \left[(f \diamond g)^{-1\diamond} \diamond d_L(f \diamond g) \right](n) \\
&= \left[(f \diamond g)^{-1\diamond} \diamond (g \diamond d_L f + f \diamond d_L g) \right](n) \\
&= \left[(f \diamond g)^{-1\diamond} \diamond (g \diamond d_L f)(n) \right] + \left[(f \diamond g)^{-1\diamond} \diamond (g \diamond d_L g)(n) \right] \\
&= \left(f^{-1\diamond} \diamond g^{-1\diamond} \diamond g \diamond d_L f \right)(n) + \left(f^{-1\diamond} \diamond g^{-1\diamond} \diamond g \diamond d_L g \right)(n) \\
&= (f^{-1\diamond} \diamond d_L f)(n) + (g^{-1\diamond} \diamond d_L g)(n) \\
&= L_\alpha f(n) + L_\alpha g(n).
\end{aligned}$$

$\square$

Remark. If $\alpha(1) = 1$ then $L_\alpha(f \diamond g)(n) = L_\alpha f(n) + L_\alpha g(n)$ for all $n \in \mathbb{N}$.

2. $\diamond-$ algebraic independence over subrings of $\mathcal{A}$.

The $\diamond-$ algebraic independence of arithmetic functions can be considered relative to a given subring $\mathcal{R}$ of $\mathcal{A}$. In particular, $(\mathcal{A}, +, \diamond)$ contains the complex numbers via the identification of a $c \cdot \alpha I(n)$ of $\mathcal{A}$.

Definition 14. Let $\mathcal{E}$ be a subring of $\mathcal{A}$. For $k > 1$ we say that $f_1, f_2, \ldots, f_k \in \mathcal{A}$. are $\diamond -$ *algebraically dependent* over $\mathcal{E}$, if there exist $P \in \mathcal{E}[f_1, f_2, \ldots, f_k] \setminus \{0\}$ such that

$$P(f_1, \ldots, f_k) = \sum_{(i)} a_i \diamond f_1^{\diamond i_1} \diamond \cdots \diamond f_k^{\diamond i_k} = 0$$

and is said to be $\diamond -$ *algebraically independent* over $\mathcal{E}$ otherwise.

We say that f_1 is $\diamond-$ algebraic over $\mathcal{E}[f_2, \ldots, f_k]$ if $f_1, f_2, \ldots, f_k$ are $\diamond-$ algebraically dependent over $\mathcal{E}$.

Definition 15. Given $f_1, f_2, \ldots, f_k$ in $\mathcal{A}$ and derivations $\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_k$ over $\mathcal{A}$, *the* $\diamond-$*Jacobian* of the f_i relative to the $\tilde{d}_i$ is given by the $k \times k$ determinant

$$\tilde{J}(f_1, \ldots, f_k/\tilde{d}_1, \ldots, \tilde{d}_k) = det(\tilde{d}_i(f_i)).$$

We use multiplication in the determinant by $\mathrm{Q}_\alpha-$ convolution. Clearly a Jacobian is an element of $\mathcal{A}$. In the case where each $\tilde{d}$ is a basic derivation $\tilde{d}_{p_i}$, corresponding to some prime p_i, we shall use the relation $\tilde{J}(f_1, \ldots, f_k/p_1, \ldots, p_k)$ for the corresponding Jacobian.

Theorem 39. *Let $\mathcal{E}$ be a subring of $\mathcal{A}$, and f a given function of $\mathcal{A}$ such that there exist a derivation $\tilde{d}$ over $\mathcal{A}$ which annihiletes all of $\mathcal{E}$ and $\tilde{d}f \neq 0$. Then f is not $\diamond-$ algebraic over $\mathcal{E}$.*

Proof. Suppose that f is $\diamond-$ algebraic over $\mathcal{E}$. Then there exist $g_i \in \mathcal{E}$ such that

$$\sum_{i=0}^{m} g_i \diamond f^{\diamond i} = 0 \tag{23}$$

with $g_m \neq 0$ and $g_i \in \mathcal{E}$ is of smallest possible degree m.
Taking the derivation $\tilde{d}$ to the equation (23), we obtain

$$\begin{aligned} 0 &= \sum_{i=0}^{m} (g_i \diamond \tilde{d}(f^{\diamond i}) + f^{\diamond i} \diamond \tilde{d}g_i) \\ &= \sum_{i=1}^{m} (g_i \diamond (f^{\diamond(i-1)} \diamond \tilde{d}f)) \\ &= \sum_{i=1}^{m} ((g_i \diamond i f^{\diamond(i-1)}) \diamond \tilde{d}f) \qquad \text{(by associative law)} \\ &= \left(\sum_{i=1}^{m} (g_i \diamond i f^{\diamond(i-1)}) \right) \diamond \tilde{d}f \qquad \text{(by distributive law).} \end{aligned}$$

Since $\tilde{d}f \neq 0$ and $(\mathcal{A}, \diamond, +)$ is an integral domain, it follows that
$\sum_{i=1}^{m} g_i \diamond i f^{\diamond(i-1)} = \sum_{i=1}^{m} i g_i \diamond f^{\diamond(i-1)} = 0$ with $i g_i \in \mathcal{E}$ which is a contradiction.
We conclude that f is not $\diamond-$ algebraic over $\mathcal{E}$. □

Theorem 40. *Let $f_1, \ldots, f_k$ be given functions of $\mathcal{A}$ and $\tilde{d}_1, \ldots, \tilde{d}_k$ derivations over $\mathcal{A}$ which annihilate all elements of the subring $\mathcal{E}$. Then if $\tilde{J}(f_1, \ldots, f_k/\tilde{d}_1, \ldots, \tilde{d}_k) \neq 0$, the $f_1, \ldots, f_k$ are $\diamond-$ algebraically independent over $\mathcal{E}$.*

Proof. We define a linear map $\tilde{d} : \mathcal{A} \to \mathcal{A}$ by $\tilde{d}(g) = \tilde{J}(g, f_2, \ldots, f_k/\tilde{d}_1, \ldots, \tilde{d}_k)$ for $g \in \mathcal{A}$. Clearly, $\tilde{d}$ is a linear combination of the derivations $\tilde{d}_1, \ldots, \tilde{d}_k$ and $\tilde{d}$ is also a derivation. For $1 < i \leq m$, we obtain $\tilde{d}(f_i) = 0$. That is $\tilde{d}$ annihilates all elements of the ring $\mathcal{E}[f_2, \ldots, f_k]$. But $\tilde{d}(f_1) \neq 0$. Thus by Theorem 39 we get f_1 is not $\diamond-$ algebraic over $\mathcal{E}[f_2, \ldots, f_k]$. By symmetry implies that the f_i are $\diamond-$ algebraically independent over $\mathcal{E}$. $\square$

Example 3. The functions $I_k(n) = n^k,\ k \in \mathbb{N} \cup \{0\}$ are $\diamond-$ algebraically independent over $\mathbb{C}$.

Let $k \in \mathbb{N} \cup \{0\}$ and $p_0, \ldots, p_k$ be distinct primes.
Using the Jacobian $\tilde{J}$ at $n = 1$,

$$\begin{aligned}
\tilde{J}(I_0, \ldots, I_k/p_0, \ldots, p_k)(1) &= \begin{vmatrix} \tilde{d}_{p_0} I_0 & \tilde{d}_{p_0} I_1 & \cdots & \tilde{d}_{p_0} I_k \\ \tilde{d}_{p_1} I_0 & \tilde{d}_{p_1} I_1 & \cdots & \tilde{d}_{p_1} I_k \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{d}_{p_k} I_0 & \tilde{d}_{p_k} I_1 & \cdots & \tilde{d}_{p_k} I_k \end{vmatrix} (1) \\
&= \frac{\alpha(1)^{k+1}}{\alpha(1)^k} \begin{vmatrix} \frac{I_0(p_0)}{\alpha(p_0)} & \frac{I_1(p_0)}{\alpha(p_0)} & \cdots & \frac{I_k(p_0)}{\alpha(p_0)} \\ \frac{I_0(p_1)}{\alpha(p_1)} & \frac{I_1(p_1)}{\alpha(p_1)} & \cdots & \frac{I_k(p_1)}{\alpha(p_1)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{I_0(p_k)}{\alpha(p_k)} & \frac{I_1(p_k)}{\alpha(p_k)} & \cdots & \frac{I_k(p_k)}{\alpha(p_k)} \end{vmatrix} \\
&= \frac{\alpha(1)}{\alpha(p_0) \cdots \alpha(p_k)} \begin{vmatrix} I_0(p_0) & I_1(p_0) & \cdots & I_k(p_0) \\ I_0(p_1) & I_1(p_1) & \cdots & I_k(p_1) \\ \vdots & \vdots & \ddots & \vdots \\ I_0(p_k) & I_1(p_k) & \cdots & I_k(p_k) \end{vmatrix} \\
&= \frac{\alpha(1)}{\alpha(p_0) \cdots \alpha(p_k)} \begin{vmatrix} p_0^0 & p_0 & \cdots & p_0^k \\ p_1^0 & p_1 & \cdots & p_1^k \\ \vdots & \vdots & \ddots & \vdots \\ p_k^0 & p_k & \cdots & p_k^k \end{vmatrix}
\end{aligned}$$

$$\tilde{J}(I_0,\ldots,I_k/p_0,\ldots,p_k)(1) = \frac{\alpha(1)}{\alpha(p_0)\cdots\alpha(p_k)}\begin{vmatrix} 1 & p_0 & \cdots & p_0^k \\ 1 & p_1 & \cdots & p_1^k \\ \vdots & \vdots & \ddots & \vdots \\ 1 & p_k & \cdots & p_k^k \end{vmatrix}$$
$$= \frac{\alpha(1)}{\alpha(p_0)\cdots\alpha(p_k)}\prod_{i>j}(p_i-p_j)$$
$$\neq 0.$$

Therefore $I_k(n) = n^k,\ k \in \mathbb{N}\cup\{0\}$ are $\diamond-$ algebraically independent over $\mathbb{C}$.

Example 4. The functions μ and $I_k(n) = n^k,\ k \in \mathbb{N}$ are $\diamond-$ algebraically independent over $\mathbb{C}$.

Let $k \in \mathbb{N}$ and $p_0,\ldots,p_k$ be distinct primes.
Putting $n = 1$ into the Jacobian $\tilde{J}$,

$$\tilde{J}(I_0,\ldots,I_k/p_0,\ldots,p_k)(1) = \begin{vmatrix} \tilde{d}_{p_0}\mu & \tilde{d}_{p_0}I_1 & \cdots & \tilde{d}_{p_0}I_k \\ \tilde{d}_{p_1}\mu & \tilde{d}_{p_1}I_1 & \cdots & \tilde{d}_{p_1}I_k \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{d}_{p_k}\mu & \tilde{d}_{p_k}I_1 & \cdots & \tilde{d}_{p_k}I_k \end{vmatrix}(1)$$
$$= \frac{\alpha(1)^{k+1}}{\alpha(1)^k}\begin{vmatrix} \frac{\mu(p_0)}{\alpha(p_0)} & \frac{I_1(p_0)}{\alpha(p_0)} & \cdots & \frac{I_k(p_0)}{\alpha(p_0)} \\ \frac{\mu(p_1)}{\alpha(p_1)} & \frac{I_1(p_1)}{\alpha(p_1)} & \cdots & \frac{I_k(p_1)}{\alpha(p_1)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\mu(p_k)}{\alpha(p_k)} & \frac{I_1(p_k)}{\alpha(p_k)} & \cdots & \frac{I_k(p_k)}{\alpha(p_k)} \end{vmatrix}$$
$$= \frac{\alpha(1)}{\alpha(p_0)\cdots\alpha(p_k)}\begin{vmatrix} -1 & p_0 & \cdots & p_0^k \\ -1 & p_1 & \cdots & p_1^k \\ \vdots & \vdots & \ddots & \vdots \\ -1 & p_k & \cdots & p_k^k \end{vmatrix}$$

$$\tilde{J}(I_0,\dots,I_k/p_0,\dots,p_k)(1) = \begin{vmatrix} 1 & p_0 & \cdots & p_0^k \\ 1 & p_1 & \cdots & p_1^k \\ \vdots & \vdots & \ddots & \vdots \\ 1 & p_k & \cdots & p_k^k \end{vmatrix}$$
$$= -\frac{\alpha(1)}{\alpha(p_0)\cdots\alpha(p_k)}\prod_{i>j}(p_i - p_j)$$
$$\neq 0.$$

Hence the functions μ and $I_k(n) = n^k,\ k \in \mathbb{N}$ are $\diamond-$ algebraically independent over $\mathbb{C}$.

Theorem 41. *Let $\mathcal{E}$ be a subring of $\mathcal{A}$ such that for some set of r distinct primes $p_1,\dots,p_r$, the corresponding p-basic derivations $\tilde{d}_{p_i}$ all annihilate $\mathcal{E}$. Then, for $2s+1 \leq r$, the functions $I_j(n), -s \leq j \leq s$, are $\diamond-$ algebraically independent over $\mathcal{E}$.*

Proof. Let $p_0,\dots,p_{2s+1}$ be distinct primes and $2s+1 \leq r$ then, for $n = 1$, we obtain

$$\tilde{J}(I_{-s},\dots,I_s/p_1,\dots,p_{2s+1})(1) = \begin{vmatrix} \tilde{d}_{p_1}I_{-s} & \tilde{d}_{p_1}I_{-s+1} & \cdots & \tilde{d}_{p_1}I_s \\ \tilde{d}_{p_2}I_{-s} & \tilde{d}_{p_2}I_{-s+1} & \cdots & \tilde{d}_{p_2}I_s \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{d}_{p_{2s+1}}I_{-s} & \tilde{d}_{p_{2s+1}}I_{-s+1} & \cdots & \tilde{d}_{p_{2s+1}}I_s \end{vmatrix}(1)$$
$$= \frac{\alpha(1)^{k+1}}{\alpha(1)^k}\begin{vmatrix} \frac{I_{-s}(p_1)}{\alpha(p_1)} & \frac{I_{-s+1}(p_1)}{\alpha(p_1)} & \cdots & \frac{I_s(p_1)}{\alpha(p_1)} \\ \frac{I_{-s}(p_2)}{\alpha(p_2)} & \frac{I_{-s+1}(p_2)}{\alpha(p_2)} & \cdots & \frac{I_s(p_2)}{\alpha(p_2)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{I_{-s}(p_{2s+1})}{\alpha(p_{2s+1})} & \frac{I_{-s+1}(p_{2s+1})}{\alpha(p_{2s+1})} & \cdots & \frac{I_s(p_{2s+1})}{\alpha(p_{2s+1})} \end{vmatrix}$$
$$= \frac{\alpha(1)}{\alpha(p_1)\cdots\alpha(p_{2s+1})}\begin{vmatrix} p_1^{-s} & p_1^{-s+1} & \cdots & p_1^s \\ p_2^{-s} & p_2^{-s+1} & \cdots & p_2^s \\ \vdots & \vdots & \ddots & \vdots \\ p_{2s+1}^{-s} & p_{2s+1}^{-s+1} & \cdots & p_{2s+1}^s \end{vmatrix}$$

$$\tilde{J}(I_{-s},\ldots,I_s/p_1,\ldots,p_{2s+1})(1) = \frac{\alpha(1)p_1^{-s}\ldots p_{2s+1}^{-s}}{\alpha(p_1)\ldots\alpha(p_{2s+1})}\begin{vmatrix} 1 & p_1 & \cdots & p_1^{2s} \\ 1 & p_2 & \cdots & p_2^{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & p_{2s+1} & \cdots & p_{2s+1}^{2s} \end{vmatrix}$$

$$= \frac{\alpha(1)p_1^{-s}\cdots p_{2s+1}^{-s}}{\alpha(p_1)\cdots\alpha(p_{2s+1})}\prod_{i>j}(p_i - p_j)$$

$$\neq 0.$$

Theorem 40 implies that the functions $I_j(n), -s \leq j \leq s$, are $\diamond-$ algebraically independent over $\mathcal{E}$. $\square$

Corollary 6. *Let $\mathcal{E}$ be a subring $\mathcal{A}$ such that for infinitely many primes p the basic derivations $\tilde{d}_p$ annihilate all of $\mathcal{E}$. Then the functions $I_j(n), j = 0, \pm1, \pm2, \ldots$ are $\diamond-$ algebraically independent over $\mathcal{E}$.*

Example 5. The functions μ, I, ϕ are $*-$ algebraically dependent over $\mathbb{C}$ but they are $\diamond-$ algebraically independent over $\mathbb{C}$.

Since $\mu * I_1 = \phi$, we have the $*$ relation $\mu * I - \phi = 0$, that is μ, I_1, ϕ are $*-$ algebraically independent over $\mathbb{C}$. Consider $\tilde{J}(\mu, I_1, \phi/p, q, r)$ with p, q, r are distinct primes.

$$\tilde{J}(\mu, I_1, \phi/p, q, r) = \begin{vmatrix} \tilde{d}_p\mu & \tilde{d}_pI_1 & \tilde{d}_p\phi \\ \tilde{d}_q\mu & \tilde{d}_qI_1 & \tilde{d}_q\phi \\ \tilde{d}_r\mu & \tilde{d}_rI_1 & \tilde{d}_r\phi \end{vmatrix}$$

$$= \tilde{d}_p\mu \diamond \tilde{d}_qI_1 \diamond \tilde{d}_r\phi + \tilde{d}_pI_1 \diamond \tilde{d}_q\phi \diamond \tilde{d}_r\mu + \tilde{d}_p\phi \diamond \tilde{d}_q\mu \diamond \tilde{d}_rI_1$$
$$- \tilde{d}_p\phi \diamond \tilde{d}_qI_1 \diamond \tilde{d}_r\mu - \tilde{d}_p\mu \diamond \tilde{d}_q\phi \diamond \tilde{d}_rI_1 - \tilde{d}_pI_1 \diamond \tilde{d}_q\mu \diamond \tilde{d}_r\phi.$$

Then,

$$\tilde{J}(\mu, I_1, \phi/p, q, r)(p)$$
$$= \frac{\mu(p^2)\nu_p(p^2)I_1(q)\nu_q(q)\phi(r)\nu_r(r)}{\alpha(p^2)\alpha(q)\alpha(r)} + \frac{I_1(p^2)\nu_p(p^2)\phi(q)\nu_q(q)\mu(r)\nu_r(r)}{\alpha(p^2)\alpha(q)\alpha(r)}$$
$$+ \frac{\phi(p^2)\nu_p(p^2)\mu(q)\nu_q(q)I_1(r)\nu_r(r)}{\alpha(p^2)\alpha(q)\alpha(r)} - \frac{\phi(p^2)\nu_p(p^2)I_1(q)\nu_q(q)\mu(r)\nu_r(r)}{\alpha(p^2)\alpha(q)\alpha(r)}$$

$$
\begin{aligned}
&-\frac{\mu(p^2)\nu_p(p^2)\phi(q)\nu_q(q)I_1(r)\nu_r(r)}{\alpha(p^2)\alpha(q)\alpha(r)}-\frac{I_1(p^2)\nu_p(p^2)\mu(q)\nu_q(q)\phi(r)\nu_r(r)}{\alpha(p^2)\alpha(q)\alpha(r)}\\
&+\frac{\mu(p)\nu_p(p)I_1(pq)\nu_q(pq)\phi(r)\nu_r(r)}{\alpha(p)\alpha(pq)\alpha(r)}+\frac{I_1(p)\nu_p(p)\phi(pq)\nu_q(pq)\mu(r)\nu_r(r)}{\alpha(p)\alpha(pq)\alpha(r)}\\
&+\frac{\phi(p)\nu_p(p)\mu(pq)\nu_q(pq)I_1(r)\nu_r(r)}{\alpha(p)\alpha(pq)\alpha(r)}-\frac{\phi(p)\nu_p(p)I_1(pq)\nu_q(pq)\mu(r)\nu_r(r)}{\alpha(p)\alpha(pq)\alpha(r)}\\
&-\frac{\mu(p)\nu_p(p)\phi(pq)\nu_q(pq)I_1(r)\nu_r(r)}{\alpha(p)\alpha(pq)\alpha(r)}-\frac{I_1(p)\nu_p(p)\mu(pq)\nu_q(pq)\phi(r)\nu_r(r)}{\alpha(p)\alpha(pq)\alpha(r)}\\
&+\frac{\mu(p)\nu_p(p)I_1(q)\nu_q(q)\phi(pr)\nu_r(pr)}{\alpha(p)\alpha(q)\alpha(pr)}+\frac{I_1(p)\nu_p(p)\phi(q)\nu_q(q)\mu(pr)\nu_r(pr)}{\alpha(p)\alpha(pq)\alpha(r)}\\
&+\frac{\phi(p)\nu_p(p)\mu(q)\nu_q(q)I_1(pr)\nu_r(pr)}{\alpha(p)\alpha(q)\alpha(pr)}-\frac{\phi(p)\nu_p(p)I_1(q)\nu_q(q)\mu(pr)\nu_r(pr)}{\alpha(p)\alpha(pq)\alpha(r)}\\
&-\frac{\mu(p)\nu_p(p)\phi(q)\nu_q(q)I_1(pr)\nu_r(pr)}{\alpha(p)\alpha(q)\alpha(pr)}-\frac{I_1(p)\nu_p(p)\mu(q)\nu_q(q)\phi(pr)\nu_r(pr)}{\alpha(p)\alpha(q)\alpha(pr)}\\
=&\frac{2p(r-q)}{\alpha(p^2)\alpha(q)\alpha(r)}+\frac{(p-r)(p+q)}{\alpha(p)\alpha(pq)\alpha(r)}+\frac{(q-p)(p+r)}{\alpha(p)\alpha(q)\alpha(pr)}
\end{aligned}
$$

Choosing

$$
\alpha(n)=\begin{cases}1, & n \text{ is a squarefree;}\\ -1, & \text{else,}\end{cases}
$$

thus

$$
\begin{aligned}
\tilde{J}(\mu, I_1, \phi/p, q, r)(p) =& -2pr+2pq+p^2+pq-pr-qr+pq+qr-p^2-pr\\
=&4pq-4pr\\
=&4p(q-r)\\
\neq&0.
\end{aligned}
$$

It follows that the functions μ, I, ϕ are $\diamond-$ algebraically independent over $\mathbb{C}$.

This example shows that the functions μ, I, ϕ are $\diamond-$ algebraically independent but they are not $*-$ algebraically independent over $\mathbb{C}$. Therefore, the $*-$ algebraic independence may not be related to the $\diamond-$ algebraic independence over subring of $\mathcal{A}$.

Theorem 42. *Let $\mathcal{E}$ be a subring of $\mathcal{A}$ such that given any finite subset $\mathcal{E}^* \subset \mathcal{E}$ there are infinitely many primes p such the derivations $\tilde{d}_p$ annihilate all of $\mathcal{E}^*$. Then given any sequence of complex numbers $r_i, i = 1, 2, \ldots,$ with distinct real parts, and any sequence of integers s_j (not necessarily distinct), the functions*

$$f_{ij}(n) = \alpha(n)n^{r_i}(log\ n)^{s_j}$$

are $\diamond-$ algebraically independent over $\mathcal{E}$.

Proof. Assume that there is a finite subset of $\{f_{ij}\}$ which are $\diamond-$ algebraically dependent over $\mathcal{E}$ and this set is $\{f_{11}, \ldots, f_{kl}\}$. Let $\mathcal{E}^*$ ($\subset \mathcal{E}$) be the finite set of all coeffients in this $\diamond-$ algebraic relationship. Then, for all sufficiently large primes p such that $\tilde{d}_p$ annihiletes the set $\mathcal{E}^*$ and the subring $\langle \mathcal{E}^* \rangle$, we get that $f_{11}, \ldots, f_{kl}$ are $\diamond-$ algebraically dependent over $\langle \mathcal{E}^* \rangle$. If we can choose primes p_{ij} among these so that

$$\tilde{J}(f_{11}, \ldots, f_{kl}/p_{11}, \ldots, p_{kl}) \neq 0,$$

by Theorem 40 we have $f_{11}, \ldots, f_{kl}$ are $\diamond-$ algebraically independent over $\langle \mathcal{E}^* \rangle$, which is a contradiction.

Without loss of generality, assume that t_j ($-s \leq t_j \leq s$ for all $j \in \{1, \ldots, l\}$ and s is a fixed positive integer) is an integers. We instead the set $\{f_{11}, \ldots, f_{kl}\}$ by the set $\{f_{ij} | i \in \{1, \ldots, k\}, j \in \{-s, \ldots, s\}\}$. Let $T = (2s+1)k$, then for any sequence of sufficiently large primes, $p_1, \ldots, p_T$, we get

$$\begin{aligned}
\tilde{J}(f_{1,-s}, \ldots, f_{1,s}, \ldots, f_{k,-s}, \ldots, f_{k,s}/p_1, \ldots, p_T)(n) &= det(\tilde{d}_{p_a}(f_{ij}))(n) \\
&= det(\alpha(n)\frac{f_{ij}}{\alpha}(np_a)\nu_{p_a}(np_a)) \\
&= det\left(\alpha(n)\frac{\alpha(np_a)(np_a)^{r_i}(log\ np_a)^j}{\alpha(np_a)}\nu_{p_a}(np_a)\right) \\
&= det\left(\alpha(n)(np_a)^{r_i}(log\ np_a)^j\nu_{p_a}(np_a)\right)
\end{aligned}$$

where $a = 1, \ldots, T;\ i \in \{1, \ldots, k\};\ j \in \{-s, \ldots, s\}$. That is

$$\begin{aligned}
\tilde{J}(f_{1,-s}, \ldots, f_{1,s}, \ldots, f_{k,-s}, \ldots, f_{k,s}/p_1, \ldots, p_T)(1) &= det(\alpha(1)(p_a)^{r_i}(log\ p_a)^j) \\
&= \alpha(1)^T det({p_a}^{r_i}(log\ p_a)^j) \\
&= \alpha(1)^T J'
\end{aligned}$$

(Note that $J' = det({p_a}^{r_i}(log\ p_a)^j)$.)

Let $t(\vec{p},\vec{r},\vec{j}) := \{1,\ldots,k\}\,; j_1,\ldots,j_T \in \{-s,\ldots,s\}$ be a typical term in the expansion of the determinant defining J'. We may assume that
$\mathcal{R}e(r_1) > \mathcal{R}e(r_2) > \ldots > \mathcal{R}e(r_k)$. We consider the first row, the column which has the unique largest absolute value is $p_1^{r_1}(log\ p_1)^s$, so we shall exchange the first column with this column. We next consider the second row, the column which has the next unique largest absolute value (after the first column), similarly we exchange the second column with this column. Afterwards we will continue this process. We assume that in the final determinant, by choosing $p_1 > p_2 > \cdots > p_T$ sufficiently large the term with largest absolute value is the main diagonal term

$$Y := a_{11}a_{22}\cdots a_{TT} = p_1^{r_1}(log\ p_1)^s p_2^{(r)_2}(log\ p_2)^{(s)_2}\cdots p_T^{(r)_T}(log\ p_T)^{(s)_T},$$

where $(r)_i, (s)_i$ denote the diagonal exponents. Let

$$a_{j^*} := a_{1_{j_1}}\cdots a_{T_{j_T}} = p_1^{\alpha_1}(log\ p_1)^{\beta_1}\cdots p_T^{\alpha_T}(log\ p_T)^{\beta_T}$$

be any term in the determinant expansions. There are three possibilities.
(i) If $r_1 \neq \alpha_1$ $(\mathcal{R}e(r_1) > \mathcal{R}e(\alpha_1))$, then choosing p_1 sufficiently large in comparison with other p_i 's, we see that $p_1^{r_1} >> p_1^{\alpha_1}$ which leads to $|Y| > |a_{j^*}|$.
(ii) If $r_1 = \alpha_1,\ s > \beta_1$, then as in (i), $(log\ p_1)^s >> (log\ p_1)^{\beta_1}$ and $|Y| > |a_{j^*}|$.
(iii) If $r_1 = \alpha_1,\ s = \beta_1$(i.e. both terms arise from the expansion of the $(1,1)$ term), repeating the same arguments as above we see that the next largest term must come from the main diagonal.

Moreover, we can even choose the primes $p_1 > \cdots > p_T$ so large that
$\left|\dfrac{t(\vec{p},\vec{r},\vec{j})}{Y}\right| < \dfrac{1}{T!}$ for each $t(\vec{p},\vec{r},\vec{j}) \neq Y$.

Thus $J' = 1 + ((T!-1)$ terms each with absolute value $< \frac{1}{T!}) \neq 0$. This show that there are sets of primes p such that $J' \neq 0$, yielding $\tilde{J}(1) \neq 0$, as required. $\square$

In 1986, H. N. Shapiro and G. H. Sparer discovered relations between the Riemanm zeta function and its algebraic independence with respect to the Dirichlet convolution. We now establish this relation with respect to the $Q_\alpha-$ convolution.

Corollary 7. *Let $r_i, i = 1, 2, \ldots, L$ be complex numbers with distinct real parts, and m_i any non-negative integers. Then, the functions*

$$\zeta^{(m_1)}(s - r_1), \ldots, \zeta^{(m_L)}(s - r_L)$$

are $\diamond-$ algebraically independent over $\mathbb{C}$.

Proof. Let $\mathcal{E}^*$ be a subring of $\mathbb{C}$ and $c \in \mathbb{C}$. Since $c = cI$, thus for all primes p we obtain $\tilde{d}_p(c)(n) = \tilde{d}_p(cI)(n) = c\tilde{d}_p(I)(n) = \alpha(n)\frac{I}{\alpha}(np)\nu_p(np) = 0$.
From $\zeta(s) = \sum_{n \in \mathbb{N}} n^{-s};\ s \in \mathbb{C}$ and $\tilde{d}_p(n^{-s}) = 0$. Hence there exist infinite many prime p such that $\tilde{d}_p\zeta = 0$. It follows from Theorem 39 that the functions

$$\zeta^{(m_1)}(s - r_1), \ldots, \zeta^{(m_L)}(s - r_L)$$

are $\diamond-$ algebraically independent over $\mathbb{C}$. □

Corollary 8. *The zeta function does not satisfy any nontrivial algebraic differential difference equation over $\mathbb{C}$.*

Proof. We will show that there is no polynomial $F(s, z_1, \ldots, z_L)$ over $\mathbb{C}$, not identically 0, such that for all $s \in \mathbb{C}$

$$F(\zeta^{(m_1)}(s - r_1), \ldots, \zeta^{(m_L)}(s - r_L)) = 0 \tag{24}$$

where the m_i and r_i are fixed integers ($m_i \geq 0$) and the pairs (m_i, r_i) are distinct. If relation (24) existed there would be a similar one in which the explicit presence of the variable s is missing from the polymial, then by Theorem 42 we have a contradiction. □

3. Solutions to arithmetic convolution equations.

In 2007, H. Glöckner, L. G. Lucht and Š. Porubský solved the polynomial convolution equation

$$Tg = a_d * g^{*d} + a_{d-1} * g^{*(d-1)} + \cdots + a_1 * g + a_0 = 0 \tag{25}$$

with fixed coefficients $a_d, a_{d-1}, \ldots, a_1, a_0 \in \mathcal{A}$ and $a_d \neq 0$ by showing that it has a solution $g \in \mathcal{A}$ satisfying $g(1) = z_o$, if z_0 is a simple zero of the polynomial

$$f(z) = a_d(1)z^d + a_{d-1}(1)z^{d-1} + \cdots + a_1(1)z + a_0(1).$$

In the next theorem, we show that the polynomial binomial convolution equation $T_\xi g = a_d \circ g^{\circ d} + a_{d-1} \circ g^{\circ(d-1)} + \cdots + a_1 \circ g + a_0 = 0$ and the polynomial $\mathrm{Q}_\alpha-$ convolution equation $T_\alpha g = a_d \diamond g^{\diamond d} + a_{d-1} \diamond g^{\diamond(d-1)} + \cdots + a_1 \diamond g + a_0 = 0$ both have solutions $g \in \mathcal{A}$ under similar conditions.

Theorem 43. *For $d \in \mathbb{N}$, let $T_\xi : \mathcal{A} \to \mathcal{A}$ be defined by*

$$T_\xi g = a_d \circ g^{\circ d} + a_{d-1} \circ g^{\circ(d-1)} + \cdots + a_1 \circ g + a_0 \tag{26}$$

for $g \in \mathcal{A}$ with $a_d, a_{d-1}, \ldots, a_1, a_0 \in \mathcal{A}$ and $a_d \neq 0$. If z_0 is a simple zero of the polynomial

$$f_\xi(z) = \frac{a_d}{\xi}(1)z^d + \frac{a_{d-1}}{\xi}(1)z^{d-1} + \cdots + \frac{a_1}{\xi}(1)z + \frac{a_0}{\xi}(1), \tag{27}$$

then there exists a uniquely determined solution $g \in \mathcal{A}$ to the convolution equation $T_\xi g = 0$ satisfying $g(1) = z_o$.

Proof. Assume that z_0 is a simple zero of $f_\xi(z)$. From the relation $g \circ h = \xi\left(\frac{g}{\xi} * \frac{h}{\xi}\right)$, we obtain that

$$\begin{aligned} T_\xi g &= a_d \circ g^{\circ d} + a_{d-1} \circ g^{\circ(d-1)} + \cdots + a_1 \circ g + a_0 \\ &= \xi\left(\frac{a_d}{\xi} * \left(\frac{g}{\xi}\right)^{*d} + \frac{a_{d-1}}{\xi} * \left(\frac{g}{\xi}\right)^{*(d-1)} + \cdots + \frac{a_1}{\xi} * \frac{g}{\xi} + \frac{a_0}{\xi}\right). \end{aligned}$$

Let $T : \mathcal{A} \to \mathcal{A}$ be defined by

$$Th = \frac{a_d}{\xi} * h^{*d} + \frac{a_{d-1}}{\xi} * h^{*(d-1)} + \cdots + \frac{a_1}{\xi} * h + \frac{a_0}{\xi} \qquad (\forall h \in \mathcal{A})$$

and

$$f_\xi(z) = \frac{a_d}{\xi}(1)z^d + \frac{a_{d-1}}{\xi}(1)z^{d-1} + \cdots + \frac{a_1}{\xi}(1)z + \frac{a_0}{\xi}(1).$$

Since $\xi(1) = 1$, we have that $f_\xi(z) = f(z)$ and z_0 is also simple zero of $f(z)$. By Theorem 19, T has a solution $h \in \mathcal{A}$ such that $h(1) = z_0$. Let $g = h\xi$

then $T_\xi(g) = T_\xi(h\xi) = \xi\left(\frac{a_d}{\xi} * h^{*d} + \frac{a_{d-1}}{\xi} * h^{*(d-1)} + \cdots + \frac{a_1}{\xi} * h + \frac{a_0}{\xi}\right) = 0$ and $g(1) = h(1)\xi(1) = z_0$. $\square$

Example 6. Let $T_\xi g = a_3 \circ g^{\circ 3} + a_2 \circ g^{\circ 2} + a_1 \circ g + a_0$ and

$$a_3(n) = \begin{cases} 1, & n < 3; \\ 0, & \text{else}, \end{cases}$$

$$a_2(n) = \begin{cases} 2, & n < 3; \\ 0, & \text{else}, \end{cases}$$

$$a_1(n) = \begin{cases} -1, & n < 3; \\ 1, & \text{else}, \end{cases}$$

$$a_0(n) = \begin{cases} -2n, & n < 3; \\ n, & \text{else}. \end{cases}$$

Consider $f_\xi(z) = \frac{a_3}{\xi}(1)z^3 + \frac{a_2}{\xi}(1)z^2 + \frac{a_1}{\xi}(1)z + \frac{a_0}{\xi}(1)$, we get that

$$\begin{aligned} f_\xi(z) &= z^3 + 2z^2 - z - 2 \\ &= (z+1)(z+2)(z-1) \end{aligned}$$

Thus $z_0 = -1, -2, 1$ are simple zero of $f_\xi(z)$.
If $g \in \mathcal{A}$ be such that $T_\xi g = 0$ then $0 = T_\xi g(1) = f_\xi(g(1))$. Hence we can determine a solution $g \in \mathcal{A}$ of $T_\xi = 0$, e.g. if $g(1) = 1 = z_0$, then

$$g(2) = -\frac{1}{6}\left[\frac{a_0}{\xi}(2) + \frac{a_1}{\xi}(2)\frac{g}{\xi}(1) + \frac{a_2}{\xi}(2)\frac{g}{\xi}(1)^2 + \frac{a_3}{\xi}(2)\frac{g}{\xi}(1)^3\right] = \frac{1}{3},$$

$$g(n) = -\frac{1}{6}\sum_{1\le j\le 3}\left[\sum_{\substack{lm_1\ldots m_j = n \\ m_1\ldots m_j < n}} \frac{a_j}{\xi}(l)\frac{g}{\xi}(m_1)\ldots\frac{g}{\xi}(m_j)\right] + \frac{a_0}{\xi}(n) \text{ for } n > 2.$$

Theorem 44. *For $d \in \mathbb{N}$ let $T_\alpha : \mathcal{A} \to \mathcal{A}$ be defined by*

$$T_\alpha g = a_d \diamond g^{\diamond d} + a_{d-1} \diamond g^{\diamond(d-1)} + \cdots + a_1 \diamond g + a_0 \tag{28}$$

for $g \in \mathcal{A}$ with $a_d, a_{d-1}, \ldots, a_1, a_0 \in \mathcal{A}$ and $a_d \neq 0$. If z_0 is a simple zero of the polynomial

$$f_\alpha(z) = \frac{a_d}{\alpha}(1)z^d + \frac{a_{d-1}}{\alpha}(1)z^{d-1} + \cdots + \frac{a_1}{\alpha}(1)z + \frac{a_0}{\alpha}(1), \tag{29}$$

then there exists a uniquely determined solution $g \in \mathcal{A}$ to the convolution equation $T_\alpha g = 0$ satisfying $g(1) = \alpha(1)z_o$.

Proof. Assume that z_0 is a simple zero of $f_\alpha(z)$. Since $g \diamond h = \alpha\left(\frac{g}{\alpha} * \frac{h}{\alpha}\right)$,thus we get that

$$\begin{aligned} T_\alpha g &= a_d \diamond g^{\circ d} + a_{d-1} \diamond g^{\diamond(d-1)} + \cdots + a_1 \diamond g + a_0 \\ &= \alpha\left(\frac{a_d}{\alpha} * \left(\frac{g}{\alpha}\right)^{*d} + \frac{a_{d-1}}{\alpha} * \left(\frac{g}{\alpha}\right)^{*(d-1)} + \cdots + \frac{a_1}{\alpha} * \frac{g}{\alpha} + \frac{a_0}{\alpha}\right). \end{aligned}$$

Let $T : \mathcal{A} \to \mathcal{A}$ be defined by

$$Th = \frac{a_d}{\alpha} * h^{*d} + \frac{a_{d-1}}{\alpha} * h^{*(d-1)} + \cdots + \frac{a_1}{\alpha} * h + \frac{a_0}{\xi} \qquad (\forall h \in \mathcal{A})$$

and

$$f_\alpha(z) = \frac{a_d}{\alpha}(1)z^d + \frac{a_{d-1}}{\alpha}(1)z^{d-1} + \cdots + \frac{a_1}{\alpha}(1)z + \frac{a_0}{\alpha}(1).$$

Since $\alpha(1)f_\alpha(z) = f(z)$. Therefore z_0 be a simple zero of $f(z)$.

By Theorem 19, T has a solution $h \in \mathcal{A}$ such that $h(1) = z_0$. Let $g = h\alpha$ then $T_\alpha(g) = T_\alpha(h\alpha) = \alpha\left(\frac{a_d}{\alpha} * h^{*d} + \frac{a_{d-1}}{\alpha} * h^{*(d-1)} + \cdots + \frac{a_1}{\alpha} * h + \frac{a_0}{\alpha}\right) = 0$
and $g(1) = h(1)\alpha(1) = \alpha(1)z_0$. $\square$

Example 7. Let $T_\alpha g = a_3 \diamond g^{\diamond 3} + a_2 \diamond g^{\diamond 2} + a_1 \diamond g + a_0$ and

$$\alpha(n) = \begin{cases} 1, & \text{n is a prime p;} \\ -1, & \text{else,} \end{cases}$$

$$a_3(n) = \begin{cases} 1, & n < 3; \\ 0, & \text{else,} \end{cases}$$

$$a_2(n) = \begin{cases} 2, & n < 3; \\ 0, & \text{else,} \end{cases}$$

$$a_1(n) = \begin{cases} -1, & n < 3; \\ 1, & \text{else,} \end{cases}$$

$$a_0(n) = \begin{cases} -2n, & n < 3; \\ n, & \text{else.} \end{cases}$$

Consider $f_\alpha(z) = \frac{a_3}{\alpha}(1)z^3 + \frac{a_2}{\alpha}(1)z^+\frac{a_1}{\alpha}(1)z + \frac{a_0}{\alpha}(1)$, we get that

$$\begin{aligned} f_\alpha(z) &= -(z^3 + 2z^2 - z - 2) \\ &= (z+1)(z+2)(1-z). \end{aligned}$$

Thus $z_0 = -1, -2, 1$ are simple zero of $f_\alpha(z)$.

If $g \in \mathcal{A}$ be such that $T_\alpha g = 0$ then $0 = T_\alpha g(1) = f_\alpha(-g(1))$. Hence we can determine a solution $g \in \mathcal{A}$ of $T_\alpha = 0$, e.g. if $g(1) = -1 = z_0$, then

$$g(2) = \frac{1}{6}\left[\frac{a_0}{\alpha}(2) + \frac{a_1}{\alpha}(2)\frac{g}{\alpha}(1) + \frac{a_2}{\alpha}(2)g(1)^2 + \frac{a_3}{\alpha}(2)g(1)^3\right] = 1,$$

$$g(n) = \frac{1}{6}\sum_{1\le j\le 3}\left[\sum_{\substack{lm_1\ldots m_j = n \\ m_1\ldots m_j < n}} \frac{a_j}{\alpha}(l)\frac{g}{\alpha}(m_1)\ldots\frac{g}{\alpha}(m_j)\right] + \frac{a_0}{\alpha}(n) \text{ for } n > 2.$$

Some cases of polynomial convolution equation with no simple zero were also considered by H. Glöckner, L. G. Lucht and S̃. Porubský (see section 3 of Glöckner *et al.*, 2007). They examplarily derived some conditions which are necessary for the existence of the $g \in \mathcal{A}$ to $Tg = 0$. More precisely, if $Tg = 0$ has a solution $g \in \mathcal{A}$, then the coefficient functions $a_j \in \mathcal{A}$ of Tg must be subject to some severe restrictions. Now we will elaborate further on these conditions.

For $k \in \mathbb{N}$ we introduce the polynomials $f_k(z) \in \mathbb{C}[z]$ by

$$f_k(z) = a_d(k)z^d + \cdots + a_1(k)z + a_0(k).$$

Obviously, $f_1(z) = f(z)$. Suppose that g is a solution of $Tg = 0$ satisfying $g(1) = z_0$. Let $n = p_1^{n_1} \cdots p_r^{n_r}$ then

$$\begin{aligned}
Tg(n) &= (a_d * g^{*d} + a_{d-1} * g^{*(d-1)} + \cdots + a_1 * g + a_0)(n) \\
&= \sum_{1 \le i \le d} a_i * g^{*i}(n) + a_0(n) \\
&= \sum_{1 \le i \le d} \left[\sum_{m_0 m_1 \cdots m_i = n} a_i(m_0) g(m_1) \cdots g(m_i) \right] + a_0(n) \\
&= \sum_{1 \le i \le d} \left[\sum_{\substack{m_0 m_1 \cdots m_i = n \\ m_0 < n}} a_i(m_0) g(m_1) \cdots g(m_i) + a_i(n) g(1)^i \right] + a_0(n) \\
&= \sum_{1 \le i \le d} \left[\sum_{j=1}^{i} \sum_{\substack{m_0 m_1 \cdots m_j = n \\ m_1 \cdots m_j > 1}} \binom{i}{j} a_i(m_0) g(m_1) \cdots g(m_j) g(1)^{i-j} \right] \\
&\quad + \sum_{1 \le i \le d} a_i(n) g(1)^i + a_0(n) \\
&= \sum_{1 \le i \le d} \sum_{\substack{m_0 m_1 \cdots m_j = n \\ m_1 \cdots m_j > 1}} \frac{1}{j!} g(m_1) \cdots g(m_j) f_{m_0}^{(j)}(g(1)) + f_n(g(1)).
\end{aligned}$$

For example, let $Tg = g^{*d} - a$ with $a \in \mathcal{A}$, $a(1) = 0$ then $f(z) = z^d$. It follows that $z_0 = 0$ is a multiple zero of $f(z)$ i.e. $f'(0) = \cdots = f^{(d-1)}(0) = 0$, $f^{(d)}(0) = d!$. Since $f_k(z) = z^d - a(k)$ thus we get
$f_k^{(j)}(z) = d(d-1) \cdots (d-j-1) z^{d-j},\ j = 1, \ldots, d$. It follows that
$f_k'(0) = \cdots = f_k^{(d-1)}(0) = 0$ and $f_k^{(d)}(0) = d!$. Hence for $n \in \mathbb{N}$ s.t. $\Omega(n) < d$,

$$0 = Tg(n) = \sum_{\substack{m_0 m_1 \cdots m_d = n \\ m_1 \cdots m_d > 1}} \frac{1}{d!} g(m_1) \cdots g(m_d) f_{m_0}^{(d)}(0) + f_n(0) = a(n)$$

i.e. $a(n) = 0$, and for $n = p_1^{n_1} \cdots p_r^{n_r}$, $\Omega(n) = d$,

$$\begin{aligned}
0 &= Tg(n) \\
&= g(p_1)^{n_1} \cdots g(p_r)^{n_r} f_1^{\Omega(n)}(g(1)) + f_n(g(1)) \\
&= \left(\frac{1}{d!}\right) (g(p_1)^{n_1} \cdots g(p_r)^{n_r})\, d! - a(p_1^{n_1} \cdots p_r^{n_r}) \\
&= g(p_1)^{n_1} \cdots g(p_r)^{n_r} - a(p_1^{n_1} \cdots p_r^{n_r})
\end{aligned}$$

i.e. $g(p_1)^{n_1} \cdots g(p_r)^{n_r} = a(p_1^{n_1} \cdots p_r^{n_r})$ for all prime p_i.

Therefore $g(p)^d = a(p^d)$ for all prime p. It follows that

$a(p_1^{n_1} \cdots p_r^{n_r})^d = (g(p_1)^{n_1} \cdots g(p_r)^{n_r})^d = a^{n_1}(p_1^d) \cdots a^{n_r}(p_r^d)$ for all prime p_i.

CONCLUSION

In this thesis, the following results have either been derived or proved.

1. Basic properties of the $Q_\alpha-$ convolution,

2. Characterizations of completely multiplicative functions with respect to the $Q_\alpha-$ convolution.

3. Connections between the binomial convolution and $Q_\alpha-$ convolution,

4. Criteria for $\diamond-$ Algebraic independence over a subring $\mathcal{R}$ of $\mathcal{A}$.

5. The solubility of polynomial convolution equation whose characteristic polynomial has no simple zero.

6. The solubility of polynomial binomial convolution equations and the polynomial $Q_\alpha-$ convolution equations.

LITERATURE CITED

Apostol, T. M. 1971. Some properties of completely multiplicative arithmetical functions. **Amer. Math. Monthly** 78: 266-271.

Glöckner H., L. G. Lucht and S̃. Porubský. 2007. Solutions to arithmetic equations. **Amer. Math. Soc.** 135: 1619-1629.

Haukkanen, P. 1996. On a binomial convolution of arithmetical functions. **Nieuw Arch. Wisk.** 14: 209-216.

Haukkanen, P. 2001. **On characterizations of completely multiplicative arithmetical functions:** in Number Theory, Turku, de Gruyter.

Lambek, J. 1966. Arithmetical functions and distributivity. **Amer. Math. Monthly** 73: 969-973.

Langford, E. 1973. Distributivity over the Dirichlet product and completely multiplicative arithmetical functions. **Amer. Math. Monthly** 80: 411-414.

Laohakosol, V. and N. Pabhapote. 2004. Completely multiplicative functions arising from simple operations. **Int. J. Math. Math. Sci.** 9: 431-441.

Laohakosol, V., N. Pabhapote and N. Wechwiriyakul. 2002. Characterizing completely multiplicative functions by generalized Möbius functions. **Int. J. Math. Math. Sci.** 29(11): 633-639.

McCarthy, P. J. 1985. **Introduction to Arithmetical functions.** Springer-Verlag, New York.

Pabhapote, N. and V. Laohakosol. 2010. Distributive property of completely multiplicative functions. **Lith. Math. J.** 50: 312-322.

Raerick, D. 1968. Operators on algebras of arithmetic functions. **Duke Math. J.** 35: 761-766.

Shapiro, H. N. 1983. **Introduction to the theory of numbers.** John Wiley and Sons, New York.

Shapiro, H. N. and G. H. Sparer. 1986. On algebraic independence of Dirichlet series. **Comm. Pure Appl. Math.** 39: 695-745.

Subbarao, M. V. 1968. Arithmetic functions and distributivity. **Amer. Math. Monthly** 75: 984-988.

Tóth, L. and P. Haukkanen. 2009. On the binomial convolution of arithmetical functions. **J. Comb. Number Theory** 1: 31-48.

CURRICULM VITAE

NAME	: Miss Sunanta Srisopha			
BIRTH DATE	: October 23, 1987			
BIRTH PLACE	: Bangkok, Thailand			
ECUCATION	:	**YEAR**	**INSTITUTION**	**GREE/DIPLOMA**
		2008	Thammasat Univ.	B.Sc.(Mathematics)
		2012	Kasetsart Univ.	M.S.(Mathematics)