



*Original Article*

## Combined overbooking and seat inventory control for two-class revenue management model

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### Abstract

We propose a two-class revenue management (RM) model, which combines two of the most important RM strategies, namely overbooking and seat inventory control for a passenger airline. We derive a closed-form expression for an optimal overbooking limit that maximizes the expected profit, and analytically perform sensitivity analysis by changing model parameters such as a revenue, a penalty cost associated with unsatisfied demand, a show-up probability, a refund, a denied boarding cost, and a plane capacity.

**Keywords:** overbooking, static model, stochastic model, revenue management

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### 1. Introduction

In 2014, the global economic crisis damaged most economic sectors, but commercial airline could still grow 2.6% and were worth \$704 billion (Flight global, 2015b). According to 2014 aerospace industry financial data, the commercial airline industry would continue to grow in the next ten years (Flight global, 2015a). One of the keys to success is revenue management (RM) (Sabre Corporation, 2015). RM can be defined as “selling the right product to the right customer at the right time for the right price” (Cross, 1997). Major research areas in RM can be categorized into 1) seat inventory control, 2) overbooking, 3) pricing and 4) demand forecasting (McGill and van Ryzin, 1999). In this paper, we focus on two RM strategies, namely overbooking and seat inventory control, practiced by a passenger airline.

Overbooking means that the airline intentionally sells more reservations for a flight than physical capacity on the aircraft to compensate for cancellations and no-shows. The seat inventory control problem concerns with mixing passengers in different fare classes in the same aircraft

compartment. In this paper, we combine two strategies and propose a static two-class overbooking model, in which low fare (class-2) customers arrive before high fare (class-1) customers. The airline incurs a penalty cost for each rejected booking request. The penalties are different for the two classes. The airline may overbook class-2 customers. The two fare classes may have different show-up rates. We want to find an optimal overbooking limit that maximizes the total expected profit.

There are many multiple-class booking control models which allow overbooking; see, e.g., Brumelle and McGill (1989), Subramanian *et al.* (1999), Gosavi *et al.* (2002), Lan *et al.* (2011), Aydin *et al.* (2012), and Lan *et al.* (2015). These models are formulated as Markov decision processes, and most do not possess the closed-form solutions except Aydin *et al.* (2012). Aydin *et al.* (2012) assume that the random vector of booking request follows a multinomial distribution. Our model assumes general distribution for the booking request. In practice most commercial RM systems are based upon the two-class model, instead of the multi-class model. In this article, a closed-form solution of the proposed two-class model is obtained.

The two-class model, which focuses on the booking control problem, dates back to Littlewood (1972). All booking

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requests show up at the time of service; i.e., there are no cancellations or no-shows: Littlewood (1972) does not allow overbooking. Shlifer and Vardi (1975) study the two-class overbooking model but does not include the booking control problem. The two-class model, which includes both overbooking and booking control problem, can be found in Sawaki (1989) and Ringbom and Shy (2002). Sawaki (1989) and Ringbom and Shy (2002) extend Littlewood (1972) to allow no-show passengers. In Sawaki (1989), the booking requests of the two classes are assumed to be continuous, whereas those in Ringbom and Shy (2002) follow bivariate normal. The show-up passenger in Sawaki (1989) and Ringbom and Shy (2002) follows a binomial distribution. In ours, the booking request needs not be normal, it can be any non-negative integer-valued random variable with a general distribution. In Ringbom and Shy (2002), the refund is fully given to class 1 and the class 2 receives no refund, whereas in ours, the refunds are given to both classes, and the refund needs not be fully given (i.e., the refund can be expressed as a percentage of the fare). Similar to other overbooking models, we accept the booking requests up to the overbooking limit, and additional requests are rejected. In Sawaki (1989), the airline incurs a penalty (loss-of-goodwill) cost for only class-1 rejected booking request, whereas in ours, the penalty cost is given to each rejected booking request. Our refund and penalty scheme are more general and fit more cases in practice. This is the first to include the refund cost and penalty cost simultaneously into the two-class RM model.

The rest of the paper is organized as follows. The model is formulated in Section 2 and analyzed in Section 3. Section 4 concludes our paper and provides future research problems. All proofs are shown in Appendix.

**2. Formulation**

Let  $\mathbb{R}$  be the set of real numbers and  $\mathbb{Z}_+$  be the set of non-negative integers. Let  $(y)^+ = \max(0, y)$  for  $y \in \mathbb{R}$ . The quantile function of distribution function of random variable  $D$  is denoted as  $F_D^{-1}(a) = \inf \{x : P(D \leq x) \geq a\}$ .

Consider an airline with fixed capacity  $\kappa$  and two customer classes with fares  $p_1 > p_2 > 0$ . We assume that all class-2 reservations arrive before class-1 start reservation. For each  $i = 1, 2$ , the airline earns revenue of  $p_i$  when a class- $i$  customer is accepted; on the other hand, if rejected the airline incurs a penalty cost  $g_i$  where  $g_1 > g_2 > 0$ . The penalty cost when a customer is rejected includes, e.g., the loss-of-goodwill cost, which measures customer satisfaction, and the opportunity cost, which measures future revenue loss. The loss-of-goodwill cost may be intangible and can be difficult to estimate in practice. The opportunity cost depend on what happens after the lost sales occur. If a customer is likely to return to make a booking request, then the opportunity cost is the expected revenue loss from this event; however, if a customer never returns to make any bookings with the airline, then the opportunity cost includes all future revenues the customer might have brought to the airline.

Let  $x \in \mathbb{Z}_+$  be an overbooking limit of class 2: Class-2 booking requests are accepted up to  $x$ . We allow overbooking; i.e., can be greater than capacity  $\kappa$ . Let  $D_i$  be class- $i$  demand, the number of class- $i$  booking requests. Assume that  $D_1$  and  $D_2$  are two independent  $\mathbb{Z}_+$ -valued random variables. The number of class-2 reservations is  $\min(x, D_2)$ , and the number of class-2 booking requests rejected is  $(D_2 - x)^+$ .

After class-2 reservations all arrive, class-1 customers start their booking. The remaining capacity after class-2 arrives is  $(\kappa - \min(x, D_2))^+$ . We do not overbook class-1 because class-1 passengers are of high priority or extremely high penalty cost. Class-1 customers are accepted up to the remaining capacity. For  $i = 1, 2$ , let  $B_i(x)$  be the number of class- $i$  reservations:

$$B_2(x) = \min(x, D_2), \quad B_1(x) = \min((\kappa - B_2(x))^+, D_1).$$

Some reservations may cancel prior to or do not show up at the time of service. In this model, we assume that cancellation and no-show passengers are the same. Given that the number of class- $i$  reservations is  $B_i(x) = y_i$ , the number of class- $i$  show-ups, denoted by  $W_i(y_i)$ , is assumed to follow a binomial distribution with parameters  $y_i$  and  $\theta_i$  where  $\theta_i \in (0, 1]$  is the show-up probability of class- $i$ . Note that when the show-up probability of class- $i$  is equal to 1 ( $\theta_i = 1$ ) it means that all passengers of class- $i$  show up at the time of service. That the binomial distribution is an adequate model for the show-ups distribution has been showed in Tasman Empire Airways (Thompson, 1961). Each class- $i$  reservation that does not show up receives a refund  $r_i$ , which is a proportion  $\gamma_i$  of revenue cost where  $\gamma_i \in (0, 1)$ ;  $r_i = \gamma_i p_i$  for  $i = 1, 2$ .

At the time of service, the number of show-up passengers may be over capacity. Recall that we overbook only class-2 passenger, so all denied boarding passengers are class-2. The airline pays a compensation  $h$  to each denied boarding passenger where  $h > p_2$ . This compensation may include a fare of a higher booking class on a next flight, vouchers for cash or tickets for future travel, and/or hotel accommodation. The airline wants to choose an optimal overbooking limit  $x^*$  that maximizes its expected profit:

$$\begin{aligned} \pi(x) = & E \left[ \sum_{i=1}^2 [p_i B_i(x) - r_i (B_i(x) - W_i(B_i(x)))] \right] \\ & - E \left[ h (W_2(B_2(x)) - \kappa)^+ \right] \\ & - E \left[ g_2 (D_2 - x)^+ + g_1 (D_1 - (\kappa - B_2(x))^+)^+ \right]. \quad (1) \end{aligned}$$

The first term in (1) is the expected of revenue  $p_i B_i(x)$  minus the expected refund cost paid to reservations with no shows  $r_i (B_i(x) - W_i(B_i(x)))$ . The second term is the expected denied boarding cost, the company pays to the denied boarding passengers when the number of show-ups is more than capacity. Recall that we do not overbook class-1 customer, so all denied boarding passengers are class-2. The last term is the expected penalty cost, the expected revenue lost when

we reject the booking requests of class 1 and 2. Note that in (1) there are two sources of uncertainty, namely demand uncertainty and the number of show-ups. In practice, an optimal overbooking limit is re-solved periodically to account for change in show-up probability and proportion of refund cost over time, resulting in overbooking limits that vary over time. The airlines accept the reservations at any time up to the current overbooking limits. Typically, airlines may only monthly update the optimal overbooking limit for a flight before departure at least six months (Phillips, 2005). Most airlines recalculate the optimal overbooking limit everyday during the last week before departure.

### 3. Analyses

In this section, increasing (respectively; decreasing) means non-decreasing (respectively; non-increasing). For  $i = 1, 2$ , let  $\alpha_i = p_i + g_i - r_i + r_i\theta_i$  and  $\tau = \alpha_2 / \alpha_1$ . Let  $\bar{F}(t; x, \theta_2) = 1 - \sum_{j=0}^x \binom{x}{j} \theta_2^j (1 - \theta_2)^{x-j}$  is the tail-sum probability (complementary cumulative distribution function) of binomial distribution with parameters  $x$  and  $\theta_2$ .

**Theorem 1.** The expected profit function  $\pi(x)$  is piecewise on  $x = 0, 1, \dots, \kappa - 2$  and  $x = \kappa, \kappa + 1, \dots$ , and it is unimodal in each piece.

1. For  $x = 0, 1, \dots, \kappa - 2$ , the expected profit  $\pi(x)$  has a local maximum point  $x'$  given by

$$x' = \begin{cases} 0 & ; 0 \leq \tau < P(D_1 > \kappa - 1) \\ \kappa - F_{D_1}^{-1}(1 - \tau) & ; P(D_1 > \kappa - 1) \leq \tau \leq P(D_1 > 0) \\ \kappa - 2 & ; P(D_1 > 0) < \tau \leq 1 \end{cases} \quad (2)$$

2. For  $x = \kappa, \kappa + 1, \dots$ , if  $0 < \alpha_2 / (h\theta_2) < \bar{F}(\kappa - 1; \kappa, \theta_2)$ , then the expected profit  $\pi(x)$  has a local maximum point  $x''$  given by

$$x'' = \arg \min \{x \in \{\kappa, \kappa + 1, \dots\} : \bar{F}(\kappa - 1; x, \theta_2) > \frac{\alpha_2}{h\theta_2}\}. \quad (3)$$

Otherwise, the expected profit function is increasing.

From Theorem 1, we can find the optimal overbooking limit  $x^*$  from three points;  $x', \kappa - 1$  and  $x''$ . Suppose that  $0 < \alpha_2 / (h\theta_2) < \bar{F}(\kappa - 1; \kappa, \theta_2)$ . Then

$$x^* = \arg \max \{\pi(x'), \pi(\kappa - 1), \pi(x'')\}.$$

Suppose that  $\alpha_2 / (h\theta_2) \geq \bar{F}(\kappa - 1; \kappa, \theta_2)$ .

If  $\lim_{x \rightarrow \infty} \pi(x) < \max \{\pi(x'), \pi(\kappa - 1)\}$ , then  $x^* = \arg \max \{\pi(x'), \pi(\kappa - 1)\}$ .

Otherwise,  $\pi(x)$  is increasing, and the airline should set the optimal overbooking limit to be as large as possible. The optimal overbooking limit in Theorem 1 has a closed-form that is easy to calculate. This solution can be extended to heuristic method in multiple fare classes model which is better than using Markov decision process.

Different shapes of the expected profit function are shown in Figures 1 and 2. In Figures 1 and 2, demand,  $D_i$ , is assumed to be Poisson random variable with mean  $\lambda_i$  for  $i = 1, 2$ , and capacity is  $\kappa = 100$ . Although Poisson random variable is assumed in numerical experiments, the proof of Theorem 1 does not need to assume Poisson distribution. This theorem holds for any non-negative random variable. We set the penalty cost equal to the revenue; i.e.,  $g_1 = p_1$  and  $g_2 = p_2$ . We assume that the opportunity cost is the lost revenue from rejecting a reservation from that particular class.

In Figure 1, we do not overbook; i.e.,  $x^* < \kappa$ . The maximum expected profit in the first piece (in Theorem 1) is greater than that the second piece. In Figure 1d, the optimal overbooking limit is  $x^* = 0$ ; this correspond to the first case in (2). It means that we do not accept class-2 reservations when mean demand of class-1 is large. In Figure 1a, 1b, 1c, the optimal overbooking limit  $x^*$  is given in the second case in (2). Note that as the mean demand of class-1  $\lambda_1$  increases, the optimal overbooking limit decreases.

In Figure 2, overbooking occurs; i.e.,  $x^* > \kappa$ . The maximum expected profit in the second piece (in Theorem 1) is greater than that in the first piece; i.e.,  $\pi(x'') > \pi(x')$ . Note that as the denied boarding cost  $h$  increases, the optimal overbooking limit decreases.

From Figures 1 and 2, we see that the optimal overbooking limits change the location when some model parameters change. We formally perform sensitivity analysis in the next Corollaries.

**Corollary 1.** For  $x = 0, 1, \dots, \kappa - 2$ , suppose that the ratio  $\alpha_2 / \alpha_1$  or capacity  $\kappa$  increases or  $D_1$  decreases with respect to usual stochastic order. Then, the local maximum point  $x'$  increases.

Suppose that we do not overbook; for instance, (i) mean demand of class-1 is larger than capacity, (ii) mean demand of class-2 is much lower than capacity. There are many cases that increase the optimal overbooking limit with corresponding to  $\alpha_2 / \alpha_1$  increases, e.g. (1) revenue cost of class-2 increases (or revenue cost of class-1 decreases), (2) penalty cost of class-2 increases (or penalty cost of class-1 decreases), (3) show-up probability of class-2 increases (or show-up probability of class-1 decreases), (4) refund cost of class-2 decreases (or refund cost of class-1 increases).

In Corollary 2, we indicate how the local maximum point  $x''$  changes, when the model parameter of class-2 and capacity are varied.

**Corollary 2.** For  $x = \kappa, \kappa + 1, \dots$ , suppose that the ratio  $\alpha_2 / (h\theta_2)$  or capacity  $\kappa$  increases. Then, the local maximum point  $x''$  increases.

Suppose that we overbook; for instance, (i) class-2 revenue is close to class-1 revenue, (ii) mean demand of class-2 is much higher than capacity. There are many cases that increase the optimal overbooking limit with correspond-

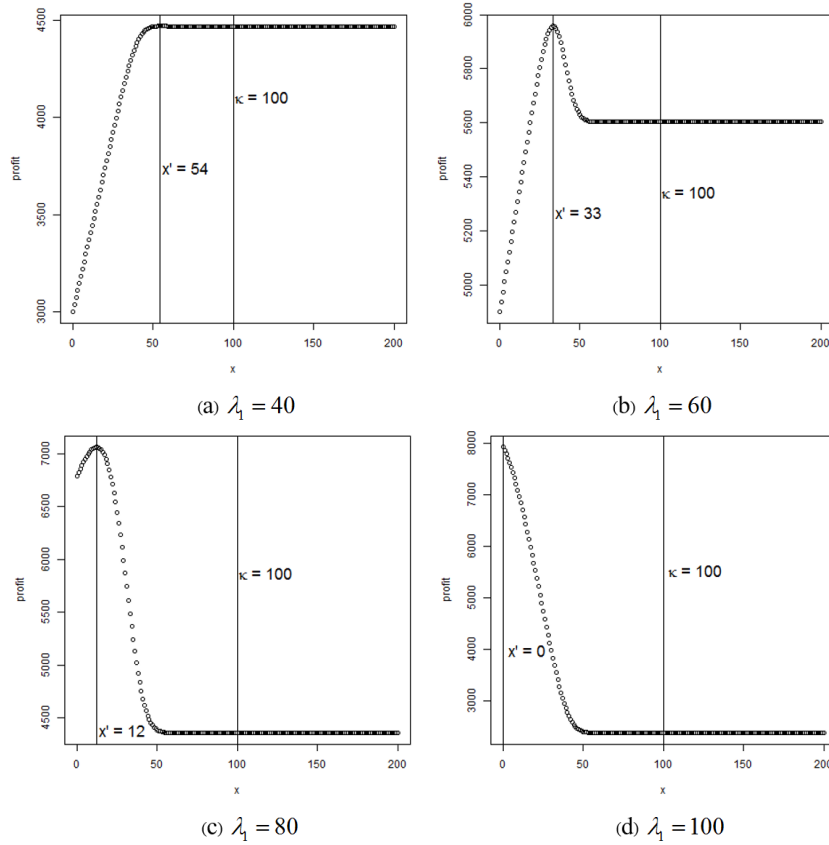


Figure 1. The expected profit when varying  $\lambda_1 = 40, 60, 80$  and  $100$ . Other parameters are  $p_1 = g_1 = 100, p_2 = g_2 = 20, r_1 = 50, r_2 = 10, \theta_1 = 0.9, \theta_2 = 0.7$  and  $h = 300$ .

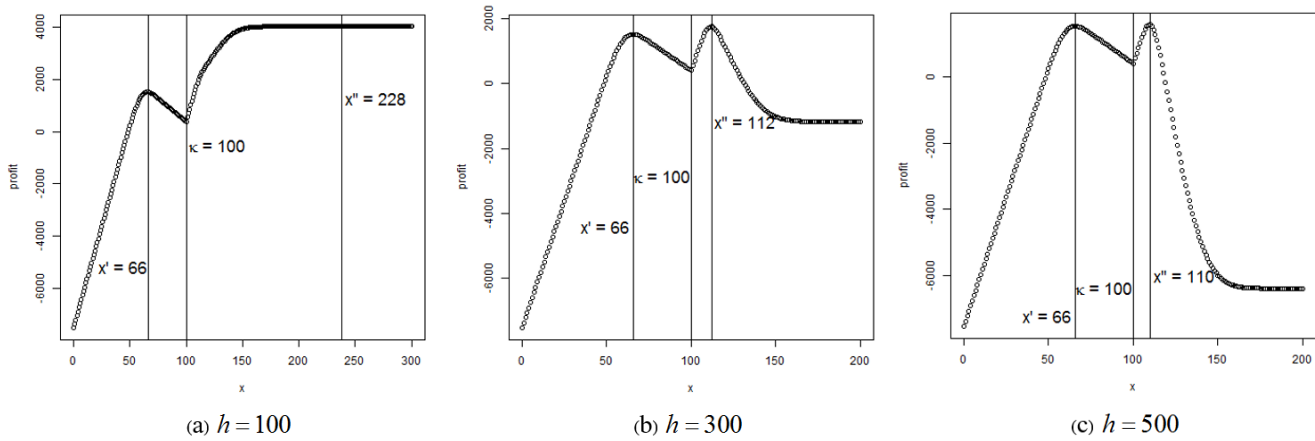


Figure 2. The expected profit when varying  $h = 100, 300$  and  $500$ . Other parameters are  $p_1 = g_1 = 100, p_2 = g_2 = 80, r_1 = 80, r_2 = 40, \theta_1 = 0.9, \theta_2 = 0.7$  and  $\lambda_1 = 40$ .

ing to  $\alpha_2 / (h\theta_2)$  increases, e.g. (1) revenue cost of class-2 increases, (2) penalty cost of class-2 increases, (3) show-up probability of class-2 decreases, (4) refund cost of class-2 decreases, (5) denied boarding cost decreases.

Corollary 2 implies that the airline may need to update the optimal overbooking limit when there is an unusual situa-

tion such as disaster, insurgence or demonstration which affects some parameters in the model. For instance, Bangkok bomb at Erawan shrine on 17 August 2015 may decrease a show-up probability from tourists who plan to travel to Bangkok. When the show-up probability decreases, the airline may need to set a higher overbooking limit. On the other

hand, during long holiday such as Christmas, new year festival, Songkran festival the show-up probability may be higher; consequently the airline may decrease the overbooking limit.

#### 4. Conclusions

In this article, we propose a static two-class overbooking model and derive an optimal overbooking limit that maximizes the expected profit. The parameters in the model are revenue cost, penalty cost, refund cost, show-up probability, denied boarding cost and capacity. Sensitivity analyses with respect to changes in model parameters are performed: If it is optimal not to overbook, then the booking limit is affected by the demand of class 1 and all of model parameters except for denied boarding cost. If it is optimal to overbook, then the overbooking limit is affected by all of model parameters of class-2 including denied boarding cost and capacity.

It is possible to extend the study as follows: Estimating the parameters in the model when demand is censored. The model, in which demands are dependent, could be studied. Moreover, we can allow overbooking on class 1; this model would have two overbooking limits. We hope to pursue some of these related problems in the future.

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### Appendix

**Lemma 1.** The expected profit  $\pi(x)$  can be written as follows:

For  $x = 0$ ,

$$\pi(x) = \alpha_1 E[B_1(x)] - \sum_{i=1}^2 g_i E[D_i]. \tag{4}$$

For  $x = 1, 2, \dots, \kappa - 1$ ,

$$\pi(x) = \sum_{i=1}^2 [\alpha_i E[B_i(x)] - g_i E[D_i]]. \tag{5}$$

For  $x = \kappa, \kappa + 1, \dots$ ,

$$\pi(x) = (\alpha_2 - h\theta_2) E[B_2(x)] + \alpha_1 E[B_1(x)] - \sum_{i=1}^2 g_i E[D_i] + h \sum_{t=0}^{\kappa-1} P(W_2(B_2(x)) > t), \tag{6}$$

where the expected number of class-2 reservations is

$$E[B_2(x)] = \begin{cases} 0 & ; x = 0 \\ \sum_{t=0}^{x-1} P(D_2 > t) & ; x = 1, 2, \dots \end{cases}$$

and the expected number of class-1 reservations is

$$E[B_1(x)] = \begin{cases} \sum_{t=0}^{\kappa-1} P(D_1 > t) & ; x = 0 \\ \sum_{t=0}^{\kappa-x-1} P(D_1 > t) + \sum_{t=\kappa-x}^{\kappa-1} P(D_2 < \kappa - t) P(D_1 > t) & ; x = 1, 2, \dots, \kappa - 1 \\ \sum_{t=0}^{\kappa-1} P(D_2 < \kappa - t) P(D_1 > t) & ; x = \kappa, \kappa + 1, \dots \end{cases}$$

**Proof.** From (1), we obtain

$$\begin{aligned} \pi(x) = & \sum_{i=1}^2 [(p_i - r_i) E[B_i(x)] + r_i E[W_i(B_i(x))]] - h E[(W_2(B_2(x)) - \kappa)^+] \\ & - g_2 E[(D_2 - x)^+] - g_1 E[(D_1 - (\kappa - B_2(x)))^+]. \end{aligned} \tag{7}$$

If  $x > 0$ , we use a tail-sum formula for expectation to find the expected number of class-2 reservations

$$E[B_2(x)] = \sum_{t=0}^{\infty} P(\min(x, D_2) > t) = \sum_{t=0}^{x-1} P(D_2 > t).$$

Similarly,

$$\begin{aligned} E[B_1(x)] &= \sum_{t=0}^{\infty} P(\min((\kappa - B_2(x))^+, D_1) > t) \\ &= \sum_{t=0}^{\infty} P((\kappa - B_2(x))^+ > t) P(D_1 > t) \\ &= \sum_{t=0}^{\kappa} [1 - P((\kappa - B_2(x))^+ \leq t)] P(D_1 > t). \end{aligned} \tag{8}$$

Clearly, if  $x = 0$ , then  $E[B_2(x)] = 0$ .

If  $x = 0$ , then  $E[B_1(x)] = \sum_{t=0}^{\kappa-1} P(D_1 > t)$ .

If  $x = 1, 2, \dots, \kappa - 1$ , the probability mass function of  $\kappa - B_2(x)$  is given as

$$P(B_2(x) = \kappa - k) = \begin{cases} P(D_2 \geq x) & ; k = \kappa - x \\ P(D_2 = \kappa - k) & ; k = \kappa - x + 1, \dots, \kappa \\ 0 & ; \text{otherwise} \end{cases} \tag{9}$$

If  $x = \kappa, \kappa + 1, \dots$ , the probability mass function of  $(\kappa - B_2(x))^+$  is given as

$$P((\kappa - B_2(x))^+ = k) = \begin{cases} P(D_2 \geq x) & ; k = 0 \\ P(D_2 = \kappa - k) & ; k = 1, 2, \dots, \kappa \\ 0 & ; \text{otherwise} \end{cases} \tag{10}$$

Substitution (9) and (10) into (8), we obtain

$$E[B_1(x)] = \begin{cases} \sum_{t=0}^{\kappa-1} P(D_1 > t) & ; x = 0 \\ \sum_{t=0}^{\kappa-x-1} P(D_1 > t) + \sum_{t=\kappa-x}^{\kappa-1} P(D_2 < \kappa - t)P(D_1 > t) & ; x = 1, 2, \dots, \kappa - 1 \\ \sum_{t=0}^{\kappa-1} P(D_2 < \kappa - t)P(D_1 > t) & ; x = \kappa, \kappa + 1, \dots \end{cases}$$

The number of class- $i$  show-ups,  $W_i(y_i)$ , has binomial distribution with parameters  $y_i$  and  $\theta_i \in (0, 1]$  where  $y_i$  is the number of class- $i$  reservations and  $\theta_i$  is the show-up probability of class- $i$ .

Then,  $E[W_i(B_i(x)) | B_i(x) = y_i] = \theta_i y_i$ . So,

$$E[W_i(B_i(x))] = \theta_i E[B_i(x)] \quad ; i = 1, 2. \tag{11}$$

We know that  $(a - b)^+ = a - \min(a, b)$ . Thus,

$$E[(D_2 - x)^+] = E[D_2 - \min(x, D_2)] = E[D_2] - E[B_2(x)]. \tag{12}$$

Similarly,

$$E[(D_1 - (\kappa - B_2(x))^+)^+] = E[D_1] - E[B_1(x)]. \tag{13}$$

The expected number of class-2 passenger who are denied boarding is

$$\begin{aligned} E[(W_2(B_2(x)) - \kappa)^+] &= E[W_2(B_2(x))] - E[\min(W_2(B_2(x)), \kappa)] \\ &= \theta_2 E[B_2(x)] - \sum_{t=0}^{\kappa-1} P(W_2(B_2(x)) > t). \end{aligned} \tag{14}$$

Substitution (11), (12), (13) and (14) into (7), we obtain

$$\pi(x) = \sum_{i=1}^2 [\alpha_i E[B_i(x)] - g_i E[D_i]] - h \left( \theta_2 E[B_2(x)] - \sum_{t=0}^{\kappa-1} P(W_2(B_2(x)) > t) \right). \tag{15}$$

After substitute  $E[B_2(x)]$  and  $E[B_1(x)]$  into (15), the expected profit becomes (4) - (6). □

**Proof of Theorem 1.** For  $x = 0, 1, 2, \dots$ , let  $\delta(x) = \pi(x+1) - \pi(x)$  be the forward difference of the expected profit. Denote  $G_i(x) = B_i(x+1) - B_i(x)$ ;  $i = 1, 2$ . Clearly,

$$E[G_2(x)] = P(D_2 > x) \quad \forall x \in \mathbb{Z}_+$$

and

$$E[G_1(x)] = \begin{cases} -P(D_2 > x)P(D_1 > \kappa - x - 1) & ; x = 0, 1, \dots, \kappa - 1 \\ 0 & ; x = \kappa, \kappa + 1, \dots \end{cases}$$

After some tedious algebra, we obtain the expression for the difference as follows.

For  $x = 0, 1, \dots, \kappa - 2$ ,

$$\delta(x) = P(D_2 > x)[\alpha_2 - \alpha_1 P(D_1 > \kappa - x - 1)].$$

For  $x = \kappa - 1$ ,

$$\delta(x) = P(D_2 > x)[\alpha_2 - \alpha_1 P(D_1 > \kappa - x - 1)] - hE[(W_2(B_2(x)) - \kappa)^+]$$

For  $x = \kappa, \kappa + 1, \dots$ ,

$$\delta(x) = P(D_2 > x)[\alpha_2 - h\theta_2 \bar{F}(\kappa - 1; x, \theta_2)],$$

where  $\bar{F}(t; x, \theta_2) = 1 - \sum_{j=0}^t \binom{x}{j} \theta_2^j (1 - \theta_2)^{x-j}$  is the tail-sum probability of binomial distribution with parameters  $x$  and  $\theta_2$ .

We will consider two piece:  $x = 0, 1, \dots, \kappa - 2$  and  $x = \kappa, \kappa + 1, \dots$ .

Define  $\eta(x) = \alpha_2 - \alpha_1 P(D_1 > \kappa - x - 1)$ . Consider the first piece,

$$\delta(x) = P(D_2 > x)[\alpha_2 - \alpha_1 P(D_1 > \kappa - x - 1)] = P(D_2 > x)\eta(x).$$

We observe that  $\delta(x)$  has the same sign as term  $\eta(x)$ . Then

1. If  $\delta(x) \geq 0$  for  $x = 0, 1, \dots, \kappa - 2$ , then the expected profit  $\pi(x)$  is increasing and a local maximum point is  $\kappa - 2$
2. If  $\delta(x) \leq 0$  for  $x = 0, 1, \dots, \kappa - 2$ , then the expected profit  $\pi(x)$  is decreasing and a local maximum point is 0.
3. We will show that  $\eta(x)$  is decreasing in  $x$ . If  $\delta(x) > 0, \forall x < x'$  and  $\delta(x) \leq 0, \forall x \geq x'$ , then there exists a local maximum point is  $x'$  such that  $\pi(x)$  is increasing for  $x < x'$  and decreasing for  $x \geq x'$ . A local maximum point is at  $x'$ .

If  $P(D_1 > 0) < \alpha_2 / \alpha_1 < 1$ , then  $\eta(x) \geq 0 \frac{1}{2}$  for all  $x = 0, 1, \dots, \kappa - 2$ , so  $\delta(x) \geq 0$  for  $x = 0, 1, \dots, \kappa - 2$ . The expected profit function is increasing in  $x$ . A local maximum point is  $\kappa - 2$ .

If  $0 < \alpha_2 / \alpha_1 < P(D_1 > \kappa - 1)$ , then  $\eta(x) \leq 0$  for all  $x = 0, 1, \dots, \kappa - 2$ , so  $\delta(x) \leq 0$  for  $x = 0, 1, \dots, \kappa - 2$ . The expected profit function is decreasing in  $x$ . A local maximum point is 0.

Recall that  $P(D_1 > \kappa - x - 1)$  is increasing in  $x$  so  $\eta(x)$  is decreasing in  $x$ . If  $P(D_1 > \kappa - 1) < \alpha_2 / \alpha_1 < P(D_1 > 1)$ , then  $\eta(0) > 0$  and  $\eta(\kappa - 2) < 0$ , i.e., there exists a local maximum point  $x'$  such that  $\eta(x) > 0, \forall x < x'$  and  $\eta(x) < 0, \forall x \geq x'$ . So,  $\delta(x) > 0, \forall x < x'$  and  $\delta(x) < 0, \forall x \geq x'$ , i.e., a local maximum point  $x'$  given by

$$x' = \arg \min \{x \in \{0, 1, \dots, \kappa - 2\} : P(D_1 > \kappa - x - 1) > \tau\} \tag{16}$$

Let  $y = \kappa - x - 1$ . Then, for  $x = 0, 1, \dots, \kappa - 2$ , we have that  $y = 1, 2, \dots, \kappa - 1$ . Also,  $y' = \kappa - x' - 1$ . A local maximum point condition (16) becomes

$$y' = \arg \max \{y \in \{1, 2, \dots, \kappa - 1\} : P(D_1 \leq y) < 1 - \tau\} = F_{D_1}^{-1}(1 - \tau) - 1$$

So,

$$x' = \kappa - F_{D_1}^{-1}(1 - \tau).$$

Next, consider the second piece,  $x = \kappa, \kappa + 1, \dots$ .



Let  $\zeta(x) = \alpha_2 - h\theta_2 \bar{F}(\kappa - 1; x, \theta_2)$ .

$$\delta(x) = P(D_2 > x)[\alpha_2 - h\theta_2 \bar{F}(\kappa - 1; x, \theta_2)] = P(D_2 > x)\zeta(x).$$

We find that  $\delta(x)$  has the same sign as term  $\zeta(x)$ . Then

1. If  $\delta(x) \geq 0$  for  $x = \kappa, \kappa + 1, \dots$ , then the expected profit  $\pi(x)$  is increasing and a local maximum point is set as large as possible.

2. We will show that  $\zeta(x)$  is decreasing in  $x$ . If  $\delta(x) > 0, \forall x < x''$  and  $\delta(x) \leq 0, \forall x \geq x''$ , then there exists a local maximum point is  $x''$  such that  $\pi(x)$  is increasing for  $x < x''$  and decreasing for  $x \geq x''$ . A local maximum point is at  $x''$ .

If  $\alpha_2 / (h\theta_2) > 1$ , then  $\zeta(x) \geq 0$  for all  $x = \kappa, \kappa + 1, \dots$ , so  $\delta(x) \geq 0$  for  $x = \kappa, \kappa + 1, \dots$ . The expected profit function is increasing in  $x$ . A local maximum point is set as large as possible.

Recall that  $\bar{F}(\kappa - 1; x, \theta_2)$  is increasing in  $x$ , so  $\zeta(x)$  is decreasing in  $x$ . If  $\bar{F}(\kappa - 1; \kappa, \theta_2) < \alpha_2 / (h\theta_2) < 1$  then  $\zeta(0) > 0$  and  $\lim_{x \rightarrow \infty} \zeta(x) < 0$ , i.e., there exists a local maximum point  $x''$  such that  $\zeta(x) > 0, \forall x < x''$  and  $\zeta(x) < 0, \forall x \geq x''$ . So,  $\delta(x) > 0, \forall x < x''$  and  $\delta(x) < 0, \forall x \geq x''$ , i.e., a local maximum point is given by

$$x'' = \arg \min \{x \in \{\kappa, \kappa + 1, \dots\} : \bar{F}(\kappa - 1; x, \theta_2) > \frac{\alpha_2}{h\theta_2}\}. \tag{17}$$

□

**Proof of corollary 1.** Note that a function  $P(D_1 > \kappa - x - 1)$  is increasing in  $x$ . The directional change of  $x'$  with respect to  $\tau = \alpha_2 / \alpha_1$  is obvious in equation (16). Let  $\tau$  and  $\hat{\tau}$  be a function that has a local maximum point  $x'$  and  $\hat{x}'$  respectively. Since,  $\tau < \hat{\tau}$ , then  $x' \leq \hat{x}'$ .

Note that  $P(D_1 \leq \kappa - x - 1)$  is increasing in  $\kappa$ . Consider the directional change of  $x'$  with respect to  $\kappa$ . Let  $\kappa$  and  $\hat{\kappa}$  be capacity such that  $\kappa < \hat{\kappa}$ ,  $y = \kappa - x - 1$  and  $\hat{y} = \hat{\kappa} - x - 1$ .

$$P(D_1 \leq y) \leq P(D_1 \leq \hat{y})$$

$$\begin{aligned} \kappa - F_{D_1}^{-1}(1 - \tau; \kappa) &\leq \kappa - F_{D_1}^{-1}(1 - \tau; \hat{\kappa}) \\ &\leq \hat{\kappa} - F_{D_1}^{-1}(1 - \tau; \hat{\kappa}) \end{aligned}$$

Thus,  $x' \leq \hat{x}'$ .

Consider the directional change of  $x'$  with respect to  $D_1$ . Assume that  $D_1 \leq_{st} \hat{D}_1$ , i.e.,  $D_1$  smaller than  $\hat{D}_1$  with respect to usual stochastic order (Muller and Stoyan, 2002).

Let  $F$  and  $G$  be distribution functions of  $D_1$  and  $\hat{D}_1$ , and  $\bar{F}(t) = P(D_1 > t)$  and  $\bar{G}(t) = P(\hat{D}_1 > t)$ . So,  $\bar{F}(\kappa - x - 1) \leq \bar{G}(\kappa - x - 1)$  for all  $x \in \{0, 1, \dots, \kappa - 2\}$ . Let  $x'$  and  $\hat{x}'$  be local maximum points of  $D_1$  and  $\hat{D}_1$  respectively. Since,  $\bar{F}$  and  $\bar{G}$  are increasing in  $x$ . Thus  $x' \geq \hat{x}'$ . □

**Proof of Corollary 2.** Note that a function  $\bar{F}(\kappa - 1; x, \theta_2)$  is increasing in  $x$ . The directional change of  $x''$  with respect to  $\alpha_2 / (h\theta_2)$  is obvious in (17). Let  $\xi = \alpha_2 / (h\theta_2)$  and let  $x''$  be a local maximum point associate with  $\xi$ . Similarly, let  $\hat{\xi} = \hat{\alpha}_2 / (h\hat{\theta}_2)$  and let  $\hat{x}''$  be a local maximum point associate with  $\hat{\xi}$ . Since,  $\xi < \hat{\xi}$ , then  $x'' \leq \hat{x}''$ .

Recall that  $\bar{F}(\kappa - 1; x, \theta_2)$  is decreasing in  $\kappa$ . Consider the directional change of  $x''$  with respect to  $\kappa$ . Let  $\kappa$  and  $\hat{\kappa}$  be capacity such that  $\kappa < \hat{\kappa}$ ,

$$\bar{F}(\kappa - 1; x, \theta_2) \geq \bar{F}(\hat{\kappa} - 1; x, \theta_2).$$

Thus,  $x'' \leq \hat{x}''$ . □