

APPENDICES

APPENDIX A

A.1 Derivations of proposition 3.1: $x_1(P_1)$ as $P_1 \in (0, \omega_0]$

Under the assumptions of $\beta(1+a) \leq 1$, $\beta = 1/(1+r)$ and $b = 1-b$, it can be obtained according to (3.29) that ,

$$\begin{aligned} A &= \frac{\beta(1-a)(1-b)(a+\beta(a+1))}{a} \cdot P_1 \\ B &= -w(1+\beta)(1-b)(a+1+\beta(1-a)P_1) \\ C &= w^2a(1-b)(1+\beta) \end{aligned}$$

Rearranging the term and obtaining that,

$$\begin{aligned} \frac{C}{A} &= \frac{w^2a^2(1+\beta)}{\beta(1-a)(a+\beta(a+1))} \cdot \frac{1}{P_1} \\ \frac{B}{A} &= -\frac{wa(1+\beta)}{2\beta(1-a)(a+\beta(a+1))} \left(\frac{a+1}{P_1} + \beta(1-a) \right) \end{aligned}$$

Therefore, the solution of x becomes

$$\begin{aligned} x &= -\frac{B}{2A} \pm \sqrt{\left(\frac{B}{2A}\right)^2 - \frac{C}{A}} \\ x &= \frac{wa(1+\beta)}{2\beta(1-a)(a+\beta(a+1))} \cdot \left(\frac{a+1}{P_1} + \beta(1-a) \right) \\ &\quad \pm \sqrt{\frac{w^2a^2(1+\beta)^2}{4\beta^2(1-a)^2(a+\beta(a+1))^2} \cdot \left(\frac{a+1}{P_1} + \beta(1-a) \right)^2 - \frac{w^2a^2(1+\beta)}{\beta(1-a)(a+\beta(a+1))} \cdot \frac{1}{P_1}} \end{aligned}$$

$$x = \frac{wa(1+\beta)}{2\beta(1-a)(a+\beta(a+1))} \cdot \left(\frac{a+1}{P_1} + \beta(1-a) \pm \sqrt{\left(\frac{a+1}{P_1} + \beta(1-a) \right)^2 - \frac{4\beta(1-a)((a+\beta(a+1)))}{(1+\beta)} \cdot \frac{1}{P_1}} \right)$$

Let $\gamma_1 = a+1$, $\gamma_2 = \beta(1-a)$, and $\gamma_3 = (4\beta(1-a)(a+\beta(1+1)))/(1+\beta)$, then

$$x = \frac{wa(1+\beta)}{2\beta(1-a)(a+\beta(a+1))} \cdot \left(\frac{a+1}{P_1} + \beta(1-a) \pm \sqrt{\left(\frac{a+1}{P_1} + \beta(1-a) \right)^2 - \frac{4\beta(1-a)((a+\beta(a+1)))}{(1+\beta)} \cdot \frac{1}{P_1}} \right)$$

$$x = \frac{2wa}{\gamma_3} \left(\frac{\gamma_1}{P_1} + \gamma_2 \pm \sqrt{\left(\frac{\gamma_1}{P_1} + \gamma_2 \right)^2 - \frac{\gamma_3}{P_1}} \right)$$

Next it will be proved that the case of single or complex solution never occurs. For the case of single solution, it would be disproved by supposing for a while that the term $\left(\frac{\gamma_1}{P_1} + \gamma_2 \right)^2 - \frac{\gamma_3}{P_1}$ equals to zero and then the contradiction will be derived:

By Supposing that

$$\left(\frac{\gamma_1}{P_1} + \gamma_2 \right)^2 - \frac{\gamma_3}{P_1} = 0,$$

then it can be obtain that

$$P_1 = \frac{-2\gamma_1\gamma_2 + \gamma_3 \pm \sqrt{(2\gamma_1\gamma_2 - \gamma_3)^2 - 4\gamma_1^2\gamma_2^2}}{2\gamma_2^2}$$

This means that P_1 will have real solutions if

$$(2\gamma_1\gamma_2 - \gamma_3)^2 - 4\gamma_1^2\gamma_2^2 \geq 0$$

By plugging $\gamma_1 = a+1$, $\gamma_2 = \beta(1-a)$, and $\gamma_3 = (4\beta(1-a)(a+\beta(1+1)))/(1+\beta)$ in the above equations, it implies $-1 \geq 0$ which is the contradiction. Therefore, the case of single solution can be rules out and it can be obtained by products that the term of

$$\gamma_1\gamma_2 - \frac{\gamma_3}{2} \geq 0$$

is always true and it will be used in later proves.

Next, to prove the case of two different solutions, it would be shown that

$\left(\frac{\gamma_1}{P_1} + \gamma_2\right)^2 - \frac{\gamma_3}{P_1} > 0$ for any $P_1 > 0$. To begin with, since it is always true that

$$\left(\frac{a+1}{P_1} - \beta(1-a)\right)^2 \geq 0,$$

$$\text{then } \left(\frac{a+1}{P_1} + \beta(1-a)\right)^2 \geq 4(a+1)\beta(1-a) \cdot \frac{1}{P_1}$$

According to the assumption of $\beta(1+a) \leq 1$,

$$\left(\frac{a+1}{P_1} + \beta(1-a)\right)^2 \geq 4(a+\beta(1+a))\beta(1-a) \cdot \frac{1}{P_1}$$

$$\text{With } (1+\beta) > 0, \quad \left(\frac{a+1}{P_1} + \beta(1-a)\right)^2 \geq \frac{4}{(1+\beta)}(a+\beta(1+a))\beta(1-a) \cdot \frac{1}{P_1}$$

$$\text{That is, } \quad \left(\frac{\gamma_1}{P_1} + \gamma_2\right)^2 \geq \frac{\gamma_3}{P_1}$$

$$\left(\frac{\gamma_1}{P_1} + \gamma_2\right)^2 - \frac{\gamma_3}{P_1} \geq 0$$

Since it is already shown before that the case $\left(\frac{\gamma_1}{P_1} + \gamma_2\right)^2 - \frac{\gamma_3}{P_1} = 0$ never occurs,

then

$$\left(\frac{\gamma_1}{P_1} + \gamma_2\right)^2 - \frac{\gamma_3}{P_1} > 0$$

The proof is done. Therefore x has two different solutions, which are

$$x(P_1) = \frac{2wa}{\gamma_3} \left(\frac{\gamma_1}{P_1} + \gamma_2 \pm \sqrt{\left(\frac{\gamma_1}{P_1} + \gamma_2\right)^2 - \frac{\gamma_3}{P_1}} \right).$$

The above function $x(P_1)$ is differentiable on R^{++} . For convenience, x^{high} and x^{low} are defined to stand for the solution of higher value and that of lower one respectively. Next, it will be proved that both x^{high} and x^{low} are monotonic decreasing in P_1 .

For x^{high} , taking first order derivatives of x^{high} with respect to P_1 and obtaining

$$\frac{\partial x^{high}}{\partial P_1} = \frac{2wa}{\gamma_3} \left(-\frac{\gamma_1}{P_1^2} - \frac{\left(\frac{\gamma_1^2}{P_1^3} + \left(\gamma_1\gamma_2 - \frac{\gamma_3}{2}\right) \cdot \frac{1}{P_1^2}\right)}{\sqrt{\left(\frac{\gamma_1}{P_1} + \gamma_2\right)^2 - \frac{\gamma_3}{P_1}}} \right) \text{ for } P_1 > 0$$

Since it is previously derived that $\gamma_1\gamma_2 - \frac{\gamma_3}{2} > 0$, it can be obviously seen that the function $\frac{\partial x^{high}}{\partial P_1}$ is always less than zero. This implies that x^{high} is monotonic decreasing in P_1 .

For x^{low} , taking first order derivatives of x^{low} with respect to P_1 and obtaining

$$\frac{\partial x^{low}}{\partial P_1} = \frac{2wa}{\gamma_3} \left(-\frac{\gamma_1}{P_1^2} + \frac{\left(\frac{\gamma_1^2}{P_1^3} + \left(\gamma_1 \gamma_2 - \frac{\gamma_3}{2} \right) \cdot \frac{1}{P_1^2} \right)}{\sqrt{\left(\frac{\gamma_1}{P_1} + \gamma_2 \right)^2 - \frac{\gamma_3}{P_1}}} \right) \text{ for } P_1 > 0$$

For $P_1 > 0$, it is always true that

$$\gamma_1 \gamma_2 - \frac{\gamma_3}{2} < \gamma_1 \gamma_2 \quad (\text{since } \gamma_1, \gamma_2, \gamma_3 > 0)$$

$$\gamma_1^4 + 2\gamma_1^2 \left(\gamma_1 \gamma_2 - \frac{\gamma_3}{2} \right) P_1 + \left(\gamma_1 \gamma_2 - \frac{\gamma_3}{2} \right)^2 P_1^2 < \gamma_1^4 + 2\gamma_1^2 \left(\gamma_1 \gamma_2 - \frac{\gamma_3}{2} \right) P_1 + (\gamma_1 \gamma_2)^2 P_1^2$$

$$\gamma_1^2 + \left(\gamma_1 \gamma_2 - \frac{\gamma_3}{2} \right) P_1 < \gamma_1 \sqrt{\gamma_1^2 + 2 \left(\gamma_1 \gamma_2 - \frac{\gamma_3}{2} \right) P_1 + \gamma_2^2 P_1^2} \quad (\text{since } \gamma_1 \gamma_2 - \frac{\gamma_3}{2} > 0)$$

$$\frac{\gamma_1^2}{P_1} + \left(\gamma_1 \gamma_2 - \frac{\gamma_3}{2} \right) < \gamma_1 \sqrt{\left(\frac{\gamma_1}{P_1} + \gamma_2 \right)^2 - \frac{\gamma_3}{P_1}}$$

$$-\frac{\gamma_1}{P_1^2} + \frac{\left(\frac{\gamma_1^2}{P_1^3} + \left(\gamma_1 \gamma_2 - \frac{\gamma_3}{2} \right) \cdot \frac{1}{P_1^2} \right)}{\sqrt{\left(\frac{\gamma_1}{P_1} + \gamma_2 \right)^2 - \frac{\gamma_3}{P_1}}} < 0.$$

Therefore, $\frac{\partial x^{low}}{\partial P_1}$ is always negative; that is, x^{low} is monotonic decreasing in P_1 .

Due to being monotonic decrease in P_1 of x^{high} and x^{low} , it is implied that the maximum value of x^{high} occurs as P_1 approaches 0^+ and its value is ∞ ¹. Moreover, the minimum value of x^{high} is $4wa(\gamma_2)/(\gamma_3)$ as $P_1 \rightarrow \infty$ ².

The maximum value of x^{low} occurs as x^{low} approaches 0^+ ³ and its value is wa/γ_1 . Moreover, the minimum value of x^{low} is 0 as $P_1 \rightarrow \infty$ ⁴.

Therefore, the amount of purchased market inputs

$$x_1 \in \{x^{low}, x^{high}\} = (0, wa/(a+1)) \cup ((wa(1+\beta))/(a+\beta(a+1)), \infty)$$

For the range of $x_1(P_1)$, it must satisfy the following

1) The time constraint of the first period, which is

$$0 \leq t_{R_1} \leq 1$$

$$0 \leq 1 + \frac{1}{1+r} - \frac{1}{w} \left(1 + \frac{\beta(a(1-b)+b)}{a(1-b)} \right) \cdot x_1 \leq 1 \quad (\text{from 3.30})$$

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$$\lim_{P_1 \rightarrow 0} x^{high} = \lim_{P_1 \rightarrow 0} \frac{2wa}{\gamma_3} \left(\frac{\gamma_1}{P_1} + \gamma_2 + \sqrt{\left(\frac{\gamma_1}{P_1} + \gamma_2 \right)^2 - \frac{\gamma_3}{P_1}} \right) = \lim_{P_1 \rightarrow 0} \frac{2wa}{\gamma_3} \left(\frac{\gamma_3}{\gamma_1 + \gamma_2 P_1 - \sqrt{(\gamma_1 + \gamma_2 P_1)^2 - \gamma_3 P_1}} \right) = \infty$$

$$2 \quad \lim_{P_1 \rightarrow \infty} x^{high} = \lim_{P_1 \rightarrow \infty} \frac{2wa}{\gamma_3} \left(\frac{\gamma_1}{P_1} + \gamma_2 + \sqrt{\left(\frac{\gamma_1}{P_1} + \gamma_2 \right)^2 - \frac{\gamma_3}{P_1}} \right) = \lim_{P_1 \rightarrow \infty} \frac{2wa}{\gamma_3} (2\gamma_2) = \frac{4wa\gamma_2}{\gamma_3}$$

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$$\lim_{P_1 \rightarrow 0} x^{low} = \lim_{P_1 \rightarrow 0} \frac{2wa}{\gamma_3} \left(\frac{\gamma_1}{P_1} + \gamma_2 - \sqrt{\left(\frac{\gamma_1}{P_1} + \gamma_2 \right)^2 - \frac{\gamma_3}{P_1}} \right) = \lim_{P_1 \rightarrow 0} \frac{2wa}{\gamma_3} \left(\frac{\gamma_3}{\gamma_1 + \gamma_2 P_1 + \sqrt{(\gamma_1 + \gamma_2 P_1)^2 - \gamma_3 P_1}} \right) = \frac{wa}{\gamma_1}$$

$$4 \quad \lim_{P_1 \rightarrow \infty} x^{low} = \lim_{P_1 \rightarrow \infty} \frac{2wa}{\gamma_3} \left(\frac{\gamma_1}{P_1} + \gamma_2 - \sqrt{\left(\frac{\gamma_1}{P_1} + \gamma_2 \right)^2 - \frac{\gamma_3}{P_1}} \right) = \lim_{P_1 \rightarrow \infty} \frac{2wa}{\gamma_3} (0) = 0$$

$$0 \leq 1 + \beta - \frac{1}{w} \left(1 + \frac{\beta(a+1)}{a} \right) \cdot x_1 \leq 1 \quad (\beta = 1/(1+r),$$

$b = 1 - b)$

$$\frac{\beta w}{\left(1 + \frac{\beta(a+1)}{a} \right)} \leq x_1 \leq \frac{(1+\beta)w}{\left(1 + \frac{\beta(a+1)}{a} \right)}$$

$$\frac{\beta a w}{(a + \beta(a+1))} \leq x_1 \leq \frac{(1+\beta)aw}{(a + \beta(a+1))}$$

2) The time constraint of the second period, which is

$$0 \leq t_{R_2} \leq 1$$

$$0 \leq \frac{\chi - 1}{\chi} \cdot \frac{\beta(a(1-b) + b)}{a(1-b)} \cdot x_1 \leq 1 \quad (\text{from 3.32})$$

$$0 \leq x_1 \leq \frac{a}{\beta} \quad (\chi = 1 + (b/(a(1-b))), b = 1 - b)$$

3) the positive domain of logarithmic function

$$w + (1+r)b_2 > 0 \quad (\text{from 3.23})$$

$$w + (1+r) \left(\frac{\beta(a(1-b) + b)}{a(1-b)} \cdot x_1 - \frac{w}{1+r} \right) > 0$$

$$x_1 > 0$$

4) the solution of equation (3.29)

$$x_1 \in \{x^{low}, x^{high}\} = (0, wa/(a+1)) \cup ((wa(1+\beta))/(a + \beta(a+1)), \infty)$$

Therefore, it can be obtained that

$$\{x_1(P_1)\} = \{x^{low}, x^{high}\} \cap \left[\frac{\beta aw}{(a + \beta(a+1))}, \frac{(1+\beta)aw}{(a + \beta(a+1))} \right] \cap \left[0, \frac{a}{\beta} \right]$$

Since $\frac{\beta aw}{(a + \beta(a+1))} \leq \frac{wa}{a+1} \leq \frac{(1+\beta)aw}{(a + \beta(a+1))}$ ⁵ and $\frac{wa}{a+1} \leq \frac{a}{\beta}$ is claimed⁶ to be true

for any values of $0 < a < 1, 0 < \beta < 1$. Accordingly, $x_1(P_1)$ can be written as

$$x_1(P_1) \in \left[\frac{\beta aw}{(a + \beta(a+1))}, \frac{wa}{(a+1)} \right].$$

Therefore, the solution $x_1(P_1)$ is unique and

$$x_1(P_1) = x^{low}(P_1) = \frac{2wa}{\gamma_3} \left(\frac{\gamma_1}{P_1} + \gamma_2 - \sqrt{\left(\frac{\gamma_1}{P_1} + \gamma_2 \right)^2 - \frac{\gamma_3}{P_1}} \right).$$

⁵ It has to be proved that $\frac{\beta aw}{(a + \beta(a+1))} \leq \frac{wa}{a+1}$ and that $\frac{wa}{a+1} \leq \frac{(1+\beta)aw}{(a + \beta(a+1))}$.

For the first one, it can be rearranged the term to be $\beta(a+1) \leq a + \beta(a+1)$ which always true. Next, for the second one, it can be obviously seen that

$$\begin{aligned} a + \beta(a+1) &\leq a+1 + \beta(a+1) \text{ therefore,} \\ a + \beta(a+1) &\leq (a+1)(1+\beta) \\ 1/(a+1) &\leq (1+\beta)/(a + \beta(a+1)) \\ wa/(a+1) &\leq (1+\beta)wa/(a + \beta(a+1)). \end{aligned}$$

⁶ Its proof is by letting w equal to the maximum value, which is 1, and then show that $\beta a \leq a(a+1)$.

From the assumption of

$$\beta \leq 1.$$

Therefore,

$$\begin{aligned} \beta a &\leq a \\ \beta a &\leq a(a+1). \end{aligned}$$

According to equation (3.34), it can be solved that 1)

$$P_1 = \omega_0 = \left(2\gamma_1\gamma_2\left(\frac{(1-\beta)}{(1+\beta)}\right) + \gamma_3\right) / \gamma_2^2 \left(1 - \left(\frac{(1-\beta)}{(1+\beta)}\right)^2\right) \text{ as}$$

$$x_1(P_1) = \beta aw / (a + \beta(a+1)) \text{ and that 2) } P_1 = 0 \text{ as } x_1(P_1) = aw / (a+1).$$

Since the function $\frac{2wa}{\gamma_3} \left(\frac{\gamma_1}{P_1} + \gamma_2 - \sqrt{\left(\frac{\gamma_1}{P_1} + \gamma_2\right)^2 - \frac{\gamma_3}{P_1}} \right)$ is monotonic decreasing and

differentiable in R^{++} , the domain of the function

$$x_1(P_1) = \frac{2wa}{\gamma_3} \left(\frac{\gamma_1}{P_1} + \gamma_2 - \sqrt{\left(\frac{\gamma_1}{P_1} + \gamma_2\right)^2 - \frac{\gamma_3}{P_1}} \right) \text{ according to equation (3.34) is } (0, \omega_0].$$

A.2 The proof of $x_1(P_1)$ as $P_1 > \omega_0$

For $P_1 > \omega_0$, the religion time of period 1 always equals to one. The following are the objective function and the constraint of period 1 as $t_{R_1} = 1$, which are based on equations (3.23)-(3.25). In period 1 problem, an agent has to maximize

$$V(P_1) = \max_{x_1, P_2, b_2} a(1-b) \log x_1 + (1-a)(1-b) \log P_1 + \beta V_0(P_2, b_2)$$

$$\text{where } V_0(P_2, b_2) = \alpha + (a(1-b) + b) \log(w + (1+r)b_2) + (1-a)(1-b)P_1$$

Subject to 1) the first-period budget constraint

$$x_1 = -b_2$$

and 2) the constraint of non-negative value of $t_{R,2}$ and x_2 according to the equation(3.22) which implies as $b_2 \geq -\beta w$ or

$$x_1 \leq \beta w$$

The above is equivalent to the following problem of unconstrained maximization:

$$V(P_1) = \max_{x_1, P_2, b_2} a(1-b)\log x_1 + (1-a)(1-b)\log P_1 + \beta V_0(P_2, b_2)$$

$$\text{where } V_0(P_2, b_2) = \alpha + (a(1-b) + b)\log(w - (1+r)x_1) + (1-a)(1-b)\log P_2$$

FOC with respect to x_1 ,

$$\frac{a(1-b)}{x_1} = \frac{\beta(1+r)(a(1-b) + b)}{(w - (1+r)x_1)}$$

It can be obtained that $x_1(P_1) = \beta aw / (a + \beta(a+1))$ which is obviously lower than βw . Therefore, $x_1(P_1) = \beta aw / (a + \beta(a+1))$ is the amount of x_1 when $P_1 > \omega_0$.