THESIS APPROVAL

GRADUATE SCHOOL, KASETSART UNIVERSITY

Doctor of Philosophy (Statistics)

DEGREE

Statistics — **FIELD**

Statistics — **DEPARTMENT**

TITLE: A Mixture of Exponentiated Inverted Weibull Distribution

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THESIS

A MIXTURE OF EXPONENTIATED INVERTED WEIBULL DISTRIBUTION

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A Thesis Submitted in Partial Fulfillment of
the Requirements for the Degree of
Doctor of Philosophy (Statistics)
Graduate School, Kasetsart University
2014

Palakorn Seenoi 2014: A Mixture of Exponentiated Inverted Weibull Distribution. Doctor of Philosophy (Statistics), Major Field: Statistics, Department of Statistics. Thesis Advisor: Assistant Professor Winai Bodhisuwan, Ph.D. 168 pages.

The Weibull distribution is the most widely used distribution to analyze the lifetime data. It provides vast impact of reliability and quality control. In some cases, the Weibull distribution may not be fit to real data corrected from applications. However, there are some distributions, e.g., inverse Gaussian, gamma, inverse gamma, lognormal that can be alternatively applied. In this thesis, the length-biased exponentiated inverted Weibull (LBEIW) and the mixture exponentiated inverted Weibull (MEIW) distributions are introduced. Important probabilistic properties will be discussed, i.e., probability density function, cumulative distribution function and survival function and hazard rate. Some structural properties of these distributions, such as moment about the origin, mean, variance, coefficient of skewness and coefficient of kurtosis are presented. The LBEIW distribution includes the length-biased inverted Weibull (LBIW) distribution as a sub-model. The MEIW distribution also includes the LBEIW, LBIW and EIW distributions as sub-models. Parameter estimation is derived by the maximum likelihood estimation and Bayesian approach. Monte Carlo simulation study is also used to demonstrate the parameter estimation based on some specified parameters. Some real data sets will be applied by using LBEIW and MEIW distributions.

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ACKNOWLEDGEMENTS

This thesis would never have been successfully completed without the support and encouragement of many wonderful people and organizations.

First, I owe a debt of gratitude to Assistant Professor Dr. Winai Bodhisuwan my thesis advisor, who worked very hard for his helpful supervision during the preparation and completion of this thesis. In addition, I would also like to thank Dr. Thidaporn Supapakorn, my thesis co-advisor, and Dr. Chookait Pudprommarat for contributing their time and providing me with helpful suggestion. I would sincerely like to thank Associate Professor Prasit Payakkapong and Assistant Professor Dr. Uraiwan Jaroengeratikun my committees for their valuable comments and suggestion.

My special thanks are extended to the Faculty of Science Kasetsart University Postgraduate Studentship, Kasetsart University and Department of Statistics, Faculty of Science, Khon Kaen University for their financial support during my study.

Finally, I am grateful to my family, my teachers and my friends for their encouragement and admirable support during the period of my study.

Palakorn Seenoi

November 2014

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LIST OF ABBREVIATIONS

IW	=	inverse Weibull
cdf	=	cumulative distribution function
EIW	=	exponentiated inverted Weibull
LBEIW	=	length-biased exponentiated inverted Weibull
MEIW	=	mixture exponentiated inverted Weibull
MLE	=	maximum likelihood estimation
pdf	=	probability density function
iid	=	independent and identically distributed
BE	=	Bayesian estimation
MCMC	=	Markov chain Monte Carlo
MSE	=	mean square error
AD	=	Anderson-Darling
AIC	=	Akaike information criterion
LBIW	=	length-biased inverted Weibull
ecdf	=	empirical cumulative distribution function

A MIXTURE OF EXPONENTIATED INVERTED WEIBULL DISTRIBUTION

INTRODUCTION

The Weibull distribution was introduced by Wallodi Weibull, Swedish scientist, in 1951 (Weibull, 1951). It is the most widely used distribution to analyze the lifetime data. It provides vast impact of reliability and quality control, such as ball bearings, automobile components, concrete bridges, demography, actuarial study, and electrical insulation. It is also used in biological and medical applications, for example, in the studies of the occurrence of tumors in human populations and laboratory animals. Therefore, applications to the lifetime or durability of manufactured items are common.

The Weibull distribution is fairly flexible in providing a good description of many types of lifetime data (Lawless, 2003) such as complete data, censored data. In addition, censored data include right, left, or interval censoring, type I or type II censoring, and single or multiple censoring (Murthy *et al.*, 2004).

In some cases, the Weibull distribution may not be fit to some datasets, however, alternatively, the inverse Gaussian, gamma, inverse gamma, lognormal distributions can be a good representative. Many studies proposed various lifetime distributions. Keller and Kamath studied the shapes of the density and failure rate functions for the basic inverse model in 1982 (Keller and Kamath, 1982). Also, the inverse Weibull (IW) distribution was proposed and applied to a wide range of situations including applications in medicine, reliability and ecology. Many researchers applied IW distribution to their works. For example, Erto (1989) has shown that the IW distribution gave a good fit to life testing data. Furthermore, Khan *et al.* (2008) explained the flexibility of the three-parameter IW distribution and its interested properties.

An exponentiated distribution is a generalization of the distribution through adding a new shape parameter $\lambda \in \Re^+$, i.e., $0 < \lambda < \infty$, by exponentiating the cumulative distribution function (cdf) F in the form F^λ. The exponentiated Weibull distribution was proposed by Mudholkar and Srivastava (1993) as an extension of the Weibull family obtained by adding the second shape parameter. Mudholkar *et al.* (1996) applied the exponentiated Weibull distribution to serve survival data and showed that the hazard rate function are increasing, decreasing, bathtub shape, and unimodal. The exponentiated exponential distribution which is proposed by Gupta and Kundu (1999), is a special case of the exponentiated Weibull distribution. Flaih *et al.* (2012) extended the IW distribution to the exponentiated inverted Weibull (EIW) distribution by adding another shape parameter. Based on likelihood ratio test, his study suggested that the EIW distribution can provide a better fit to the real dataset than the IW distribution.

In this research, we propose the new distributions for lifetime data, which are the length-biased exponentiated inverted Weibull (LBEIW) and the mixture exponentiated inverted Weibull (MEIW) distributions. We expect that the LBEIW is more flexible alternative than the EIW and Weibull distributions, and the MEIW distribution is more flexible alternative than the LBEIW, EIW and Weibull distributions for applications in lifetime data analysis. The parameters of the LBEIW and MEIW distributions are estimated by the maximum likelihood estimation (MLE) and Bayesian estimation methods.

OBJECTIVES

1. To propose the LBEIW distribution and the MEIW distribution.

2. To derive the parameter estimation of the LBEIW distribution and the MEIW distribution.

3. To compare the efficiencies of two proposed distributions with other distributions for applications in lifetime data analysis.

LITERATURE REVIEW

This chapter presents the background necessary for understanding the thesis including the statistical model and the statistical method.

1. Introduction and basic utility notions

Suppose the distribution of a continuous random variable T has the parameter set $\theta = \{\theta_1, \theta_2, \ldots, \theta_p\}$. Let the probability density function (pdf) of T be given by $f(t|\theta)$.

The cdf of T is defined to be

$$F(t|\theta) = \int_0^t f(x|\theta)\,dx. \tag{1}$$

The survival function, also known as reliability function, is defined as a probability that the individual will be functioning based on the specified environment survives and time. In the same meaning, the survival function is used in a broader range of applications, including human mortality while the reliability function is common in engineering. The survival function, denoted as $S(t|\theta)$, is defined as

$$S(t|\theta) = P(T > t) = 1 - F(t|\theta). \tag{2}$$

The hazard rate of T can be interpreted as the instantaneous failure rate or the conditional probability density of failure at time t, given that the unit has survived until time t (Finkelstein, 2002). The hazard rate $h(t|\theta)$ is

$$h(t|\theta) = \lim_{\Delta t \to 0} \frac{P(t \le T \le t + \Delta t)}{\Delta t} = \frac{f(t|\theta)}{1 - F(t|\theta)}. \tag{3}$$

Some relationships can be presented as follows:

The pdf can be derived from $S(t|\theta)$ as

$$f(t|\theta) = \frac{d}{dt}[1 - S(t|\theta)] = -S'(t|\theta). \tag{4}$$

The hazard rate $h(t|\theta)$ is also expressed by

$$h(t|\theta) = \frac{f(t|\theta)}{S(t|\theta)}, \tag{5}$$

hence,

$$h(t|\theta) = -\frac{-S'(t|\theta)}{S(t|\theta)} = -\frac{d}{dt}\log S(t|\theta). \tag{6}$$

The survival function can be represented as

$$S(t|\theta) = \exp\left[-\int_0^t h(x|\theta)\,dx\right]. \tag{7}$$

1.1 Definition of some statistical properties

Some statistical properties of a continuous random variable T can be defined as follows:

The k^{th} moment about the origin

$$E(T^k) = \int_0^\infty t^k f(t|\theta)\,dt. \tag{8}$$

The 1^{st} moment about the origin

$$E(T) = \int_0^\infty t f(t|\theta)\,dt. \tag{9}$$

The 2nd moment about the origin

$$E(T^2) = \int_0^\infty t^2 f(t|\theta)\,dt. \tag{10}$$

The 3rd moment about the origin

$$E(T^3) = \int_0^\infty t^3 f(t|\theta)\,dt. \tag{11}$$

The 4th moment about the origin

$$E(T^4) = \int_0^\infty t^4 f(t|\theta)\,dt. \tag{12}$$

The variance

$$Var(T) = E(T^2) - [E(T)]^2. \tag{13}$$

The coefficient of skewness

$$CS(T) = \frac{E[T - E(T)]^3}{[Var(T)]^{\frac{3}{2}}}. \tag{14}$$

The coefficient of kurtosis

$$CK(T) = \frac{E[T - E(T)]^4}{[Var(T)]^2} - 3. \tag{15}$$

1.2 Useful functions

The gamma function is given by

$$\Gamma(t) = \int_0^\infty x^{t-1} \exp(-x)\, dx; \quad t > 0, \tag{16}$$

with an important property

$$\Gamma(t+1) = t\,\Gamma(t). \tag{17}$$

We also include the lower incomplete gamma function and the upper incomplete gamma function are

$$\gamma(\theta, t) = \int_0^t x^{\theta-1} \exp(-x)\, dx, \tag{18}$$

and

$$\Gamma(\theta, t) = \int_t^\infty x^{\theta-1} \exp(-x)\, dx, \tag{19}$$

respectively.

Consequently, the digamma function is defined by

$$\psi(t) = \frac{d}{dt} \log\Gamma(t) = \frac{\Gamma'(t)}{\Gamma(t)}, \tag{20}$$

which is the logarithmic derivative of the gamma function.

In addition, the beta function and the incomplete beta function are

$$B(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}; \quad a > 0, b > 0, \tag{21}$$

and

$$B_t(a,b) = \int_0^t x^{a-1}(1-x)^{b-1} dx, \tag{22}$$

respectively.

2. The Weibull distribution

In probability theory and statistics, the Weibull distribution is a continuous probability distribution. It is named after Waloddi Weibull, who described it in detail in 1951, although it was first identified by Frechet (1927) and first applied by Rosin and Rammler (1933) to describe a particle size distribution.

2.1 Three-parameter Weibull distribution

A random variable T be a three-parameter Weibull distribution with parameter τ, α and β if its pdf is given by

$$f(t|\tau,\alpha,\beta) = \frac{\beta}{\alpha}\left(\frac{t-\tau}{\alpha}\right)^{\beta-1} \exp\left\{-\left(\frac{t-\tau}{\alpha}\right)^{\beta}\right\}; \quad t \geq \tau. \tag{23}$$

This is the most general form of the Weibull distribution (Rinne, 2008). The fact that the distribution of T given by Eq.(23) is noted as $T \sim \text{Weibull}(\tau,\alpha,\beta)$ for short.

The first parameter τ is defined on $\Re$, i.e., $-\infty < \tau < \infty$, and is measured in the same unit as the realization t of T, which is a normal a unit of time (second, minute, hour, day, month). In the context of T being a lifetime τ is called delay, guarantee time, minimum life, safe life and shelf age, more generally it is termed origin or threshold. Therefore, the domain of support for $f(t|\tau,\alpha,\beta)$ is $t \geq \tau$. For t being a duration which normally cannot be negative and τ being the minimum duration, the domain of τ will not be $\Re$ but be the smaller interval $[0,\infty)$. From a statistical point of view, τ is a location parameter. Changing τ when the other parameters are held constant will result in a parallel movement of the density curve over the abscissa. According to Figure 1 the three-parameter Weibull densities, $f(t|\tau,\alpha,\beta)$, is depicted for $\tau = 0$, 0.5 and 1. Enlarging τ causes a movement of the density to the right (to the left) so that τ is called shift parameter or translation parameter.

The second parameter α, called a scale parameter, has the domain $(0,\infty)$

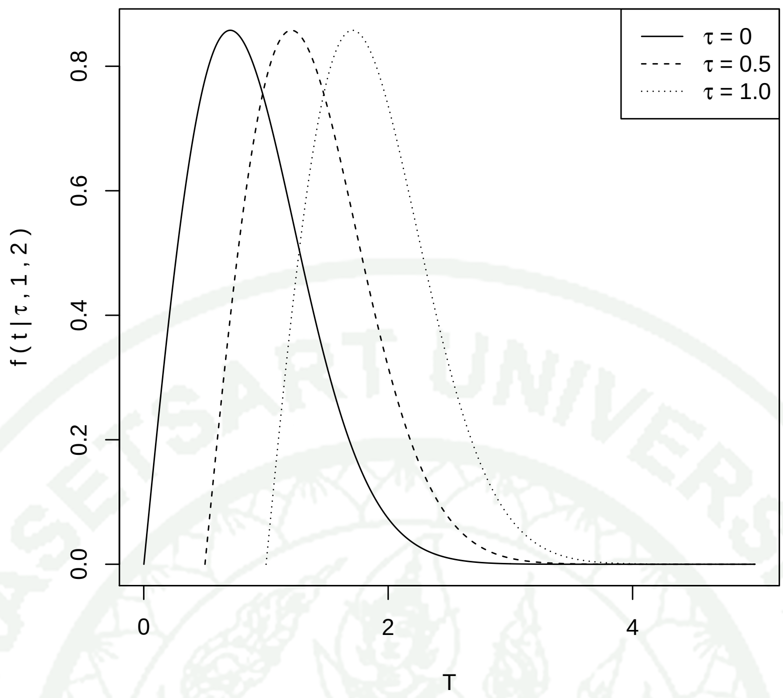

Figure 1 The three-patameter Weibull densities with specified values of location parameter

which is measured in the same unit as t. When α is changed and the values of τ as well as β are still constant, this situation will change density at t in the direction of the ordinate (see Figure 2 where $f(t|\tau,\alpha,\beta)$ is depicted for $\alpha = 0.5$, 1 and 2). Enlarging α will cause a compression or reduction of the density and reducing α will magnify or stretch the density while the scale on the abscissa goes into the opposite direction. This means that a growing (shrinking) α will cause the variation of T to become large (small).

The third parameter β has domain $(0,\infty)$ and bears no dimension. It is called Weibull-slope because it gives the slope of the pdf. From the statistical point of view, β is a shape parameter. When β is changed, the form of pdf is vary as shown in Figure 3. There $f(t|\tau,\alpha,\beta)$ is depicted for $\beta = 0.8$, 1, 2, 3.6 and 6.5. The shape parameter is responsible for the appearance of a Weibull density. For $\beta < 1$, the exponential part of the density dominates, and the curve is J-shaped. When $\beta > 1$, the effect of the polynomial part of the density becomes more pronounced, and the density curve becomes skewed unimodal.

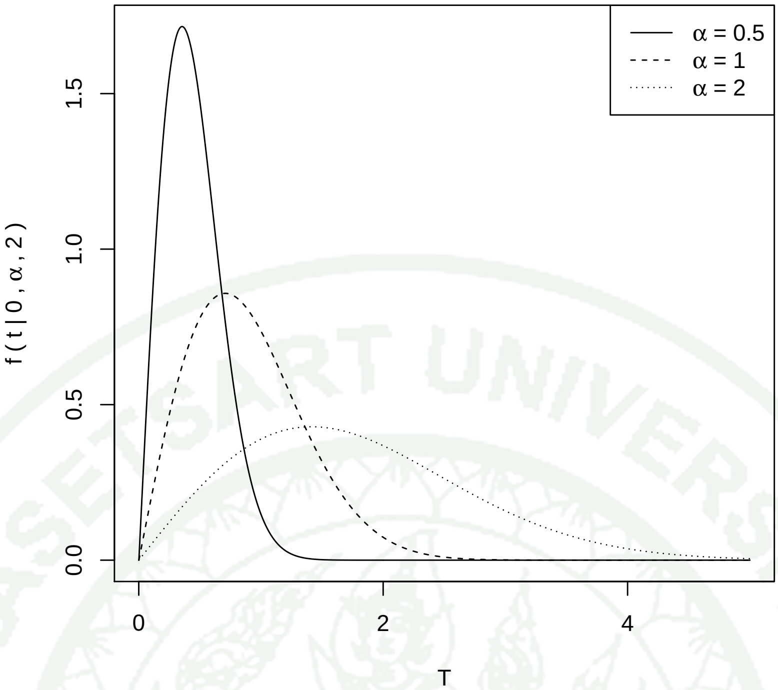

Figure 2 The three-parameter Weibull densities with specified values of scale parameter

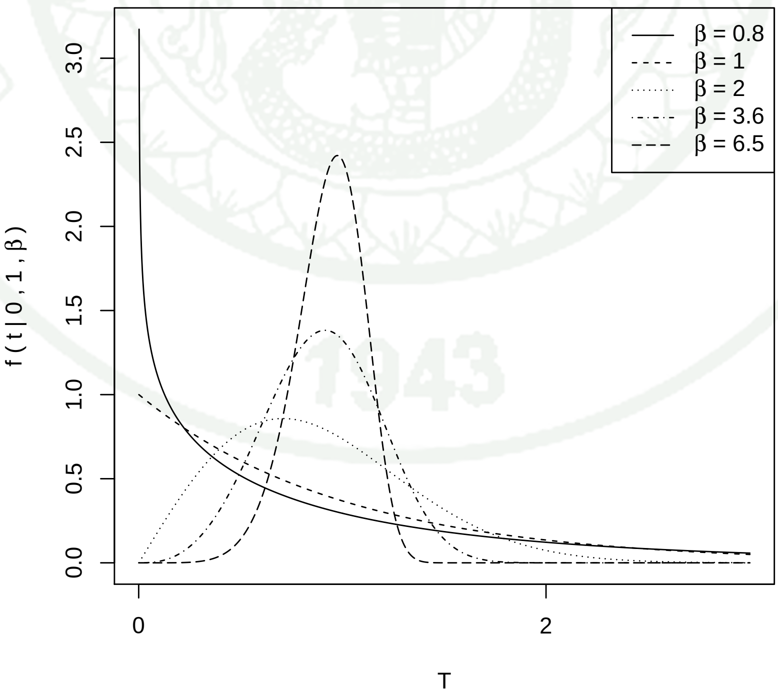

Figure 3 The three-parameter Weibull densities with specified values of shape parameter

The cdf for the three-parameter Weibull distribution is

$$F(t|\tau,\alpha,\beta) = 1 - \exp\left\{-\left(\frac{t-\tau}{\alpha}\right)^{\beta}\right\}. \tag{24}$$

The survival function is the complementary function of $F(t|\tau,\alpha,\beta)$

$$S(t|\tau,\alpha,\beta) = \exp\left\{-\left(\frac{t-\tau}{\alpha}\right)^{\beta}\right\}. \tag{25}$$

The hazard rate (or failure rate) is given by

$$h(t|\tau,\alpha,\beta) = \frac{\beta}{\alpha}\left(\frac{t-\tau}{\alpha}\right)^{\beta-1}. \tag{26}$$

The hazard rate plots of the three-parameter Weibull distribution are shown in Figure 4.

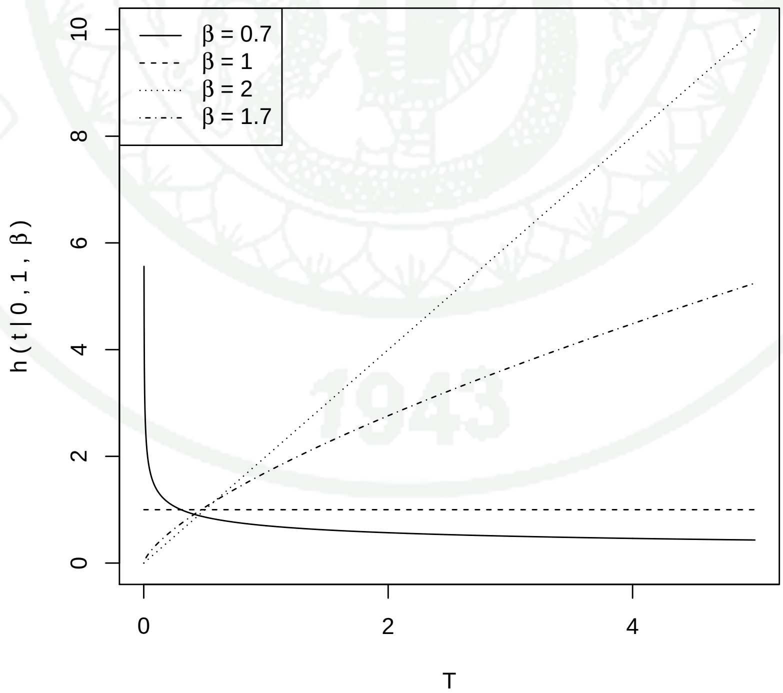

Figure 4 The hazard rate of the three-parameter Weibull distribution with specified values of shape parameter

If the quantity T is a time-to-failure, the three-parameter Weibull distribution gives the hazard rate is proportional to a power of time. The shape parameter, β, is that power plus one, and so this parameter can be interpreted directly as follows:

1) A value of $\beta < 1$ indicates that the hazard rate decreases over time. This happens if there is significant infant mortality, or defective items failing early and the failure rate decreasing over time as the defective items are weeded out of the population.

2) A value of $\beta = 1$ indicates that the hazard rate is constant over time. This might suggest that random external events cause mortality, or failure.

3) A value of $\beta > 1$ indicates that the hazard rate increases with time. This happens if there is an aging process, or parts that are more likely to fail as time goes on.

The quantile (inverse distribution) function for the three-parameter Weibull distribution is

$$Q(p) = \tau + \alpha[-\ln(1-p)]^{1/\beta}; \quad 0 \leq p < 1. \tag{27}$$

In particular, the k^{th} moment about the origin of T is given by

$$E[T^k] = \tau^k + \alpha^k \Gamma\left(1 + \frac{k}{\beta}\right). \tag{28}$$

The mean and variance of a three-parameter Weibull random variable can be expressed as

$$E[T] = \tau + \alpha\Gamma\left(1 + \frac{1}{\beta}\right), \tag{29}$$

and

$$Var(T) = \alpha^2 \left[\Gamma\left(1+\frac{2}{\beta}\right) - \left(\Gamma\left(1+\frac{1}{\beta}\right)\right)^2 \right], \tag{30}$$

respectively.

The coefficient of skewness is given by

$$CS(T) = \frac{\Gamma\left(1+\frac{3}{\beta}\right)\alpha^3 - 3\mu\sigma^2 - \mu^3}{\sigma^3}, \tag{31}$$

where the mean is denoted by μ and the standard deviation is denoted by σ.

The coefficient of kurtosis is written as

$$CK(T) = \frac{\alpha^4\Gamma(1+\frac{4}{\beta}) - 4\gamma_1\sigma^3\mu - 6\mu^2\sigma^2 - \mu^4}{\sigma^4} - 3. \tag{32}$$

2.2 Two-parameter Weibull distribution

The two-parameter Weibull distribution is a special case of three-parameter Weibull distribution with pdf and cdf, respectively, as follows:

$$f(t|\alpha,\beta) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} \exp\left\{-\left(\frac{t}{\alpha}\right)^{\beta}\right\}; \quad t \geq 0, \alpha > 0, \beta > 0, \tag{33}$$

and

$$F(t|\alpha,\beta) = 1 - \exp\left\{-\left(\frac{t}{\alpha}\right)^{\beta}\right\}. \tag{34}$$

2.3 One-parameter Weibull distribution

When $\tau = 0$ and $\alpha = 1$, the two-parameter Weibull distribution reduced to one-parameter Weibull distribution with pdf and cdf obtained as:

$$f(t|\beta) = \beta t^{\beta-1}\exp(-t^{\beta}); \quad t \geq 0, \beta > 0, \tag{35}$$

and

$$F(t|\beta) = 1 - \exp(-t^{\beta}), \tag{36}$$

respectively.

3. The inverted Weibull distribution

The IW distribution has been introduced by Keller and Kamath (1982) as a model to describe degradation phenomena of mechanical components (pistons, crankshafts) of diesel engines. Other names for this distribution are complementary Weibull distribution (Drapella, 1993), reciprocal Weibull distribution (Mudholkara and Kolliab, 1994) and reverse Weibull distribution (Murthy *et al.*, 2004).

Definition 1 Let T be a random variable of the three-parameter Weibull distribution with parameters $\tau \leq t$, $\alpha > 0$ and $\beta > 0$. A random variable X is distributed as two-parameter IW distribution with parameters α and β if

$$X = \frac{\alpha^2}{T - \tau}.$$

Theorem 1 Let X is distributed as two-parameter IW distribution with parameters α and β, the pdf of X is given by

$$f(x|\alpha,\beta) = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{-\beta-1}\exp\left\{-\left(\frac{x}{\alpha}\right)^{-\beta}\right\}; \quad x \geq 0, \alpha > 0, \beta > 0. \tag{37}$$

Proof. Let $T \sim \text{Weibull}(\tau, \alpha, \beta)$. If $X = \dfrac{\alpha^2}{T-\tau}$, then $T = \dfrac{\alpha^2}{X} + \tau$ and $\dfrac{dT}{dX} = -\dfrac{\alpha^2}{X^2}$. The pdf of X can be obtained by

$$
\begin{aligned}
f(x|\alpha,\beta) &= \frac{\beta}{\alpha}\left(\frac{\alpha^2/x+\tau-\tau}{\alpha}\right)^{\beta-1}\exp\left\{-\left(\frac{\alpha^2/x+\tau-\tau}{\alpha}\right)^{\beta}\right\}\left|-\frac{\alpha^2}{x^2}\right| \\
&= \frac{\beta}{\alpha}\left(\frac{\alpha}{x}\right)^{\beta-1}\exp\left\{-\left(\frac{\alpha}{x}\right)^{\beta}\right\}\frac{\alpha^2}{x^2} \\
&= \frac{\beta}{\alpha}\left(\frac{\alpha}{x}\right)^{\beta+1}\exp\left\{-\left(\frac{\alpha}{x}\right)^{\beta}\right\} \\
&= \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{-(\beta+1)}\exp\left\{-\left(\frac{x}{\alpha}\right)^{-\beta}\right\}. \quad \square
\end{aligned}
$$

Theorem 2 Let X be two-parameter IW distribution, the cdf of X is written as

$$
F(x|\alpha,\beta) = \exp\left\{-\left(\frac{x}{\alpha}\right)^{-\beta}\right\}. \tag{38}
$$

Proof. If X has two-parameter IW distribution. The cdf of X can be verified as

$$
\begin{aligned}
F(x|\alpha,\beta) &= \int_0^t f(x|\alpha,\beta)\,dx \\
&= \int_0^t \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{-\beta-1}\exp\left\{-\left(\frac{x}{\alpha}\right)^{-\beta}\right\}dx \\
&= \int_{-\infty}^{-(t/\alpha)^{-\beta}} \exp(u)\,du \\
&= \exp\left[-\left(\frac{t}{\alpha}\right)^{-\beta}\right] - \exp(-\infty) \\
&= \exp\left[-\left(\frac{t}{\alpha}\right)^{-\beta}\right] - 0 \\
&= \exp\left[-\left(\frac{t}{\alpha}\right)^{-\beta}\right]. \quad \square
\end{aligned}
$$

The hazard rate of the IW distribution is written by

$$h(x|\alpha,\beta) = \frac{\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{-\beta-1}\exp\left\{-\left(\frac{x}{\alpha}\right)^{-\beta}\right\}}{1-\exp\left\{-\left(\frac{x}{\alpha}\right)^{-\beta}\right\}}. \tag{39}$$

Inverting the cdf $F(x|\alpha,\beta)$, Eq.(38), leads to the following quantile function

$$Q(p) = F^{-1}(p) = \alpha(-\ln p)^{1/\beta}.$$

The k^{th} moment about the origin of X is given by

$$\begin{aligned} E(X^k) &= \int_0^\infty x^k f(x|\alpha,\beta)\,dx \\ &= \int_0^\infty x^k \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{-\beta-1}\exp\left\{-\left(\frac{x}{\alpha}\right)^{-\beta}\right\}dx \\ &= \alpha^k\Gamma(1-\frac{k}{\beta}); \quad k<\beta. \end{aligned} \tag{40}$$

Corollary 1 For $\alpha = 1$, we get one-parameter IW distribution with pdf

$$f(x|\beta) = \beta x^{-\beta-1}\exp\left\{-x^{-\beta}\right\}; \quad x \geq 0, \beta > 0, \tag{41}$$

and the cdf

$$F(x|\beta) = \exp\left\{-x^{-\beta}\right\}. \tag{42}$$

4. The exponentiated inverted Weibull distribution

The EIW distribution was proposed by Flaih *et al.* (2012). Based on likelihood ratio test, his study suggest that the EIW distribution provides a better fit to the real dataset than the one-parameter inverted Weibull distribution.

If X has EIW distribution with two shape parameters β and λ, we denote it by $\text{EIW}(\beta,\lambda)$ and its pdf is

$$f_E(x|\beta,\lambda) = \lambda\beta x^{-(\beta+1)}\{\exp(-x^{-\beta})\}^{\lambda}; \quad x \geq 0,\ \beta > 0,\ \lambda > 0. \tag{43}$$

The pdf plots of the EIW distribution, shown in Figure 5 can be seen as a unimodal.

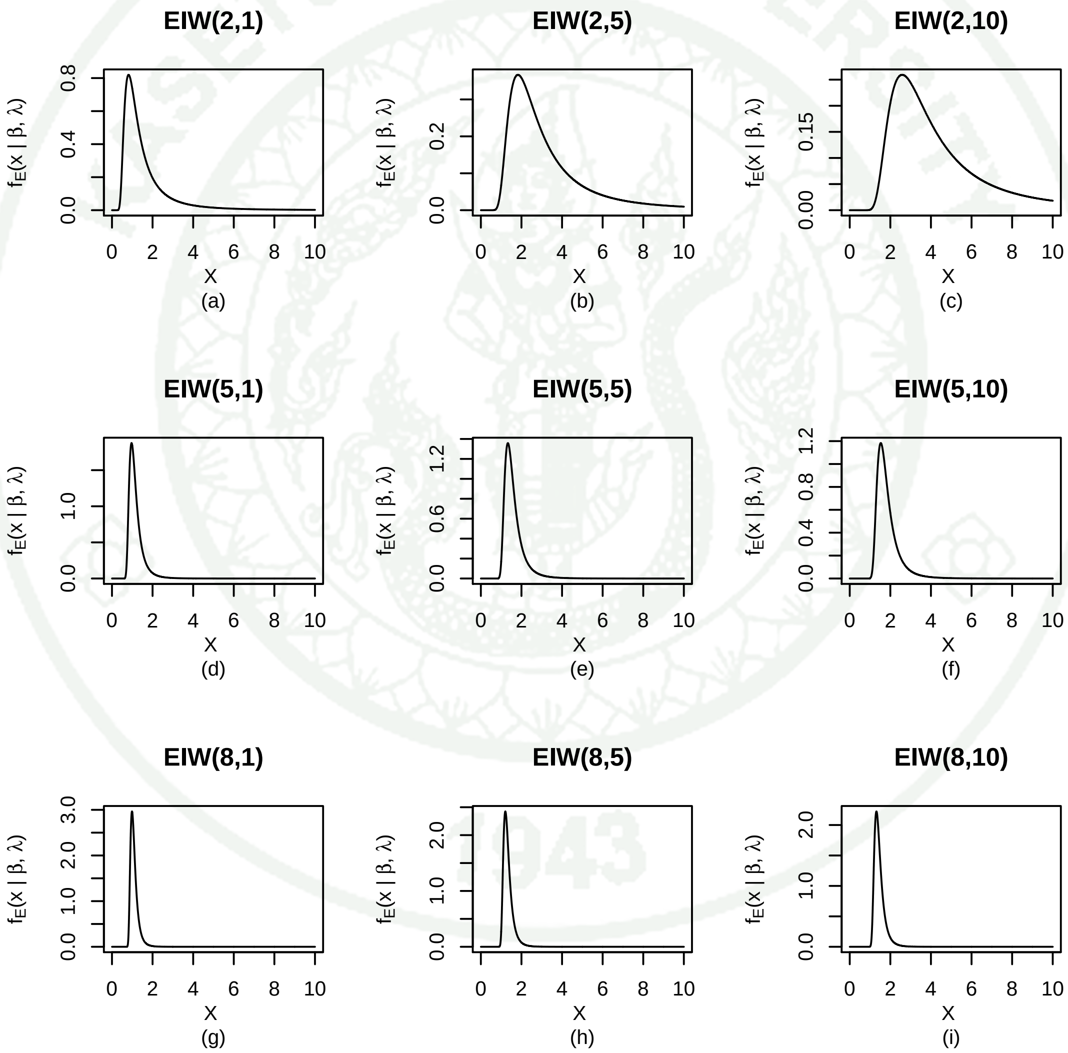

Figure 5 The pdf of the EIW distribution with specified values of β and λ

The cdf of X is

$$F_E(x|\beta,\lambda) = \{\exp(-x^{-\beta})\}^{\lambda}. \tag{44}$$

The survival function is

$$S_E(x|\beta,\lambda) = 1 - \{\exp(-x^{-\beta})\}^{\lambda}. \tag{45}$$

Some survival function plots of the EIW distribution with specified parameter values are shown in Figure 6.

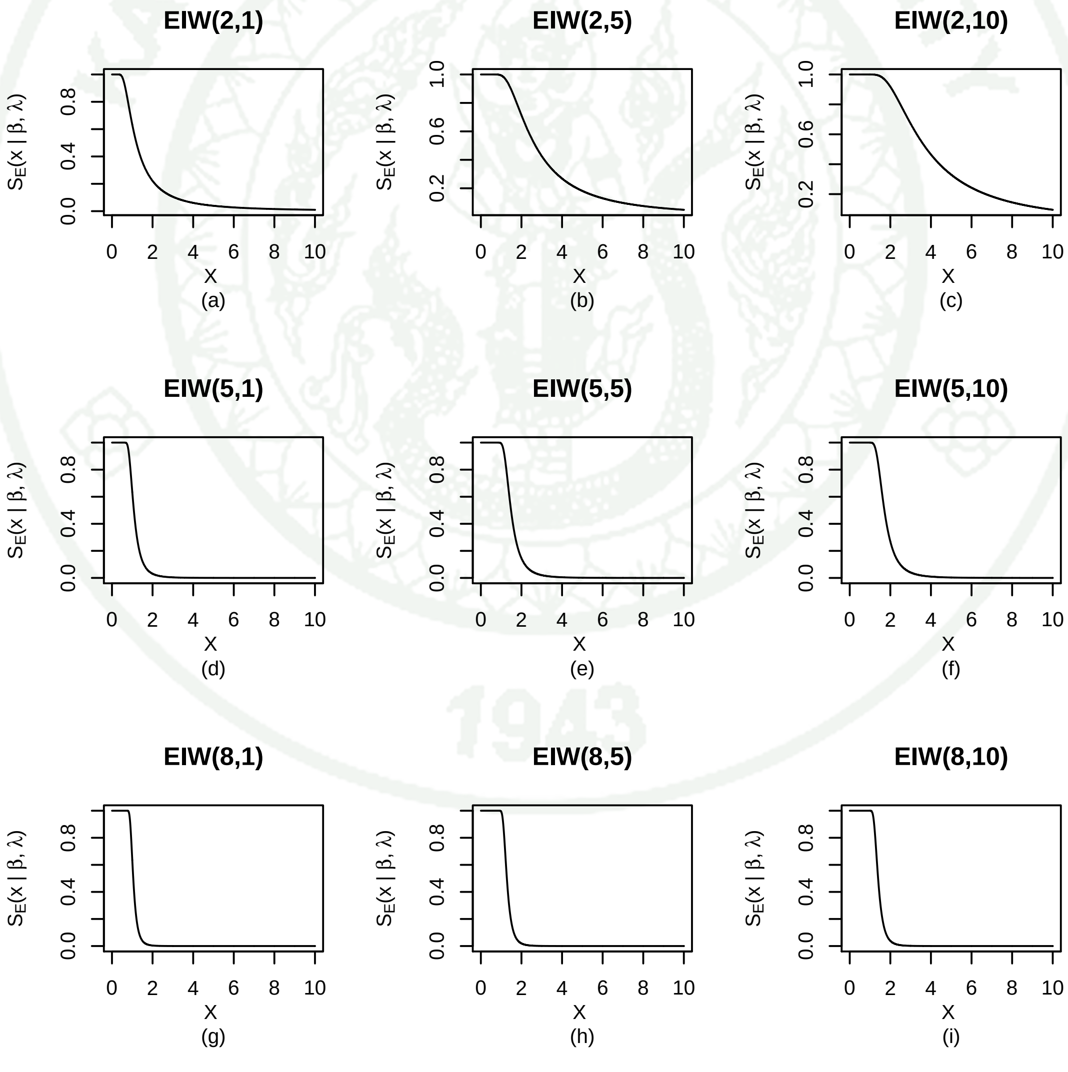

Figure 6 The survival function of the EIW distribution with specified values of β and λ

The hazard rate of the EIW distribution is given by:

$$h_E(x|\beta,\lambda) = \frac{\lambda\beta x^{-(\beta+1)}\{\exp(-x^{-\beta})\}^{\lambda}}{1-\{\exp(-x^{-\beta})\}^{\lambda}}. \tag{46}$$

Some plots of hazard rate of the EIW distribution with specified parameter values are shown in Figure 7.

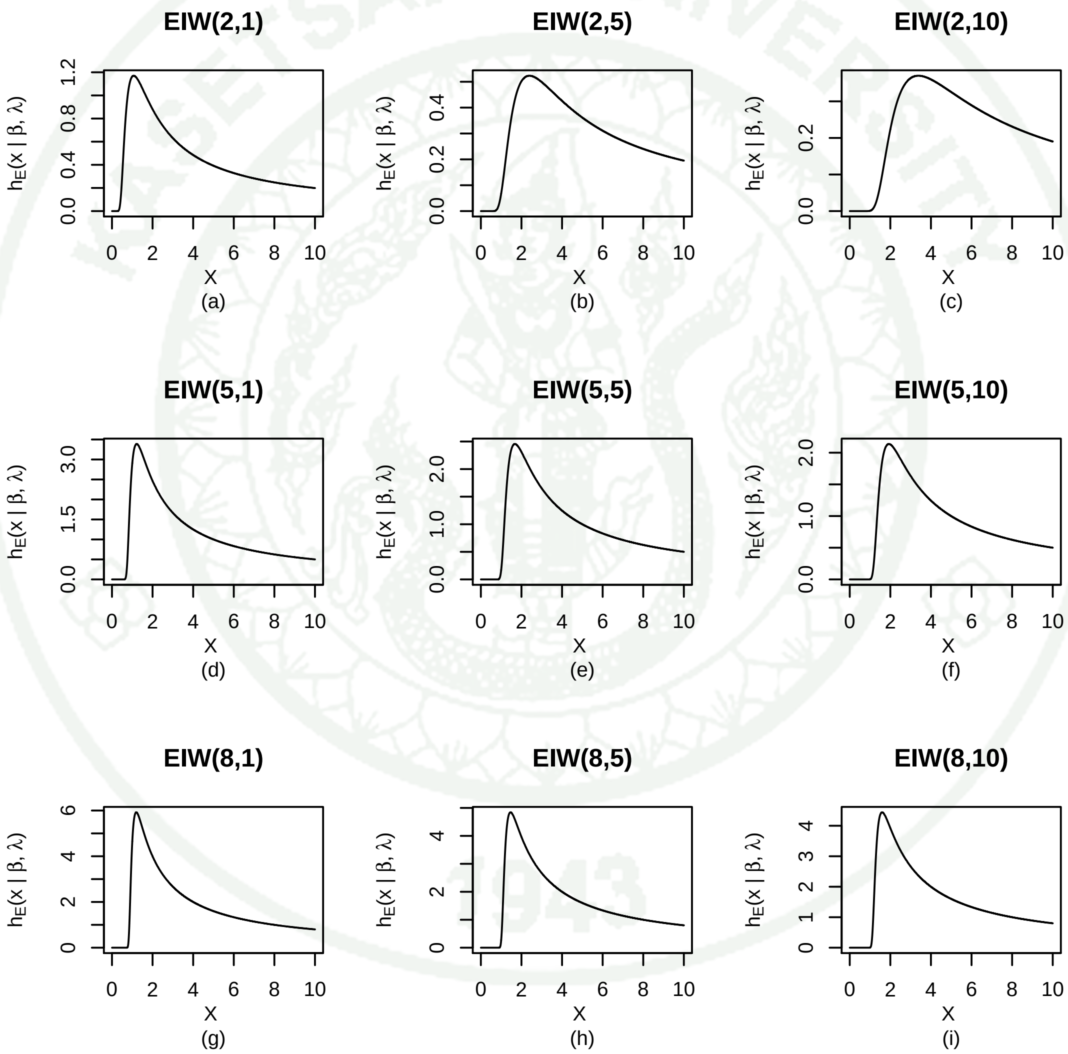

Figure 7 The hazard rate of the EIW distribution with specified values of β and λ

The k^{th} moment about the origin of the EIW distribution is given as follows:

$$E(X^k) = \lambda^{\frac{k}{\beta}} \Gamma(1 - \frac{k}{\beta}); \quad \beta > k. \tag{47}$$

Then, the mean of X is obtained as:

$$E(X) = \lambda^{\frac{1}{\beta}} \Gamma\left(1 - \frac{1}{\beta}\right); \quad \beta > 1. \tag{48}$$

For $\lambda = 1$, the EIW distribution represents the one-parameter IW distribution, when $\beta = 1$ it represents the exponentiated inverted exponential distribution. Thus, the EIW distribution is a generalization of the exponentiated inverted exponential distribution as well as the one-parameter IW distribution.

5. The gamma distribution

The gamma distribution is widely used in survival analysis and life testing (Lawless, 2003) and it is a good alternative to the popular Weibull distribution. It is a flexible distribution that commonly offers a good fit to any variables such as ecology, meteorology, climatology and other physical situations (Moala *et al.*, 2013). From a computational point of view, gamma models fit very well into survival models, because it is easy to derive the formulas for any number of events. This is due to the simplicity of the derivatives of the Laplace transform. This is also the reason why this distribution has been applied in most of the applications published until now.

In Bayesian analysis, the gamma distribution is used as a conjugate prior distribution for various types of parameters, such as exponential, inverse gamma, normal and Weibull distributions.

The random variable X follows gamma distribution with the shape and scale parameters as $\alpha > 0$ and $\beta > 0$ respectively, denoted by Gamma(α, β), if it has the following pdf

$$f(x|\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x); \quad x \geq 0. \tag{49}$$

The formula for the cdf of the gamma distribution is

$$F(x|\alpha,\beta) = \frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}. \tag{50}$$

The mean and variance of the Gamma(α,β) are respectively

$$E(X) = \frac{\alpha}{\beta}, \tag{51}$$

and

$$Var(X) = \frac{\alpha}{\beta^2}. \tag{52}$$

The gamma distribution includes the chi-squared, Erlang, and exponential distributions as special cases, but the shape parameter of the gamma is not confined to integer values. The gamma distribution starts at the origin and has a flexible shape. The parameters are easy to estimate by maching moments.

6. The beta distribution

In probability theory and statistics, the beta distribution is a family of continuous probability distributions defined on the interval [0, 1] and parameterized by two positive shape parameters, denoted by *a* and *b*, which appear as exponents of the random variable and control the shape of the distribution.

The beta distribution has been applied to model the behavior of random variables limited to intervals of finite length in a wide variety of disciplines. For example, it has been used as a statistical description of allele frequencies in population genetics; time allocation in project management control systems, sunshine data, variability of soil properties, proportions of the minerals in rocks in stratigraphy and heterogeneity in the probability of HIV transmission (Sulaiman *et al.*, 1999).

In Bayesian inference, the beta distribution is the conjugate prior probability distribution for the Bernoulli, binomial and geometric distributions. For example, the beta distribution can be used in Bayesian analysis to describe initial knowledge concerning probability of success such as the probability that a space vehicle will successfully complete a specified mission. The beta distribution is a suitable model for the random behavior of percentages and proportions (Evans *et al.*, 2000).

In the following, that a random variable X is beta distributed with parameters a and b will be denoted by $X \sim \text{Beta}(a,b)$.

The pdf and cdf of the beta distribution, for $0 \leq x \leq 1$, and shape parameters including $a > 0$ and $b > 0$, are follows, respectively,

$$f(x|a,b) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}, \tag{53}$$

and

$$F(x|a,b) = \frac{B_x(a,b)}{B(a,b)}. \tag{54}$$

Mean and variance of $\text{Beta}(a,b)$ are

$$E(X) = \frac{a}{a+b}, \tag{55}$$

and

$$Var(X) = \frac{ab}{(a+b)^2(a+b+1)}, \tag{56}$$

respectively.

This $\text{Beta}(a,b)$ is U shaped if $a < 1$, $b < 1$ and J shaped if $(a-1)(b-1) < 0$, and is otherwise unimodal (Evans *et al.*, 2000). For the special case $a = b = 1$, the Beta(a,b) reduces to a continuous uniform distribution.

7. The lognormal distribution

The three-parameter lognormal distribution is a skewed distribution, which is useful for modeling continuous positive random variables. Although the lognormal distribution is used for modeling positively skewed data, depending on the values of its parameters, the lognormal distribution can have various shapes including a bell-curve similar to the normal distribution. Limpert *et al.* (2001) illustrated how the lognormal distributions are widespread through the science, such as geology, medicine, environmental science, food technology, ecology, linguistics, social sciences, operation research, finance and it has been used in other diverse areas of the sciences.

Let X be a random variable of the three-parameter lognormal distribution with parameters γ, μ and σ. The pdf of X can be expressed as

$$f(x|\gamma,\mu,\sigma) = \frac{1}{(x-\gamma)\sigma\sqrt{2\pi}} \exp\left\{-\frac{[\log(x-\gamma-\mu)]^2}{2\sigma^2}\right\}, \tag{57}$$

where $x > 0,\ 0 \leq \gamma < x,\ -\infty < \mu < \infty$ and $\sigma > 0$.

The cdf of the three-parameter lognormal distribution is obtained by

$$F(x|\gamma,\mu,\sigma) = \theta\left[\frac{\log(x-\gamma)-\mu}{\sigma}\right], \tag{58}$$

where $\theta(z)$ is a cdf of standard normal distribution.

For γ is the threshold parameter or location parameter defining the point where the support set of the distribution begins; μ is the scale parameter that stretchs or shrinks the distribution and σ is the shape parameter that affects the shape of the distribution.

Yuan (1933) derived the mean, variance, coefficient of skewness and coefficient of kurtosis of the lognormal distribution as a function of μ, σ and γ as follows

$$E(X) = \gamma + \exp\left(\mu + \frac{\sigma^2}{2}\right), \tag{59}$$

$$Var(X) = \gamma + \exp\left(\mu + \frac{\sigma^2}{2}\right), \tag{60}$$

$$CS(X) = (\exp(\sigma^2) + 2)\sqrt{(\exp(\sigma^2) - 1)}, \tag{61}$$

$$CK(X) = \exp(4\sigma^2) + 2\exp(3\sigma^3) + 3\exp(2\sigma^3) - 3. \tag{62}$$

The lognormal distribution with two parameters μ and σ is a special case of the three-parameter lognormal distribution when $\gamma = 0$.

Moreover, if X is a random variable that has a three-parameter lognormal distribution with parameters γ, μ and σ, then $Y = \log(X - \gamma)$ has a normal distribution with mean μ and variance σ^2.

8. Length-biased distribution

The concept of length-biased method was introduced by Cox (1962). This method is found in various applications in biomedical area such as family history and disease, survival analysis, intermediate events and latency period of AIDS due to blood transfusion (Gupta and Akman, 1995). Many studies were done to characterize relationships between original distributions and their length-biased versions. Patill and Rao expressed some basic distributions and their length-biased forms such as lognormal, gamma, pareto and beta distributions (Patil and Rao, 1978). Recently, many researches are applied to length-biased method for lifetime distribution, length-biased weighted generalized Rayleigh distribution (Das and Roy, 2011a), length-biased weighted Weibull distribution (Das and Roy, 2011b), length-biased beta distribution

(Mir *et al.*, 2013), and Bayes estimation of length-biased Weibull distribution (Pandya *et al.*, 2013).

The general definition of a length-biased distribution provide as follows:

Definition 2 If X has a lifetime distribution with pdf $f(x)$ and expected value, $E(X) < \infty$, the pdf of length-biased distribution of X can be defined as:

$$f_L(x) = \frac{xf(x)}{E(X)}. \tag{63}$$

9. Mixture distribution

Mixture distributions provide powerful and popular tools for generating flexible distributions with attractive probabilistic properties (McLachlan and Peel, 2000). Many applications of finite mixture distributions is applied to available data such as life testing, reliability, economics, physical sciences (Everitt and Hand, 1981) (Titterington *et al.*, 1985) (McLachlan and Basford, 1988). The mixture distribution is compounding of distributions, a new mixture may be obtained by mixing of different parameter set in a distribution or mixing of two distributions.

Over three decades or so, many mixture distribution has been developed, which are as follows: Jorgensen *et al.* (1991) obtained a new three-parameter family of the generalized inverse Gaussian distribution, which includes the inverse Gaussian and the reciprocal inverse Gaussian distributions as special cases, while preserving some of the interesting properties of the inverse Gaussian distribution. They also consider estimation and inference properties for the family, and show that it may have applications for positive right-skewed unimodal data and, in particular, duration or failure-time data. Sultan *et al.* (2007) developed the mixture model of two inverse Weibull distributions. They discussed some properties of the model with some graphs of the density and hazard rate. The identifiability property of the proposed distribution is proved. In addition, the estimates of the unknown parameters via the EM algorithm are obtained. The performance of the findings in the paper is showed by demonstrating some numerical illustrations through Monte Carlo simulations. Mubarak (2011) introduced the mixture model of two Frechet distributions. Some properties of the model with hazard rate are discussed. In addition, the estimates of the unknown parameters via the EM algorithm are obtained. The performance of the findings in the paper is showed by demonstrating some numerical illustration through Monte Carlo simulation. Lafta *et al.* (2013) obtained the new probability distribution from mixing Pareto distribution with Weibull distribution. The researchers constructed the pdf, and cdf, reliability function, hazard rate and moment about the origin. They obtained the parameters estimated by moment method and maximum likelihood estimation method using simulation procedure taking different sample size and replicate for each experiment. Jamal *et al.* (2014) introduced a mixture of modified inverse Weibull distribution and studied its different properties. The estimates of the unknown parameters via the EM algorithm are obtained.

The general definition of a mixture distribution is presented as follows:

Definition 3 If $0 \leq p \leq 1$ is a mixing parameter, $f_{X_1}(x)$ and $f_{X_2}(x)$ are the pdf of the variables X_1 and X_2, respectively. Then the pdf of the random variable X expressed by mixture between X_1 and X_2 is

$$f_X(x) = p f_{X_1}(x) + (1-p) f_{X_2}(x), \tag{64}$$

where $x \geq 0$ and $0 \leq p \leq 1$.

Definition 4 Let $F_{X_1}(x)$ and $F_{X_2}(x)$ are the cdf of the variables X_1 and X_2, respectively. Then the cdf of the random variable X resulted from mixture between X_1 and X_2 is

$$F_X(x) = p F_{X_1}(x) + (1-p) F_{X_2}(x). \tag{65}$$

10. Parameter estimation methods

The estimation of the parameters of a statistical distribution is one of the fundamental issues in statistics. If the phenomenon is modeled using a parametric model, it is necessary to assign values to the parameters. Choosing an appropriate estimator is an important task, hence, several optimal criterions are considered. We introduce two methods of parameter estimation: the MLE and Bayesian estimation, which are the most frequently used for parameter estimation.

10.1 Maximum likelihood estimation

The MLE is the most popular method of estimation and inference for parametric models. Let $X_1, X_2, \ldots, X_n$ be an independent and identically distributed (iid) sample from a population with pdf $f(x|\theta_1, \theta_2, \ldots, \theta_p)$. Therefore, the likelihood function can be written as

$$L(\theta_1, \theta_2, \ldots, \theta_p|x) = \prod_{i=1}^{n} f(x_i|\theta_1, \theta_2, \ldots, \theta_p).$$

The log-likelihood function is given by

$$l(\theta_1, \theta_2, \ldots, \theta_p|x) = \log L(\theta_1, \theta_2, \ldots, \theta_p|x) = \sum_{i=1}^{n} \log f(x_i|\theta_1, \theta_2, \ldots, \theta_p).$$

If the likelihood function is differentiable, the possible candidates for the maximum likelihood estimators are the values of $(\theta_1, \theta_2, \ldots, \theta_p)$ solved by

$$\begin{aligned} \frac{\partial}{\partial \theta_i} L(\theta_1, \theta_2, \ldots, \theta_p|x) &= 0, \quad i = 1, 2, \ldots, p, \\ \text{or} \qquad \frac{\partial}{\partial \theta_i} l(\theta_1, \theta_2, \ldots, \theta_p|x) &= 0, \quad i = 1, 2, \ldots, p. \end{aligned}$$

10.2 Bayesian estimation

The Bayesian estimation (BE) approach to statistics is different from the classical estimation approach that we have been taking. The main difference between the classical statistical theory and the BE is that the later considers parameters as random variables characterized by prior distribution. This prior distribution is combined with the traditional likelihood to obtain the posterior of the interested parameter on which the statistical inference is based. Nevertheless, some aspects of the BE can be helpful to other statistical approaches.

Definition 5 Let $\pi(\theta)$ be a prior distribution which is a probability distribution over the space of possible parameter values.

Definition 6 The joint distribution has a pdf which is given by

$$\pi(x_1, x_2, \ldots, x_n, \theta) = f(x_1, x_2, \ldots, x_n|\theta)\pi(\theta) = \prod_{i=1}^{n} f(x_i|\theta)\pi(\theta).$$

Definition 7 The marginal distribution of $X_1, X_2, \ldots, X_n$ has a pdf which is given by

$$\pi(x_1, x_2, \ldots, x_n) = \int_{\theta} f(x_1, x_2, \ldots, x_n|\theta)\pi(\theta)\, d\theta = \int_{\theta} \prod_{i=1}^{n} f(x_i|\theta)\pi(\theta)\, d\theta.$$

Definition 8 Let $\pi(\theta|x_1, x_2, \ldots, x_n)$ be a posterior distribution which is the conditional probability distribution of the parameter given for the observed data.

Theorem 3 The posterior distribution can be computed as

$$\pi(\theta|x_1, x_2, \ldots, x_n) = \frac{\prod_{i=1}^{n} f(x_i|\theta)\pi(\theta)}{\int_{\theta} \prod_{i=1}^{n} f(x_i|\theta)\pi(\theta)\, d\theta}.$$

Proof. The Baye's theorem provides an expression for the conditional probability of A given B, provided that $\mathrm{P}(B) \neq 0$ given by

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(B|A)\mathrm{P}(A)}{P(B)}.$$

In Bayesian inference, the probability of evidence B is constant for all A_n. The posterior can be expressed as proportional to the numerator

$$\mathrm{P}(A_n|B) = \frac{\mathrm{P}(B|A_n)\mathrm{P}(A_n)}{\mathrm{P}(B)}.$$

For some partition $\{A_i\}$ of the event space, the event space is given in terms of $\mathrm{P}(A_i)$ and $\mathrm{P}(B|A_i)$. It is then useful to eliminate $\mathrm{P}(B)$ using the law of total probability which $\mathrm{P}(B)$ can be written as

$$\mathrm{P}(B) = \sum_{i=1}^{n} \mathrm{P}(B|A_i)\mathrm{P}(A_i).$$

Now, if we set $A = \{\theta\}$ and $B = \{x_1, x_2, \ldots, x_n\}$, then

$$\pi(\theta|x_1, x_2, \ldots, x_n) = \frac{\prod_{i=1}^{n} f(x_i|\theta)\pi(\theta)}{\int_{\theta} \prod_{i=1}^{n} f(x_i|\theta)\pi(\theta)\, d\theta}. \quad \square$$

Definition 9 Let $L(\hat{\theta}, \theta)$ be a loss function which specifies the loss incurred if the true value of the parameter is θ, which is estimated as $\hat{\theta}_{BE}$.

Definition 10 The Bayesian approach estimator for a given loss function is the one that minimizes the expected loss given the posterior distribution which is defined as

$$\hat{\theta}_{BE} = \underset{\hat{\theta} \in \mathbb{R}}{\operatorname{argmin}}\, E[L(\hat{\theta}, \theta)] = \underset{\hat{\theta} \in \mathbb{R}}{\operatorname{argmin}} \int_{\theta} L(\hat{\theta}, \theta)\pi(\theta|x_1, x_2, \ldots, x_n)\, d\theta.$$

Theorem 4 The Bayesian estimator using squared error loss is the mean of posterior distribution which is given by

$$\hat{\theta}_{BE} = E[\pi(\theta|x_1, x_2, \ldots, x_n)].$$

Proof. From Definition 10, Bayesian approach estimator for a given loss function minimizes the expected loss which can be written as

$$\begin{aligned}\hat{\theta}_{BE} &= \underset{\hat{\theta}\in\mathbb{R}}{\operatorname{argmin}}\, E[L(\hat{\theta},\theta)] \\ &= \underset{\hat{\theta}\in\mathbb{R}}{\operatorname{argmin}} \int_{\theta} L(\hat{\theta},\theta)\pi(\theta|x_1, x_2, \ldots, x_n)\, d\theta. \end{aligned} \tag{66}$$

By substituting squared error loss into Eq.(66), we obtain

$$\int_{\theta} L(\hat{\theta},\theta)\pi(\theta|x_1, x_2, \ldots, x_n)\, d\theta = \int_{\theta} (\theta-\hat{\theta})^2\pi(\theta|x_1, x_2, \ldots, x_n)\, d\theta.$$

The gradient of $E[L(\hat{\theta},\theta)]$ is

$$\begin{aligned}\frac{\partial}{\partial\hat{\theta}} E[L(\hat{\theta},\theta)] &= \frac{\partial}{\partial\hat{\theta}}\int_{\theta} (\theta-\hat{\theta})^2\pi(\theta|x_1, x_2, \ldots, x_n)\, d\theta \\ &= \int_{\theta} \frac{\partial}{\partial\hat{\theta}}(\theta-\hat{\theta})^2\pi(\lambda|x_1, x_2, \ldots, x_n)\, d\theta \\ &= -2\int_{\theta} (\theta-\hat{\theta})\pi(\theta|x_1, x_2, \ldots, x_n)\, d\theta,\end{aligned}$$

and the second derivative is

$$\frac{\partial^2}{\partial\hat{\theta}^2} E[L(\hat{\theta},\theta)] = \frac{\partial}{\partial\hat{\theta}}\left[-2\int_{\theta} (\theta-\hat{\theta})\pi(\theta|x_1, x_2, \ldots, x_n)\, d\theta\right] > 0.$$

The Bayesian approach estimator using squared error loss now becomes

$$
\begin{aligned}
-2\int_{\theta}(\theta-\hat{\theta})\pi(\theta|x_1,x_2,\ldots,x_n)\,d\theta &= 0 \\
\int_{\theta}\hat{\theta}\,\pi(\theta|x_1,x_2,\ldots,x_n)\,d\theta &= \int_{\theta}\theta\,\pi(\theta|x_1,x_2,\ldots,x_n)\,d\theta \\
\hat{\theta}_{BE} &= \int_{\theta}\theta\,\pi(\theta|x_1,x_2,\ldots,x_n)\,d\theta \\
\hat{\theta}_{BE} &= E[\pi(\theta|x_1,x_2,\ldots,x_n)]. \quad \square
\end{aligned}
$$

Some Bayesian estimators cannot be expressed in explicit forms, we use the Markov chain Monte Carlo (MCMC) techniques to generate samples from the posterior distributions and in turn to compute the Bayesian estimators. The posterior density functions match quite well with the histograms of the samples obtained by MCMC methods.

MCMC techniques have been popular since the early 1990s. In 1953, MCMC were introduced into physics in a simplified version (Ntzoufras, 2009). Intermediate landmark publications include the generalization of the Metropolis algorithm proposed by Hastings in 1970 and the development of the Gibbs sampling introduced by German and German in 1984. Nevertheless, it took about 35 years until MCMC methods were rediscovered by Bayesian scientists and became one of the main computational tools in modern statistical inference. The most popular MCMC methods are the Metropolis-Hastings algorithm and Gibbs sampling.

10.2.1 The Metropolis-Hastings algorithm

Metropolis *et al.* (1953) originally formulated the Metropolis algorithm, by introducing the Markov-chain-based simulation methods used in science (Ntzoufras, 2009). Later, Hastings (1970) generalized the original method, which is known as the Metropolis-Hastings algorithm. Metropolis-Hastings is a MCMC technique for obtaining a sequence of random samples from a probability distribution which generates a random walk using a proposal density and a method for rejecting proposed moves. The algorithm can be summarized by the following steps:

Algorithm 1 The Metropolis-Hastings algorithm

Set initial values $\theta^{(0)}$
For $r = 1, \ldots, R$ repeat the following steps
    Set $\theta = \theta^{(r-1)}$
    Generate new candidate parameter values $\theta^{'}$ from a proposal distribution $q(\theta^{'}|\theta)$
    Calculate
$$\alpha = min\left(1, \frac{f(\theta^{'}|x)q(\theta|\theta^{'})}{f(\theta|x)q(\theta^{'}|\theta)}\right) = min\left(1, \frac{f(x|\theta^{'})f(\theta^{'})q(\theta|\theta^{'})}{f(x|\theta)f(\theta)q(\theta^{'}|\theta)}\right)$$
    Update $\theta^{(r)} = \theta^{'}$ with probability α ; otherwise set $\theta^{(r)} = \theta$.

10.2.2 The Gibbs sampling algorithm

The Gibbs sampling was introduced by German and German (1984). It is a special case of Metropolis-Hastings algorithm. Gibbs sampling is another MCMC technique for sampling from posterior distributions. Gibbs sampling is applicable when the joint distribution is not known explicitly, but the conditional distribution of each variable is known. The Gibbs sampling algorithm generates a sequence of draws from the distribution of each variable as follows:

Algorithm 2 The Gibbs sampling algorithm

Set initial values $\theta^{(0)}$

For $r = 1, \ldots, R$ repeat the following steps

Set $\theta = \theta^{(r-1)}$

For $j = 1, \ldots, d$, update θ_j from

$\theta_j \sim f(\theta_j|\theta_{(-j)}, x) = f(\theta_j|\theta_1^{(r)}, \ldots, \theta_{j-1}^{(r)}, \theta_{j+1}^{(r-1)}, \ldots, \theta_p^{(r-1)}, x) \propto f(\theta|x)$

Set $\theta^{(r)} = \theta$ and save it as the generated set of values at $r+1$ iteration of the algorithm

Hence, given a particular state of the chain $\theta^{(r)}$, we generate the new parameter values by

$$
\begin{aligned}
\theta_1^{(r)} &= f(\theta_1|\theta_2^{(r-1)}, \theta_3^{(r-1)}, \ldots, \theta_p^{(r-1)}, x), \\
\theta_2^{(r)} &= f(\theta_2|\theta_1^{(r)}, \theta_3^{(r-1)}, \ldots, \theta_p^{(r-1)}, x), \\
\theta_3^{(r)} &= f(\theta_3|\theta_1^{(r)}, \theta_2^{(r)}, \theta_4^{(r-1)}, \ldots, \theta_p^{(r-1)}, x), \\
&\vdots \\
\theta_j^{(r)} &= f(\theta_j|\theta_1^{(r)}, \theta_2^{(r)}, \ldots, \theta_{j-1}^{(r)}, \theta_{j+1}^{(r-1)}, \ldots, \theta_p^{(r-1)}, x), \\
&\vdots \\
\theta_p^{(r)} &= f(\theta_p|\theta_1^{(r)}, \theta_2^{(r)}, \ldots, \theta_{p-1}^{(r)}, x).
\end{aligned}
$$

11. Criterion for a comparison of parameter estimation methods

Here, we compare the performance of the MLE and Bayesian approach methods for the parameter of the proposed distributions. The comparison for each parameter, $\theta_1,\ldots,\theta_p$, is based on the mean, variance and mean square error (MSE) criterion defined by, respectively

$$Mean(\hat{\theta}_i) = \bar{\hat{\theta}}_i = \frac{1}{m}\sum_{j=1}^{m}\hat{\theta}_{ij}, \tag{67}$$

$$Var(\hat{\theta}_i) = \frac{1}{m-1}\sum_{j=1}^{m}\left(\hat{\theta}_{ij}-\bar{\hat{\theta}}_i\right)^2, \tag{68}$$

$$MSE(\hat{\theta}_i) = \frac{1}{m}\sum_{j=1}^{m}\left(\hat{\theta}_{ij}-\theta_i\right)^2, \tag{69}$$

where $\hat{\theta}_{ij}$ is the estimated value for the i^{th} parameter and the j^{th} sample,
θ_i is the true value of the i^{th} parameter,
p is the number of parameters, $i = 1,2,\ldots,p$,
and m is the number of samples, $j = 1,2,...,m$.

MATERIALS AND METHODS

Materials

The materials used for this study are as follows:

1. A high performance personal computer (Intel® Core™2 Duo Processor P8700, 2.0 GB of RAM) for running the coded program.

2. R program version 3.10.1 (R Core Team, 2013) is used for the simulation studies and application studies in this research.

Methods

The methods of the research are as follows:

1. Investigate probability functions and some properties of the LBEIW and MEIW distributions.

2. Derive estimated parameters of the LBEIW and MEIW distributions by MLE and Bayesian estimation.

3. Generate random variate of the LBEIW and MEIW for simulation study.

4. Simulation study is applied in order to compare, for some situations, numerical results under estimating parameters of the LBEIW distribution and MEIW distribution with MLE and Bayesian estimation by using MSE criteria.

5. Apply with real data sets by comparing the efficiencies of fitting distribution among the LBEIW, MEIW, EIW, LBIW, Weibull and lognormal distributions by using Anderson-Darling (AD) test (Chen and Balakrishnan, 1995) and Akaike information criterion (AIC) (Akaike, 1974) for the goodness of fit purpose.

RESULTS AND DISCUSSION

Results

This section presents the results of this thesis. The results are divided into two parts: the theorems of the LBEIW distribution and the MEIW distribution. The statistical probability functions, some statistical properties, simulation study for parameter estimation and application to a real dataset have been included in each part.

1. The length-biased exponentiated inverted Weibull distribution

In this section, we propose a new distribution which is a LBEIW distribution. We first provide a theorem of the LBEIW distribution which will subsequently reveal its pdf.

Theorem 5 Let X be a random variable of an EIW distribution with pdf $f_E(x|\beta,\lambda)$ and mean $E(X)$. Then $f_L(x|\beta,\lambda) = \dfrac{xf_E(x|\beta,\lambda)}{E(X)}$ is a pdf of the LBEIW distribution with two shape parameters β and λ. The notation for X with the LBEIW distribution is denoted as $X \sim \text{LBEIW}(\beta,\lambda)$. The pdf of X is given by

$$f_L(x|\beta,\lambda) = \frac{\beta\lambda^{1-\frac{1}{\beta}}}{\Gamma(1-\frac{1}{\beta})}x^{-\beta}\{\exp(-x^{-\beta})\}^{\lambda};\ x>0, \beta>1, \lambda>0. \tag{70}$$

Proof. By Definition 2, substitute $f_E(x|\beta,\lambda)$, Eq.(43), and $E(X)$, Eq.(48), into $f_L(x)$, Eq.(63), then the pdf for the LBEIW distribution can be obtained by

$$\begin{aligned} f_L(x|\beta,\lambda) &= \frac{x}{\lambda^{\frac{1}{\beta}}\Gamma(1-\frac{1}{\beta})}\lambda\beta x^{-(\beta+1)}\{\exp(-x^{-\beta})\}^{\lambda} \\ &= \frac{\beta\lambda^{1-\frac{1}{\beta}}}{\Gamma(1-\frac{1}{\beta})}x^{-\beta}\{\exp(-x^{-\beta})\}^{\lambda}. \quad \square \end{aligned}$$

The pdf plots of the LBEIW distribution are shown in Figure 8 which can be seen as a unimodal and positively skewed.

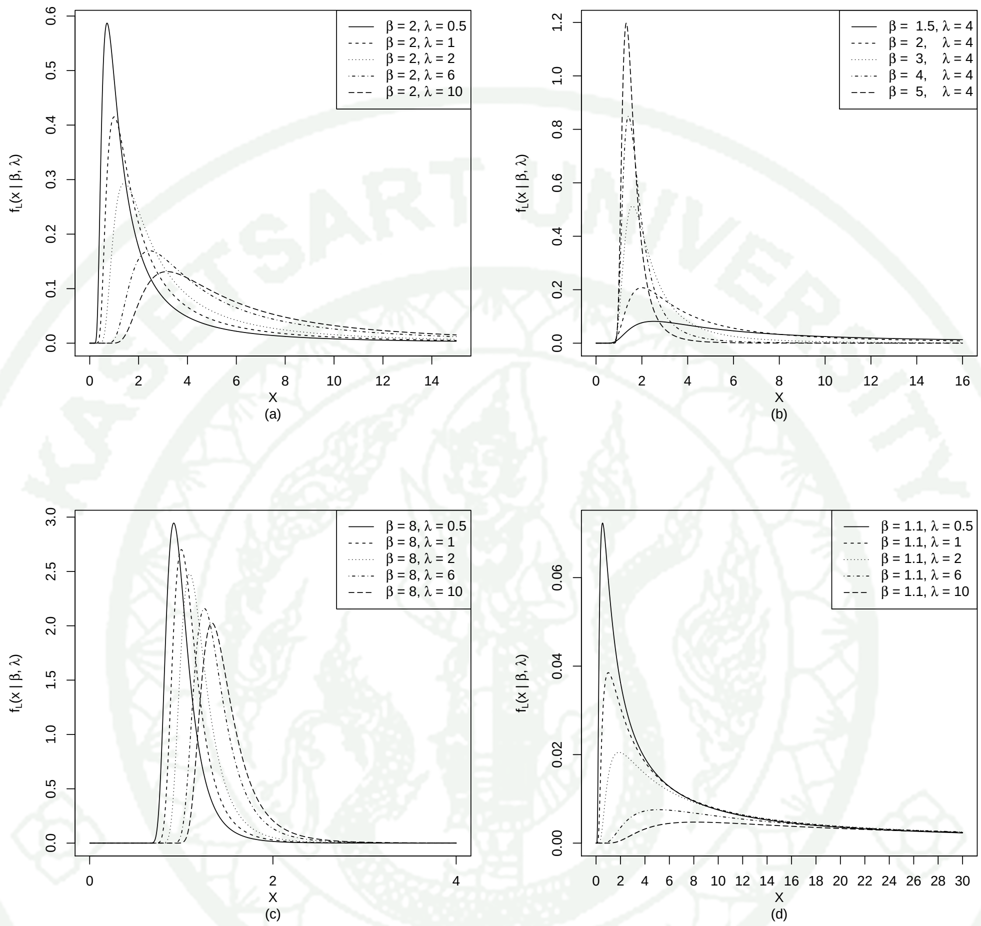

Figure 8 The pdf of the LBEIW distribution for some specified values of β and λ

Theorem 6 Let X be a random variable of the LBEIW distribution with parameters β and λ. The cdf of the LBEIW distribution is written as

$$F_L(x|\beta,\lambda) = \frac{\Gamma(1-\frac{1}{\beta},\frac{\lambda}{x^\beta})}{\Gamma(1-\frac{1}{\beta})}. \tag{71}$$

Proof. By Substituting $f_L(x|\beta,\lambda)$, Eq.(70), into definition of cdf, Eq.(1), then

$$F_L(x|\beta,\lambda) = \int_0^x \frac{\beta\lambda^{1-\frac{1}{\beta}}}{\Gamma(1-\frac{1}{\beta})} t^{-\beta}\{\exp(-t^{-\beta})\}^{\lambda}\,dt.$$

By setting $u=\lambda t^{-\beta}$, $\dfrac{\lambda}{x^\beta} < u < \infty$, the above integration becomes:

$$\begin{aligned} F_L(x|\beta,\lambda) &= \frac{1}{\Gamma(1-\frac{1}{\beta})\lambda^{\frac{1}{\beta}}}\int_{\frac{\lambda}{x^\beta}}^{\infty} \frac{\lambda^{\frac{1}{\beta}}}{u^{\frac{1}{\beta}}}\exp(-u)\,du \\ &= \frac{1}{\Gamma(1-\frac{1}{\beta})}\int_{\frac{\lambda}{x^\beta}}^{\infty} u^{-\frac{1}{\beta}}\exp(-u)\,du \\ &= \frac{1}{\Gamma(1-\frac{1}{\beta})}\int_{\frac{\lambda}{x^\beta}}^{\infty} u^{\left(1-\frac{1}{\beta}\right)-1}\exp(-u)\,du \\ &= \frac{\Gamma(1-\frac{1}{\beta},\frac{\lambda}{x^\beta})}{\Gamma(1-\frac{1}{\beta})}. \quad \square \end{aligned}$$

Some plots of the LBEIW cdf with specified parameter values are shown in Figure 9.

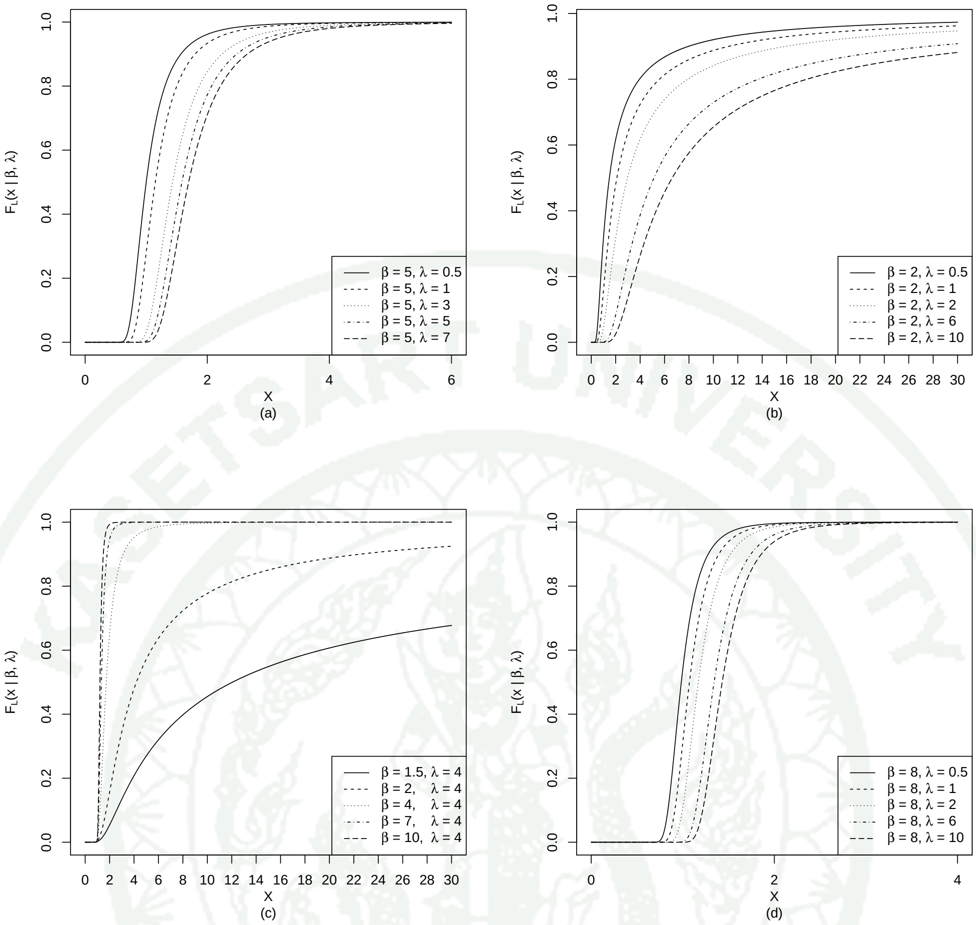

Figure 9 The cdf of the LBEIW distribution for some specified values of β and λ

Theorem 7 Let X be a random variable of the LBEIW distribution with parameters β and λ. The survival function of the LBEIW distribution can be defined as

$$S_L(x|\beta,\lambda) = \frac{\gamma(1-\frac{1}{\beta},\frac{\lambda}{x^{\beta}})}{\Gamma(1-\frac{1}{\beta})}. \tag{72}$$

Proof. By substituting $F_L(x|\beta,\lambda)$, Eq.(71), into equation of the survival function, Eq.(2), the survival function of the LBEIW distribution can be expressed by

$$\begin{aligned} S_L(x|\beta,\lambda) &= 1-\frac{\Gamma(1-\frac{1}{\beta},\frac{\lambda}{x^{\beta}})}{\Gamma(1-\frac{1}{\beta})} \\ &= \frac{\Gamma(1-\frac{1}{\beta})-\Gamma(1-\frac{1}{\beta},\frac{\lambda}{x^{\beta}})}{\Gamma(1-\frac{1}{\beta})} \\ &= \frac{\gamma(1-\frac{1}{\beta},\frac{\lambda}{x^{\beta}})}{\Gamma(1-\frac{1}{\beta})}. \quad \square \end{aligned}$$

Some survival function plots of the LBEIW distribution with specified parameter values are shown in Figure 10.

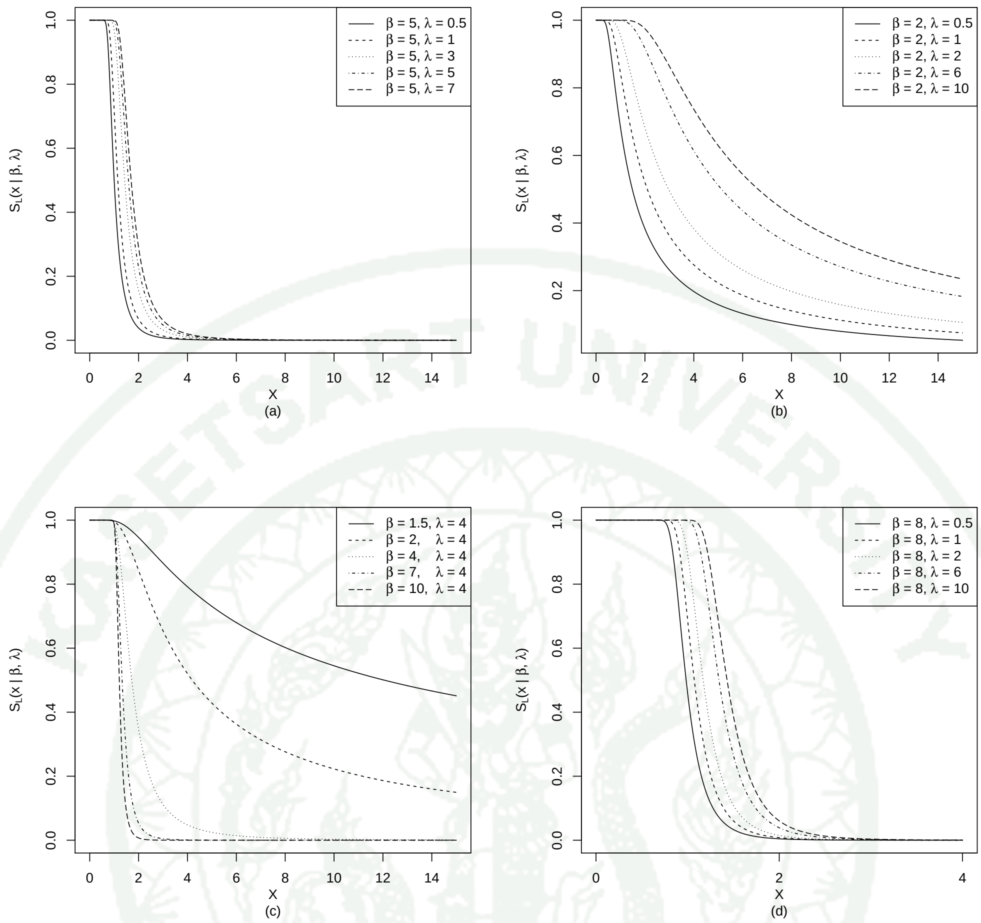

Figure 10 The LBEIW survival functions for some specified values of β and λ

Theorem 8 If $X \sim$ LBEIW(β,λ), the hazard rate of the LBEIW distribution is

$$h_L(x|\beta,\lambda) = \frac{\beta\lambda^{1-\frac{1}{\beta}}x^{-\beta}\{\exp(-x^{-\beta})\}^{\lambda}}{\gamma(1-\frac{1}{\beta},\frac{\lambda}{x^{\beta}})}. \tag{73}$$

Proof. By substituting $f_L(x|\beta,\lambda)$, Eq.(70), and $S_L(x|\beta,\lambda)$, Eq.(72), into definition of hazard rate, Eq.(3), we obtain

$$\begin{aligned} h_L(x|\beta,\lambda) &= \frac{\beta\lambda^{1-\frac{1}{\beta}}x^{-\beta}\{\exp(-x^{-\beta})\}^{\lambda}/\Gamma(1-\frac{1}{\beta})}{\gamma(1-\frac{1}{\beta},\frac{\lambda}{x^{\beta}})/\Gamma(1-\frac{1}{\beta})} \\ &= \frac{\beta\lambda^{1-\frac{1}{\beta}}x^{-\beta}\{\exp(-x^{-\beta})\}^{\lambda}}{\gamma(1-\frac{1}{\beta},\frac{\lambda}{x^{\beta}})}. \quad \square \end{aligned}$$

Some hazard rate plots of the LBEIW distribution with specified parameter values are displayed in Figure 11.

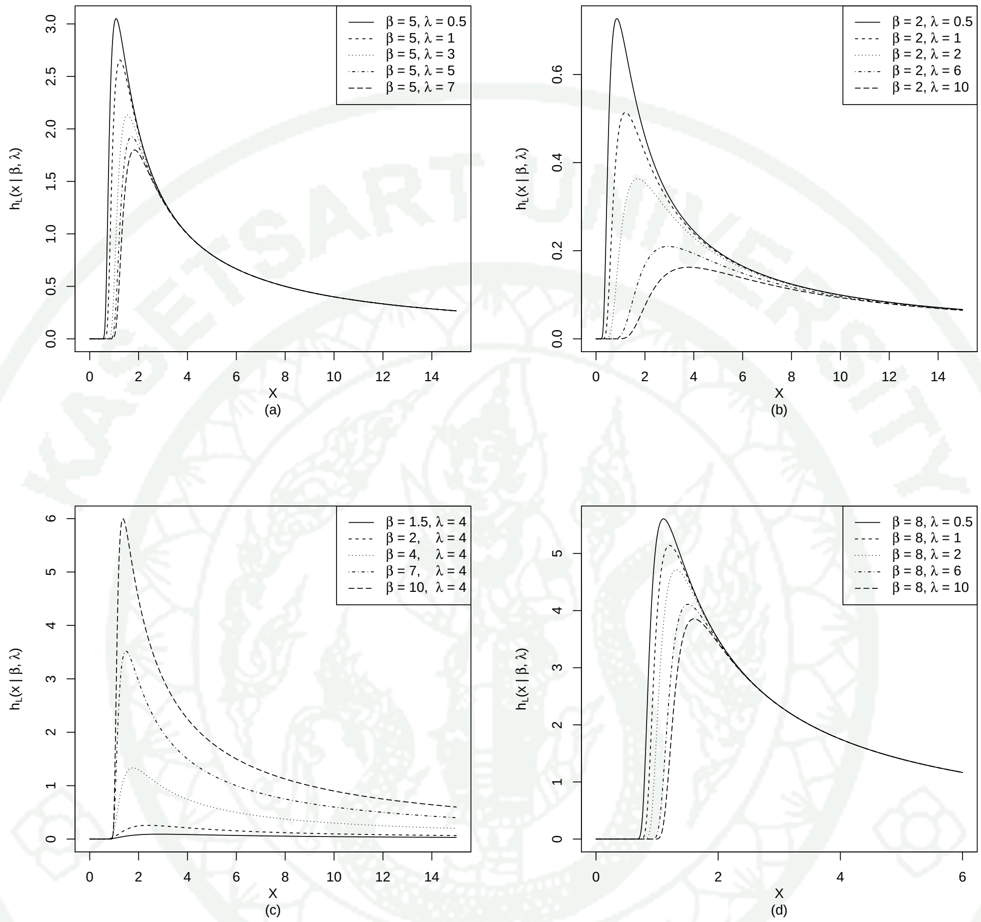

Figure 11 The LBEIW hazard rate for some specified values of β and λ

1.1 Some properties of the LBEIW distribution

The result of this section gives the k^{th} moment about the origin, first four moments about the origin, mean, variance, coefficient of skewness and coefficient of kurtosis of LBEIW distribution.

The following theorem provides the k^{th} moment about the origin of the LBEIW distribution.

Theorem 9 If $X \sim \text{LBEIW}(\beta, \lambda)$, then the k^{th} moment about the origin of X is given by:

$$E(X^k) = \lambda^{\frac{k}{\beta}} \frac{\Gamma(1-\frac{k+1}{\beta})}{\Gamma(1-\frac{1}{\beta})}, \tag{74}$$

where $\lambda > 0$, $\beta > k+1$ and $k = 1, 2, 3, \ldots$.

Proof. From the pdf of the LBEIW distribution, Eq.(70), and definition of the k^{th} moment about the origin, Eq.(8), then $E(X^k)$ can be written as:

$$E(X^k) \;=\; \int_0^\infty x^k \frac{\beta\lambda^{1-\frac{1}{\beta}}}{\Gamma(1-\frac{1}{\beta})} x^{-\beta}\{\exp(-x^{-\beta})\}^{\lambda}\, dx.$$

By setting $u = \lambda x^{-\beta}$ the above integration becomes:

$$\begin{aligned}
E(X^k) &= \frac{(-1)}{\Gamma(1-\frac{1}{\beta})\lambda^{\frac{1}{\beta}}} \int_\infty^0 \frac{\lambda^{\frac{k+1}{\beta}}}{u^{\frac{k+1}{\beta}}} \exp(-u)\, du \\
&= \frac{\lambda^{\frac{k}{\beta}}}{\Gamma(1-\frac{1}{\beta})} \int_0^\infty u^{-\frac{k+1}{\beta}} \exp(-u)\, du \\
&= \frac{\lambda^{\frac{k}{\beta}}}{\Gamma(1-\frac{1}{\beta})} \int_0^\infty u^{\left(1-\frac{k+1}{\beta}\right)-1} \exp(-u)\, du \\
&= \lambda^{\frac{k}{\beta}} \frac{\Gamma(1-\frac{k+1}{\beta})}{\Gamma(1-\frac{1}{\beta})}. \quad \square
\end{aligned}$$

From the k^{th} moment about the origin of the LBEIW distribution, it is straightforward to deduce the first four moments about the origin, respectively

$$E(X) = \lambda^{\frac{1}{\beta}} \frac{\Gamma(1-\frac{2}{\beta})}{\Gamma(1-\frac{1}{\beta})}, \tag{75}$$

$$E(X^2) = \lambda^{\frac{2}{\beta}} \frac{\Gamma(1-\frac{3}{\beta})}{\Gamma(1-\frac{1}{\beta})}, \tag{76}$$

$$E(X^3) = \lambda^{\frac{3}{\beta}} \frac{\Gamma(1-\frac{4}{\beta})}{\Gamma(1-\frac{1}{\beta})}, \tag{77}$$

$$E(X^4) = \lambda^{\frac{4}{\beta}} \frac{\Gamma(1-\frac{5}{\beta})}{\Gamma(1-\frac{1}{\beta})}. \tag{78}$$

By the first four moments about the origin, we can calculate the mean, variance, coefficient of skewness and coefficient of kurtosis of the LBEIW distribution as following:

Corollary 2 From the 1^{st} moment about the origin of LBEIW random variable, Eq.(75), we obtain the mean of the LBEIW distribution as

$$E(X) = \lambda^{\frac{1}{\beta}} \frac{\Gamma(1-\frac{2}{\beta})}{\Gamma(1-\frac{1}{\beta})}; \quad \beta > 2. \tag{79}$$

Corollary 3 The variance of $X \sim$ LBEIW (β, λ) is given by

$$Var(X) = \frac{\lambda^{\frac{2}{\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)} \left[\Gamma\left(1-\frac{3}{\beta}\right) - \frac{\left[\Gamma\left(1-\frac{2}{\beta}\right)\right]^2}{\Gamma\left(1-\frac{1}{\beta}\right)} \right], \tag{80}$$

where $\beta > 3$ and $\lambda > 0$.

Proof. By definition of variance, substitute $E(X^2)$, Eq.(76), and $E(X)$, Eq.(75), into Eq.(13), we get

$$\begin{aligned} Var(X) &= \lambda^{\frac{2}{\beta}} \frac{\Gamma(1-\frac{3}{\beta})}{\Gamma(1-\frac{1}{\beta})} - \left[\lambda^{\frac{1}{\beta}} \frac{\Gamma(1-\frac{2}{\beta})}{\Gamma(1-\frac{1}{\beta})}\right]^2 \\ &= \frac{\lambda^{\frac{2}{\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)} \left[\Gamma\left(1-\frac{3}{\beta}\right) - \frac{\left[\Gamma\left(1-\frac{2}{\beta}\right)\right]^2}{\Gamma\left(1-\frac{1}{\beta}\right)}\right]. \quad \square \end{aligned}$$

Corollary 4 The coefficient of skewness of the LBEIW distribution is given by

$$CS(X) = \frac{\Gamma_1^{\frac{1}{2}}\left[\Gamma_4 - 3\frac{\Gamma_2\Gamma_3}{\Gamma_1} + 2\frac{\Gamma_2^3}{\Gamma_1^2}\right]}{\left(\Gamma_3 - \frac{\Gamma_2^2}{\Gamma_1}\right)^{\frac{3}{2}}}, \tag{81}$$

where $\beta > 4$, $\lambda > 0$ and $\Gamma_i = \Gamma\left(1-\frac{i}{\beta}\right)$.

Proof. By substituting Eq.(75) - Eq.(77) and Eq.(80) into coefficient of skewness definition in Eq.(14), we get

$$\begin{aligned} CS(X) &= \frac{\lambda^{\frac{3}{\beta}}\frac{\Gamma_4}{\Gamma_1} - 3\lambda^{\frac{1}{\beta}}\frac{\Gamma_2}{\Gamma_1}\lambda^{\frac{2}{\beta}}\frac{\Gamma_3}{\Gamma_1} + 2\left[\lambda^{\frac{1}{\beta}}\frac{\Gamma_2}{\Gamma_1}\right]^3}{\left[\frac{\lambda^{\frac{2}{\beta}}}{\Gamma_1}\left(\Gamma_3 - \frac{\Gamma_2^2}{\Gamma_1}\right)\right]^{\frac{3}{2}}} \\ &= \frac{\frac{\lambda^{\frac{3}{\beta}}}{\Gamma_1}\left[\Gamma_4 - 3\frac{\Gamma_2\Gamma_3}{\Gamma_1} + 2\frac{\Gamma_2^3}{\Gamma_1^2}\right]}{\frac{\lambda^{\frac{3}{\beta}}}{\Gamma_1^{\frac{3}{2}}}\left(\Gamma_3 - \frac{\Gamma_2^2}{\Gamma_1}\right)^{\frac{3}{2}}} \\ &= \frac{\Gamma_1^{\frac{1}{2}}\left[\Gamma_4 - 3\frac{\Gamma_2\Gamma_3}{\Gamma_1} + 2\frac{\Gamma_2^3}{\Gamma_1^2}\right]}{\left(\Gamma_3 - \frac{\Gamma_2^2}{\Gamma_1}\right)^{\frac{3}{2}}}. \quad \square \end{aligned}$$

Corollary 5 The coefficient of kurtosis of the LBEIW distribution with parameters β and λ can be written as

$$CK(X) = \frac{\Gamma_1\Gamma_5 - 4\Gamma_2\Gamma_4 + 6\frac{\Gamma_2^2\Gamma_3}{\Gamma_1} - 3\frac{\Gamma_2^4}{\Gamma_1^2}}{\left[\Gamma_3 - \frac{\Gamma_2^2}{\Gamma_1}\right]^2} - 3, \tag{82}$$

where $\beta > 5$, $\lambda > 0$ and $\Gamma_i = \Gamma\left(1 - \frac{i}{\beta}\right)$.

Proof. From Eq.(15), we obtain

$$\begin{aligned}
CK(X) &= \frac{\lambda^{\frac{4}{\beta}}\frac{\Gamma_5}{\Gamma_1} - 4\lambda^{\frac{1}{\beta}}\frac{\Gamma_2}{\Gamma_1}\frac{\Gamma_4}{\Gamma_1} + 6\left[\lambda^{\frac{1}{\beta}}\frac{\Gamma_2}{\Gamma_1}\right]^2\lambda^{\frac{2}{\beta}}\frac{\Gamma_3}{\Gamma_1} - 3\left[\lambda^{\frac{1}{\beta}}\frac{\Gamma_2}{\Gamma_1}\right]^3}{\left[\frac{\lambda^{\frac{2}{\beta}}}{\Gamma_1}\left(\Gamma_3 - \frac{\Gamma_2^2}{\Gamma_1}\right)\right]^2} - 3 \\
&= \frac{\frac{\lambda^{\frac{4}{\beta}}}{\Gamma_1}\left[\Gamma_5 - 4\frac{\Gamma_2\Gamma_4}{\Gamma_1} + 6\frac{\Gamma_2^2\Gamma_3}{\Gamma_1^2} - 3\frac{\Gamma_2^4}{\Gamma_1^3}\right]}{\frac{\lambda^{\frac{4}{\beta}}}{\Gamma_1^2}\left[\Gamma_3 - \frac{\Gamma_2^2}{\Gamma_1}\right]^2} - 3 \\
&= \frac{\Gamma_1\left[\Gamma_5 - 4\frac{\Gamma_2\Gamma_4}{\Gamma_1} + 6\frac{\Gamma_2^2\Gamma_3}{\Gamma_1^2} - 3\frac{\Gamma_2^4}{\Gamma_1^3}\right]}{\left[\Gamma_3 - \frac{\Gamma_2^2}{\Gamma_1}\right]^2} - 3 \\
&= \frac{\Gamma_1\Gamma_5 - 4\Gamma_2\Gamma_4 + 6\frac{\Gamma_2^2\Gamma_3}{\Gamma_1} - 3\frac{\Gamma_2^4}{\Gamma_1^2}}{\left[\Gamma_3 - \frac{\Gamma_2^2}{\Gamma_1}\right]^2} - 3. \quad \square
\end{aligned}$$

1.2 A special case of the LBEIW distribution

The following corollary present a special case of the LBEIW distribution when $\lambda = 1$.

Corollary 6 For $\lambda = 1$, we get the length-biased inverted Weibull (LBIW) distribution with pdf given by

$$g_L(x|\beta) = \frac{\beta}{\Gamma(1-\frac{1}{\beta})} x^{-\beta}\{\exp(-x^{-\beta})\}. \tag{83}$$

Proof. By substituting $\lambda = 1$ in Eq.(70), we get

$$\begin{aligned} g_L(x|\beta,\lambda=1) &= \frac{\beta 1^{1-\frac{1}{\beta}}}{\Gamma(1-\frac{1}{\beta})} x^{-\beta}\{\exp(-x^{-\beta})\}^1 \\ &= \frac{\beta}{\Gamma(1-\frac{1}{\beta})} x^{-\beta}\{\exp(-x^{-\beta})\}. \quad \square \end{aligned}$$

1.3 Parameter estimation

In this section, we derive the methods, MLE and BE, for parameter estimation of the LBEIW distribution.

1.3.1 Maximum likelihood estimation

Now, the estimation of parameters for the LBEIW distribution via the MLE method procedure will be discussed. Suppose that $X_1, X_2, \ldots, X_n$ is a sample of size n obtained from the LBEIW distribution with parameters β and λ. It is assumed that both parameters β and λ are unknown, the likelihood function for the parameters β and λ is given by

$$L(\beta,\lambda|x) = \frac{\beta^n \lambda^{n(1-\frac{1}{\beta})}}{\left[\Gamma(1-\frac{1}{\beta})\right]^n} \prod_{i=1}^{n} x_i^{-\beta} \exp(-\lambda \sum_{i=1}^{n} x_i^{-\beta}). \quad (84)$$

By taking logarithmic of $L(\beta,\lambda|x)$, it is so called the log-likelihood function of X written as

$$\begin{aligned} l(\beta,\lambda|x) &= \log L(\beta,\lambda|x) \\ &= n\log\beta + n\log\lambda - \frac{n}{\beta}\log\lambda - \beta\sum_{i=1}^{n}\log x_i - \lambda\sum_{i=1}^{n} x_i^{-\beta} \\ &\quad -n\log\Gamma\left(1-\frac{1}{\beta}\right). \end{aligned} \quad (85)$$

The MLE solutions of parameters β and λ are obtained by setting the first partial derivatives of $l(\beta,\lambda|x)$ to zero with respect to β and λ, respectively. These simultaneous equations are

$$\frac{\partial l(\beta,\lambda|x)}{\partial\beta} = \frac{n}{\beta} + \frac{n}{\beta^2}\log\lambda - \sum_{i=1}^{n}\log x_i + \lambda\sum_{i=1}^{n} x_i^{-\beta}\log x_i - n\psi\left(1-\frac{1}{\beta}\right), \quad (86)$$

$$\frac{\partial l(\beta,\lambda|x)}{\partial\lambda} = \frac{n}{\lambda} - \frac{n}{\beta\lambda} - \sum_{i=1}^{n} x_i^{-\beta}. \quad (87)$$

It may be noted that this is an implicit equations in $\hat{\beta}_{MLE}$ and $\hat{\lambda}_{MLE}$, so it cannot be solved analytically. We apply the Newton-Raphson method to solve for MLE estimators.

1.3.2 Bayesian estimation

Parameter estimation of the LBEIW distribution using BE method under square error loss function will be discussed. Since the parameters β and λ are assumed to be unknown, the prior distribution for β and λ are taken to be a gamma distribution, given by

$$\begin{aligned} \beta \sim \text{Gamma}(a,b), \text{which pdf } \pi(\beta) &= \frac{b^a}{\Gamma(a)}\beta^{a-1}e^{-b\beta}, \qquad a,b>0, \\ \lambda \sim \text{Gamma}(c,d), \text{which pdf } \pi(\lambda) &= \frac{d^c}{\Gamma(c)}\lambda^{c-1}e^{-d\lambda}, \qquad c,d>0, \end{aligned}$$

where a, b, c and d are known parameters, called prior parameters.

Since β and λ are assumed to be independence, then the joint prior distribution of β and λ is of the form

$$\pi(\beta,\lambda) = \frac{b^a}{\Gamma(a)}\beta^{a-1}e^{-b\beta} \times \frac{d^c}{\Gamma(c)}\lambda^{c-1}e^{-d\lambda}. \tag{88}$$

Combining the prior with likelihood given by Eq.(88) and Eq.(84) respectively, we get the joint distribution has a pdf which is obtain by

$$\begin{aligned} \pi(x,\beta,\lambda) = &\frac{\beta^n\lambda^{n(1-\frac{1}{\beta})}}{\left[\Gamma(1-\frac{1}{\beta})\right]^n}\prod_{i=1}^{n}x_i^{-\beta}\exp(-\lambda\sum_{i=1}^{n}x_i^{-\beta}) \\ &\times\frac{b^a}{\Gamma(a)}\beta^{a-1}e^{-b\beta}\times\frac{d^c}{\Gamma(c)}\lambda^{c-1}e^{-d\lambda}. \end{aligned} \tag{89}$$

In addition, the marginal distribution can be written as

$$\pi(x) = \int_0^\infty \int_0^\infty \pi(x, \beta, \lambda)\, d\beta\, d\lambda. \tag{90}$$

Therefore, we can easily obtain the posterior distribution of β and λ as

$$\pi(\beta, \lambda | x) = \frac{\pi(x, \beta, \lambda)}{\pi(x)}. \tag{91}$$

As a result, the Bayes estimates of β and λ based on the square error loss function are respectively obtained as

$$\hat{\beta}_{BE} = E[\beta | x] = E_\beta[\beta, \lambda | x], \tag{92}$$

and

$$\hat{\lambda}_{BE} = E[\lambda | x] = E_\lambda[\beta, \lambda | x]. \tag{93}$$

The posterior distribution, Eq.(91), cannot be expressed in an explicit form. Therefore, the Bayesian estimator can be calculated by using numerical integration which is the MCMC technique. In this study, the Metropolis-Hastings algorithm (Ntzoufras, 2009) is applied for obtaining posterior distribution to estimated the parameters of the LBEIW distribution. The computational method is presented in Algorithm 3.

Algorithm 3 MCMC for estimating the parameters of the LBEIW distribution

1 Set initial values β_0 and λ_0.

2 For $r = 1, 2, 3, ..., R$ repeat the following steps

3 Propose new values β'_r and λ'_r from $\beta'_r = \beta_{r-1} + e_r$ and $\lambda'_r = \lambda_{r-1} + e_r$ where $e_r \sim N(0, \sigma^2 I)$

4 Substitute the β'_r and λ'_r obtained in step 3 and the β_{r-1} and λ_{r-1} from the previous iteration into $\pi(\beta, \lambda|x)$ in order to compute the acceptance probability; viz.
$\alpha = \min(1, A)$
where $A = \frac{\pi(\beta'_r, \lambda'_r|x)\pi(\beta'_r, \lambda'_r)}{\pi(\beta_{r-1}, \lambda_{r-1}|x)\pi(\beta_{r-1}, \lambda_{r-1})}$

5 Generate a realization $X \sim U(0, 1)$,
where $U(0, 1)$ represents a uniform distribution on the interval (0,1).

6 Set $\beta_r = \beta'_r$ and $\lambda_r = \lambda'_r$ if $x < \alpha$
otherwise set $\beta_r = \beta_{r-1}$ and $\lambda_r = \lambda_{r-1}$.

7 End for

8 Calculate the Bayesian estimator of the LBEIW distribution by
$\hat{\beta}_{BE} = \sum_{r=1}^{R} \frac{\beta_r}{R}$ and $\hat{\lambda}_{BE} = \sum_{r=1}^{R} \frac{\lambda_r}{R}$.

1.4 Simulation study

In this section, we present a simulation study for the estimation of parameter. Our objective here is to compare the true values of the parameters of the LBEIW(β, λ) and their estimates from the MLE and BE methods. In this study, the specified parameter values are composed of β = 2, 5, 8 and λ = 1, 5, 10. The summary of the specified parameter set is 9 cases. All cases in the simulation study were then generated from this LBEIW distribution with sample sizes n = 15, 30, 50, 100, 150, 200, 300 and 500, respectively. Simulation study of the all possible cases mentioned above is running via R program (R Core Team, 2013).

1.4.1 The Generation of the LBEIW random variate

We use inverse transformation technique to generate random data from LBEIW distribution by setting

$$U = F_L^{-1}(x), \tag{94}$$

where U is distributed as uniform distribution on (0,1), denoted as $U(0,1)$.

The cdf of the LBEIW distribution from Eq.(71) can be written in term of

$$F_L(x) = \frac{\Gamma(1-\frac{1}{\beta}, \frac{\lambda}{x^{\beta}})}{\Gamma(1-\frac{1}{\beta})}.$$

To generate random data $X_i, i = 1, \ldots, n$, from LBEIW(β, λ), one can use the following steps,

1) Generate $U_i, i = 1, ..., n$, from $U(0,1)$.

2) Set $U_i = F_L(x_i)$, then $U_i = \dfrac{\Gamma(1-\frac{1}{\beta}, \frac{\lambda}{x_i{}^{\beta}})}{\Gamma(1-\frac{1}{\beta})}$

and $U_i\Gamma(1-\dfrac{1}{\beta}) = \Gamma(1-\dfrac{1}{\beta}, \dfrac{\lambda}{x_i{}^{\beta}})$.

3) Assigned $v_i = U_i\Gamma(1-\dfrac{1}{\beta})$.

4) Set $a_i = Igamma.Inv(1-\dfrac{1}{\beta}, v_i)$, when $Igamma.Inv$ is inverted of incomplete gamma function in R program.

5) Set $a_i = \dfrac{\lambda}{x_i{}^{\beta}}$.

6) Then, $X_i = (\dfrac{\lambda}{a_i})^{\frac{1}{\beta}}$.

1.4.2 A comparison of the parameter estimation methods

In this section, comparison of parameter estimation of the LBEIW distribution use the MLE and BE methods in a Monte Carlo simulation. The sample mean, variance and MSE of the estimated parameter are calculated by Eq.(67), Eq.(68) and Eq.(69), respectively. The MSE is illustrated to compare the performances of the parameter estimation methods.

The sample mean, variance and MSE of the estimates are calculated based on 1,000 Monte Carlo simulation and the results are illustrated in Table A1 - Table A9 (see Appendix A). The MSE plots of the both MLE and BE estimates shown in Figure 12 - Figure 20. We found that in most of the considered cases, the BE relative to square error loss function are better than their corresponding MLE. As to be expected, the MSE of the estimated parameters decrease as sample sizes, n, increases.

Obviously, the MSE of $\hat{\beta}$ and $\hat{\lambda}$ from MLE are very different from

BE in cases of small sample sizes. When sample sizes increase, the MSE of $\hat{\beta}$ and $\hat{\lambda}$ from MLE and BE are very close.

However, the MLE sometimes does not converge in the Newton-Raphson procedure and fails to estimate parameters, as the BE does not show this problem.

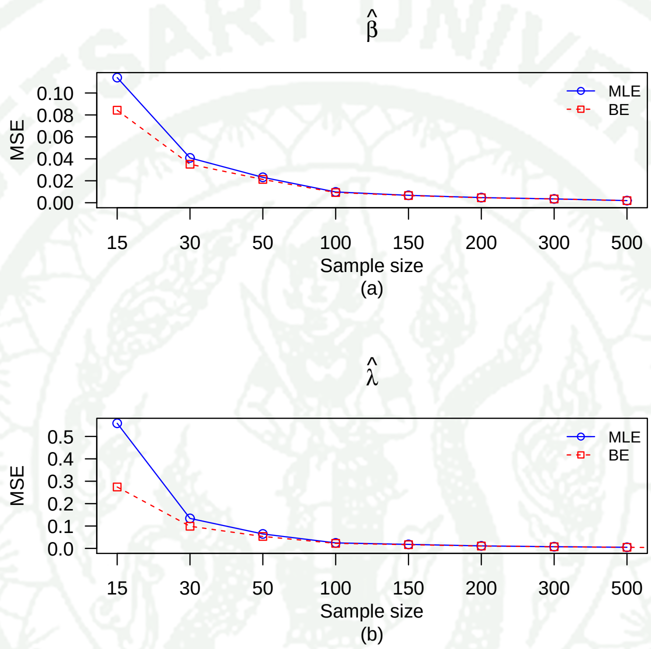

Figure 12 The estimated MSE of $\hat{\beta}$ and $\hat{\lambda}$ from MLE and BE obtained from $X \sim \text{LBEIW}(2,1)$

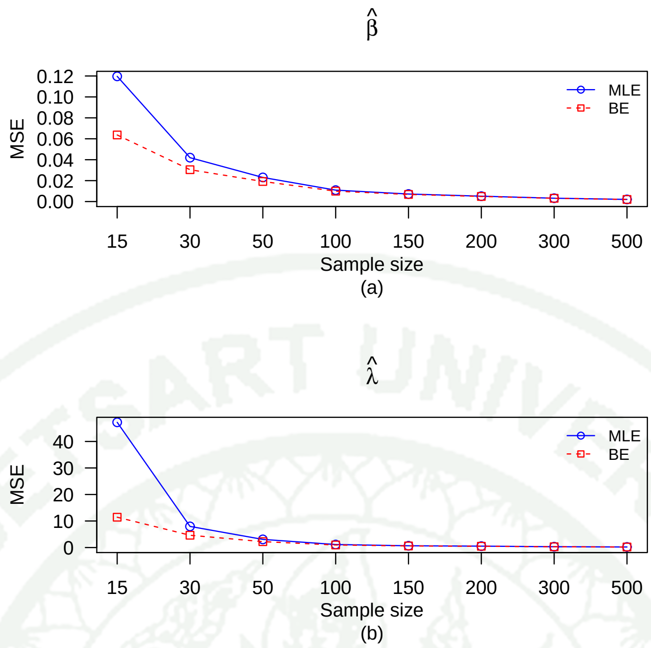

Figure 13 The estimated MSE of $\hat{\beta}$ and $\hat{\lambda}$ from MLE and BE obtained from $X \sim \text{LBEIW}(2,5)$

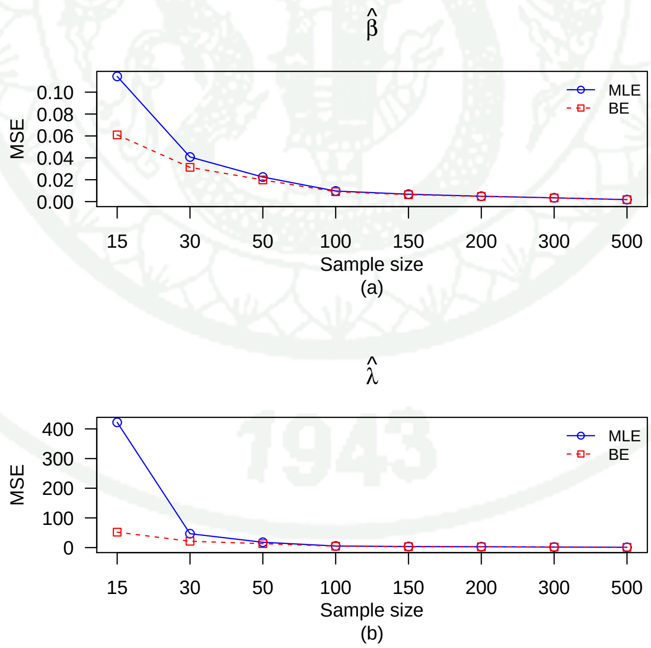

Figure 14 The estimated MSE of $\hat{\beta}$ and $\hat{\lambda}$ from MLE and BE obtained from $X \sim \text{LBEIW}(2,10)$

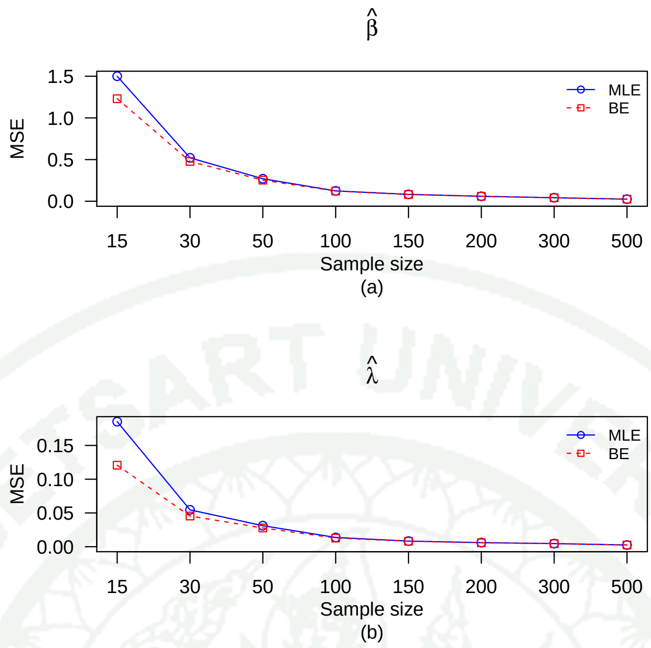

Figure 15 The estimated MSE of $\hat{\beta}$ and $\hat{\lambda}$ from MLE and BE obtained from $X \sim \text{LBEIW}(5,1)$

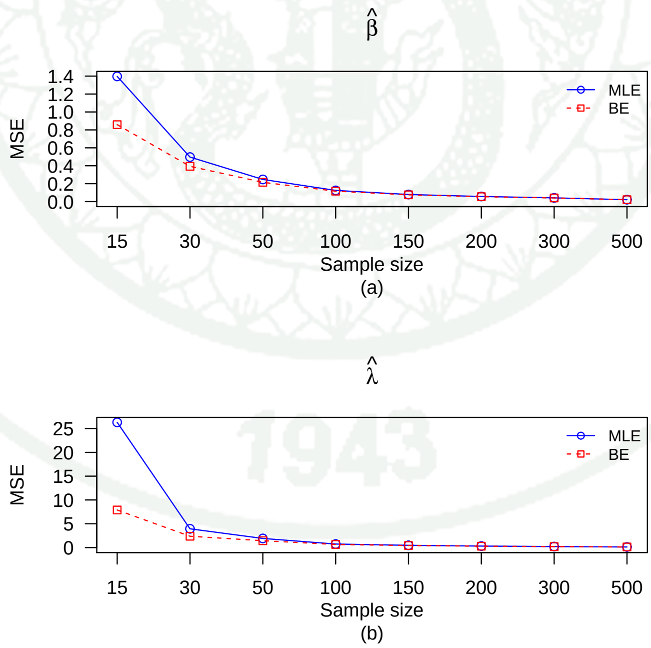

Figure 16 The estimated MSE of $\hat{\beta}$ and $\hat{\lambda}$ from MLE and BE obtained from $X \sim \text{LBEIW}(5,5)$

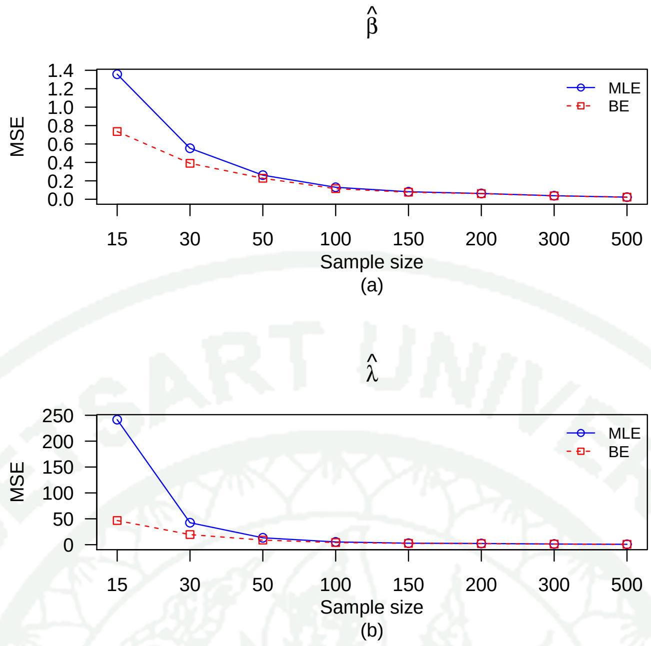

Figure 17 The estimated MSE of $\hat{\beta}$ and $\hat{\lambda}$ from MLE and BE obtained from $X \sim \text{LBEIW}(5, 10)$

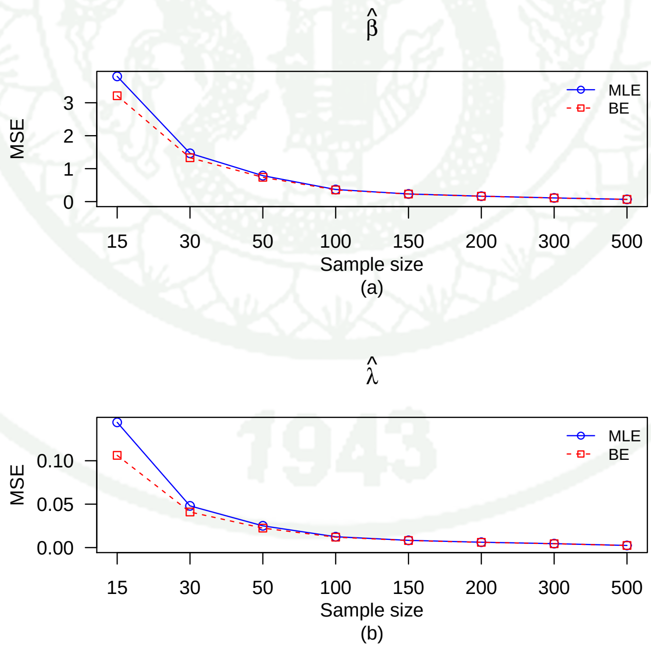

Figure 18 The estimated MSE of $\hat{\beta}$ and $\hat{\lambda}$ from MLE and BE obtained from $X \sim \text{LBEIW}(8, 1)$

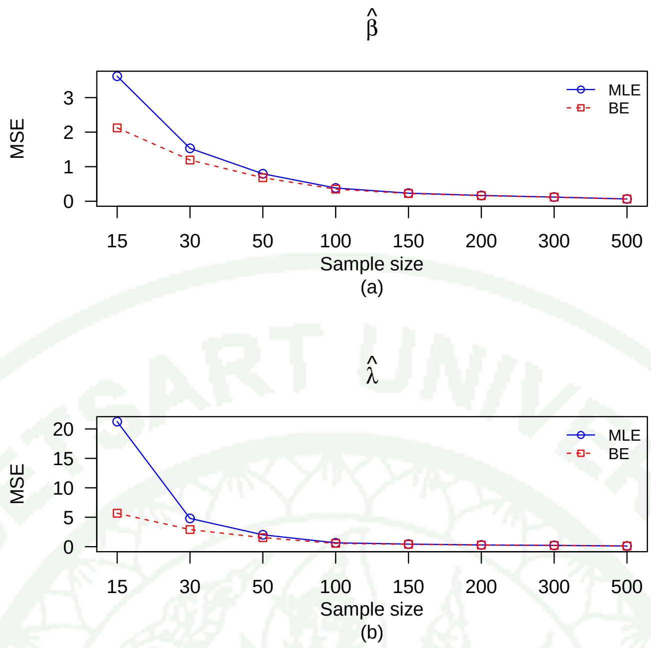

Figure 19 The estimated MSE of $\hat{\beta}$ and $\hat{\lambda}$ from MLE and BE obtained from $X \sim \text{LBEIW}(8,5)$

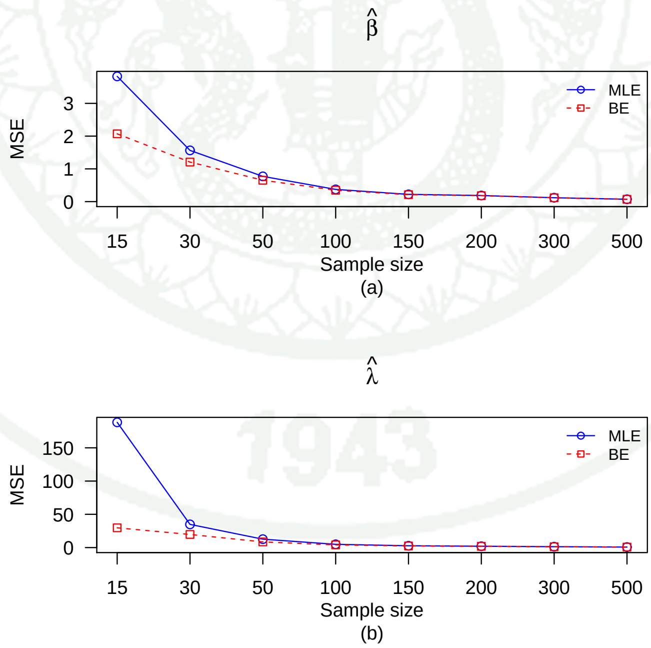

Figure 20 The estimated MSE of $\hat{\beta}$ and $\hat{\lambda}$ from MLE and BE obtained from $X \sim \text{LBEIW}(8,10)$

1.5 Application study

In this study, some comparison of model fitting based on the LBEIW, EIW, LBIW and two-parameter Weibull distributions will be discussed. We consider uncensored data of 24 observed on distance between cracks in a pipe dataset (Lawless, 2003). The AD test and the AIC are performed in order to verify which distribution fits the data. Table 1 lists the value of the AD test, AIC and MLE estimates of the parameters. The p-value of AD test for the LBEIW distribution is close to these for EIW distribution that same as the AIC value. Then the LBEIW distribution fits the data as well as the EIW distribution.

Table 1 Goodness of fit summary of the distance between cracks in a pipe dataset

Fitting Distribution	**AD test**		**AIC**	**Estimate**	
	Statistic	**p-value**			
LBEIW	0.2905	0.9447	176.7056	$\hat{\beta} = 3.3891$,	$\hat{\lambda} = 9508.9505$
EIW	0.2213	0.9835	176.0836	$\hat{\beta} = 2.7347$,	$\hat{\lambda} = 2384.5601$
LBIW	7.7953	0.0002	245.7286	$\hat{\beta} = 1.3484$	
Weibull	0.7467	0.5196	181.7820	$\hat{\beta} = 26.0230$,	$\hat{\alpha} = 2.3089$

The empirical cumulative distribution function (ecdf) plots (step line) of this dataset with the estimated cdf curves (dot line) based on LBEIW, EIW, LBIW and two-parameter Weibull distributions are shown in Figure 21.

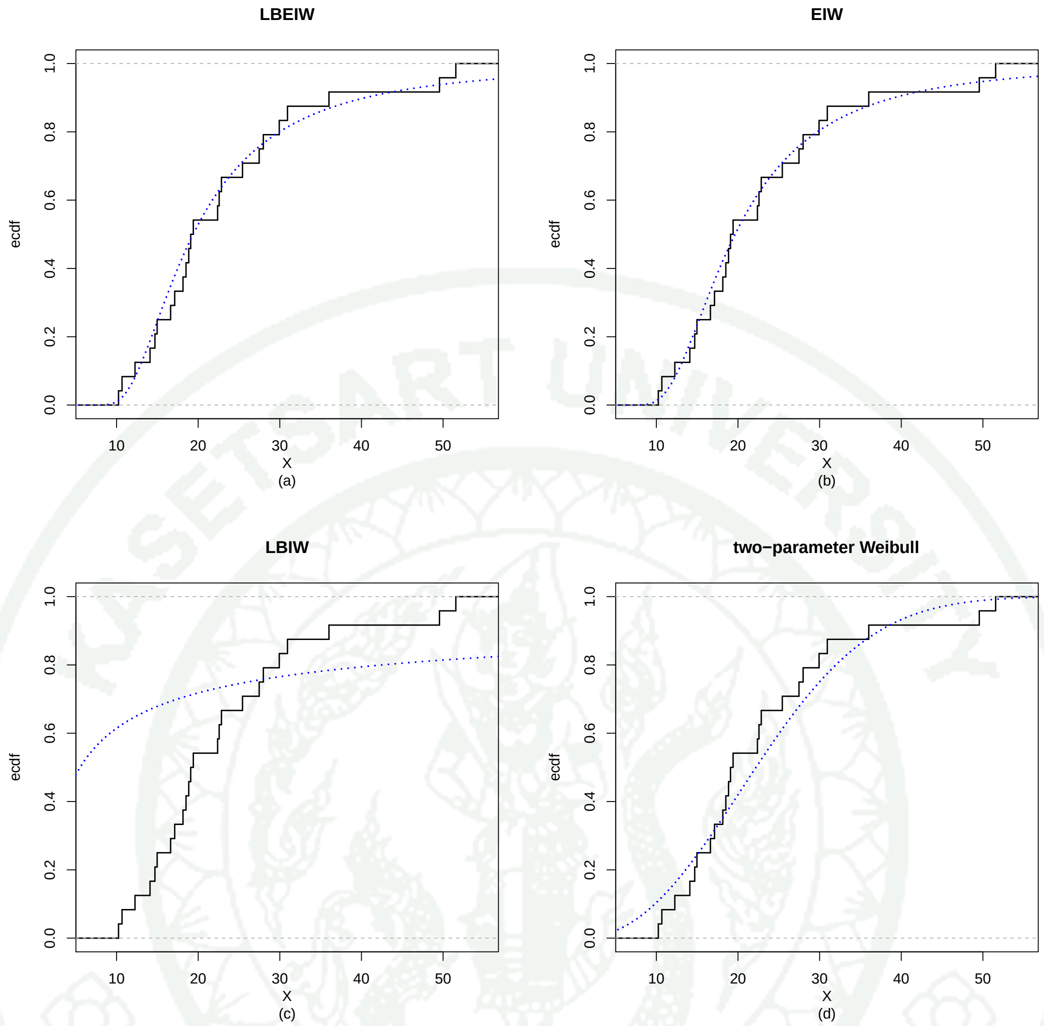

Figure 21 The ecdf plots (step line) with the estimated cdf curves (dot line) based on (a) LBEIW, (b) EIW, (c) LBIW and (d) two-parameter Weibull distributions

In addition, the probability plot of the LBEIW distribution with parameter estimation corresponded to MLE for this dataset, is shown in the Figure 22.

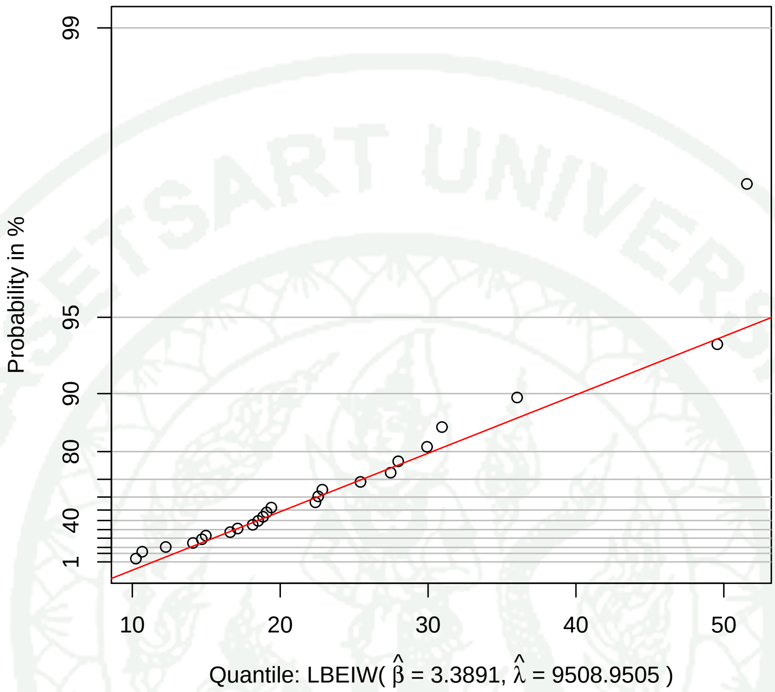

Figure 22 The probability plot for the model based on the LBEIW distribution applied to the distance between cracks in a pipe dataset.

Moreover, we compare the MLE method with the BE method in the parameter estimation for this dataset. The estimators obtained by both approaches are shown in Table 2. The BE gives the p-value of the AD test, which is slightly larger than those of the MLE. Similarly, the AIC value of BE is also lower than these of the MLE. That is the BE outperforms the MLE for the distance between cracks in a pipe dataset. Furthermore, histogram of the breaking strengths of single carbon fibers dataset and pdf curve of the MLE and BE estimates based on LBEIW distribution corresponding to this dataset shown in the Figure 23.

Table 2 Comparison of estimators obtained by MLE and BE methods for the distance between cracks in a pipe dataset

Method	AD test		AIC	Estimate
	Statistic	p-value		
MLE	0.2905	0.9447	176.7056	$\hat{\beta} = 3.3891, \quad \hat{\lambda} = 9508.9505$
BE	0.2795	0.9523	176.5734	$\hat{\beta} = 3.3127, \quad \hat{\lambda} = 7667.7310$

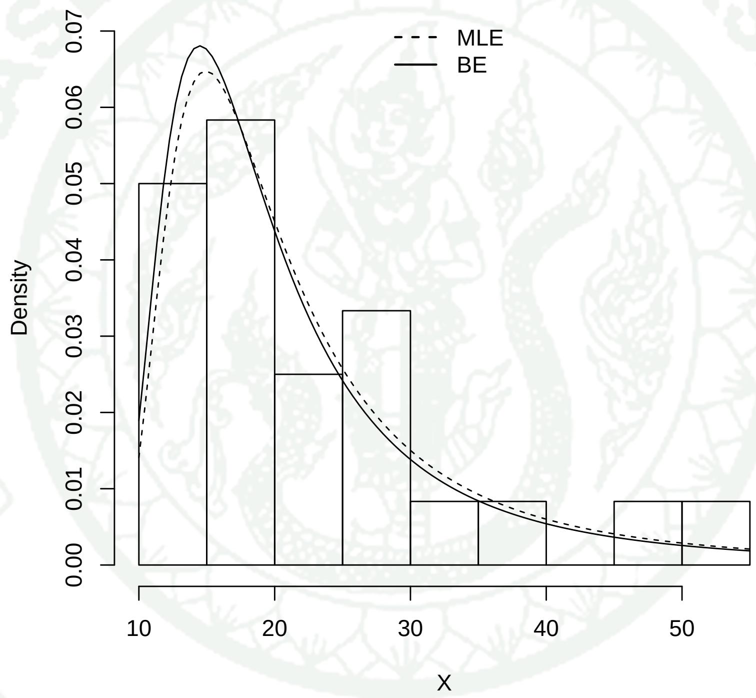

Figure 23 The goodness of fit plots of the MLE and BE estimates based on the MEIW distribution for the distance between cracks in a pipe dataset

2. The mixture exponentiated inverted Weibull distribution

The MEIW distribution is obtained by mixing between EIW(β,λ) and LBEIW(β,λ). It has three parameters β, λ and p, denoted by MEIW(β,λ,p). Statistical probability functions with some graphs, statistical properties, parameter estimation and application of the MEIW(β,λ,p) are discussed.

Theorem 10 Let $f_E(x|\beta,\lambda)$ and $f_L(x|\beta,\lambda)$ are pdf of the EIW and LBEIW distributions, respectively. The pdf of the MEIW distribution is given by

$$f(x|\beta,\lambda,p) = \left[p + (1-p)\frac{\lambda^{-\frac{1}{\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)}x\right]\beta\lambda x^{-(\beta+1)}\left\{\exp(-x^{-\beta})\right\}^{\lambda}, \tag{95}$$

where $\beta > 1$, $\lambda > 0$ and $0 \leq p \leq 1$.

Proof. By substituting $f_E(x|\beta,\lambda)$, Eq.(43), and $f_L(x|\beta,\lambda)$, Eq.(70), into Eq.(64), then the pdf of the MEIW(β,λ,p) can be written as

$$\begin{aligned}
f(x|\beta,\lambda,p) &= p\lambda\beta x^{-(\beta+1)}\{\exp(-x^{-\beta})\}^{\lambda} + (1-p)\frac{\beta\lambda^{1-\frac{1}{\beta}}}{\Gamma(1-\frac{1}{\beta})}x^{-\beta}\{\exp(-x^{-\beta})\}^{\lambda}, \\
&= \left[p\lambda + (1-p)\frac{\lambda^{1-\frac{1}{\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)}x\right]\beta x^{-(\beta+1)}\left\{\exp(-x^{-\beta})\right\}^{\lambda}, \\
&= \left[p + (1-p)\frac{\lambda^{-\frac{1}{\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)}x\right]\beta\lambda x^{-(\beta+1)}\left\{\exp(-x^{-\beta})\right\}^{\lambda}. \quad \square
\end{aligned}$$

The pdf plots of the MEIW distribution are shown in Figure 24 which can be seen as a unimodal and positively skewed.

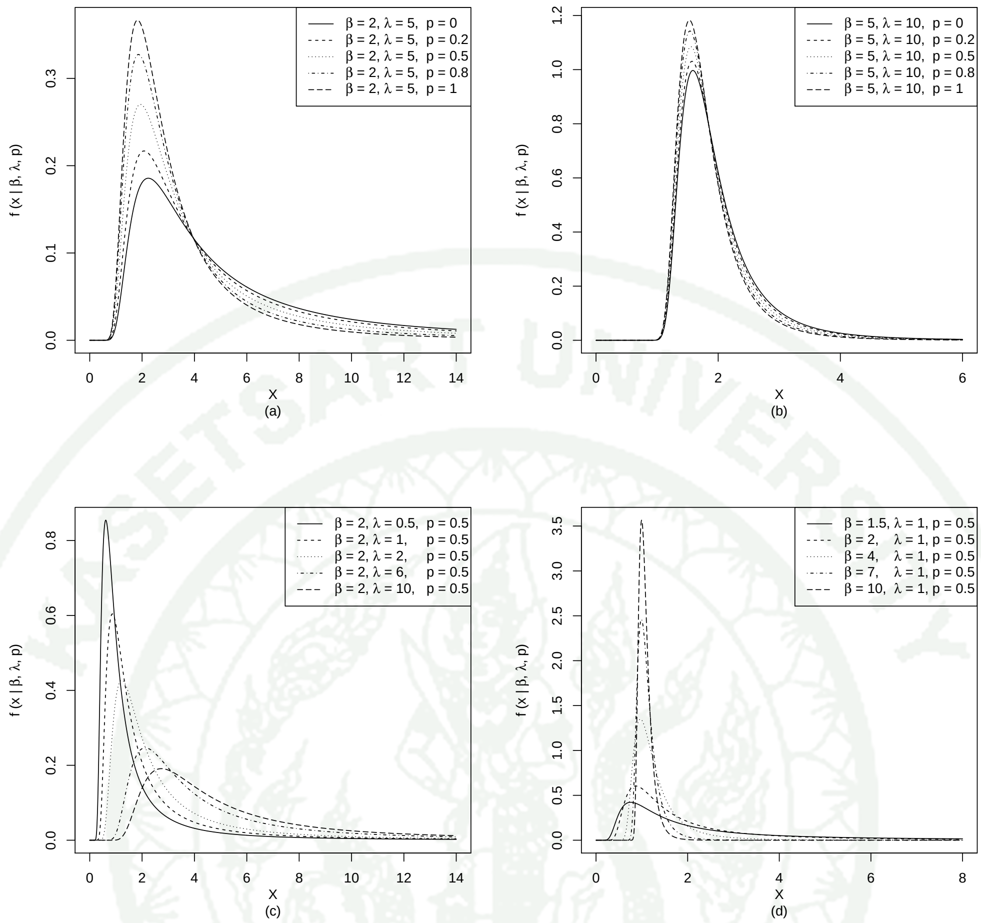

Figure 24 The pdf plots of the MEIW distribution for selected values of β, λ and p

Theorem 11 Let $X \sim \text{MEIW}(\beta, \lambda, p)$. The MEIW cdf can be written as

$$F(x|\beta,\lambda,p) = p\left\{\exp(-x^{-\beta})\right\}^{\lambda} + (1-p)\,\frac{\Gamma\left(1-\frac{1}{\beta},\frac{\lambda}{x^{\beta}}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)}. \tag{96}$$

Proof. By substituting the cdf of the EIW and LBEIW distributions into Eq.(65), the cdf of the MEIW distribution is given as

$$F(x|\beta,\lambda,p) \;=\; p\left\{\exp(-x^{-\beta})\right\}^{\lambda} + (1-p)\,\frac{\Gamma\left(1-\frac{1}{\beta},\frac{\lambda}{x^{\beta}}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)}. \quad \square$$

Some plots of the MEIW cdf with specified parameter values are shown in Figure 25.

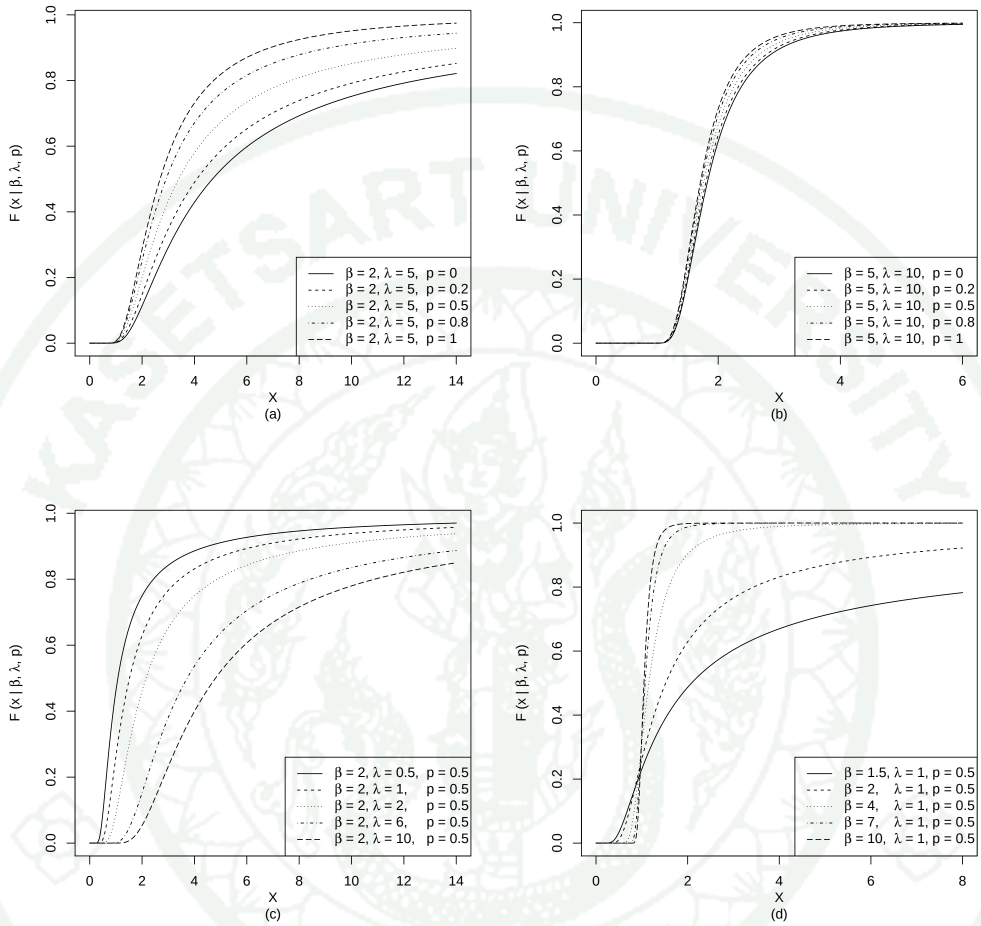

Figure 25 The MEIW cdf for some specified values of β, λ and *p*

Theorem 12 If $X \sim \text{MEIW}(\beta,\lambda,p)$, then the survival function of X is given by

$$S(x|\beta,\lambda,p) = 1 - \left[p\left\{\exp(-x^{-\beta})\right\}^{\lambda} + (1-p)\frac{\Gamma\left(1-\frac{1}{\beta},\frac{\lambda}{x^{\beta}}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)}\right]. \tag{97}$$

Proof. By substituting the cdf $F(x|\beta,\lambda,p)$ into survival function definition, Eq.(2), we get

$$S(x|\beta,\lambda,p) = 1 - \left[p\left\{\exp(-x^{-\beta})\right\}^{\lambda} + (1-p)\frac{\Gamma\left(1-\frac{1}{\beta},\frac{\lambda}{x^{\beta}}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)}\right]. \quad \square$$

Some survival function plots of the MEIW distribution with specified parameter values are shown in Figure 26.

Theorem 13 If X is distributed as MEIW with parameters β, λ and p, then the hazard rate of X can be written as

$$h(x|\beta,\lambda,p) = \frac{\left[p + (1-p)\frac{\lambda^{-\frac{1}{\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)}x\right]\beta\lambda x^{-(\beta+1)}\left\{\exp(-x^{-\beta})\right\}^{\lambda}}{1 - \left[p\left\{\exp(-x^{-\beta})\right\}^{\lambda} + (1-p)\frac{\Gamma\left(1-\frac{1}{\beta},\frac{\lambda}{x^{\beta}}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)}\right]}. \tag{98}$$

Proof. By substituting pdf and cdf of the MEIW distribution into Eq.(3), then the hazard rate of the MEIW distribution can be obtained by

$$h(x|\beta,\lambda,p) = \frac{\left[p + (1-p)\frac{\lambda^{-\frac{1}{\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)}x\right]\beta\lambda x^{-(\beta+1)}\left\{\exp(-x^{-\beta})\right\}^{\lambda}}{1 - \left[p\left\{\exp(-x^{-\beta})\right\}^{\lambda} + (1-p)\frac{\Gamma\left(1-\frac{1}{\beta},\frac{\lambda}{x^{\beta}}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)}\right]}. \quad \square$$

Some hazard rate plots of the MEIW distribution with specified parameter values are displayed in Figure 27.

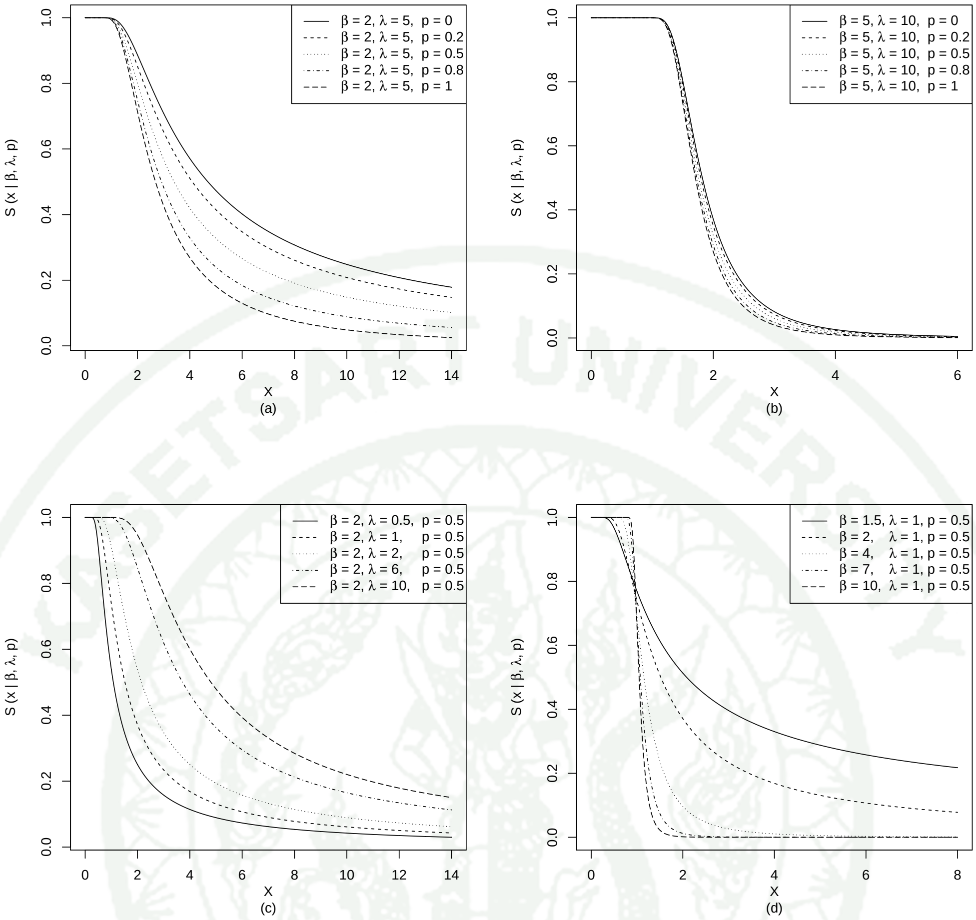

Figure 26 The MEIW survival functions for some specified values of β, λ and *p*

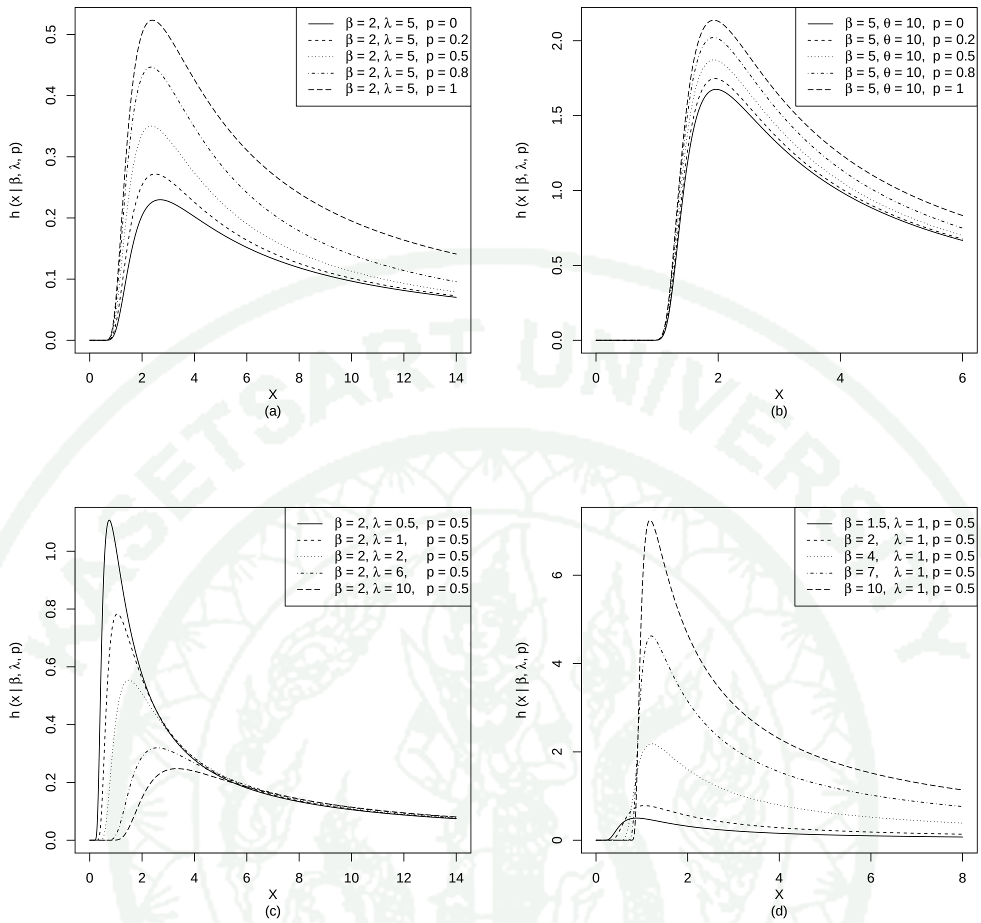

Figure 27 The MEIW hazard rate for some specified values of β, λ and *p*

2.1 Some properties of the MEIW distribution

This section shows the k^{th} moment about the origin, first four moments about the origin, mean, variance, coefficient of skewness and coefficient of kurtosis of the MEIW distribution.

The k^{th} moment about the origin of the MEIW distribution is defined as following theorem:

Theorem 14 The k^{th} moment about the origin of the MEIW distribution is given by

$$E(X^k) = \lambda^{\frac{k}{\beta}}\left[p\Gamma\left(1-\frac{k}{\beta}\right)+(1-p)\frac{\Gamma\left(1-\frac{k+1}{\beta}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)}\right], \tag{99}$$

where $\lambda > 0$, $0 \le p \le 1$, $\beta > k+1$ and $k = 1,2,3,\ldots$.

Proof. If X is distributed as MEIW(β,λ,p), the k^{th} moment about the origin of X is

$$\begin{aligned}
E(X^k) &= \int_0^\infty x^k f(x|\beta,\lambda,p)\,dx \\
&= \int_0^\infty x^k\left(pf_E(x|\beta,\lambda)+(1-p)f_L(x|\beta,\lambda)\right)dx \\
&= p\int_0^\infty x^k f_E(x|\beta,\lambda)\,dx+(1-p)\int_0^\infty x^k f_L(x|\beta,\lambda)\,dx \\
&= p\lambda^{\frac{k}{\beta}}\Gamma\left(1-\frac{k}{\beta}\right)+(1-p)\lambda^{\frac{k}{\beta}}\frac{\Gamma\left(1-\frac{k+1}{\beta}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)} \\
&= \lambda^{\frac{k}{\beta}}\left[p\Gamma\left(1-\frac{k}{\beta}\right)+(1-p)\frac{\Gamma\left(1-\frac{k+1}{\beta}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)}\right]. \quad \square
\end{aligned}$$

Consequently, the first four moments about the origin are written as

$$E(X) = \lambda^{\frac{1}{\beta}} \left[p\Gamma\left(1 - \frac{1}{\beta}\right) + (1-p)\frac{\Gamma\left(1 - \frac{2}{\beta}\right)}{\Gamma\left(1 - \frac{1}{\beta}\right)} \right], \tag{100}$$

$$E(X^2) = \lambda^{\frac{2}{\beta}} \left[p\Gamma\left(1 - \frac{2}{\beta}\right) + (1-p)\frac{\Gamma\left(1 - \frac{3}{\beta}\right)}{\Gamma\left(1 - \frac{1}{\beta}\right)} \right], \tag{101}$$

$$E(X^3) = \lambda^{\frac{3}{\beta}} \left[p\Gamma\left(1 - \frac{3}{\beta}\right) + (1-p)\frac{\Gamma\left(1 - \frac{4}{\beta}\right)}{\Gamma\left(1 - \frac{1}{\beta}\right)} \right], \tag{102}$$

$$E(X^4) = \lambda^{\frac{4}{\beta}} \left[p\Gamma\left(1 - \frac{4}{\beta}\right) + (1-p)\frac{\Gamma\left(1 - \frac{5}{\beta}\right)}{\Gamma\left(1 - \frac{1}{\beta}\right)} \right]. \tag{103}$$

Therefore, the mean, variance, coefficient of skewness and coefficient of kurtosis belonging to the MEIW distribution can be derived as follows:

Corollary 7 Set $k = 1$ in k^{th} moment about the origin of the MEIW distribution, the mean of the MEIW distribution is given by

$$E(X) = \lambda^{\frac{1}{\beta}} \left[p\Gamma\left(1 - \frac{1}{\beta}\right) + (1-p)\frac{\Gamma\left(1 - \frac{2}{\beta}\right)}{\Gamma\left(1 - \frac{1}{\beta}\right)} \right], \tag{104}$$

where $\beta > 2$, $\lambda > 0$ and $0 \leq p \leq 1$.

Corollary 8 The variance of $X \sim \text{MEIW}(\beta, \lambda, p)$ is derived as

$$Var(X) = \lambda^{\frac{2}{\beta}} \left\{ \left[p\Gamma\left(1-\frac{2}{\beta}\right) + (1-p)\frac{\Gamma\left(1-\frac{3}{\beta}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)} \right] \right.$$
$$\left. - \left[p\Gamma\left(1-\frac{1}{\beta}\right) + (1-p)\frac{\Gamma\left(1-\frac{2}{\beta}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)} \right]^2 \right\}, \tag{105}$$

where $\beta > 3$, $\lambda > 0$ and $0 \le p \le 1$.

Proof. By substituting $E(X^2)$, Eq.(101), and $E(X)$, Eq.(100), into variance definition, Eq.(13), we obtain

$$\begin{aligned} Var(X) &= \lambda^{\frac{2}{\beta}} \left[p\Gamma\left(1-\frac{2}{\beta}\right) + (1-p)\frac{\Gamma\left(1-\frac{3}{\beta}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)} \right] \\ &\quad - \left\{ \lambda^{\frac{1}{\beta}} \left[p\Gamma\left(1-\frac{1}{\beta}\right) + (1-p)\frac{\Gamma\left(1-\frac{2}{\beta}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)} \right] \right\}^2 \\ &= \lambda^{\frac{2}{\beta}} \left\{ \left[p\Gamma\left(1-\frac{2}{\beta}\right) + (1-p)\frac{\Gamma\left(1-\frac{3}{\beta}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)} \right] \right. \\ &\quad \left. - \left[p\Gamma\left(1-\frac{1}{\beta}\right) + (1-p)\frac{\Gamma\left(1-\frac{2}{\beta}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)} \right]^2 \right\}. \quad \square \end{aligned}$$

Corollary 9 The coefficient of skewness of the MEIW distribution is given by

$$CS(X) = \frac{\Gamma_3(p) - 3\Gamma_1(p)\Gamma_2(p) + 2\Gamma_1^3(p)}{\left[\Gamma_2(p) - \Gamma_1^2(p)\right]^{\frac{3}{2}}}, \tag{106}$$

where $\beta > 4$, $\lambda > 0$, $0 \le p \le 1$ and $\Gamma_i(p) = p\Gamma\left(1-\frac{i}{\beta}\right) - (1-p)\frac{\Gamma\left(1-\frac{i+1}{\beta}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)}$.

Proof. By substituting the first three moments about the origin and variance of the MEIW distribution into coefficient of skewness definition, Eq.(14), we obtain

$$
\begin{aligned}
CS(X) &= \frac{\lambda^{\frac{3}{\beta}}\Gamma_3(p) - 3\lambda^{\frac{3}{\beta}}\Gamma_1(p)\Gamma_2(p) + 2\lambda^{\frac{3}{\beta}}\Gamma_1^3(p)}{\left[\lambda^{\frac{2}{\beta}}\left(\Gamma_2(p) - \Gamma_1^2(p)\right)\right]^{\frac{3}{2}}} \\
&= \frac{\lambda^{\frac{3}{\beta}}\left[\Gamma_3(p) - 3\Gamma_1(p)\Gamma_2(p) + 2\Gamma_1^3(p)\right]}{\lambda^{\frac{3}{\beta}}\left[\Gamma_2(p) - \Gamma_1^2(p)\right]^{\frac{3}{2}}} \\
&= \frac{\Gamma_3(p) - 3\Gamma_1(p)\Gamma_2(p) + 2\Gamma_1^3(p)}{\left[\Gamma_2(p) - \Gamma_1^2(p)\right]^{\frac{3}{2}}}. \quad \square
\end{aligned}
$$

Corollary 10 The coefficient of kurtosis of the MEIW distribution with parameters β, λ and p is written as

$$CK(X) = \frac{\Gamma_4(p) - 4\Gamma_1(p)\Gamma_4(p) + 6\Gamma_1^2(p)\Gamma_2(p) - 3\Gamma_1^4(p)}{\left[\Gamma_2(p) - \Gamma_1^2(p)\right]^2} - 3 \qquad (107)$$

where $\beta > 5$, $\lambda > 0$ and $0 \leq p \leq 1$.

Proof. By substituting the first four moments about the origin, Eq.(100) - Eq.(103), and variance of the MEIW distribution, Eq.(105), into coefficient of kurtosis definition, we get

$$
\begin{aligned}
CK(X) &= \frac{\lambda^{\frac{4}{\beta}}\Gamma_4(p) - 4\lambda^{\frac{4}{\beta}}\Gamma_1(p)\Gamma_3(p) + 6\lambda^{\frac{4}{\beta}}\Gamma_1^2(p)\Gamma_2(p) - 3\lambda^{\frac{4}{\beta}}\Gamma_1^4(p)}{\lambda^{\frac{4}{\beta}}\left[\Gamma_2(p) - \Gamma_1^2(p)\right]^2} - 3 \\
&= \frac{\lambda^{\frac{4}{\beta}}\left[\Gamma_4(p) - 4\Gamma_1(p)\Gamma_3(p) + 6\Gamma_1^2(p)\Gamma_2(p) - 3\Gamma_1^4(p)\right]}{\lambda^{\frac{4}{\beta}}\left[\Gamma_2(p) - \Gamma_1^2(p)\right]^2} - 3 \\
&= \frac{\Gamma_4(p) - 4\Gamma_1(p)\Gamma_3(p) + 6\Gamma_1^2(p)\Gamma_2(p) - 3\Gamma_1^4(p)}{\left[\Gamma_2(p) - \Gamma_1^2(p)\right]^2} - 3. \quad \square
\end{aligned}
$$

2.2 A special case of the MEIW distribution

In Corollary 11 and Corollary 12 presents two special case of the MEIW distribution, when $p = 1$ and $p = 0$:

Corollary 11 Let $X \sim \text{MEIW}(\beta, \lambda, p)$. When $p = 1$, it can be reduced to the EIW distribution with the pdf given by

$$f_E(x|\beta,\lambda) = \beta\lambda x^{-(\beta+1)}\left\{\exp(-x^{-\beta})\right\}^{\lambda}. \tag{108}$$

Proof. By substituting $p = 1$ in $f(x|\beta,\lambda,p)$, Eq.(95), we get

$$\begin{aligned} f(x|\beta,\lambda,p=1) &= \left[1+(1-1)\frac{\lambda^{-\frac{1}{\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)}x\right]\beta\lambda x^{-(\beta+1)}\left\{\exp(-x^{-\beta})\right\}^{\lambda}, \\ &= \beta\lambda x^{-(\beta+1)}\left\{\exp(-x^{-\beta})\right\}^{\lambda}. \quad \square \end{aligned}$$

Corollary 12 Let $X \sim \text{MEIW}(\beta, \lambda, p)$. When $p = 0$, this distribution also called the LBEIW distribution with the pdf written as

$$f_L(x|\beta,\lambda) = \frac{\beta\lambda^{-\frac{1}{\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)}x^{-\beta}\left\{\exp(-x^{-\beta})\right\}^{\lambda}. \tag{109}$$

Proof. By substituting $p = 0$ in pdf of the $\text{MEIW}(\beta,\lambda,p)$, Eq.(95)

$$\begin{aligned} f(x|\beta,\lambda,p=0) &= \left[0+(1-0)\frac{\lambda^{-\frac{1}{\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)}x\right]\beta\lambda x^{-(\beta+1)}\left\{\exp(-x^{-\beta})\right\}^{\lambda}, \\ &= \left[\frac{\lambda^{-\frac{1}{\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)}x\right]\beta\lambda x^{-(\beta+1)}\left\{\exp(-x^{-\beta})\right\}^{\lambda}, \\ &= \frac{\beta\lambda^{-\frac{1}{\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)}x^{-\beta}\left\{\exp(-x^{-\beta})\right\}^{\lambda}. \quad \square \end{aligned}$$

2.3 Parameter estimation

In this section, we derive the methods for estimating the parameters of the MEIW distribution by two methods: the MLE and the Bayesian estimation.

2.3.1 Maximum likelihood estimation

Suppose that $X_1,\ldots,X_n$ is a random sample of size n obtained from the MEIW(β,λ,p). It is assumed that parameters β, λ and p are unknown, then the likelihood function for the parameters β, λ and p is

$$L(\beta,\lambda,p|x)=\prod_{i=1}^{n}\left\{\left[p+(1-p)\frac{\lambda^{-\frac{1}{\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)}x_i\right]\beta\lambda x_i{}^{-(\beta+1)}\left\{\exp(-x_i{}^{-\beta})\right\}^{\lambda}\right\}. \quad (110)$$

By taking logarithmic of the likelihood function, $L(\beta,\lambda,p|x)$, the log-likelihood function is

$$\begin{aligned} l(\beta,\lambda,p|x) \quad &= \quad \sum_{i=1}^{n}\log\left[p+(1-p)\frac{\lambda^{-\frac{1}{\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)}x_i\right]+n\log(\beta)+n\log(\lambda) \\ &\quad -(\beta+1)\sum_{i=1}^{n}\log(x_i)-\lambda\sum_{i=1}^{n}x_i{}^{-\beta}. \end{aligned} \quad (111)$$

The maximum likelihood estimates of β, λ and p are obtained by setting the first partial derivatives of the log-likelihood function, Eq.(111), to zero with respect to β, λ and p, respectively. These simultaneous equations are

$$\begin{aligned} \frac{\partial l(\beta,\lambda,p|x)}{\partial\beta} \quad &= \quad \sum_{i=1}^{n}\left\{\frac{(1-p)x_i\left(\log(\lambda)\beta^{-2}-\left[\Gamma\left(1-\frac{1}{\beta}\right)\right]^{-1}\Gamma'\left(1-\frac{1}{\beta}\right)\right)}{\lambda^{\frac{1}{\beta}}p\Gamma\left(1-\frac{1}{\beta}\right)+(1-p)\lambda^{-\frac{1}{\beta}}x_i}\right\} \\ &\quad +\frac{n}{\beta}-\sum_{i=1}^{n}\log(x_i)+\lambda\sum_{i=1}^{n}x_i{}^{-\beta}\log(x_i), \end{aligned}$$

$$\frac{\partial l(\beta,\lambda,p|x)}{\partial \lambda} = -\sum_{i=1}^{n}\left\{\frac{(1-p)\beta^{-1}\lambda^{-\frac{(\beta+1)}{\beta}}x_i}{p\Gamma\left(1-\frac{1}{\beta}\right)+(1-p)\lambda^{-\frac{1}{\beta}}x_i}\right\}+\frac{n}{\lambda}-\sum_{i=1}^{n}x_i{}^{-\beta},$$

$$\frac{\partial l(\beta,\lambda,p|x)}{\partial p} = \sum_{i=1}^{n}\left\{\frac{\Gamma\left(1-\frac{1}{\beta}\right)-\lambda^{-\frac{1}{\beta}}x_i}{p\Gamma\left(1-\frac{1}{\beta}\right)+(1-p)\lambda^{-\frac{1}{\beta}}x_i}\right\}.$$

A numerical procedure with the Newton-Raphson method is applied for solving an implicit equations in $\hat{\beta}_{MLE}$, $\hat{\lambda}_{MLE}$ and $\hat{p}_{MLE}$.

2.3.2 Bayesian estimation

This case proposes an approach to parameter estimation of the MEIW distribution using a BE method. We assume a gamma prior distribution with parameters β and λ and also assume a beta prior distribution for parameter p. The prior distributions are written as

$$\begin{aligned}
\beta \sim \text{Gamma}(a,b), \text{which pdf } \pi(\beta) &= \frac{b^a}{\Gamma(a)}\beta^{a-1}e^{-b\beta}, & a,b>0,\\
\lambda \sim \text{Gamma}(c,d), \text{which pdf } \pi(\lambda) &= \frac{d^c}{\Gamma(c)}\lambda^{c-1}e^{-d\lambda}, & c,d>0,\\
\text{and } p \sim \text{Beta}(e,f), \quad \text{which pdf } \pi(p) &= \frac{1}{B(e,f)}p^{e-1}(1-p)^{f-1}, & e,f>0.
\end{aligned}$$

Then, the joint prior density of β, λ and p can be obtained as

$$\pi(\beta,\lambda,p) = \frac{b^a}{\Gamma(a)}\beta^{a-1}e^{-b\beta}\times\frac{d^c}{\Gamma(c)}\lambda^{c-1}e^{-d\lambda}\times\frac{1}{B(e,f)}p^{e-1}(1-p)^{f-1}. \qquad (112)$$

Combining the prior given as $\pi(\beta,\lambda,p)$, Eq.(112), with likelihood given as $L(\beta,\lambda,p|x)$, Eq.(110), we get the joint distribution has a pdf which is following

$$\begin{aligned}\pi(x,\beta,\lambda,p) &= L(\beta,\lambda,p|x)\pi(\beta,\lambda,p)\\ &= \prod_{i=1}^{n}\left\{\left[p+(1-p)\frac{\lambda^{-\frac{1}{\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)}x_i\right]\beta\lambda x_i^{-(\beta+1)}\left\{\exp(-x_i^{-\beta})\right\}^{\lambda}\right\}\\ &\quad\times\frac{b^a}{\Gamma(a)}\beta^{a-1}e^{-b\beta}\times\frac{d^c}{\Gamma(c)}\lambda^{c-1}e^{-d\lambda}\times\frac{1}{B(e,f)}p^{e-1}(1-p)^{f-1}. \quad (113)\end{aligned}$$

In the addition, the marginal density is shown as

$$\pi(x) = \int_0^1\int_0^\infty\int_0^\infty \pi(x,\beta,\lambda,p)\,d\beta\,d\lambda\,dp. \quad (114)$$

Therefore, pdf of the posterior distribution of (β,λ,p) is given by

$$\pi(\beta,\lambda,p|x) = \frac{\pi(x,\beta,\lambda,p)}{\pi(x)}. \quad (115)$$

The Bayes estimates of β, λ and p against the square error loss function are respectively obtained as

$$\hat{\beta}_{BE} = E_\beta[\beta,\lambda,p|x], \quad (116)$$

$$\hat{\lambda}_{BE} = E_\lambda[\beta,\lambda,p|x], \quad (117)$$

and

$$\hat{p}_{BE} = E_p[\beta,\lambda,p|x]. \quad (118)$$

The Metropolis-Hastings algorithm (Ntzoufras, 2009) is applied for obtaining posterior distribution to estimated the parameters of the MEIW distribution, which is illustrated in Algorithm 4.

Algorithm 4 MCMC for estimating the parameters of the MEIW distribution

1 Set initial values β_0, λ_0 and p_0.
2 For $r = 1, 2, 3, \ldots, R$ repeat the following steps
3 Propose new values β'_r, λ'_r and p'_r from
$\beta'_r = \beta_{r-1} + e_r,\ \lambda'_r = \lambda_{r-1} + e_r$ and $p'_r = p_{r-1} + e_r$
where $e_r \sim N(0, \sigma^2 I)$
4 Substitute the β'_r, λ'_r and p'_r obtained in step 3 and the $\beta_{r-1}, \lambda_{r-1}$ and p_{r-1}
from the previous iteration into $\pi(\beta, \lambda, p|x)$ in order to compute
the acceptance probability; viz.
$\alpha = \min(1, A)$
where $A = \frac{\pi(\beta'_r, \lambda'_r, p'_r|x)\pi(\beta'_r, \lambda'_r, p'_r)}{\pi(\beta_{r-1}, \lambda_{r-1}, p_{r-1}|x)\pi(\beta_{r-1}, \lambda_{r-1}, p_{r-1})}$
5 Generate a realization $X \sim U(0, 1)$.
6 Set $\beta_r = \beta'_r,\ \lambda_r = \lambda'_r$ and $p_r = p'_r$ if $x < \alpha$
otherwise set $\beta_r = \beta_{r-1},\ \lambda_r = \lambda_{r-1}$ and $p_r = p_{r-1}$.
7 End for
8 Calculate the Bayesian estimator of the MEIW distribution by
$\hat{\beta}_{BE} = \sum_{r=1}^{R} \frac{\beta_r}{R},\ \hat{\lambda}_{BE} = \sum_{r=1}^{R} \frac{\lambda_r}{R}$ and $\hat{p}_{BE} = \sum_{r=1}^{R} \frac{p_r}{R}$.

2.4 Simulation study

Our objective here is to compare the true value of the parameters of the MEIW(β, λ, p) and their estimates from MLE and BE methods. The true parameter values are varied in the case of $\beta = 2, 5, 8$, $\lambda = 1, 5, 10$ and $p = 0.2, 0.5, 0.8$. The summary of the specified parameter set is 27 cases. All cases in the simulation study were then generated from the MEIW distribution with sample sizes n = 15, 30, 50, 100, 150, 200, 300 and 500, respectively. We use the R program (R Core Team, 2013) to generate each sample of a fixed size and repeated this for 1,000 trials.

2.4.1 The Generation of the MEIW random variate

The random samples of the MEIW(β, λ, p) are generated as follows:

1) Generate two uniform variables U_1 and U_2 on the interval (0,1).

2) If $U_1 < p$, then use U_2 to generate a random variable X from the EIW(β, λ) by using Eq.(44) as $X = F_E^{-1}(U_2)$.

3) If $U_1 \geq p$, then use U_2 to generate a random variate X from the LBEIW(β, λ) by using Eq.(71) as $X = F_L^{-1}(U_2)$.

2.4.2 A comparison of the parameter estimation methods

In this section, we compare the estimates of the MEIW distribution with parameters β, λ and p by using the MLE and BE methods in a Monte Carlo simulation. The MSE is calculated to compare the performances of the methods of parameter estimation.

The sample mean, variance and MSE of the estimates are calculated based on 1,000 Monte Carlo simulation and the results are illustrated in Table B1 - Table B27 (see Appendix B). The MSE plots of the both MLE and BE shown in

Figure 28 - Figure 54. We see that in most of the considered cases, the BE outperforms the MLE. As to be expected, the MSE of the estimated parameters decrease as sample sizes, n, increases.

Obviously, the MSE of $\hat{\beta}$ and $\hat{\lambda}$ from MLE are very different from BE for sample sizes as small. When sample size increase, the MSE of $\hat{\beta}$ and $\hat{\lambda}$ from MLE and BE are very close. In all case of sample sizes, the BE is lower MSE than BE. In sometimes, the MLE does not converge in the Newton-Raphson procedure and fails to estimate parameters, as the BE does not show this problem.

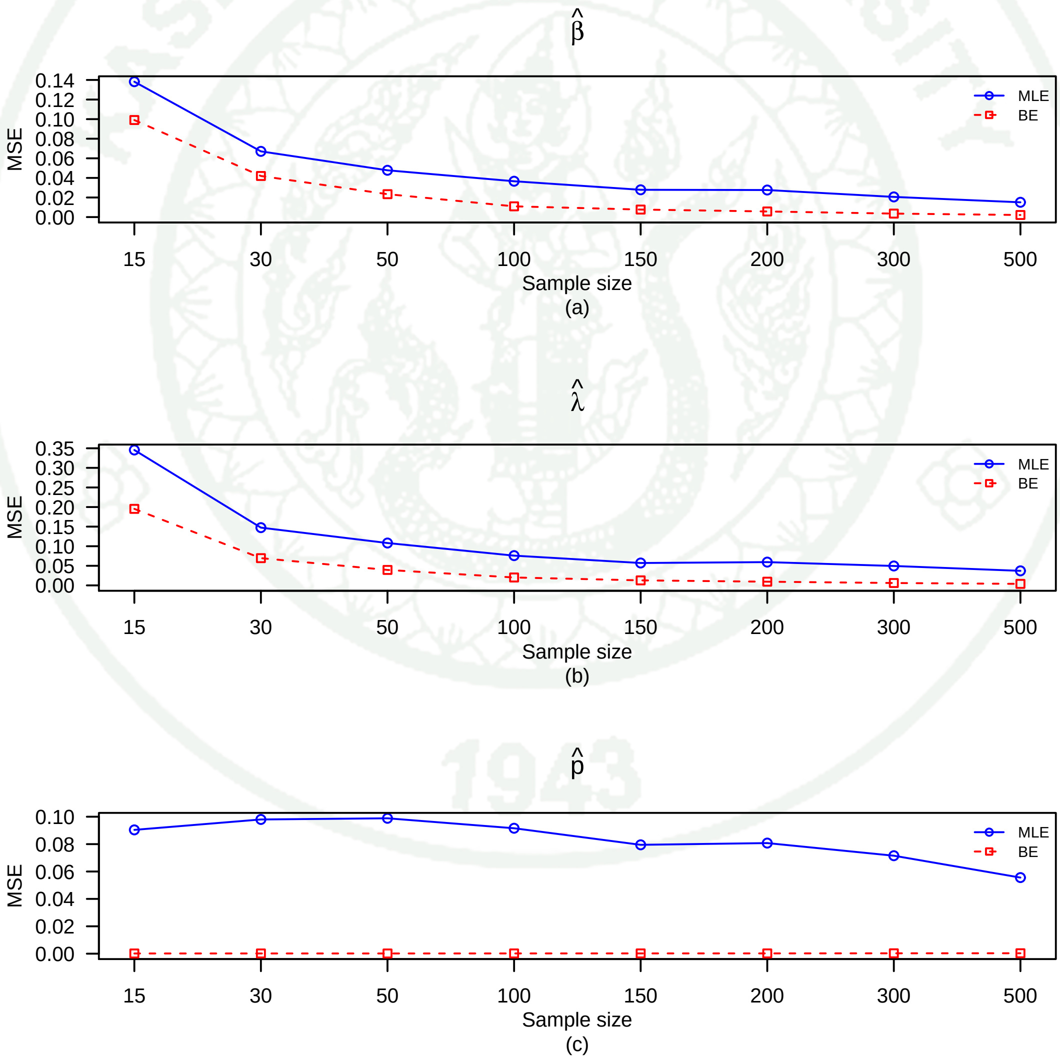

Figure 28 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \text{MEIW}(2,1,0.2)$

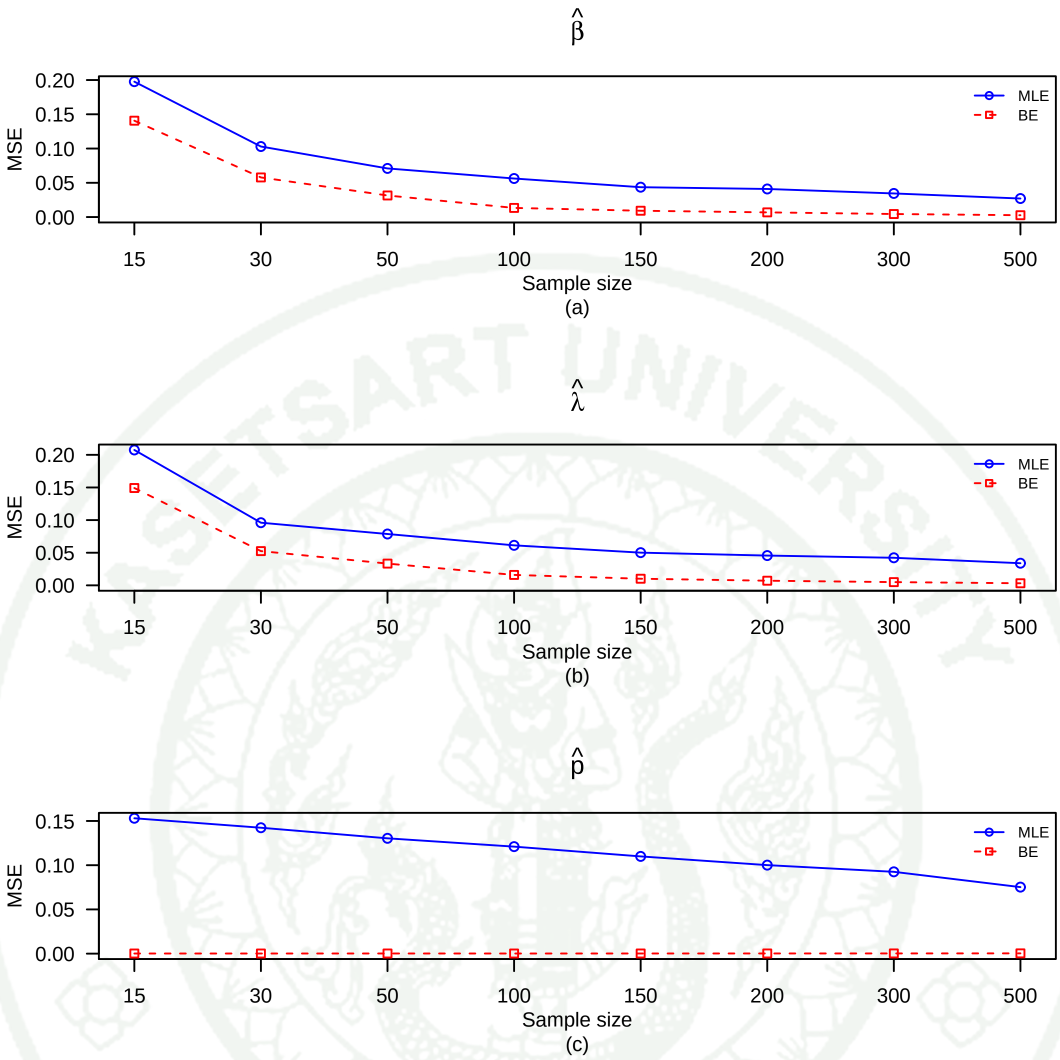

Figure 29 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \mathrm{MEIW}(2,1,0.5)$

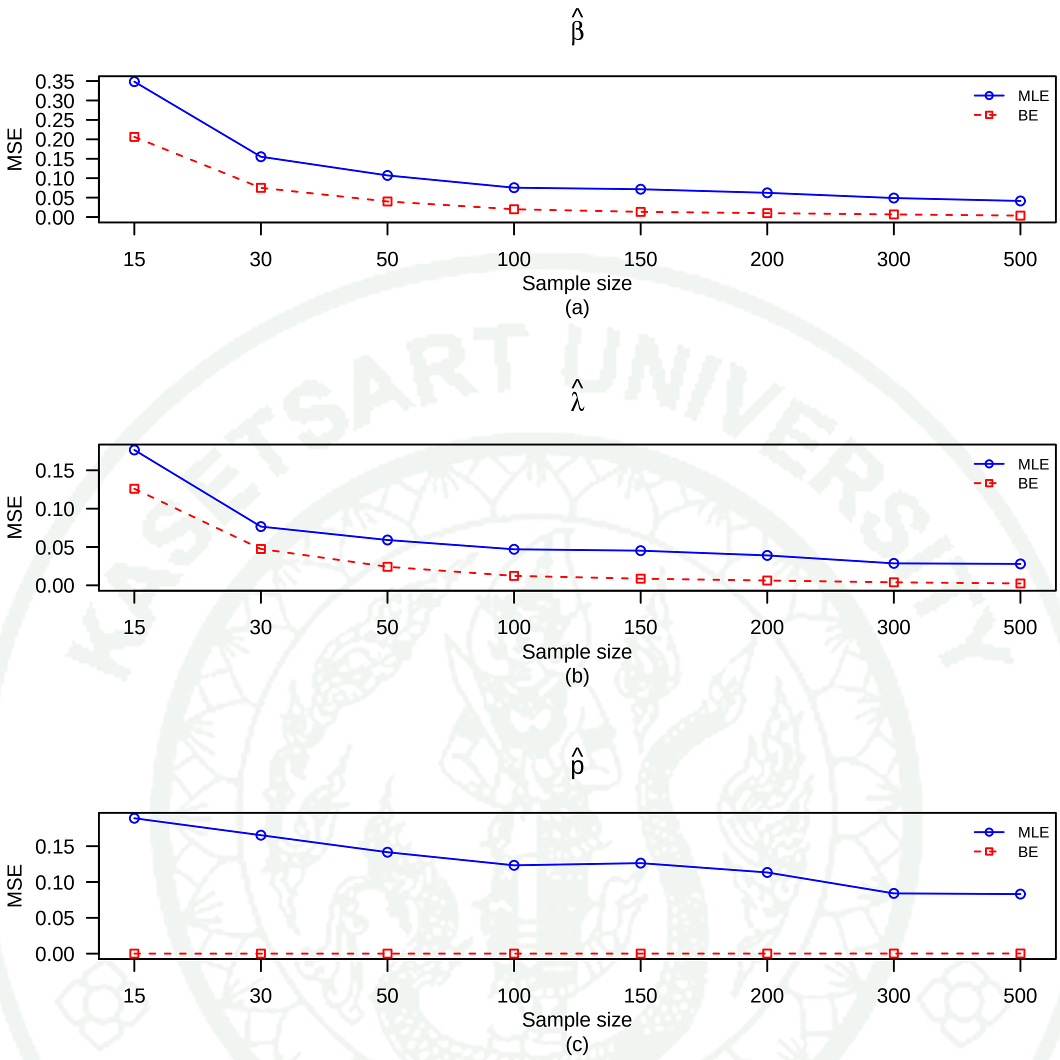

Figure 30 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \mathrm{MEIW}(2,1,0.8)$

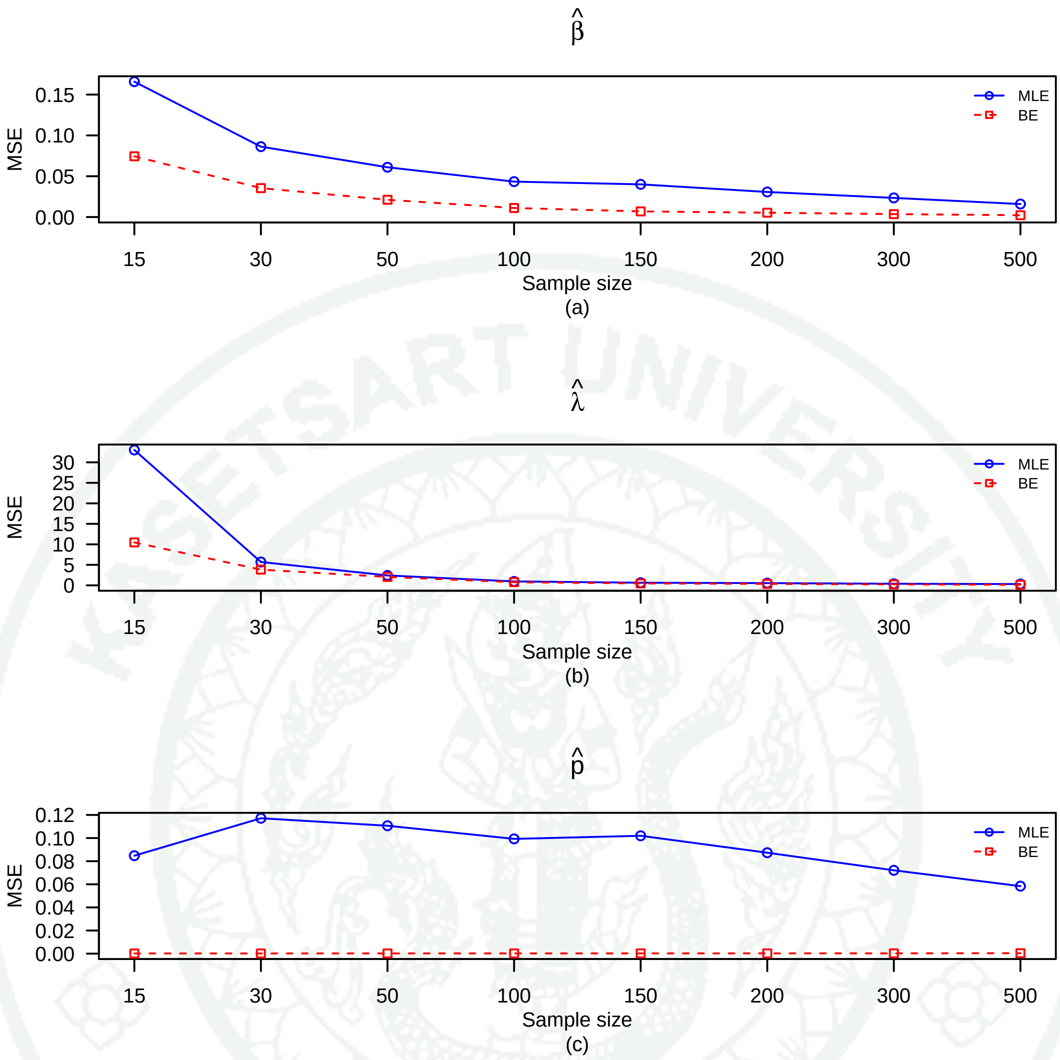

Figure 31 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \text{MEIW}(2,5,0.2)$

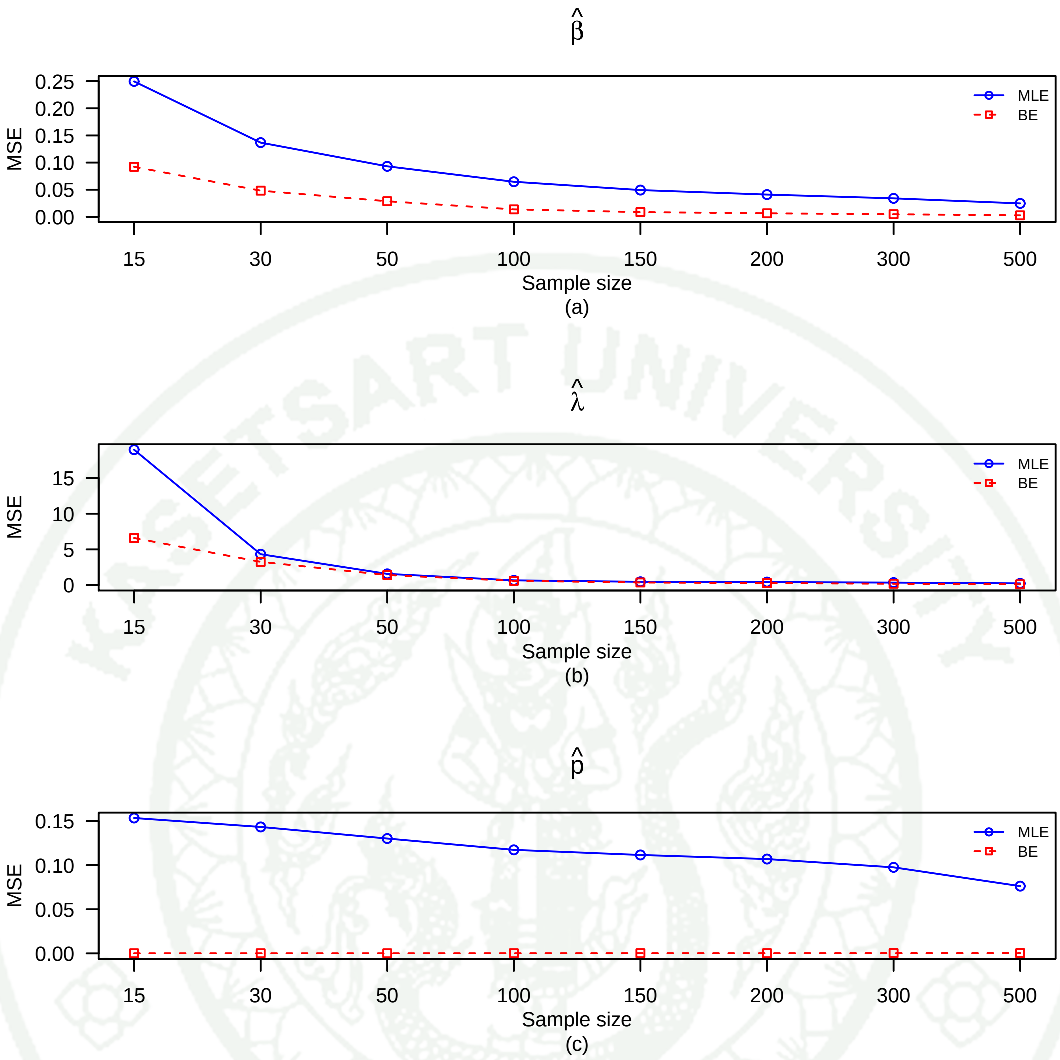

Figure 32 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \mathrm{MEIW}(2,5,0.5)$

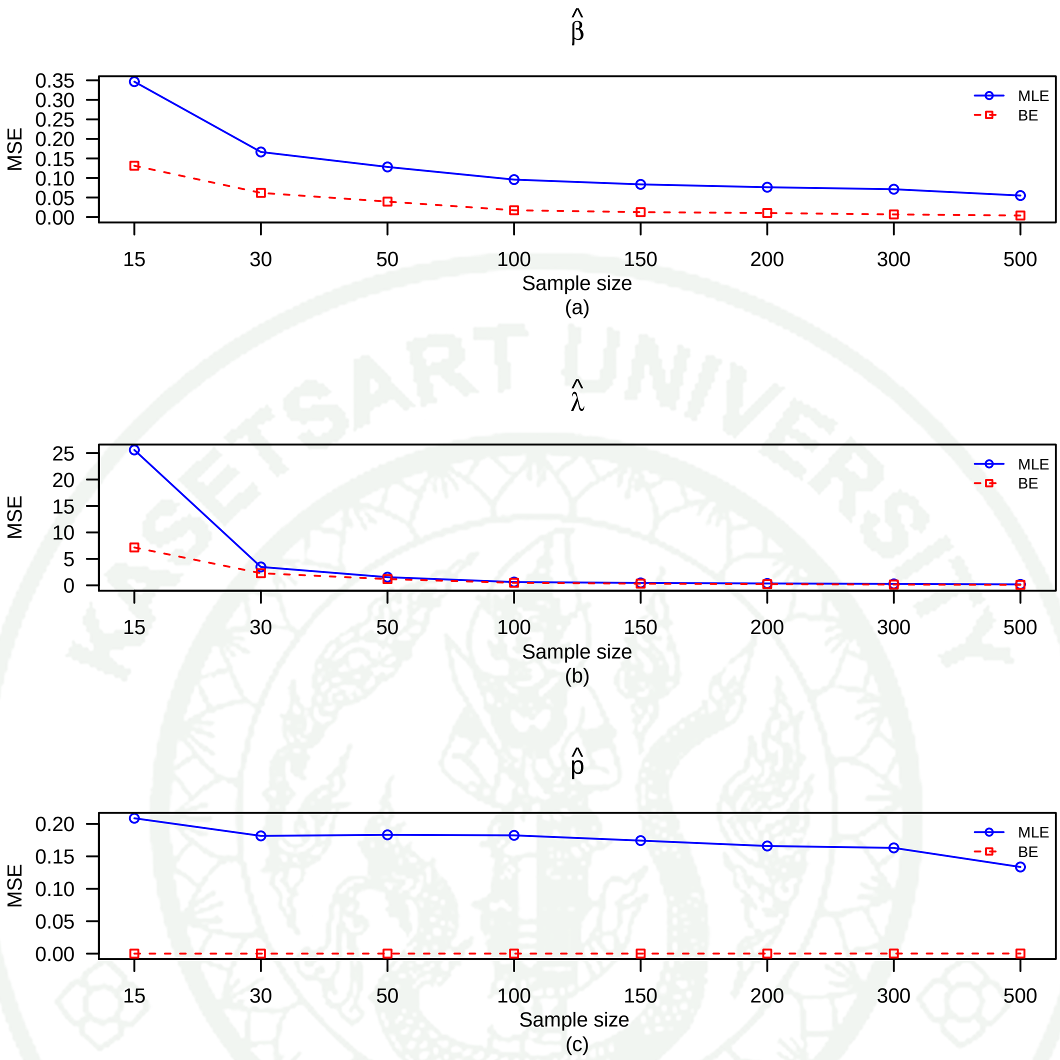

Figure 33 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \text{MEIW}(2,5,0.8)$

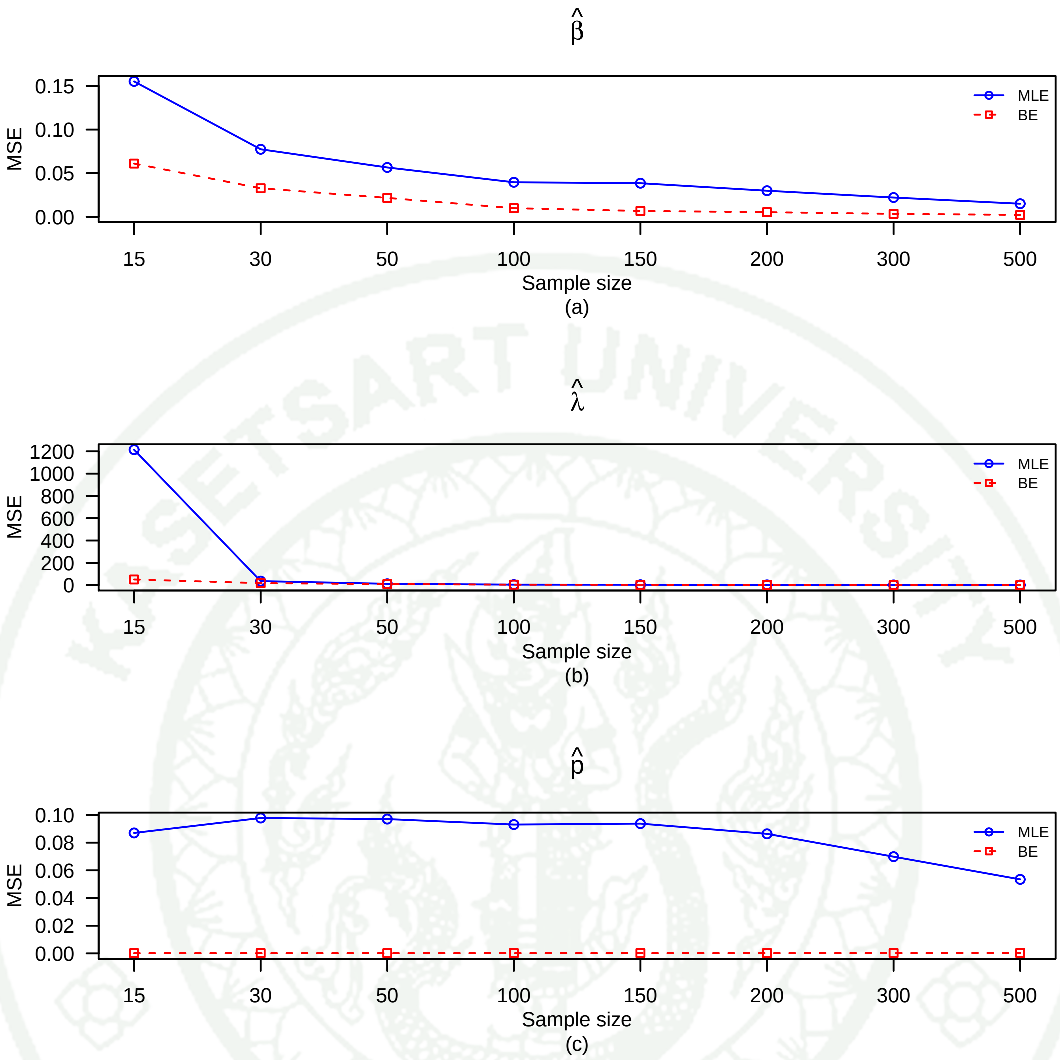

Figure 34 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \mathrm{MEIW}(2, 10, 0.2)$

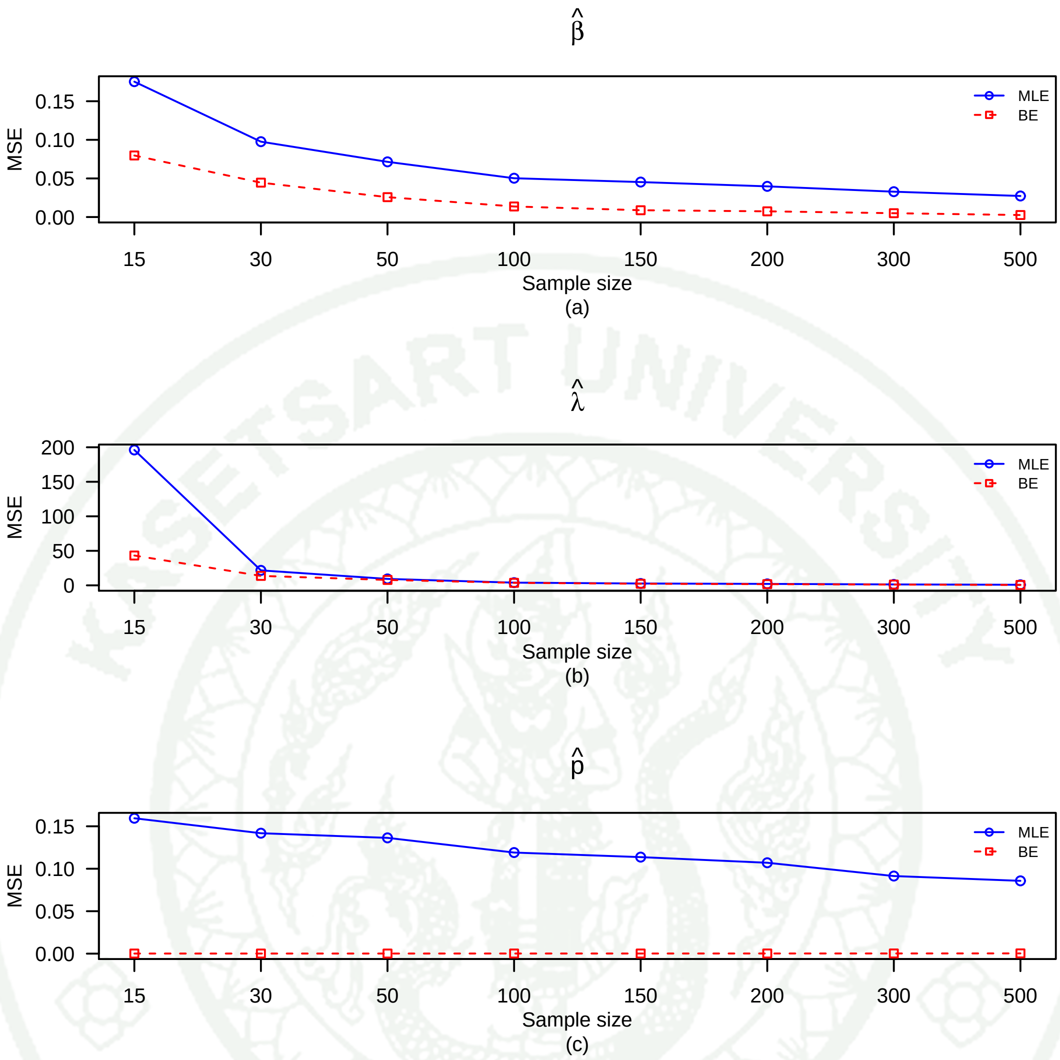

Figure 35 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \mathrm{MEIW}(2, 10, 0.5)$

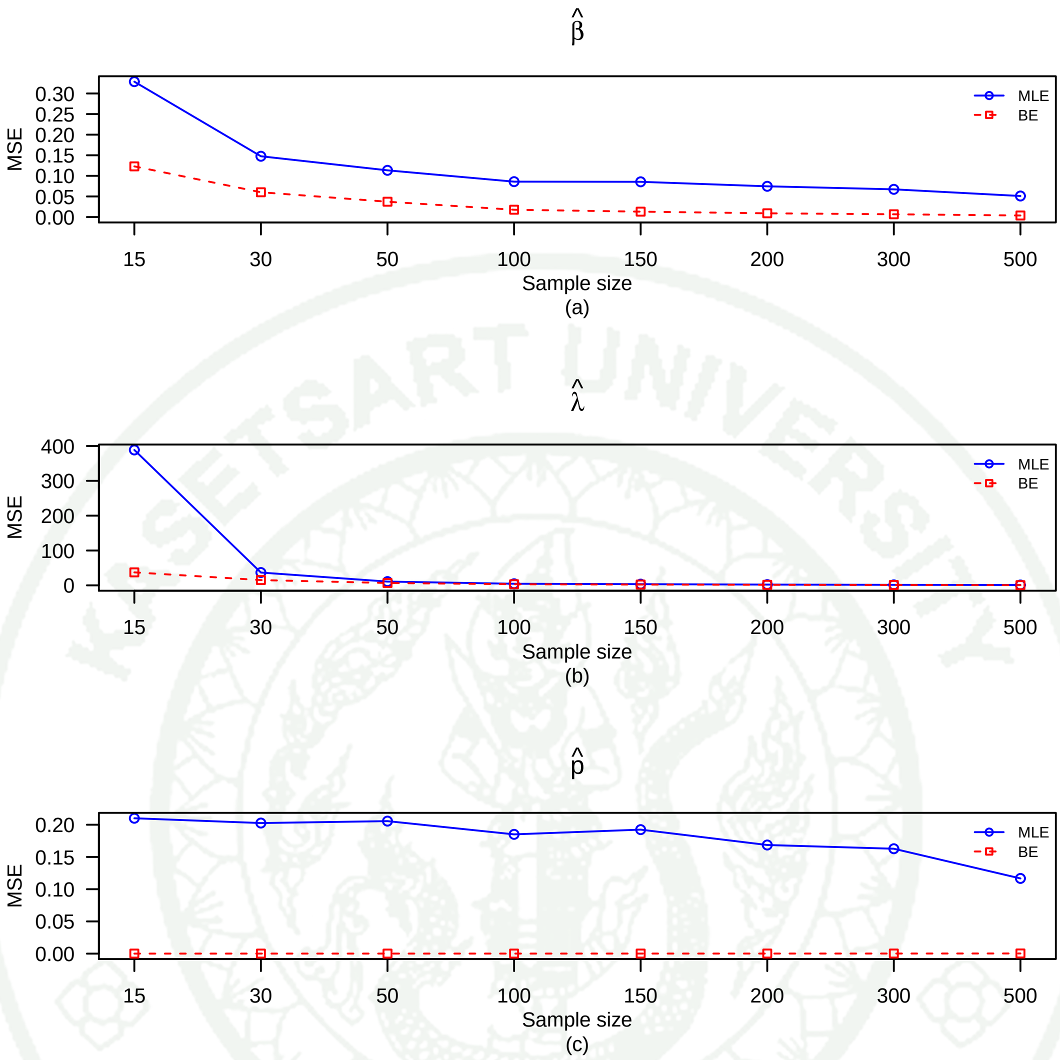

Figure 36 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \mathrm{MEIW}(2,10,0.8)$

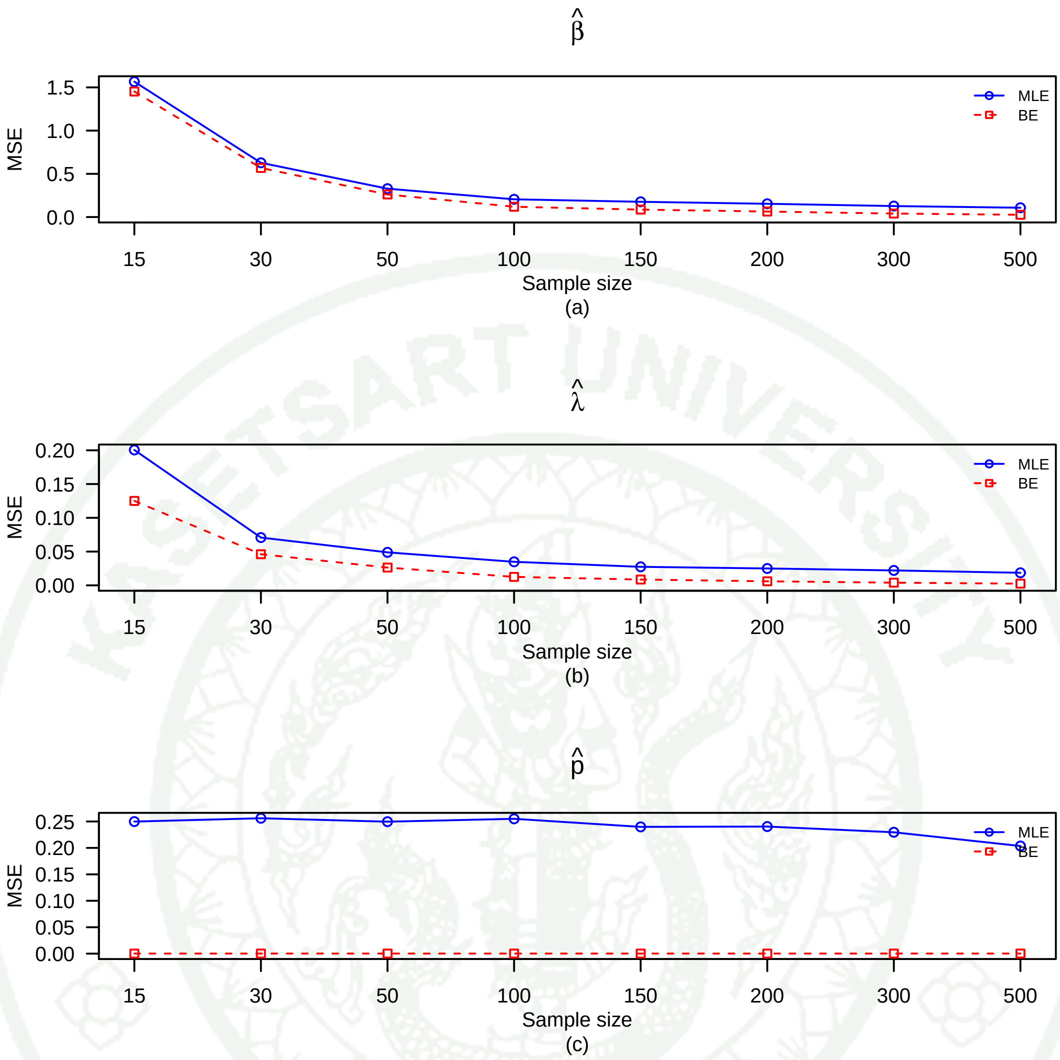

Figure 37 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \text{MEIW}(5, 1, 0.2)$

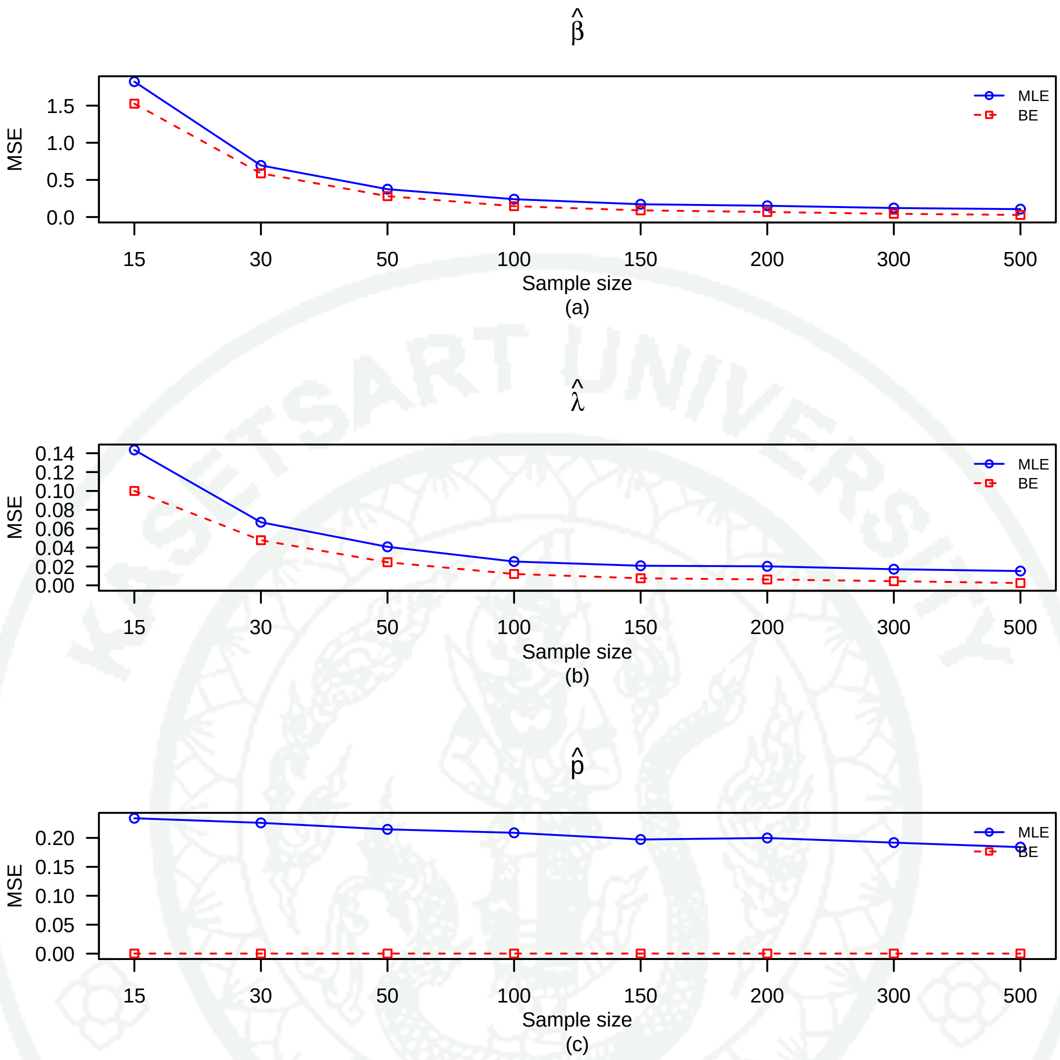

Figure 38 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \text{MEIW}(5, 1, 0.5)$

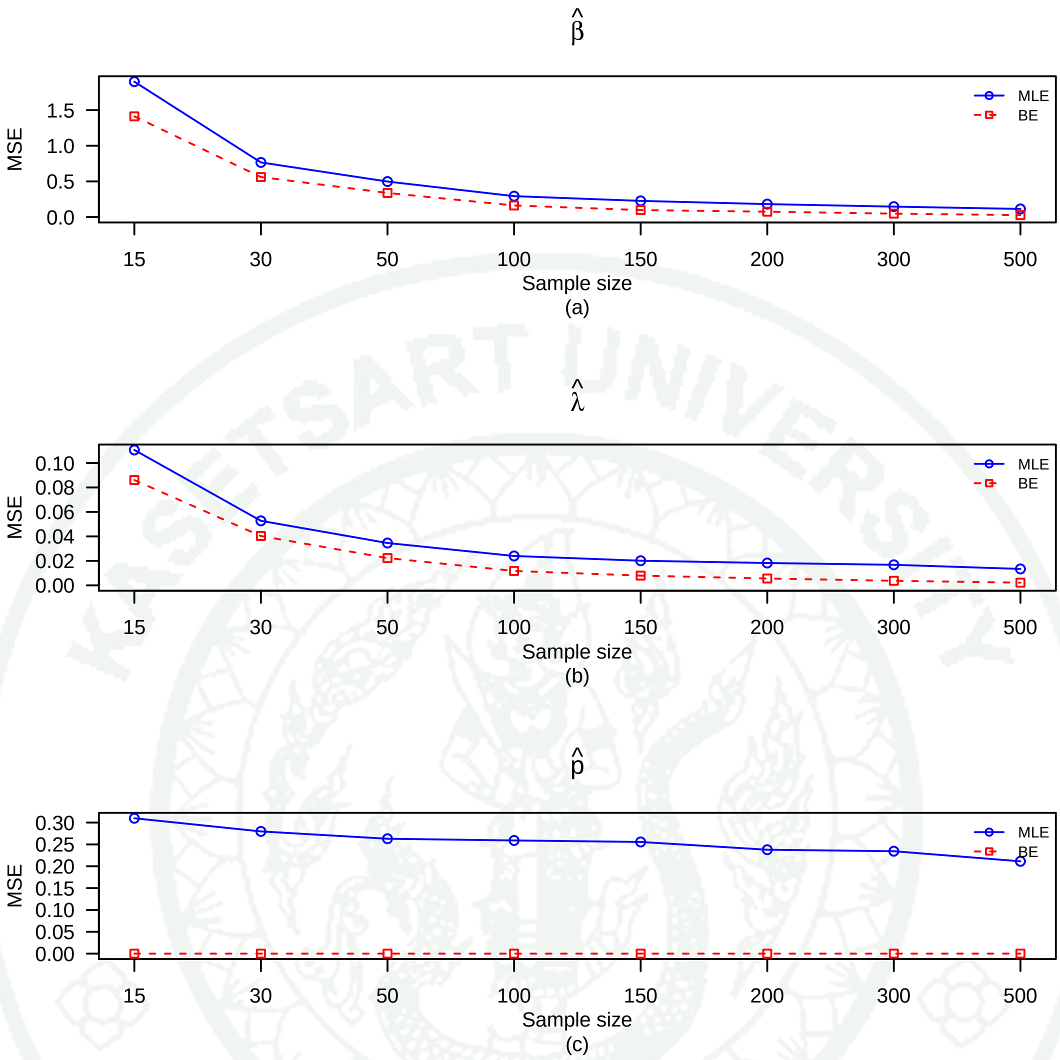

Figure 39 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \text{MEIW}(5, 1, 0.8)$

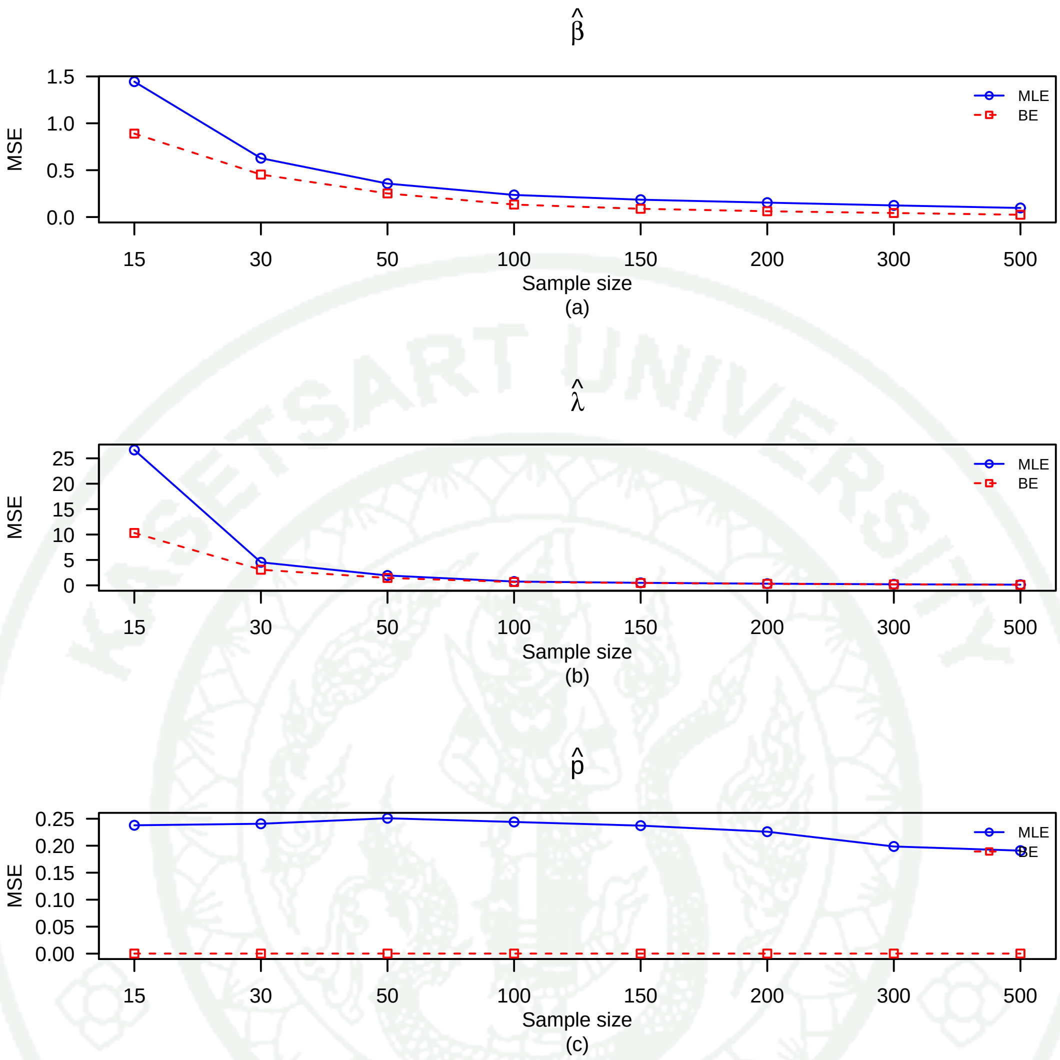

Figure 40 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \text{MEIW}(5,5,0.2)$

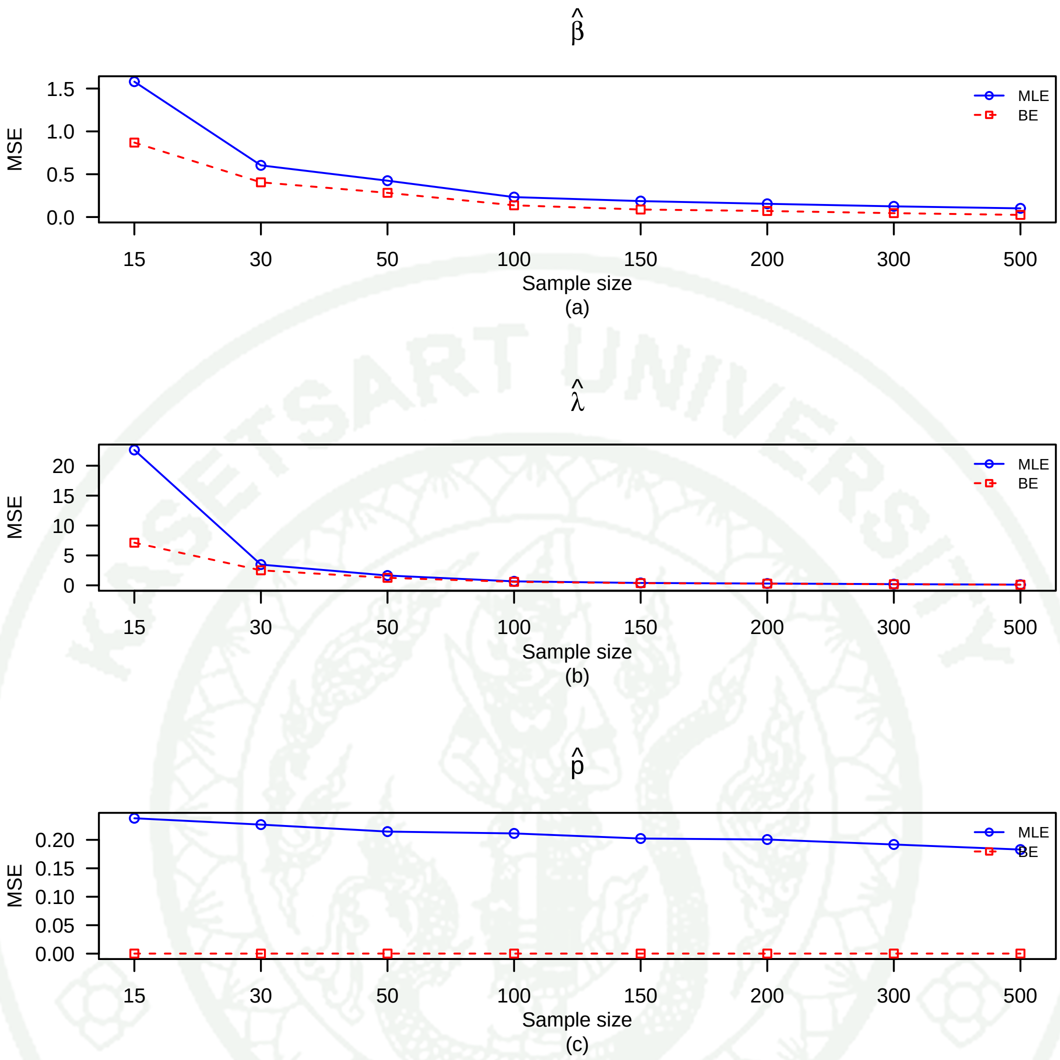

Figure 41 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \text{MEIW}(5,5,0.5)$

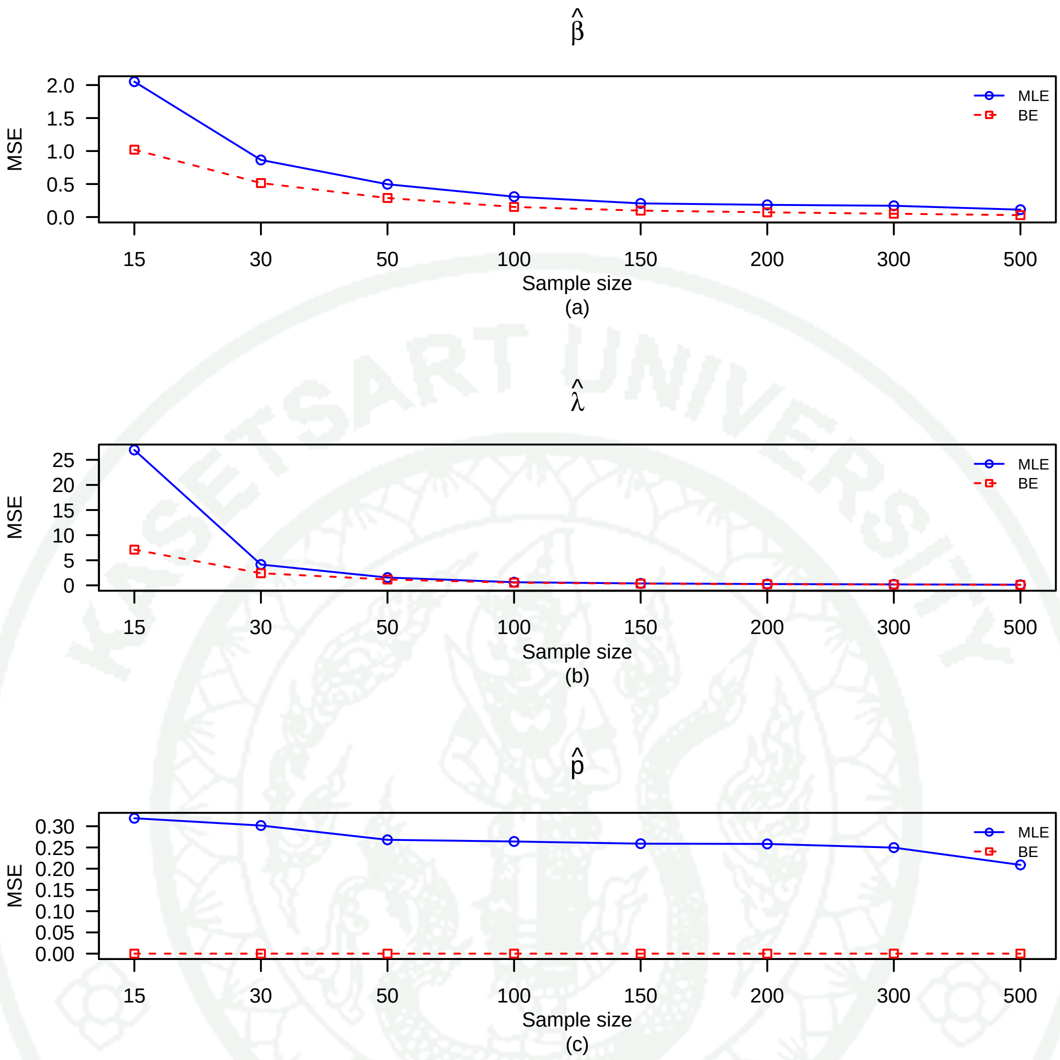

Figure 42 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \mathrm{MEIW}(5, 5, 0.8)$

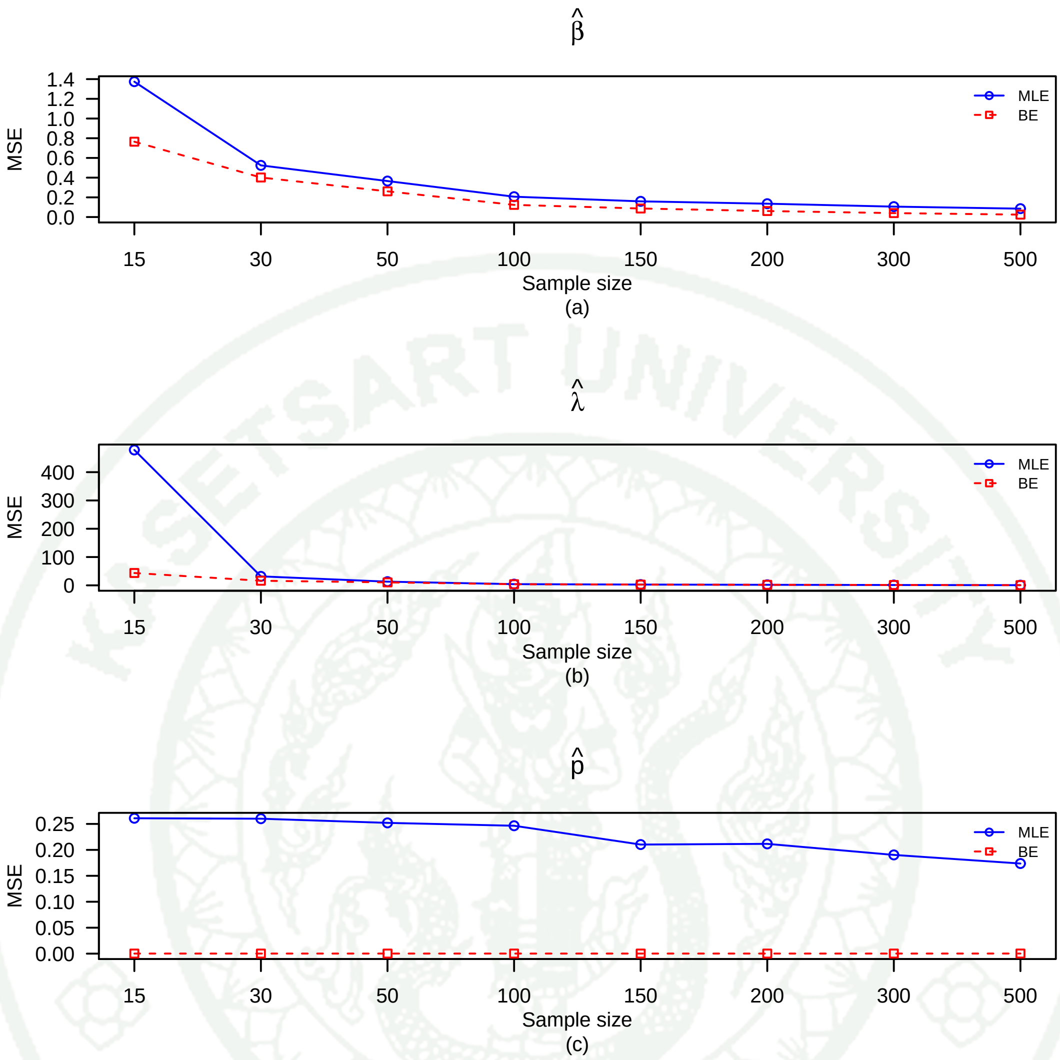

Figure 43 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \text{MEIW}(5, 10, 0.2)$

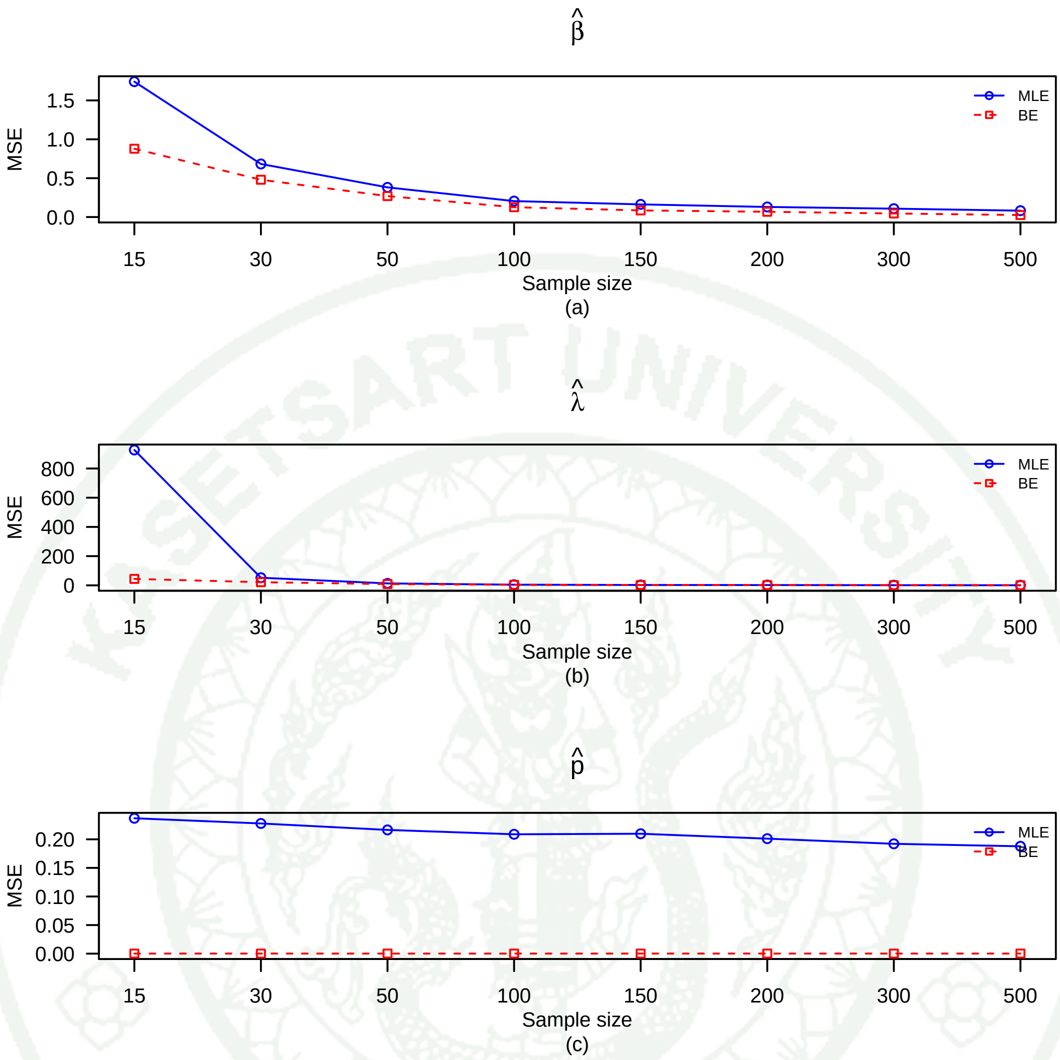

Figure 44 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \text{MEIW}(5, 10, 0.5)$

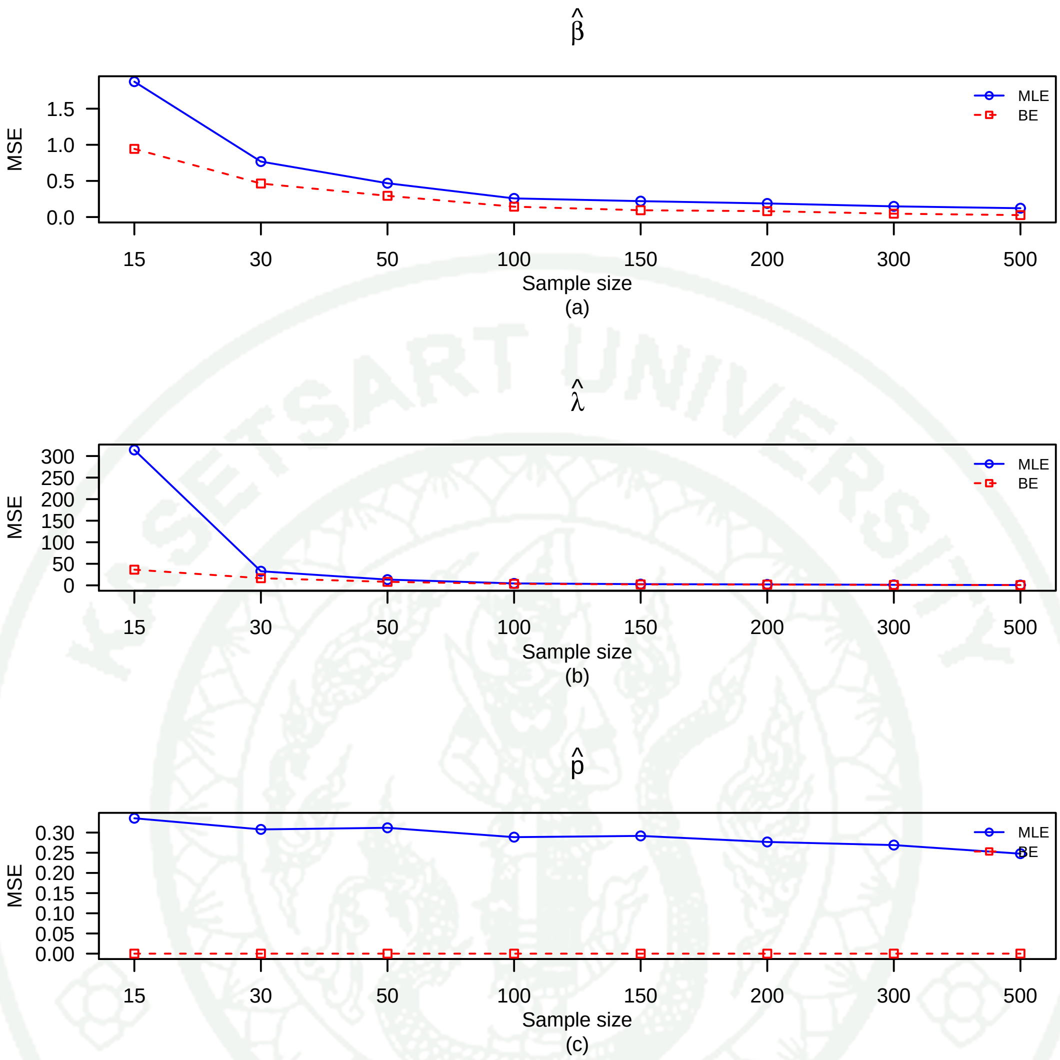

Figure 45 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \mathrm{MEIW}(5, 10, 0.8)$

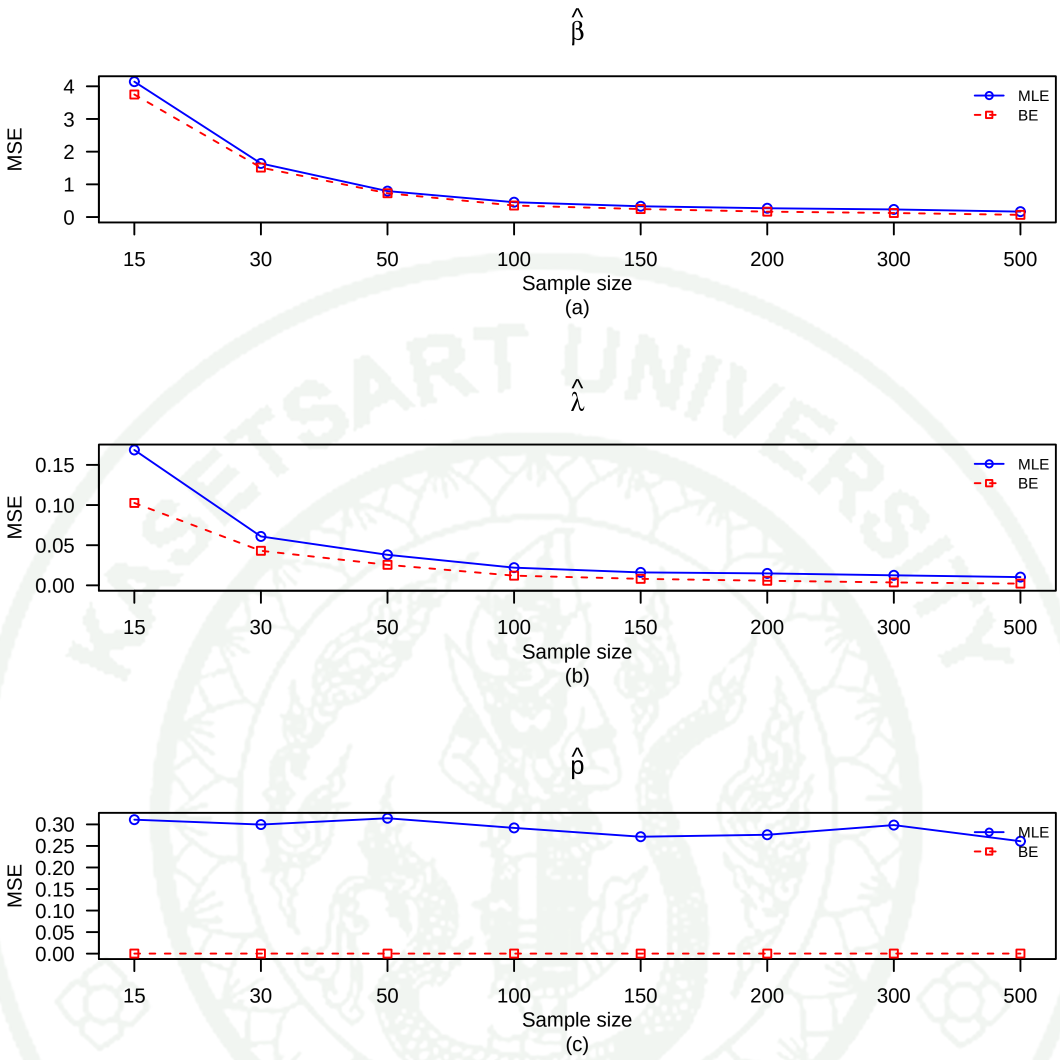

Figure 46 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \text{MEIW}(8, 1, 0.2)$

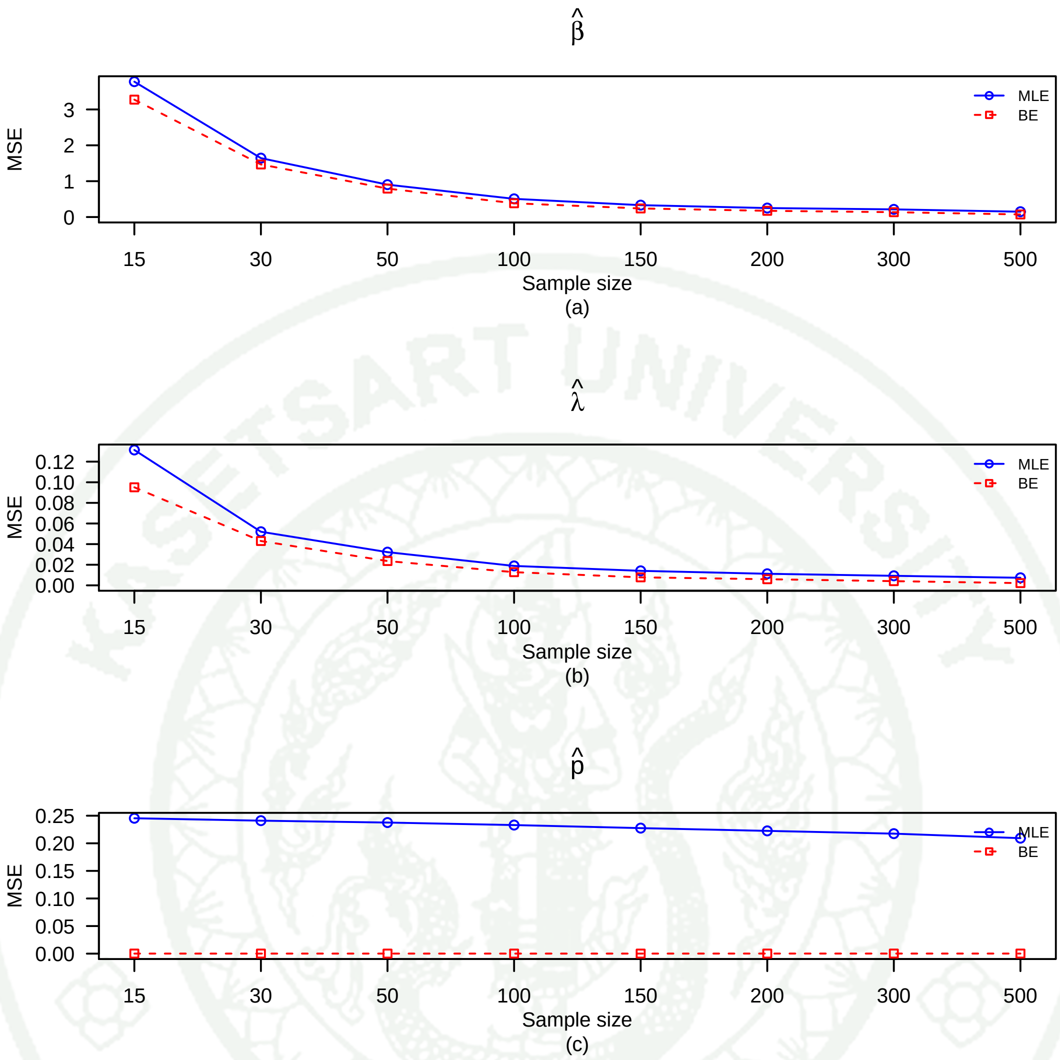

Figure 47 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \text{MEIW}(8,1,0.5)$

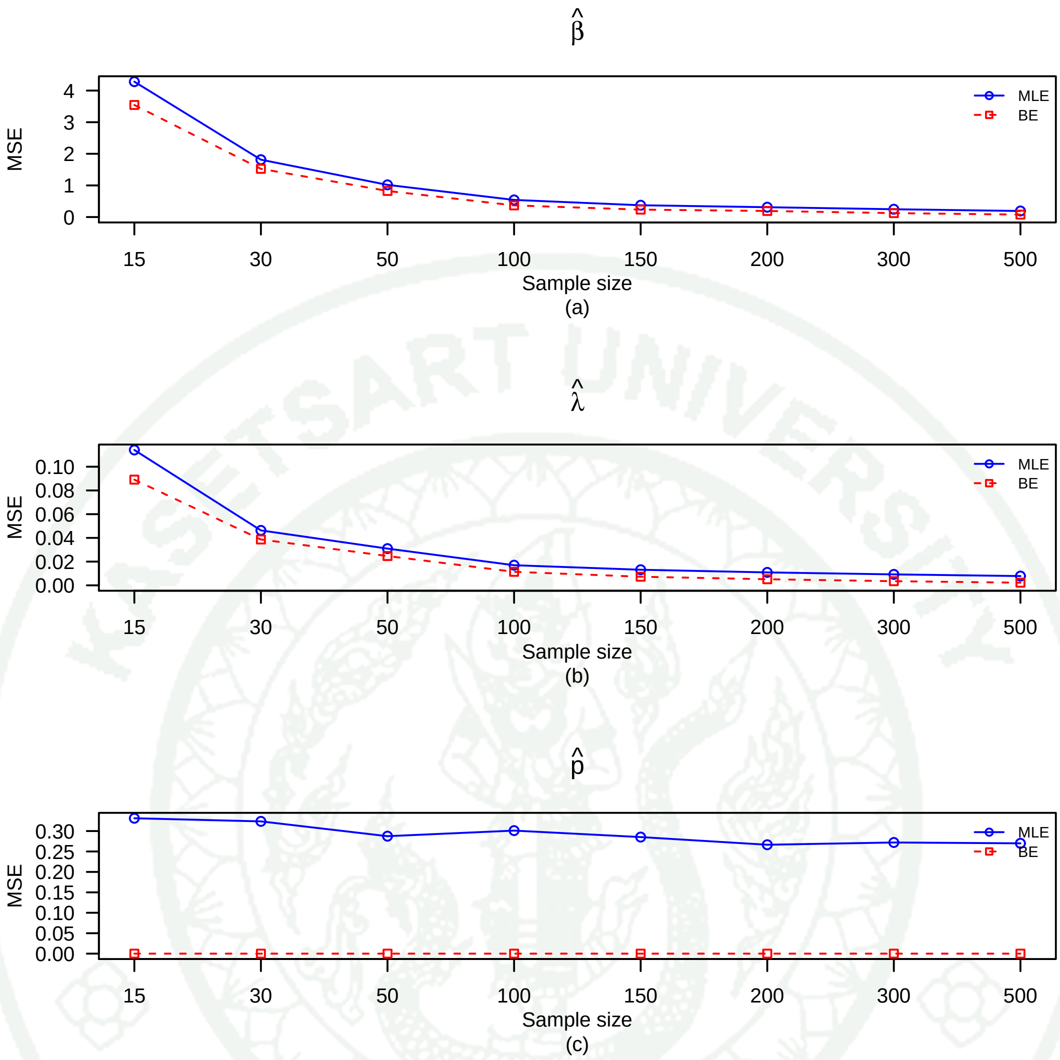

Figure 48 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \text{MEIW}(8, 1, 0.8)$

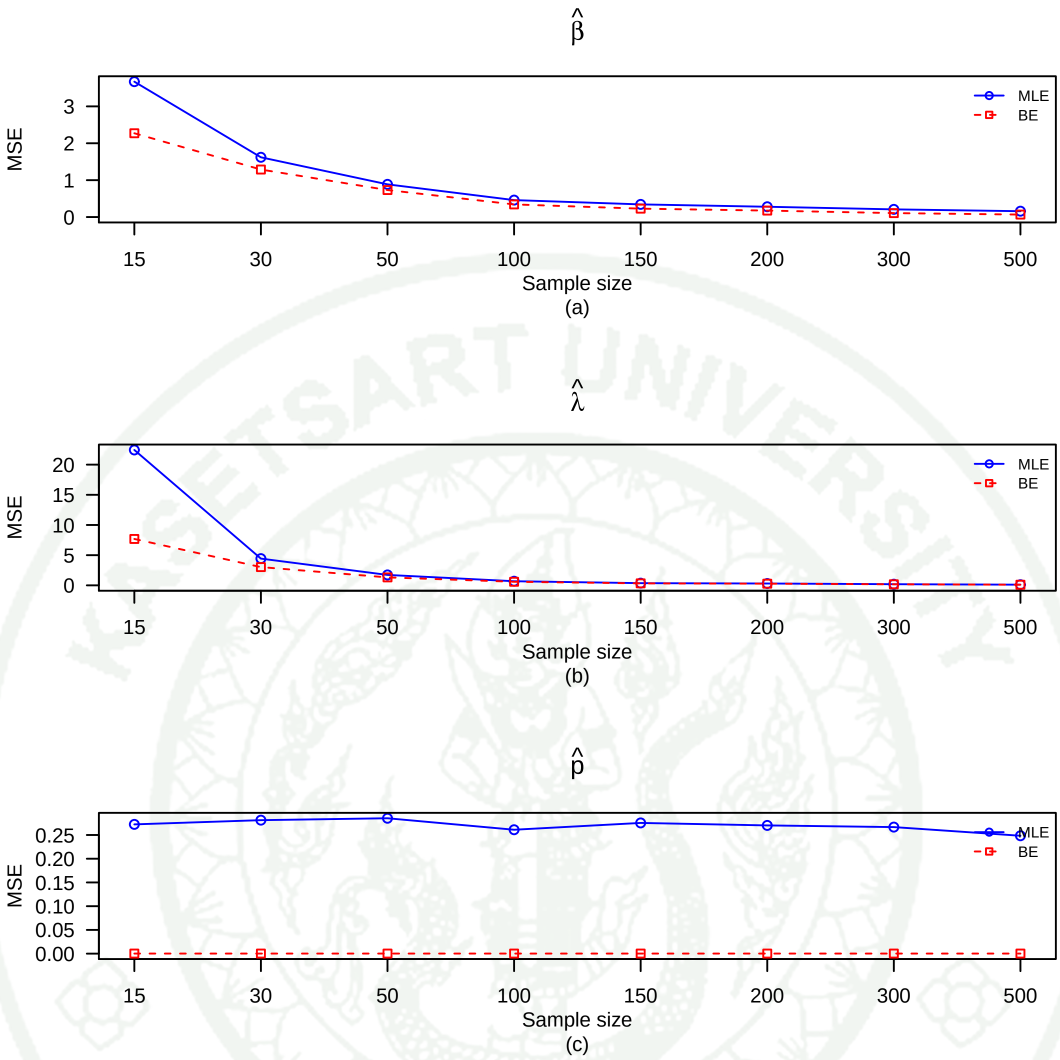

Figure 49 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \text{MEIW}(8, 5, 0.2)$

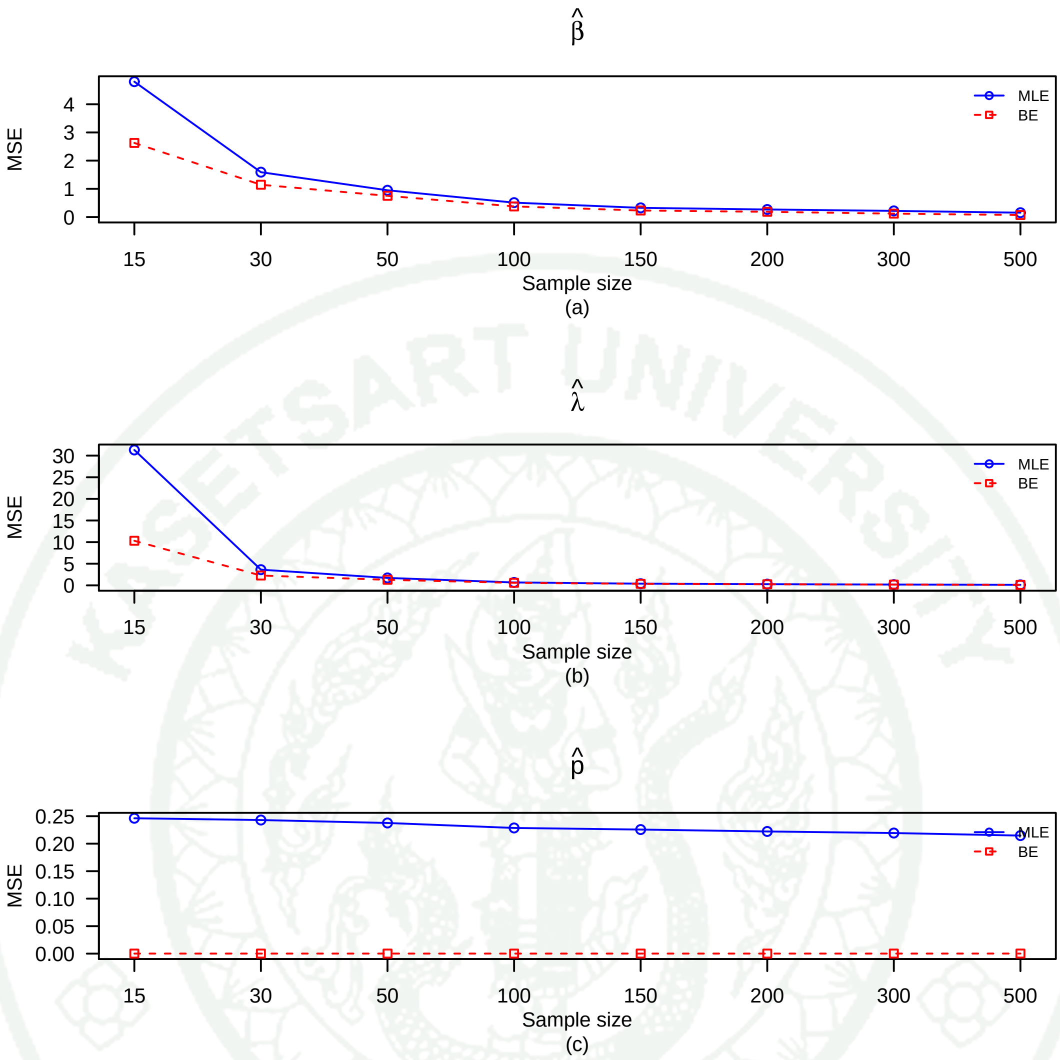

Figure 50 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \text{MEIW}(8, 5, 0.5)$

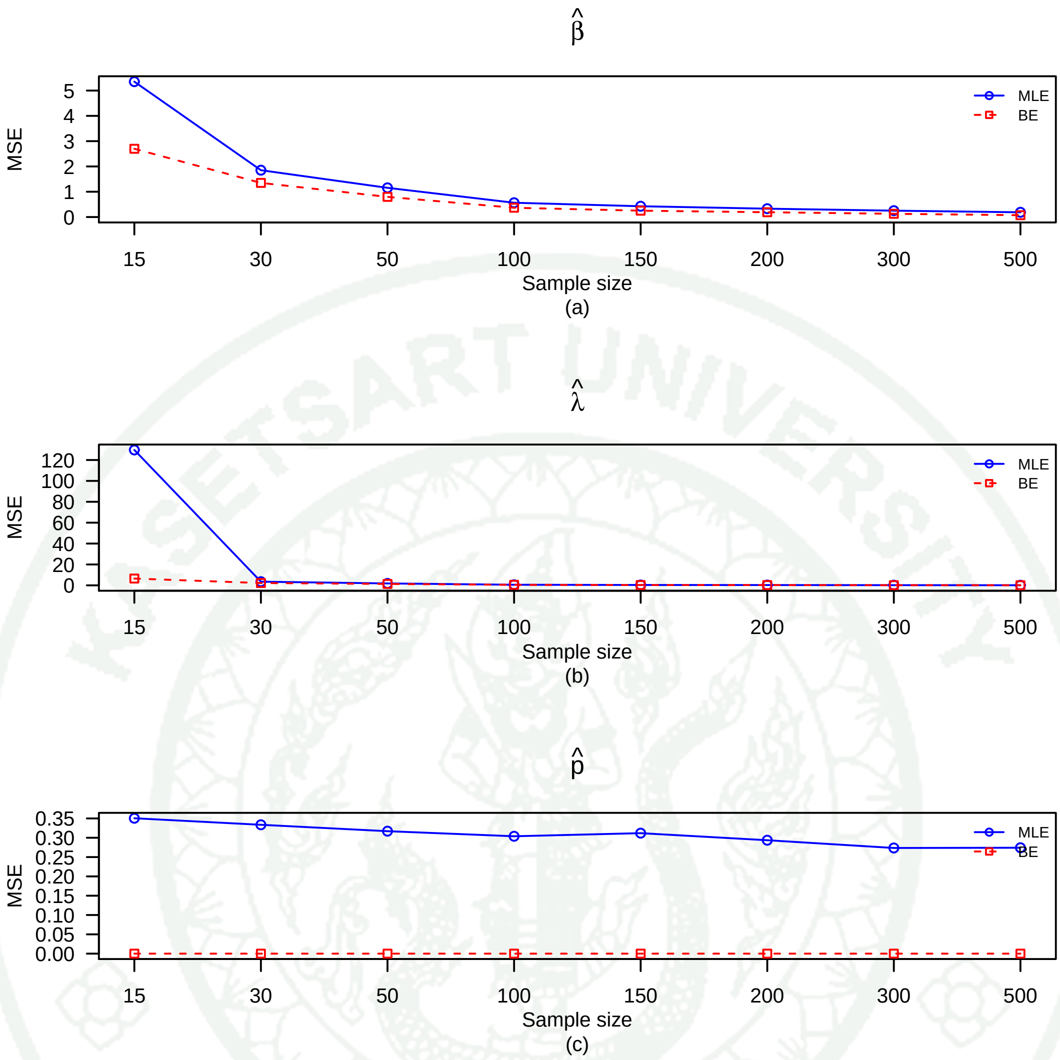

Figure 51 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \text{MEIW}(8, 5, 0.8)$

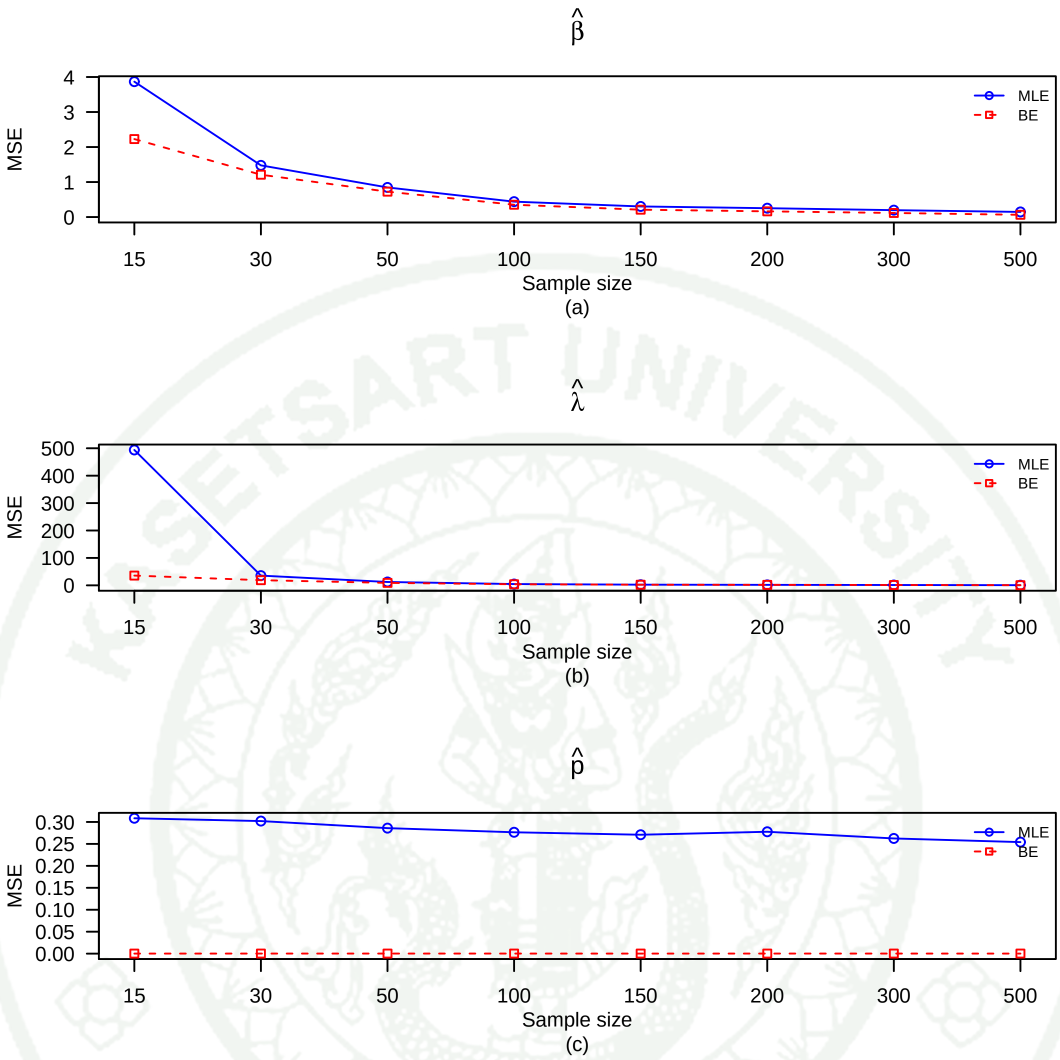

Figure 52 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \text{MEIW}(8, 10, 0.2)$

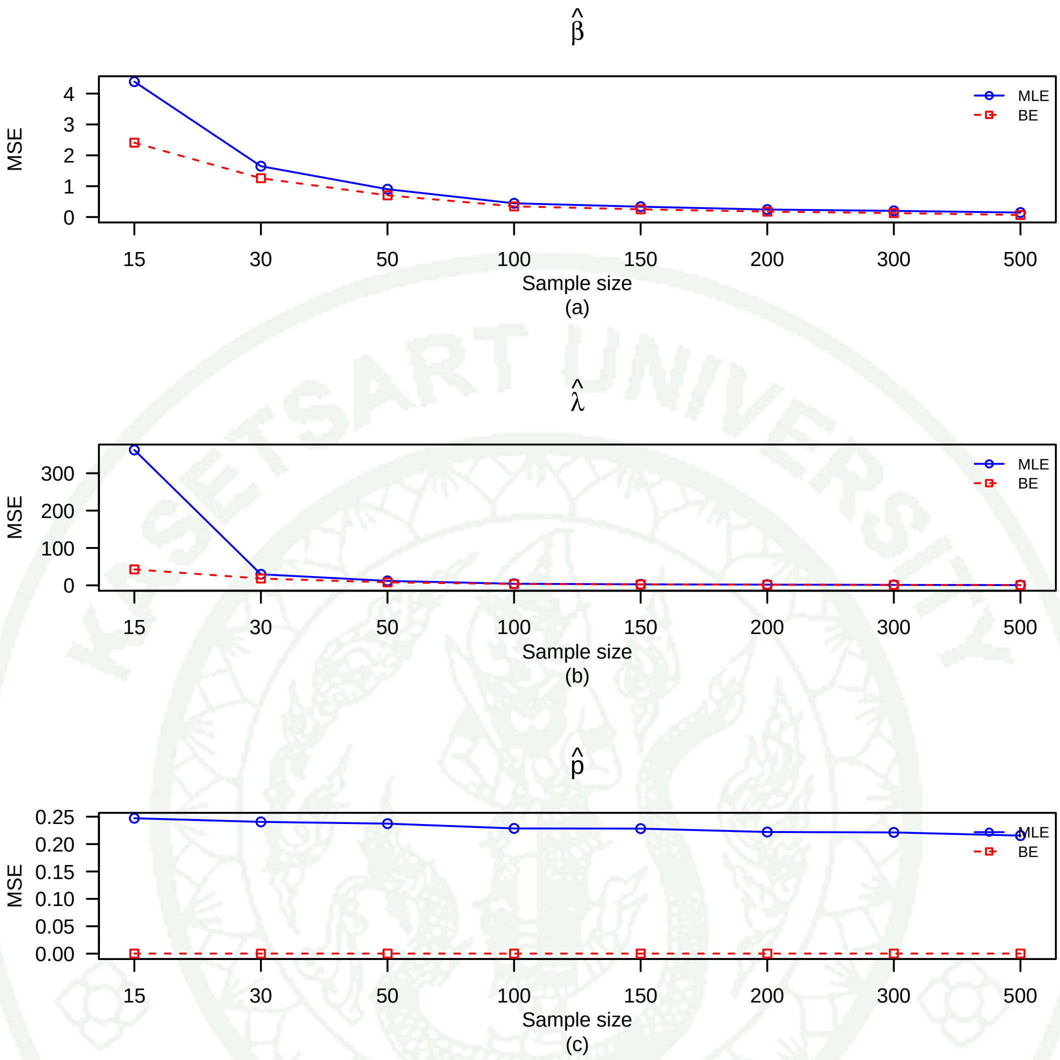

Figure 53 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \mathrm{MEIW}(8, 10, 0.5)$

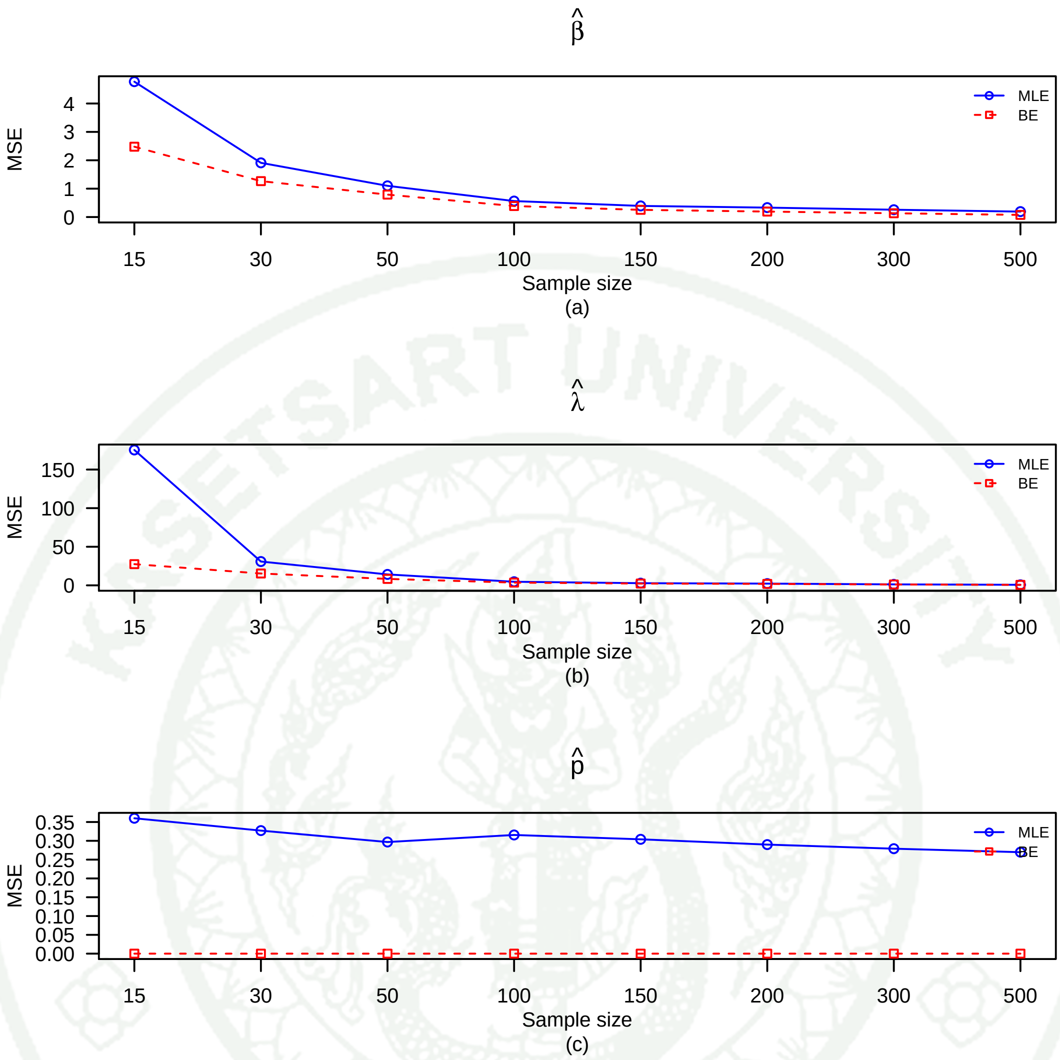

Figure 54 The estimated MSE of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{p}$ from MLE and BE obtained from $X \sim \text{MEIW}(8,10,0.8)$

2.5 Application study

Watson and Smith (1985) had previously used the breaking strengths of single carbon fibers data to assess the classical weakest link theory of breakage. This is a random sample of 200 failure stresses observed in hybrid boundless of parallel carbon fibres, listed in Crowder (2000).

We fit the breaking strengths of single carbon fibers dataset under the MEIW, LBEIW, EIW, three-parameter Weibull and three-parameter lognormal distributions. The AD test and AIC are applied for goodness of fit purpose. Table 3 lists the values of the AD test, AIC and the MLE estimates of the parameters. The p-value of AD test for the MEIW distribution is 0.8571, and the LBEIW distribution has the p-value of this test is 0.8555, which is higher than that of the EIW, three-parameter Weibull and three-parameter lognormal distributions. Likewise, AIC value of MEW and LBEIW distributions is lower than that of among other distributions. Then, the MEIW distribution as well as the LBEIW distribution fit to the breaking strengths of single carbon fibers dataset is better than the among other distributions.

Table 3 Goodness of fit summary of the breaking strengths of single carbon fibers dataset

Fitting Distribution	**AD test**		**AIC**	**Estimate**
	Statistic	**p-value**		
MEIW	0.3914	0.8571	111.6807	$\hat{\beta} = 6.7908$, $\hat{\lambda} = 15.1184$, $\hat{p} = 0.2536$
LBEIW	0.3930	0.8555	111.7182	$\hat{\beta} = 6.9254$, $\hat{\lambda} = 15.2316$
EIW	0.4433	0.8048	112.5676	$\hat{\beta} = 6.2947$, $\hat{\lambda} = 14.2506$
lognorm	0.8520	0.4449	161.8734	$\hat{\gamma} = -0.5589$, $\hat{\mu} = 1.0463$, $\hat{\sigma} = 0.5632$
Weibull	4.8571	0.0034	120.3754	$\hat{\tau} = 1.1236$, $\hat{\alpha} = 0.6669$, $\hat{\beta} = 1.4169$

The ecdf plots (step line) of this dataset with the estimated cdf curves (dot line) based on MEIW, LBEIW, EIW, three-parameter Weibull and three-parameter lognormal distributions are shown in Figure 55.

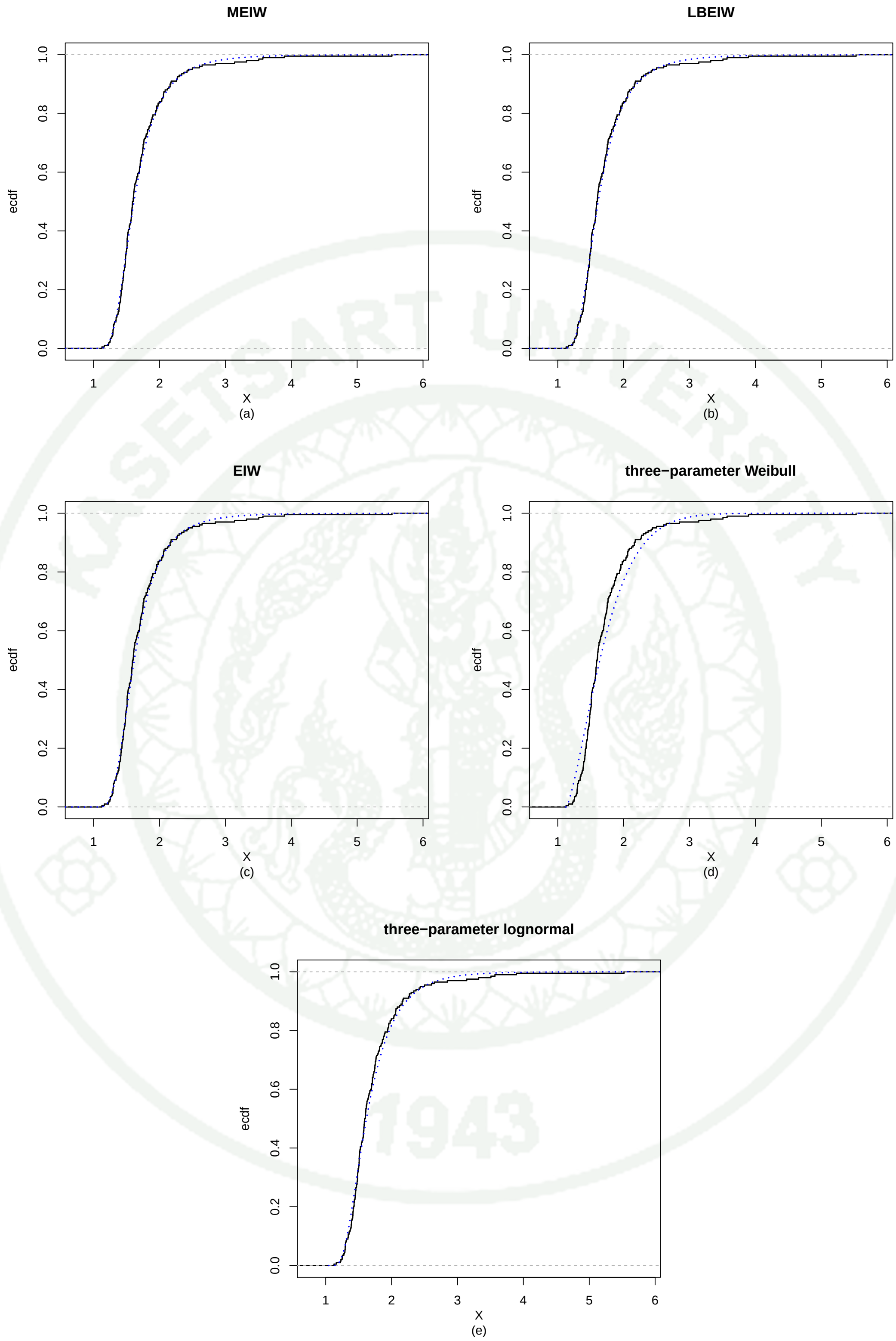

Figure 55 The ecdf plots (step line) with the estimated cdf curves (dot line) based on (a) MEIW, (b) LBEIW, (c) EIW, (d) three-parameter Weibull and (e) three-parameter lognormal distribution

In addition, the probability plot of the MEIW distribution of the MLE, corresponding to the breaking strengths of single carbon fibers dataset, is shown in the Figure 56.

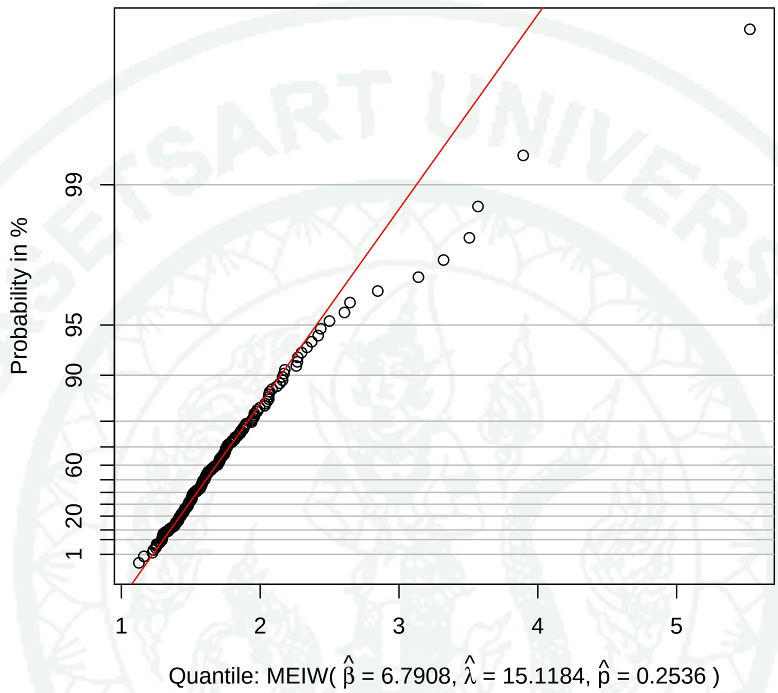

Figure 56 The probability plot for the model based on the MEIW distribution applied to the breaking strengths of single carbon fibers dataset

Moreover, we compare the MLE method with the BE method in the parameter estimation for the breaking strengths of single carbon fibers dataset. The estimators obtained by both approaches are compared in Table 4. The BE gives the p-value of the AD test, which is slightly larger than these of the MLE. Similarly, the AIC value of BE is also lower than these of the MLE. That is the BE outperforms the MLE for the breaking strengths of carbon fibers dataset. Furthermore, histogram of the breaking strengths of single carbon fibers dataset and pdf curve of the MLE and BE estimates based on MEIW distribution corresponding to this dataset shown in the Figure 57.

Table 4 Comparison of estimators obtained by MLE and BE methods for the breaking strengths of single carbon fibers dataset

Method	AD test		AIC	Estimate
	Statistic	p-value		
MLE	0.3914	0.8571	111.6807	$\hat{\beta} = 6.7908$, $\hat{\lambda} = 15.1184$, $\hat{p} = 0.2536$
BE	0.3409	0.9044	110.9079	$\hat{\beta} = 6.9441$, $\hat{\lambda} = 16.2310$, $\hat{p} = 0.2459$

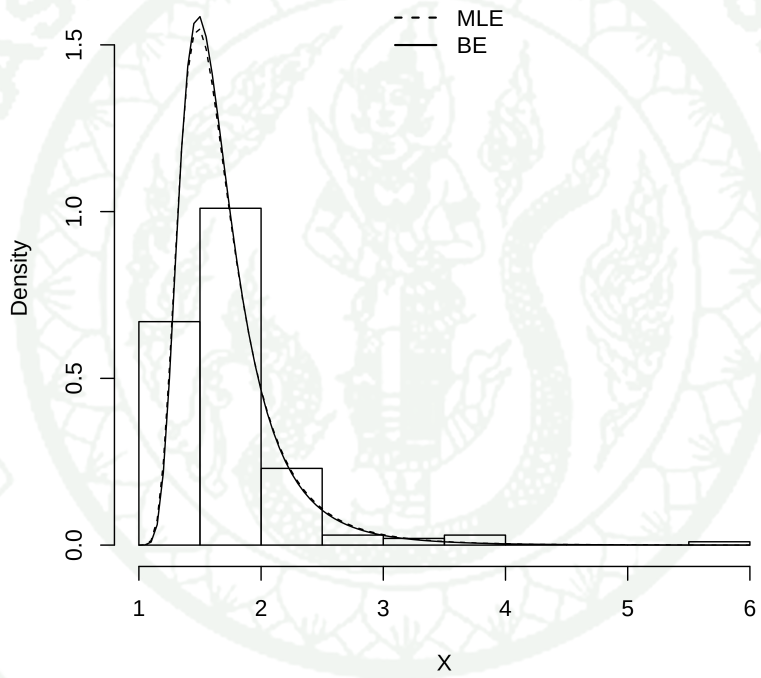

Figure 57 The goodness of fit plots of the MLE and BE estimates based on the MEIW distribution for the breaking strengths of single carbon fibers dataset

Discussion

The results of the two proposed distributions in this research shows that the LBEIW and MEIW distributions are to improve model of lifetime data analysis.

1. The length-biased exponentiated inverted Weibull distribution

The LBEIW distribution is a weighted distribution of the EIW distribution. In fact, the LBEIW includes the LBIW distribution as the sub-model. We present the probability functions and some properties of the distribution.

The simulation study for the comparison of parameter estimation of the LBEIW distribution using MLE and BE methods which assume independent parameters. The estimation of the parameters of the LBEIW distribution by the BE method is better than the MLE method in almost all of the parameter cases. The MLE method gives high values of MSE that represent low efficient estimation. In almost cases, when the sample size increases, the MSE of the MLE method is close to the BE method. Obviously, the BE gives more stationary estimators than the MLE in all cases. Moreover, we found that the MLE sometimes do not converge in the Newton-Raphson procedure although the BE does not experience this problem.

For application of the LBEIW distribution is given to show that this distribution fits the data as well as the EIW distribution for lifetime data and is more flexible than the LBIW and two-parameter Weibull distributions. We can say that the LBIEW distribution is an alternative distribution for heterogeneous populations in lifetime data analysis.

2. The mixture exponentiated inverted Weibull distribution

The MEIW distribution is a new mixed distribution to extend the EIW distribution and the LBEIW distribution. In fact, the MEIW includes the EIW distribution and the LBEIW distribution as the sub-model. We present some probability functions and some properties of the distribution.

The simulation study for the comparison of parameters estimation of the MEIW distribution using MLE and BE methods which assume independent parameters. The estimation of the parameters of the MEIW distribution by the BE method is better than the MLE method in almost all of the parameter cases. The MLE method gives high values of MSE that represent low efficient estimation, especially the parameter p. In almost cases, when the sample size increases, the MSE of the MLE method is close to the BE method. Obviously, the BE gives more stationary estimators than the MLE in all cases. Moreover, we found that the MLE sometimes do not converge in the Newton-Raphson procedure although the BE does not experience this problem.

For application of the MEIW distribution is given to show that this distribution as well as the LBEIW distribution could give a better fit to lifetime data than the EIW, three-parameter Weibull and three-parameter lognormal distributions. We can say that the MEIW distribution is an alternative distribution for heterogeneous populations in lifetime data analysis.

CONCLUSION AND RECOMMENDATIONS

Conclusion

This thesis present the LBEIW distribution which is obtained from weighted of the EIW distribution and the MEIW which is obtained by mixing the EIW and the LBEIW distributions are given by, respectively.

1. The length-biased exponentiated inverted Weibull distribution

The LBEIW distribution is obtained from weighted of the EIW distribution which the pdf, the survival function and the hazard rate are given by, respectively:

The pdf:

$$f_L(x|\beta,\lambda) = \frac{\beta\lambda^{1-\frac{1}{\beta}}}{\Gamma(1-\frac{1}{\beta})}x^{-\beta}\{\exp(-x^{-\beta})\}^{\lambda};\ x>0, \beta>1, \lambda>0. \tag{119}$$

The survival function:

$$S_L(x|\beta,\lambda) = \frac{\gamma(1-\frac{1}{\beta},\frac{\lambda}{x^{\beta}})}{\Gamma(1-\frac{1}{\beta})}. \tag{120}$$

The hazard rate:

$$h_L(x|\beta,\lambda) = \frac{\beta\lambda^{1-\frac{1}{\beta}}x^{-\beta}\{\exp(-x^{-\beta})\}^{\lambda}}{\gamma(1-\frac{1}{\beta},\frac{\lambda}{t^{\beta}})}. \tag{121}$$

Next, we find that LBIW is special case of LBEIW distribution.

In addition, we propose the properties of the LBEIW distribution, which includes k^{th} moment about the origin, mean, variance, coefficient of skewness and coefficient of kurtosis are given as follows:

The k^{th} moment about the origin:

$$E(X^k) = \lambda^{\frac{k}{\beta}} \frac{\Gamma(1-\frac{k+1}{\beta})}{\Gamma(1-\frac{1}{\beta})}. \tag{122}$$

The mean:

$$E(X) = \lambda^{\frac{1}{\beta}} \frac{\Gamma(1-\frac{2}{\beta})}{\Gamma(1-\frac{1}{\beta})}. \tag{123}$$

The variance:

$$Var(X) = \frac{\lambda^{\frac{2}{\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)} \left[\Gamma\left(1-\frac{3}{\beta}\right) - \frac{\left[\Gamma\left(1-\frac{2}{\beta}\right)\right]^2}{\Gamma\left(1-\frac{1}{\beta}\right)}\right]. \tag{124}$$

The coefficient of skewness:

$$CS(X) = \frac{\Gamma_1^{\frac{1}{2}}\left[\Gamma_4 - 3\frac{\Gamma_2\Gamma_3}{\Gamma_1} + 2\frac{\Gamma_2^3}{\Gamma_1^2}\right]}{\left(\Gamma_3 - \frac{\Gamma_2^2}{\Gamma_1}\right)^{\frac{3}{2}}}, \tag{125}$$

The coefficient of kurtosis:

$$CK(X) = \frac{\Gamma_1\Gamma_5 - 4\Gamma_2\Gamma_4 + 6\frac{\Gamma_2^2\Gamma_3}{\Gamma_1} - 3\frac{\Gamma_2^4}{\Gamma_1^2}}{\left[\Gamma_3 - \frac{\Gamma_2^2}{\Gamma_1}\right]^2} - 3. \tag{126}$$

Moreover, we derived parameter estimation of the LBEIW distribution by MLE and BE methods. For simulation study, we use Monte Carlo simulation which was carried out in order to investigate the performance of the MLE and BE methods for parameter estimation of the LBEIW distribution. We found that the BE outperform the MLE for the proposed distribution.

For application of the proposed distribution, we compared efficiencies of the LBEIW distribution with the EIW, LBIW and two-parameter Weibull distributions, fitting among distribution by using real data. The data shown distance between cracks in a pipe dataset. We found that the LBEIW distribution fit to the data as well as the EIW distribution.

2. The mixture exponentiated inverted Weibull distribution

We also propose the MEIW distribution which is obtained by mixing the EIW and the LBEIW distributions which the pdf, the survival function and the hazard rate are given by, respectively:

The pdf:

$$f(x|\beta,\lambda,p) = \left[p + (1-p)\frac{\lambda^{-\frac{1}{\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)}x\right]\beta\lambda x^{-(\beta+1)}\left\{\exp(-x^{-\beta})\right\}^{\lambda}. \tag{127}$$

The survival function:

$$S(x|\beta,\lambda,p) = 1 - \left[p\left\{\exp(-x^{-\beta})\right\}^{\lambda} + (1-p)\frac{\Gamma\left(1-\frac{1}{\beta},\frac{\lambda}{x^{\beta}}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)}\right]. \tag{128}$$

The hazard rate:

$$h(x|\beta,\lambda,p) = \frac{\left[p + (1-p)\frac{\lambda^{-\frac{1}{\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)}x\right]\beta\lambda x^{-(\beta+1)}\left\{\exp(-x^{-\beta})\right\}^{\lambda}}{1 - \left[p\left\{\exp(-x^{-\beta})\right\}^{\lambda} + (1-p)\frac{\Gamma\left(1-\frac{1}{\beta},\frac{\lambda}{x^{\beta}}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)}\right]}. \tag{129}$$

Next, we find that the EIW, LBEIW, and LEIW distributions are special cases of MEIW distribution.

In addition, we propose the properties of the MEIW distribution, which includes k^{th} moment about the origin, mean, variance, coefficient of skewness and coefficient of kurtosis are given as follows:

The k^{th} moment about the origin:

$$E(X^k) = \lambda^{\frac{k}{\beta}} \left[p\Gamma\left(1 - \frac{k}{\beta}\right) + (1-p)\frac{\Gamma\left(1 - \frac{k+1}{\beta}\right)}{\Gamma\left(1 - \frac{1}{\beta}\right)} \right]. \tag{130}$$

The mean:

$$E(X) = \lambda^{\frac{1}{\beta}} \left[p\Gamma\left(1 - \frac{1}{\beta}\right) + (1-p)\frac{\Gamma\left(1 - \frac{2}{\beta}\right)}{\Gamma\left(1 - \frac{1}{\beta}\right)} \right]. \tag{131}$$

The variance:

$$\begin{aligned} Var(X) = \lambda^{\frac{2}{\beta}} \Bigg\{ & \left[p\Gamma\left(1 - \frac{2}{\beta}\right) + (1-p)\frac{\Gamma\left(1 - \frac{3}{\beta}\right)}{\Gamma\left(1 - \frac{1}{\beta}\right)} \right] \\ & - \left[p\Gamma\left(1 - \frac{1}{\beta}\right) + (1-p)\frac{\Gamma\left(1 - \frac{2}{\beta}\right)}{\Gamma\left(1 - \frac{1}{\beta}\right)} \right]^2 \Bigg\}. \end{aligned} \tag{132}$$

The coefficient of skewness:

$$CS(X) = \frac{\Gamma_3(p) - 3\Gamma_1(p)\Gamma_2(p) + 2\Gamma_1^3(p)}{\left[\Gamma_2(p) - \Gamma_1^2(p)\right]^{\frac{3}{2}}}. \tag{133}$$

The coefficient of kurtosis:

$$CK(X) = \frac{\Gamma_4(p) - 4\Gamma_1(p)\Gamma_4(p) + 6\Gamma_1^2(p)\Gamma_2(p) - 3\Gamma_1^4(p)}{\left[\Gamma_2(p) - \Gamma_1^2(p)\right]^2} - 3. \tag{134}$$

Moreover, we derived parameter estimation of the MEIW distribution by MLE and BE methods For simulation study, we use Monte Carlo simulation which was carried out in order to investigate the performance of the MLE and BE methods for parameter estimation of the MEIW distribution. We found that the BE outperform the MLE for the proposed distribution.

For application of the proposed distribution, we compared efficiencies of the MEIW distribution with the LBEIW, EIW, three-parameter Weibull and three-parameter lognormal distributions, fitting among distribution by using real data. The data shown breaking strengths of carbon fibers dataset. We found that the LBEIW and MEIW distributions give the best result.

Recommendations

Weibull distribution is a basic distribution for fitting lifetime data. In some case the Weibull distribution may not be fit to some datasets. In addition, many studies proposed various lifetime distributions. Thus, the LBEIW and MEIW distributions are an alternative distribution for lifetime data analysis.

Further Research

In future research, we may applied the LBEIW and MEIW distributions to statistical reliability analysis and survival analysis. Next, we can build regression model for lifetime data based on the LBEIW and MEIW distributions.

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APPENDICES

Appendix A

Simulation study of the LBEIW distribution

Appendix Table A1 Simulation study of the LBEIW distribution (True parameters: $\beta = 2, \lambda = 1$)

n	Parameter	MLE			BE		
		Estimate	**Var**	**MSE**	**Estimate**	**Var**	**MSE**
15	β	2.1253	0.0984	0.1140	2.0383	0.0829	0.0843
	λ	1.2506	0.4964	0.5587	0.9916	0.2745	0.2743
30	β	2.0585	0.0375	0.0408	2.0217	0.0346	0.0351
	λ	1.1091	0.1222	0.1339	1.0002	0.0991	0.0990
50	β	2.0293	0.0223	0.0232	2.0073	0.0212	0.0212
	λ	1.0577	0.0613	0.0645	0.9938	0.0535	0.0535
100	β	2.0187	0.0095	0.0098	2.0076	0.0092	0.0093
	λ	1.0258	0.0240	0.0247	0.9946	0.0226	0.0226
150	β	2.0114	0.0066	0.0067	2.0038	0.0065	0.0065
	λ	1.0254	0.0173	0.0179	1.0035	0.0166	0.0166
200	β	2.0101	0.0046	0.0047	2.0044	0.0045	0.0045
	λ	1.0158	0.0110	0.0112	0.9996	0.0106	0.0106
300	β	2.0065	0.0034	0.0035	2.0026	0.0034	0.0034
	λ	1.0109	0.0075	0.0076	0.9999	0.0073	0.0073
500	β	2.0042	0.0019	0.0019	2.0018	0.0019	0.0019
	λ	1.0061	0.0050	0.0050	0.9992	0.0049	0.0049

Appendix Table A2 Simulation study of the LBEIW distribution (True parameters: $\beta = 2$, $\lambda = 5$)

n	Parameter	MLE			BE		
		Estimate	**Var**	**MSE**	**Estimate**	**Var**	**MSE**
15	β	2.1480	0.0979	0.1196	1.9894	0.0636	0.0637
	λ	7.5796	40.6067	47.2199	5.1589	11.4148	11.4286
30	β	2.0709	0.0370	0.0419	2.0006	0.0305	0.0304
	λ	6.0175	6.9347	7.9631	5.1291	4.6152	4.6272
50	β	2.0374	0.0217	0.0231	1.9961	0.0192	0.0192
	λ	5.5770	2.7325	3.0628	5.0800	2.2107	2.2149
100	β	2.0186	0.0106	0.0109	1.9986	0.0099	0.0099
	λ	5.2540	1.0663	1.1297	5.0223	0.9606	0.9602
150	β	2.0140	0.0070	0.0072	2.0006	0.0068	0.0068
	λ	5.1626	0.6289	0.6547	5.0112	0.5950	0.5945
200	β	2.0096	0.0051	0.0052	2.0000	0.0049	0.0049
	λ	5.1338	0.4994	0.5168	5.0252	0.4755	0.4757
300	β	2.0051	0.0032	0.0032	1.9982	0.0031	0.0031
	λ	5.0561	0.2989	0.3018	4.9785	0.2892	0.2894
500	β	2.0047	0.0020	0.0021	2.0005	0.0020	0.0020
	λ	5.0679	0.1944	0.1989	5.0214	0.1909	0.1912

Appendix Table A3 Simulation study of the LBEIW distribution (True parameters: $\beta = 2$, $\lambda = 10$)

n	Parameter	MLE			BE		
		Estimate	Var	MSE	Estimate	Var	MSE
15	β	2.1252	0.0988	0.1144	1.9342	0.0567	0.0610
	λ	16.7403	377.0209	422.0599	9.7780	51.9299	51.9272
30	β	2.0494	0.0384	0.0408	1.9645	0.0301	0.0313
	λ	12.2678	41.6895	46.7906	9.8828	21.2611	21.2535
50	β	2.0321	0.0214	0.0224	1.9854	0.0196	0.0198
	λ	11.2998	16.2086	17.8817	10.1493	13.2233	13.2324
100	β	2.0143	0.0095	0.0097	1.9906	0.0090	0.0091
	λ	10.4830	5.0584	5.2867	9.9380	4.6262	4.6254
150	β	2.0129	0.0066	0.0068	1.9968	0.0063	0.0063
	λ	10.4417	3.3393	3.5310	10.0710	3.0670	3.0690
200	β	2.0074	0.0049	0.0049	1.9952	0.0047	0.0047
	λ	10.3549	2.6372	2.7604	10.0734	2.4666	2.4696
300	β	2.0049	0.0034	0.0035	1.9967	0.0034	0.0034
	λ	10.1530	1.7352	1.7568	9.9651	1.6594	1.6590
500	β	2.0050	0.0018	0.0018	2.0000	0.0018	0.0018
	λ	10.1337	0.9048	0.9218	10.0201	0.8906	0.8901

Appendix Table A4 Simulation study of the LBEIW distribution (True parameters: $\beta = 5$, $\lambda = 1$)

n	Parameter	MLE			BE		
		Estimate	Var	MSE	Estimate	Var	MSE
15	β	5.4174	1.3272	1.5001	5.1807	1.1990	1.2304
	λ	1.1257	0.1696	0.1853	0.9953	0.1208	0.1207
30	β	5.1779	0.4897	0.5209	5.0717	0.4728	0.4775
	λ	1.0420	0.0529	0.0546	0.9833	0.0452	0.0454
50	β	5.1208	0.2555	0.2699	5.0566	0.2504	0.2533
	λ	1.0270	0.0304	0.0311	0.9928	0.0277	0.0277
100	β	5.0468	0.1220	0.1241	5.0148	0.1213	0.1214
	λ	1.0161	0.0134	0.0137	0.9985	0.0128	0.0128
150	β	5.0338	0.0816	0.0827	5.0128	0.0812	0.0812
	λ	1.0080	0.0082	0.0083	0.9961	0.0080	0.0080
200	β	5.0234	0.0591	0.0596	5.0071	0.0589	0.0589
	λ	1.0073	0.0060	0.0060	0.9984	0.0058	0.0058
300	β	5.0287	0.0417	0.0425	5.0174	0.0415	0.0418
	λ	1.0035	0.0046	0.0046	0.9974	0.0045	0.0045
500	β	5.0192	0.0248	0.0251	5.0122	0.0248	0.0249
	λ	0.9999	0.0025	0.0025	0.9961	0.0025	0.0025

Appendix Table A5 Simulation study of the LBEIW distribution (True parameters: $\beta = 5$, $\lambda = 5$)

n	Parameter	MLE			BE		
		Estimate	Var	MSE	Estimate	Var	MSE
15	β	5.4188	1.2226	1.3967	4.8426	0.8345	0.8584
	λ	6.8057	23.0658	26.3033	5.0494	7.8960	7.8905
30	β	5.1827	0.4638	0.4967	4.9194	0.3875	0.3936
	λ	5.6591	3.5066	3.9375	4.9883	2.3806	2.3784
50	β	5.1045	0.2367	0.2473	4.9546	0.2134	0.2153
	λ	5.3897	1.7678	1.9180	5.0210	1.4446	1.4436
100	β	5.0577	0.1219	0.1251	4.9815	0.1163	0.1166
	λ	5.1758	0.7117	0.7418	4.9950	0.6515	0.6509
150	β	5.0356	0.0781	0.0793	4.9843	0.0755	0.0757
	λ	5.1336	0.4557	0.4731	5.0125	0.4281	0.4278
200	β	5.0289	0.0564	0.0572	4.9903	0.0551	0.0551
	λ	5.0994	0.3038	0.3134	5.0088	0.2886	0.2884
300	β	5.0192	0.0414	0.0418	4.9935	0.0409	0.0409
	λ	5.0595	0.2060	0.2093	4.9996	0.2017	0.2015
500	β	5.0126	0.0214	0.0215	4.9963	0.0212	0.0212
	λ	5.0395	0.1225	0.1240	5.0017	0.1203	0.1202

Appendix Table A6 Simulation study of the LBEIW distribution (True parameters: $\beta = 5$, $\lambda = 10$)

n	Parameter	MLE			BE		
		Estimate	Var	MSE	Estimate	Var	MSE
15	β	5.4650	1.1421	1.3573	4.7220	0.6586	0.7352
	λ	15.8743	207.1987	241.5042	9.7670	46.6397	46.6474
30	β	5.2472	0.4930	0.5536	4.9080	0.3821	0.3902
	λ	12.3502	36.9563	42.4428	10.2214	19.4783	19.5078
50	β	5.1115	0.2493	0.2615	4.9127	0.2203	0.2277
	λ	11.0733	12.1771	13.3169	9.9517	8.8288	8.8223
100	β	5.0731	0.1247	0.1299	4.9730	0.1143	0.1149
	λ	10.6266	4.9255	5.3132	10.0807	4.1465	4.1488
150	β	5.0533	0.0787	0.0814	4.9884	0.0763	0.0763
	λ	10.4031	2.7196	2.8794	10.0620	2.5007	2.5020
200	β	5.0205	0.0624	0.0628	4.9711	0.0604	0.0612
	λ	10.2408	2.2431	2.2989	9.9813	2.0829	2.0812
300	β	5.0253	0.0377	0.0383	4.9919	0.0370	0.0370
	λ	10.2366	1.2213	1.2760	10.0626	1.1705	1.1732
500	β	5.0069	0.0227	0.0228	4.9862	0.0225	0.0227
	λ	10.1007	0.7444	0.7538	9.9936	0.7257	0.7250

Appendix Table A7 Simulation study of the LBEIW distribution (True parameters: $\beta = 8$, $\lambda = 1$)

n	Parameter	MLE			BE		
		Estimate	Var	MSE	Estimate	Var	MSE
15	β	8.6981	3.3157	3.7998	8.3451	3.0968	3.2128
	λ	1.0754	0.1387	0.1442	0.9712	0.1055	0.1062
30	β	8.3852	1.3187	1.4658	8.2183	1.2845	1.3309
	λ	1.0351	0.0468	0.0480	0.9866	0.0408	0.0410
50	β	8.1974	0.7495	0.7877	8.0942	0.7300	0.7382
	λ	1.0227	0.0245	0.0250	0.9938	0.0224	0.0224
100	β	8.1136	0.3549	0.3675	8.0634	0.3516	0.3553
	λ	1.0081	0.0124	0.0125	0.9936	0.0120	0.0120
150	β	8.0554	0.2295	0.2324	8.0219	0.2297	0.2299
	λ	1.0045	0.0083	0.0083	0.9947	0.0081	0.0081
200	β	8.0258	0.1639	0.1644	8.0004	0.1633	0.1632
	λ	1.0072	0.0061	0.0062	0.9996	0.0060	0.0060
300	β	8.0454	0.1100	0.1119	8.0285	0.1101	0.1108
	λ	1.0038	0.0045	0.0045	0.9986	0.0044	0.0044
500	β	8.0195	0.0683	0.0686	8.0095	0.0679	0.0679
	λ	1.0021	0.0024	0.0024	0.9988	0.0024	0.0024

Appendix Table A8 Simulation study of the LBEIW distribution (True parameters: $\beta = 8$, $\lambda = 5$)

n	Parameter	MLE			BE		
		Estimate	**Var**	**MSE**	**Estimate**	**Var**	**MSE**
15	β	8.7125	3.1139	3.6184	7.7287	2.0513	2.1229
	λ	6.8126	17.9703	21.2376	5.1155	5.6722	5.6798
30	β	8.3792	1.3891	1.5315	7.9265	1.1886	1.1928
	λ	5.7722	4.2075	4.7995	5.1171	2.8974	2.9082
50	β	8.2095	0.7525	0.7956	7.9447	0.6777	0.6800
	λ	5.4259	1.8374	2.0169	5.0643	1.5301	1.5327
100	β	8.1367	0.3651	0.3834	8.0064	0.3504	0.3501
	λ	5.2074	0.6145	0.6570	5.0363	0.5598	0.5606
150	β	8.0658	0.2297	0.2338	7.9810	0.2245	0.2246
	λ	5.1275	0.4220	0.4378	5.0176	0.4012	0.4011
200	β	8.0486	0.1660	0.1682	7.9836	0.1616	0.1617
	λ	5.0702	0.2844	0.2891	4.9861	0.2705	0.2705
300	β	8.0217	0.1205	0.1208	7.9779	0.1186	0.1190
	λ	5.0729	0.2164	0.2215	5.0161	0.2103	0.2104
500	β	8.0190	0.0654	0.0657	7.9916	0.0650	0.0650
	λ	5.0400	0.1116	0.1131	5.0044	0.1095	0.1094

Appendix Table A9 Simulation study of the LBEIW distribution (True parameters: $\beta = 8$, $\lambda = 10$)

n	Parameter	MLE			BE		
		Estimate	Var	MSE	Estimate	Var	MSE
15	β	8.8351	3.1298	3.8241	7.5412	1.8612	2.0698
	λ	15.5143	158.0622	188.3089	9.7797	29.6687	29.6875
30	β	8.4251	1.3853	1.5646	7.8580	1.1883	1.2073
	λ	12.2157	29.9813	34.8605	10.2919	19.6465	19.7121
50	β	8.2118	0.7266	0.7708	7.8731	0.6365	0.6520
	λ	11.1077	11.4103	12.6258	10.0226	8.5004	8.4924
100	β	8.1032	0.3627	0.3730	7.9397	0.3441	0.3474
	λ	10.6041	4.5256	4.8859	10.1096	4.0105	4.0185
150	β	8.0712	0.2164	0.2212	7.9637	0.2104	0.2115
	λ	10.3774	2.5741	2.7140	10.0609	2.3990	2.4003
200	β	8.0450	0.1842	0.1860	7.9613	0.1791	0.1805
	λ	10.2254	1.9618	2.0106	9.9789	1.8468	1.8454
300	β	8.0312	0.1174	0.1183	7.9749	0.1158	0.1163
	λ	10.2080	1.2734	1.3154	10.0415	1.2233	1.2238
500	β	8.0326	0.0708	0.0718	7.9979	0.0702	0.0701
	λ	10.1318	0.7137	0.7303	10.0303	0.7006	0.7008

Appendix B

Simulation study of the MEIW distribution

Appendix Table B1 Simulation study of the MEIW distribution (True parameters: $\beta = 2$, $\lambda = 1$, $p = 0.2$)

n	Parameter	MLE			BE		
		Estimate	Var	MSE	Estimate	Var	MSE
15	β	2.0902	0.1302	0.1382	2.0735	0.0938	0.0991
	λ	1.2111	0.3013	0.3456	1.0031	0.1955	0.1953
	p	0.2341	0.0893	0.0904	0.2003	0.0002	0.0002
30	β	2.0179	0.0668	0.0671	2.0442	0.0400	0.0420
	λ	1.1322	0.1300	0.1474	1.0073	0.0696	0.0695
	p	0.2516	0.0954	0.0980	0.2002	0.0002	0.0002
50	β	1.9733	0.0471	0.0478	2.0196	0.0230	0.0234
	λ	1.1111	0.0959	0.1081	1.0046	0.0394	0.0394
	p	0.2729	0.0936	0.0988	0.2002	0.0002	0.0002
100	β	1.9621	0.0352	0.0366	2.0100	0.0109	0.0110
	λ	1.0781	0.0699	0.0759	1.0013	0.0202	0.0202
	p	0.2639	0.0876	0.0916	0.2006	0.0002	0.0002
150	β	1.9638	0.0266	0.0279	2.0093	0.0076	0.0077
	λ	1.0583	0.0538	0.0572	0.9911	0.0127	0.0128
	p	0.2634	0.0755	0.0795	0.2002	0.0002	0.0002
200	β	1.9629	0.0263	0.0276	2.0052	0.0057	0.0057
	λ	1.0523	0.0567	0.0594	0.9969	0.0094	0.0094
	p	0.2497	0.0784	0.0808	0.1998	0.0002	0.0002
300	β	1.9602	0.0190	0.0206	1.9993	0.0036	0.0036
	λ	1.0481	0.0473	0.0496	0.9999	0.0061	0.0061
	p	0.2474	0.0694	0.0715	0.1997	0.0003	0.0003
500	β	1.9718	0.0143	0.0151	2.0005	0.0021	0.0021
	λ	1.0362	0.0359	0.0371	1.0025	0.0038	0.0038
	p	0.2330	0.0545	0.0556	0.2000	0.0003	0.0003

Appendix Table B2 Simulation study of the MEIW distribution (True parameters: $\beta = 2$, $\lambda = 1$, $p = 0.5$)

n	Parameter	MLE			BE		
		Estimate	Var	MSE	Estimate	Var	MSE
15	β	2.1299	0.1809	0.1976	2.1026	0.1303	0.1406
	λ	1.0873	0.2000	0.2075	1.0052	0.1493	0.1492
	p	0.4176	0.1464	0.1531	0.4950	0.0001	0.0001
30	β	2.0411	0.1013	0.1029	2.0531	0.0550	0.0578
	λ	1.0336	0.0950	0.0960	0.9931	0.0525	0.0525
	p	0.4454	0.1395	0.1424	0.4957	0.0001	0.0001
50	β	2.0204	0.0706	0.0710	2.0325	0.0305	0.0315
	λ	1.0116	0.0786	0.0787	0.9954	0.0333	0.0333
	p	0.4402	0.1269	0.1304	0.4945	0.0001	0.0002
100	β	2.0026	0.0563	0.0563	2.0151	0.0131	0.0133
	λ	0.9945	0.0613	0.0613	0.9944	0.0160	0.0160
	p	0.4399	0.1175	0.1210	0.4952	0.0001	0.0002
150	β	2.0047	0.0436	0.0436	2.0129	0.0090	0.0092
	λ	0.9845	0.0499	0.0501	0.9941	0.0101	0.0101
	p	0.4382	0.1062	0.1099	0.4949	0.0002	0.0002
200	β	1.9880	0.0409	0.0410	2.0079	0.0068	0.0068
	λ	0.9952	0.0457	0.0456	0.9919	0.0070	0.0071
	p	0.4626	0.0988	0.1001	0.4950	0.0002	0.0002
300	β	1.9941	0.0344	0.0344	2.0020	0.0044	0.0044
	λ	0.9838	0.0420	0.0423	0.9938	0.0049	0.0050
	p	0.4481	0.0899	0.0925	0.4947	0.0002	0.0002
500	β	1.9986	0.0270	0.0270	2.0044	0.0027	0.0027
	λ	0.9844	0.0337	0.0339	0.9933	0.0032	0.0032
	p	0.4560	0.0733	0.0752	0.4941	0.0003	0.0003

Appendix Table B3 Simulation study of the MEIW distribution (True parameters: $\beta = 2$, $\lambda = 1$, $p = 0.8$)

n	Parameter	MLE			BE		
		Estimate	Var	MSE	Estimate	Var	MSE
15	β	2.2677	0.2770	0.3484	2.1428	0.1860	0.2062
	λ	1.0125	0.1765	0.1765	0.9976	0.1261	0.1259
	p	0.6291	0.1599	0.1890	0.8040	0.0000	0.0000
30	β	2.1510	0.1326	0.1552	2.0520	0.0726	0.0752
	λ	0.9546	0.0746	0.0766	0.9908	0.0474	0.0474
	p	0.6345	0.1381	0.1653	0.8038	0.0000	0.0000
50	β	2.0783	0.1011	0.1071	2.0172	0.0398	0.0401
	λ	0.9655	0.0580	0.0591	0.9949	0.0242	0.0242
	p	0.6726	0.1255	0.1416	0.8038	0.0000	0.0001
100	β	2.0697	0.0708	0.0755	2.0133	0.0198	0.0200
	λ	0.9554	0.0451	0.0470	0.9959	0.0124	0.0124
	p	0.6800	0.1090	0.1233	0.8038	0.0001	0.0001
150	β	2.0700	0.0668	0.0716	2.0022	0.0135	0.0134
	λ	0.9469	0.0424	0.0452	1.0006	0.0087	0.0087
	p	0.6636	0.1078	0.1263	0.8031	0.0001	0.0001
200	β	2.0694	0.0575	0.0623	2.0032	0.0100	0.0100
	λ	0.9485	0.0364	0.0390	1.0019	0.0063	0.0063
	p	0.6723	0.0971	0.1133	0.8029	0.0001	0.0001
300	β	2.0467	0.0468	0.0489	2.0012	0.0068	0.0068
	λ	0.9633	0.0273	0.0287	1.0011	0.0040	0.0040
	p	0.7086	0.0758	0.0841	0.8029	0.0001	0.0001
500	β	2.0467	0.0392	0.0414	1.9965	0.0037	0.0037
	λ	0.9591	0.0263	0.0280	1.0022	0.0025	0.0025
	p	0.7046	0.0741	0.0831	0.8031	0.0002	0.0002

Appendix Table B4 Simulation study of the MEIW distribution (True parameters: $\beta = 2$, $\lambda = 5$, $p = 0.2$)

n	**Parameter**	**MLE**			**BE**		
		Estimate	**Var**	**MSE**	**Estimate**	**Var**	**MSE**
15	β	2.1041	0.1551	0.1658	1.9993	0.0745	0.0744
	λ	7.2266	28.0879	33.0177	5.4029	10.3135	10.4655
	p	0.2277	0.0841	0.0847	0.2015	0.0002	0.0002
30	β	1.9997	0.0863	0.0862	2.0125	0.0353	0.0354
	λ	5.8644	4.9509	5.6931	5.2329	3.7803	3.8307
	p	0.2900	0.1091	0.1171	0.2010	0.0002	0.0002
50	β	1.9540	0.0589	0.0610	1.9948	0.0212	0.0212
	λ	5.4076	2.2516	2.4155	5.0918	2.0423	2.0487
	p	0.2923	0.1022	0.1106	0.2008	0.0002	0.0002
100	β	1.9586	0.0417	0.0434	2.0060	0.0111	0.0111
	λ	5.2266	0.9066	0.9571	5.0861	0.7708	0.7774
	p	0.2793	0.0931	0.0993	0.2010	0.0002	0.0002
150	β	1.9393	0.0364	0.0400	1.9984	0.0069	0.0069
	λ	5.1151	0.6282	0.6408	5.0206	0.4655	0.4655
	p	0.2902	0.0939	0.1020	0.2011	0.0002	0.0002
200	β	1.9540	0.0286	0.0307	1.9996	0.0053	0.0053
	λ	5.0877	0.5323	0.5394	5.0476	0.3617	0.3636
	p	0.2624	0.0835	0.0873	0.2010	0.0002	0.0002
300	β	1.9595	0.0218	0.0234	1.9984	0.0036	0.0036
	λ	5.0432	0.3825	0.3840	5.0048	0.2320	0.2318
	p	0.2559	0.0690	0.0721	0.2010	0.0003	0.0003
500	β	1.9673	0.0149	0.0159	1.9979	0.0022	0.0022
	λ	4.9992	0.3103	0.3100	4.9923	0.1445	0.1445
	p	0.2407	0.0568	0.0584	0.2013	0.0004	0.0004

Appendix Table B5 Simulation study of the MEIW distribution (True parameters: $\beta = 2$, $\lambda = 5$, $p = 0.5$)

n	**Parameter**	**MLE**			**BE**		
		Estimate	**Var**	**MSE**	**Estimate**	**Var**	**MSE**
15	β	2.1695	0.2211	0.2496	2.0035	0.0923	0.0922
	λ	6.3650	17.1160	18.9623	5.2618	6.5322	6.5943
	p	0.3498	0.1311	0.1535	0.4965	0.0001	0.0001
30	β	2.0827	0.1301	0.1368	2.0095	0.0482	0.0482
	λ	5.4559	4.1219	4.3256	5.2282	3.2068	3.2557
	p	0.3835	0.1300	0.1434	0.4968	0.0001	0.0001
50	β	2.0457	0.0911	0.0931	2.0149	0.0285	0.0287
	λ	5.0895	1.5679	1.5744	5.0746	1.4073	1.4114
	p	0.4177	0.1236	0.1303	0.4975	0.0001	0.0001
100	β	2.0110	0.0645	0.0646	2.0070	0.0137	0.0137
	λ	4.9245	0.6597	0.6647	5.0383	0.5992	0.6001
	p	0.4383	0.1137	0.1174	0.4969	0.0002	0.0002
150	β	2.0139	0.0491	0.0493	1.9986	0.0086	0.0086
	λ	4.8224	0.4315	0.4626	4.9858	0.3686	0.3685
	p	0.4179	0.1050	0.1116	0.4977	0.0002	0.0002
200	β	2.0159	0.0408	0.0410	1.9981	0.0064	0.0064
	λ	4.8158	0.3923	0.4258	4.9987	0.2806	0.2804
	p	0.4125	0.0994	0.1070	0.4968	0.0002	0.0002
300	β	2.0186	0.0337	0.0340	2.0004	0.0047	0.0047
	λ	4.8184	0.3186	0.3512	5.0004	0.1913	0.1911
	p	0.4157	0.0906	0.0976	0.4966	0.0002	0.0002
500	β	2.0148	0.0245	0.0247	2.0012	0.0027	0.0027
	λ	4.8368	0.2157	0.2421	4.9892	0.1090	0.1091
	p	0.4318	0.0717	0.0763	0.4964	0.0003	0.0003

Appendix Table B6 Simulation study of the MEIW distribution (True parameters: $\beta = 2$, $\lambda = 5$, $p = 0.8$)

n	**Parameter**	**MLE**			**BE**		
		Estimate	**Var**	**MSE**	**Estimate**	**Var**	**MSE**
15	β	2.2793	0.2688	0.3466	1.9945	0.1314	0.1313
	λ	6.4250	23.5770	25.5840	5.3030	7.0757	7.1605
	p	0.5938	0.1664	0.2087	0.8059	0.0000	0.0001
30	β	2.1587	0.1415	0.1665	1.9834	0.0617	0.0619
	λ	5.4058	3.3154	3.4768	5.1218	2.2806	2.2931
	p	0.6114	0.1461	0.1815	0.8058	0.0000	0.0001
50	β	2.1448	0.1074	0.1282	1.9944	0.0396	0.0396
	λ	5.1701	1.5138	1.5413	5.0963	1.1740	1.1821
	p	0.6017	0.1440	0.1832	0.8059	0.0000	0.0001
100	β	2.1261	0.0800	0.0958	1.9902	0.0173	0.0174
	λ	4.9666	0.6208	0.6213	5.0412	0.4818	0.4830
	p	0.5905	0.1386	0.1824	0.8057	0.0001	0.0001
150	β	2.1206	0.0692	0.0836	1.9905	0.0125	0.0126
	λ	4.8737	0.4404	0.4559	4.9908	0.3239	0.3236
	p	0.5919	0.1311	0.1743	0.8058	0.0001	0.0001
200	β	2.1245	0.0609	0.0763	1.9917	0.0102	0.0103
	λ	4.8776	0.3414	0.3560	5.0000	0.2338	0.2336
	p	0.5903	0.1221	0.1660	0.8052	0.0001	0.0001
300	β	2.1203	0.0568	0.0712	1.9919	0.0066	0.0067
	λ	4.8586	0.2562	0.2759	4.9969	0.1542	0.1540
	p	0.5930	0.1203	0.1630	0.8057	0.0001	0.0002
500	β	2.0987	0.0453	0.0550	1.9913	0.0039	0.0039
	λ	4.8768	0.1688	0.1838	5.0008	0.0925	0.0924
	p	0.6269	0.1037	0.1336	0.8053	0.0002	0.0002

Appendix Table B7 Simulation study of the MEIW distribution (True parameters: $\beta = 2$, $\lambda = 10$, $p = 0.2$)

n	Parameter	MLE Estimate	MLE Var	MLE MSE	BE Estimate	BE Var	BE MSE
15	β	2.0976	0.1458	0.1552	1.9654	0.0599	0.0610
	λ	16.2895	1175.9965	1214.6089	10.3330	50.3933	50.4537
	p	0.2292	0.0862	0.0870	0.2010	0.0002	0.0002
30	β	2.0077	0.0774	0.0774	1.9939	0.0327	0.0327
	λ	11.9030	32.0301	35.6393	10.4177	17.4783	17.6353
	p	0.2624	0.0939	0.0978	0.2008	0.0002	0.0002
50	β	1.9843	0.0564	0.0566	1.9962	0.0217	0.0217
	λ	10.8037	11.0383	11.6732	10.2205	9.1627	9.2021
	p	0.2595	0.0936	0.0970	0.2014	0.0002	0.0002
100	β	1.9573	0.0378	0.0396	1.9952	0.0098	0.0099
	λ	10.2073	4.1801	4.2189	10.0407	3.9839	3.9816
	p	0.2753	0.0875	0.0931	0.2009	0.0002	0.0002
150	β	1.9474	0.0358	0.0385	1.9939	0.0067	0.0067
	λ	10.0889	3.0244	3.0293	10.0857	2.7562	2.7608
	p	0.2747	0.0882	0.0937	0.2016	0.0002	0.0002
200	β	1.9503	0.0274	0.0299	1.9953	0.0053	0.0053
	λ	9.9320	2.1435	2.1460	9.9665	1.9219	1.9211
	p	0.2717	0.0813	0.0864	0.2017	0.0002	0.0002
300	β	1.9673	0.0210	0.0221	1.9978	0.0034	0.0034
	λ	9.9180	1.4347	1.4400	10.0284	1.2592	1.2588
	p	0.2395	0.0684	0.0699	0.2012	0.0002	0.0003
500	β	1.9755	0.0144	0.0150	1.9978	0.0022	0.0022
	λ	9.8484	0.9752	0.9972	9.9683	0.7679	0.7681
	p	0.2260	0.0528	0.0535	0.2012	0.0003	0.0003

Appendix Table B8 Simulation study of the MEIW distribution (True parameters: $\beta = 2$, $\lambda = 10$, $p = 0.5$)

n	**Parameter**	**MLE**			**BE**		
		Estimate	**Var**	**MSE**	**Estimate**	**Var**	**MSE**
15	β	2.1971	0.1366	0.1753	1.9756	0.0792	0.0797
	λ	14.4876	176.1332	196.0960	10.6366	42.8267	43.1891
	p	0.3386	0.1335	0.1594	0.4965	0.0001	0.0001
30	β	2.0908	0.0895	0.0976	1.9876	0.0444	0.0445
	λ	11.0176	20.6617	21.6765	10.2644	13.6407	13.6970
	p	0.3676	0.1245	0.1418	0.4970	0.0001	0.0001
50	β	2.0727	0.0663	0.0715	1.9933	0.0257	0.0258
	λ	10.4006	9.1543	9.3056	10.2055	7.8085	7.8430
	p	0.3593	0.1167	0.1364	0.4971	0.0001	0.0001
100	β	2.0187	0.0500	0.0503	1.9943	0.0137	0.0137
	λ	9.8226	3.8681	3.8957	10.0373	3.6579	3.6557
	p	0.4178	0.1124	0.1191	0.4976	0.0001	0.0002
150	β	2.0411	0.0437	0.0453	1.9985	0.0088	0.0088
	λ	9.7839	2.6139	2.6580	10.0368	2.3068	2.3058
	p	0.3800	0.0994	0.1137	0.4972	0.0002	0.0002
200	β	2.0400	0.0382	0.0397	2.0003	0.0074	0.0074
	λ	9.7692	2.0442	2.0954	10.0521	1.7514	1.7523
	p	0.3842	0.0937	0.1070	0.4974	0.0002	0.0002
300	β	2.0194	0.0325	0.0328	1.9985	0.0050	0.0050
	λ	9.7372	1.2596	1.3274	10.0223	1.1154	1.1148
	p	0.4188	0.0849	0.0914	0.4975	0.0002	0.0002
500	β	2.0286	0.0265	0.0273	2.0035	0.0026	0.0026
	λ	9.7306	0.7365	0.8083	10.0377	0.5831	0.5840
	p	0.4101	0.0777	0.0857	0.4967	0.0003	0.0003

Appendix Table B9 Simulation study of the MEIW distribution (True parameters: $\beta = 2$, $\lambda = 10$, $p = 0.8$)

n	Parameter	MLE			BE		
		Estimate	Var	MSE	Estimate	Var	MSE
15	β	2.2905	0.2445	0.3286	1.9303	0.1183	0.1230
	λ	15.6041	357.5701	388.6186	10.4606	37.1047	37.2798
	p	0.5843	0.1637	0.2100	0.8069	0.0000	0.0001
30	β	2.1809	0.1150	0.1476	1.9614	0.0588	0.0602
	λ	11.7767	33.6287	36.7519	10.2818	15.0797	15.1440
	p	0.5904	0.1588	0.2026	0.8071	0.0000	0.0001
50	β	2.1657	0.0860	0.1134	1.9713	0.0364	0.0372
	λ	10.8169	10.1509	10.8081	10.0499	6.7137	6.7095
	p	0.5639	0.1500	0.2056	0.8068	0.0000	0.0001
100	β	2.1370	0.0672	0.0859	1.9856	0.0175	0.0177
	λ	10.4038	4.3693	4.5280	10.0647	3.2326	3.2336
	p	0.5834	0.1383	0.1851	0.8067	0.0001	0.0001
150	β	2.1424	0.0653	0.0856	1.9853	0.0129	0.0131
	λ	10.2549	3.3495	3.4111	9.9989	2.2727	2.2704
	p	0.5626	0.1362	0.1924	0.8071	0.0001	0.0001
200	β	2.1215	0.0599	0.0746	1.9863	0.0089	0.0091
	λ	10.1936	2.0840	2.1194	9.9954	1.4905	1.4890
	p	0.5920	0.1254	0.1686	0.8068	0.0001	0.0001
300	β	2.1215	0.0526	0.0673	1.9884	0.0066	0.0067
	λ	10.1255	1.4094	1.4238	9.9745	1.0540	1.0536
	p	0.5911	0.1192	0.1628	0.8064	0.0001	0.0002
500	β	2.0920	0.0427	0.0511	1.9926	0.0037	0.0038
	λ	10.1688	0.9137	0.9413	10.0399	0.5799	0.5809
	p	0.6448	0.0927	0.1167	0.8057	0.0002	0.0003

Appendix Table B10 Simulation study of the MEIW distribution (True parameters: $\beta = 5$, $\lambda = 1$, $p = 0.2$)

n	Parameter	MLE			BE		
		Estimate	Var	MSE	Estimate	Var	MSE
15	β	5.3766	1.4256	1.5661	5.2839	1.3729	1.4522
	λ	1.1585	0.1755	0.2005	0.9977	0.1251	0.1250
	p	0.3759	0.2191	0.2498	0.2019	0.0002	0.0002
30	β	5.1187	0.6142	0.6277	5.1541	0.5448	0.5680
	λ	1.0955	0.0617	0.0708	0.9888	0.0460	0.0461
	p	0.3924	0.2193	0.2561	0.2022	0.0002	0.0002
50	β	4.9726	0.3280	0.3284	5.0574	0.2581	0.2611
	λ	1.0804	0.0424	0.0489	0.9912	0.0263	0.0264
	p	0.3988	0.2103	0.2496	0.2017	0.0002	0.0002
100	β	4.8805	0.1910	0.2051	5.0143	0.1191	0.1192
	λ	1.0784	0.0287	0.0348	0.9951	0.0125	0.0125
	p	0.4316	0.2016	0.2550	0.2021	0.0002	0.0002
150	β	4.8893	0.1644	0.1765	5.0204	0.0858	0.0861
	λ	1.0678	0.0229	0.0274	0.9952	0.0086	0.0086
	p	0.4112	0.1954	0.2398	0.2021	0.0002	0.0002
200	β	4.8696	0.1365	0.1534	5.0117	0.0634	0.0635
	λ	1.0732	0.0197	0.0250	1.0005	0.0060	0.0060
	p	0.4216	0.1915	0.2404	0.2018	0.0002	0.0002
300	β	4.8658	0.1090	0.1269	5.0056	0.0403	0.0403
	λ	1.0616	0.0183	0.0221	0.9947	0.0039	0.0040
	p	0.4097	0.1858	0.2296	0.2017	0.0002	0.0002
500	β	4.8823	0.0937	0.1075	5.0093	0.0262	0.0262
	λ	1.0582	0.0152	0.0186	1.0002	0.0025	0.0025
	p	0.3870	0.1688	0.2036	0.2021	0.0002	0.0002

Appendix Table B11 Simulation study of the MEIW distribution (True parameters: $\beta = 5$, $\lambda = 1$, $p = 0.5$)

n	Parameter	MLE			BE		
		Estimate	Var	MSE	Estimate	Var	MSE
15	β	5.5033	1.5710	1.8227	5.3179	1.4268	1.5265
	λ	1.0682	0.1389	0.1434	0.9703	0.0993	0.1000
	p	0.4724	0.2334	0.2339	0.4954	0.0001	0.0001
30	β	5.2549	0.6306	0.6950	5.1947	0.5510	0.5884
	λ	1.0450	0.0649	0.0669	0.9903	0.0478	0.0478
	p	0.4997	0.2262	0.2260	0.4965	0.0001	0.0001
50	β	5.1048	0.3643	0.3749	5.0744	0.2760	0.2813
	λ	1.0330	0.0398	0.0408	0.9995	0.0245	0.0245
	p	0.4846	0.2147	0.2148	0.4961	0.0001	0.0001
100	β	5.0163	0.2401	0.2401	5.0307	0.1455	0.1463
	λ	1.0227	0.0247	0.0252	0.9968	0.0121	0.0121
	p	0.5091	0.2089	0.2088	0.4964	0.0001	0.0001
150	β	5.0156	0.1722	0.1722	5.0243	0.0896	0.0901
	λ	1.0103	0.0207	0.0208	0.9943	0.0074	0.0075
	p	0.4880	0.1973	0.1973	0.4959	0.0001	0.0001
200	β	5.0065	0.1521	0.1520	5.0235	0.0674	0.0679
	λ	1.0139	0.0200	0.0202	0.9990	0.0062	0.0062
	p	0.4929	0.2000	0.1998	0.4967	0.0001	0.0001
300	β	4.9843	0.1212	0.1213	5.0095	0.0435	0.0435
	λ	1.0114	0.0170	0.0171	0.9976	0.0044	0.0044
	p	0.4994	0.1920	0.1918	0.4961	0.0001	0.0001
500	β	4.9801	0.1062	0.1065	5.0137	0.0283	0.0284
	λ	1.0116	0.0150	0.0151	0.9975	0.0024	0.0024
	p	0.5075	0.1842	0.1841	0.4956	0.0001	0.0001

Appendix Table B12 Simulation study of the MEIW distribution (True parameters: $\beta = 5$, $\lambda = 1$, $p = 0.8$)

n	Parameter	MLE			BE		
		Estimate	Var	MSE	Estimate	Var	MSE
15	β	5.6355	1.4965	1.8988	5.2818	1.3345	1.4126
	λ	1.0017	0.1107	0.1106	0.9767	0.0856	0.0861
	p	0.5297	0.2370	0.3099	0.8039	0.0000	0.0000
30	β	5.3228	0.6626	0.7661	5.0936	0.5528	0.5610
	λ	0.9828	0.0525	0.0527	0.9937	0.0403	0.0403
	p	0.5650	0.2248	0.2797	0.8040	0.0000	0.0000
50	β	5.2606	0.4303	0.4977	5.0816	0.3315	0.3379
	λ	0.9717	0.0338	0.0346	0.9942	0.0223	0.0223
	p	0.5823	0.2157	0.2629	0.8037	0.0000	0.0000
100	β	5.2018	0.2525	0.2929	5.0490	0.1602	0.1625
	λ	0.9607	0.0225	0.0240	0.9958	0.0118	0.0118
	p	0.5789	0.2104	0.2591	0.8037	0.0000	0.0000
150	β	5.1806	0.1943	0.2268	5.0340	0.0967	0.0977
	λ	0.9586	0.0184	0.0201	0.9985	0.0079	0.0079
	p	0.5737	0.2046	0.2556	0.8035	0.0000	0.0000
200	β	5.1445	0.1606	0.1813	5.0148	0.0749	0.0750
	λ	0.9643	0.0170	0.0183	1.0018	0.0056	0.0056
	p	0.5954	0.1962	0.2379	0.8038	0.0000	0.0000
300	β	5.1253	0.1306	0.1461	5.0058	0.0477	0.0477
	λ	0.9622	0.0154	0.0168	1.0000	0.0037	0.0037
	p	0.6027	0.1957	0.2344	0.8038	0.0000	0.0000
500	β	5.1136	0.1009	0.1137	5.0022	0.0270	0.0269
	λ	0.9605	0.0119	0.0134	0.9985	0.0022	0.0022
	p	0.6126	0.1764	0.2113	0.8037	0.0000	0.0000

Appendix Table B13 Simulation study of the MEIW distribution (True parameters: $\beta = 5$, $\lambda = 5$, $p = 0.2$)

n	Parameter	MLE			BE		
		Estimate	Var	MSE	Estimate	Var	MSE
15	β	5.3616	1.3147	1.4441	4.8961	0.8800	0.8899
	λ	7.0533	22.4492	26.6427	5.3338	10.2069	10.3081
	p	0.3535	0.2145	0.2379	0.2009	0.0002	0.0002
30	β	5.0593	0.6248	0.6277	4.9371	0.4499	0.4534
	λ	5.7218	4.0199	4.5369	5.1044	3.0675	3.0753
	p	0.3785	0.2090	0.2407	0.2029	0.0001	0.0001
50	β	4.9851	0.3570	0.3569	4.9865	0.2520	0.2519
	λ	5.4838	1.7100	1.9423	5.1130	1.4443	1.4556
	p	0.4112	0.2065	0.2509	0.2032	0.0002	0.0002
100	β	4.9163	0.2291	0.2358	4.9886	0.1331	0.1331
	λ	5.1955	0.7092	0.7467	4.9886	0.6551	0.6545
	p	0.4120	0.1993	0.2440	0.2019	0.0002	0.0002
150	β	4.8905	0.1732	0.1850	4.9836	0.0880	0.0882
	λ	5.1518	0.4755	0.4980	5.0164	0.4656	0.4654
	p	0.3977	0.1984	0.2372	0.2024	0.0002	0.0002
200	β	4.8958	0.1430	0.1537	4.9976	0.0619	0.0619
	λ	5.1353	0.3062	0.3242	5.0255	0.2968	0.2971
	p	0.3953	0.1880	0.2259	0.2021	0.0002	0.0002
300	β	4.8992	0.1140	0.1240	4.9889	0.0432	0.0433
	λ	5.0685	0.2055	0.2100	4.9936	0.1936	0.1934
	p	0.3581	0.1737	0.1986	0.2019	0.0002	0.0002
500	β	4.8974	0.0863	0.0967	4.9971	0.0245	0.0245
	λ	5.0652	0.1258	0.1299	5.0124	0.1169	0.1169
	p	0.3601	0.1654	0.1909	0.2028	0.0002	0.0002

Appendix Table B14 Simulation study of the MEIW distribution (True parameters: $\beta = 5$, $\lambda = 5$, $p = 0.5$)

n	Parameter	MLE			BE		
		Estimate	Var	MSE	Estimate	Var	MSE
15	β	5.4663	1.3642	1.5803	4.8964	0.8591	0.8690
	λ	6.6846	19.8136	22.6315	5.2021	7.0909	7.1247
	p	0.4564	0.2363	0.2379	0.4967	0.0001	0.0001
30	β	5.2163	0.5573	0.6035	4.9701	0.4049	0.4054
	λ	5.6215	3.0877	3.4708	5.1079	2.5035	2.5126
	p	0.4683	0.2260	0.2268	0.4969	0.0001	0.0001
50	β	5.1087	0.4130	0.4244	4.9688	0.2819	0.2826
	λ	5.3287	1.5372	1.6437	5.0336	1.2598	1.2597
	p	0.4742	0.2141	0.2145	0.4964	0.0001	0.0001
100	β	5.0288	0.2322	0.2328	4.9844	0.1369	0.1370
	λ	5.1519	0.6435	0.6660	5.0391	0.5843	0.5852
	p	0.4923	0.2114	0.2113	0.4971	0.0001	0.0001
150	β	5.0298	0.1855	0.1862	4.9986	0.0878	0.0877
	λ	5.0856	0.3925	0.3995	5.0267	0.3679	0.3683
	p	0.4769	0.2020	0.2023	0.4970	0.0001	0.0001
200	β	5.0062	0.1547	0.1546	4.9978	0.0708	0.0707
	λ	5.0412	0.2991	0.3005	5.0053	0.2823	0.2821
	p	0.4920	0.2007	0.2006	0.4968	0.0001	0.0001
300	β	5.0165	0.1248	0.1250	5.0059	0.0456	0.0456
	λ	5.0358	0.1982	0.1993	5.0284	0.1864	0.1871
	p	0.4713	0.1912	0.1918	0.4970	0.0001	0.0001
500	β	5.0362	0.1002	0.1014	5.0089	0.0249	0.0250
	λ	4.9914	0.1157	0.1156	5.0138	0.1053	0.1053
	p	0.4297	0.1782	0.1829	0.4959	0.0001	0.0001

Appendix Table B15 Simulation study of the MEIW distribution (True parameters: $\beta = 5$, $\lambda = 5$, $p = 0.8$)

n	**Parameter**	**MLE**			**BE**		
		Estimate	**Var**	**MSE**	**Estimate**	**Var**	**MSE**
15	β	5.6451	1.6367	2.0513	4.8355	0.9945	1.0206
	λ	6.8557	23.5425	26.9627	5.2153	7.0733	7.1126
	p	0.5110	0.2355	0.3188	0.8044	0.0000	0.0000
30	β	5.3999	0.7059	0.8651	4.9635	0.5143	0.5151
	λ	5.6513	3.7229	4.1434	5.1317	2.4009	2.4159
	p	0.5342	0.2313	0.3017	0.8044	0.0000	0.0000
50	β	5.2739	0.4223	0.4969	4.9788	0.2884	0.2886
	λ	5.2936	1.4678	1.5526	5.0375	1.1627	1.1629
	p	0.5748	0.2175	0.2680	0.8044	0.0000	0.0000
100	β	5.1938	0.2715	0.3088	4.9828	0.1532	0.1533
	λ	5.1249	0.6073	0.6223	5.0382	0.5322	0.5331
	p	0.5718	0.2122	0.2641	0.8045	0.0000	0.0000
150	β	5.1733	0.1769	0.2068	4.9846	0.0975	0.0976
	λ	5.0451	0.3803	0.3820	5.0164	0.3353	0.3352
	p	0.5672	0.2049	0.2589	0.8045	0.0000	0.0000
200	β	5.1561	0.1608	0.1850	4.9787	0.0713	0.0717
	λ	5.0033	0.2575	0.2572	5.0001	0.2300	0.2298
	p	0.5643	0.2030	0.2583	0.8047	0.0000	0.0000
300	β	5.1646	0.1440	0.1710	5.0057	0.0500	0.0500
	λ	5.0207	0.1948	0.1950	5.0394	0.1789	0.1803
	p	0.5726	0.1981	0.2497	0.8047	0.0000	0.0000
500	β	5.1119	0.0993	0.1117	4.9926	0.0286	0.0287
	λ	4.9742	0.1149	0.1155	5.0051	0.1002	0.1001
	p	0.6192	0.1764	0.2090	0.8046	0.0000	0.0001

Appendix Table B16 Simulation study of the MEIW distribution (True parameters: $\beta = 5$, $\lambda = 10$, $p = 0.2$)

n	Parameter	MLE			BE		
		Estimate	Var	MSE	Estimate	Var	MSE
15	β	5.3088	1.2794	1.3742	4.7257	0.6902	0.7647
	λ	16.1079	441.2923	478.4110	9.9092	43.6328	43.5973
	p	0.3957	0.2227	0.2609	0.2027	0.0002	0.0002
30	β	5.0417	0.5233	0.5246	4.8501	0.3793	0.4014
	λ	11.8939	28.1346	31.6934	10.0802	16.2925	16.2827
	p	0.4144	0.2143	0.2600	0.2030	0.0001	0.0002
50	β	5.0009	0.3657	0.3654	4.9564	0.2595	0.2611
	λ	10.9872	12.0558	13.0182	10.1552	10.5185	10.5321
	p	0.4126	0.2070	0.2520	0.2025	0.0002	0.0002
100	β	4.9209	0.2004	0.2065	4.9731	0.1230	0.1236
	λ	10.4397	4.0376	4.2269	10.1275	3.9648	3.9771
	p	0.4096	0.2027	0.2465	0.2027	0.0002	0.0002
150	β	4.9248	0.1540	0.1595	4.9770	0.0864	0.0868
	λ	10.2591	2.6972	2.7617	10.0767	2.6578	2.6610
	p	0.3600	0.1849	0.2103	0.2033	0.0002	0.0002
200	β	4.9116	0.1278	0.1355	4.9880	0.0608	0.0609
	λ	10.1229	1.8546	1.8678	10.0284	1.8723	1.8713
	p	0.3723	0.1821	0.2116	0.2028	0.0002	0.0002
300	β	4.9058	0.0970	0.1058	4.9811	0.0396	0.0399
	λ	10.0140	1.1987	1.1977	9.9859	1.2535	1.2525
	p	0.3476	0.1687	0.1903	0.2033	0.0002	0.0002
500	β	4.9135	0.0778	0.0852	4.9909	0.0245	0.0245
	λ	9.9892	0.6797	0.6791	10.0037	0.6946	0.6940
	p	0.3356	0.1555	0.1737	0.2034	0.0002	0.0002

Appendix Table B17 Simulation study of the MEIW distribution (True parameters: $\beta = 5$, $\lambda = 10$, $p = 0.5$)

n	Parameter	MLE			BE		
		Estimate	**Var**	**MSE**	**Estimate**	**Var**	**MSE**
15	β	5.5462	1.4441	1.7411	4.7394	0.8115	0.8786
	λ	17.1920	875.9038	926.8438	10.2288	44.0168	44.0251
	p	0.4345	0.2327	0.2368	0.4970	0.0001	0.0001
30	β	5.2276	0.6313	0.6825	4.8982	0.4697	0.4796
	λ	12.3334	46.6931	52.0910	10.5154	21.4089	21.6531
	p	0.4646	0.2267	0.2277	0.4964	0.0001	0.0001
50	β	5.1255	0.3661	0.3815	4.9266	0.2644	0.2696
	λ	11.0529	12.1527	13.2492	10.1367	8.8082	8.8180
	p	0.4560	0.2147	0.2164	0.4971	0.0001	0.0001
100	β	5.0592	0.2018	0.2051	4.9628	0.1252	0.1265
	λ	10.3826	4.2578	4.4000	10.0263	3.7138	3.7108
	p	0.4533	0.2068	0.2088	0.4972	0.0001	0.0001
150	β	5.0641	0.1587	0.1627	4.9795	0.0850	0.0853
	λ	10.2271	2.5035	2.5525	10.0275	2.3311	2.3295
	p	0.4161	0.2029	0.2097	0.4965	0.0001	0.0001
200	β	5.0333	0.1287	0.1297	4.9671	0.0666	0.0676
	λ	10.0866	1.8977	1.9033	9.9580	1.8004	1.8004
	p	0.4233	0.1955	0.2012	0.4970	0.0001	0.0001
300	β	5.0344	0.1069	0.1080	4.9869	0.0456	0.0458
	λ	10.0318	1.1670	1.1669	9.9808	1.1282	1.1274
	p	0.4262	0.1869	0.1921	0.4968	0.0001	0.0001
500	β	5.0235	0.0815	0.0820	4.9892	0.0254	0.0255
	λ	9.9757	0.6715	0.6714	9.9863	0.6738	0.6733
	p	0.4269	0.1826	0.1878	0.4969	0.0001	0.0001

Appendix Table B18 Simulation study of the MEIW distribution (True parameters: $\beta = 5$, $\lambda = 10$, $p = 0.8$)

n	Parameter	MLE Estimate	MLE Var	MLE MSE	BE Estimate	BE Var	BE MSE
15	β	5.6547	1.4464	1.8741	4.6447	0.8183	0.9437
	λ	16.0570	277.5516	314.0664	9.7304	36.3502	36.3865
	p	0.4861	0.2371	0.3355	0.8047	0.0000	0.0000
30	β	5.3886	0.6176	0.7679	4.8529	0.4430	0.4642
	λ	12.1750	28.0171	32.7197	10.1708	16.5843	16.5969
	p	0.5203	0.2298	0.3078	0.8048	0.0000	0.0000
50	β	5.3127	0.3708	0.4682	4.9219	0.2881	0.2939
	λ	11.3131	11.5913	13.3040	10.2098	8.1370	8.1729
	p	0.5023	0.2234	0.3118	0.8047	0.0000	0.0000
100	β	5.2225	0.2088	0.2581	4.9576	0.1419	0.1436
	λ	10.5401	4.2279	4.5154	10.0298	3.4869	3.4843
	p	0.5279	0.2148	0.2887	0.8048	0.0000	0.0000
150	β	5.2024	0.1794	0.2202	4.9661	0.0931	0.0941
	λ	10.3855	2.6509	2.7969	10.0155	2.1703	2.1684
	p	0.5170	0.2119	0.2918	0.8048	0.0000	0.0000
200	β	5.1978	0.1492	0.1881	4.9860	0.0814	0.0815
	λ	10.3456	2.0533	2.1707	10.0712	1.7971	1.8004
	p	0.5290	0.2035	0.2768	0.8047	0.0000	0.0000
300	β	5.1777	0.1167	0.1481	4.9824	0.0462	0.0465
	λ	10.2189	1.2188	1.2655	10.0104	1.0784	1.0774
	p	0.5308	0.1968	0.2691	0.8047	0.0000	0.0000
500	β	5.1517	0.0992	0.1221	4.9875	0.0265	0.0266
	λ	10.1150	0.6967	0.7092	9.9808	0.6181	0.6178
	p	0.5574	0.1891	0.2477	0.8046	0.0000	0.0001

Appendix Table B19 Simulation study of the MEIW distribution (True parameters: $\beta = 8$, $\lambda = 1$, $p = 0.2$)

n	Parameter	MLE			BE		
		Estimate	Var	MSE	Estimate	Var	MSE
15	β	8.6373	3.7392	4.1416	8.4406	3.5585	3.7490
	λ	1.1255	0.1530	0.1686	0.9838	0.1025	0.1026
	p	0.4574	0.2450	0.3110	0.2020	0.0002	0.0002
30	β	8.2630	1.5721	1.6397	8.2539	1.4498	1.5128
	λ	1.0752	0.0553	0.0609	0.9873	0.0429	0.0430
	p	0.4454	0.2397	0.2997	0.2024	0.0002	0.0002
50	β	7.9918	0.7940	0.7932	8.0791	0.7224	0.7280
	λ	1.0643	0.0339	0.0380	0.9900	0.0255	0.0256
	p	0.4761	0.2382	0.3143	0.2024	0.0001	0.0002
100	β	7.9211	0.4482	0.4540	8.0412	0.3509	0.3522
	λ	1.0564	0.0189	0.0220	0.9986	0.0121	0.0121
	p	0.4525	0.2284	0.2919	0.2023	0.0002	0.0002
150	β	7.9334	0.3258	0.3299	8.0539	0.2417	0.2444
	λ	1.0441	0.0142	0.0161	0.9954	0.0081	0.0081
	p	0.4280	0.2197	0.2714	0.2026	0.0001	0.0002
200	β	7.8890	0.2569	0.2689	8.0228	0.1643	0.1647
	λ	1.0472	0.0126	0.0149	0.9989	0.0057	0.0057
	p	0.4356	0.2207	0.2760	0.2025	0.0002	0.0002
300	β	7.8467	0.2095	0.2328	8.0191	0.1231	0.1234
	λ	1.0543	0.0096	0.0125	1.0004	0.0036	0.0036
	p	0.4820	0.2191	0.2984	0.2015	0.0002	0.0002
500	β	7.8482	0.1432	0.1661	7.9937	0.0668	0.0667
	λ	1.0440	0.0083	0.0103	1.0007	0.0022	0.0022
	p	0.4313	0.2077	0.2610	0.2024	0.0002	0.0002

Appendix Table B20 Simulation study of the MEIW distribution (True parameters: $\beta = 8$, $\lambda = 1$, $p = 0.5$)

n	Parameter	MLE			BE		
		Estimate	Var	MSE	Estimate	Var	MSE
15	β	8.6846	3.3096	3.7749	8.3262	3.1705	3.2737
	λ	1.0758	0.1257	0.1313	0.9898	0.0951	0.0951
	p	0.4599	0.2440	0.2454	0.4962	0.0001	0.0001
30	β	8.3501	1.5217	1.6428	8.1998	1.4275	1.4660
	λ	1.0292	0.0512	0.0520	0.9867	0.0429	0.0430
	p	0.4827	0.2410	0.2410	0.4964	0.0001	0.0001
50	β	8.1686	0.8759	0.9035	8.0935	0.7855	0.7934
	λ	1.0281	0.0314	0.0322	0.9988	0.0236	0.0236
	p	0.4995	0.2378	0.2376	0.4963	0.0001	0.0001
100	β	8.1333	0.4907	0.5079	8.0817	0.3781	0.3844
	λ	1.0007	0.0188	0.0188	0.9897	0.0127	0.0128
	p	0.4583	0.2316	0.2331	0.4966	0.0001	0.0001
150	β	8.0579	0.3285	0.3315	8.0370	0.2386	0.2397
	λ	1.0055	0.0141	0.0141	0.9956	0.0078	0.0078
	p	0.4801	0.2274	0.2275	0.4971	0.0001	0.0001
200	β	8.0169	0.2512	0.2512	8.0301	0.1731	0.1738
	λ	1.0109	0.0111	0.0112	0.9969	0.0059	0.0059
	p	0.5208	0.2223	0.2225	0.4966	0.0001	0.0001
300	β	8.0121	0.2141	0.2140	8.0236	0.1348	0.1353
	λ	1.0072	0.0092	0.0093	0.9975	0.0041	0.0041
	p	0.5056	0.2177	0.2175	0.4961	0.0001	0.0001
500	β	8.0140	0.1490	0.1491	8.0090	0.0725	0.0725
	λ	0.9991	0.0074	0.0073	0.9969	0.0022	0.0022
	p	0.4699	0.2086	0.2093	0.4958	0.0001	0.0001

Appendix Table B21 Simulation study of the MEIW distribution (True parameters: $\beta = 8$, $\lambda = 1$, $p = 0.8$)

n	**Parameter**	**MLE**			**BE**		
		Estimate	**Var**	**MSE**	**Estimate**	**Var**	**MSE**
15	β	8.8705	3.5271	4.2813	8.3559	3.4205	3.5438
	λ	1.0216	0.1138	0.1141	0.9830	0.0889	0.0891
	p	0.5115	0.2483	0.3313	0.8039	0.0000	0.0000
30	β	8.5121	1.5535	1.8142	8.1832	1.4928	1.5249
	λ	0.9852	0.0461	0.0463	0.9881	0.0385	0.0386
	p	0.5127	0.2414	0.3237	0.8037	0.0000	0.0000
50	β	8.3524	0.8948	1.0181	8.1216	0.8112	0.8251
	λ	0.9773	0.0305	0.0310	0.9879	0.0246	0.0247
	p	0.5650	0.2324	0.2874	0.8038	0.0000	0.0000
100	β	8.2696	0.4694	0.5416	8.0670	0.3640	0.3681
	λ	0.9774	0.0165	0.0170	1.0031	0.0114	0.0114
	p	0.5370	0.2322	0.3011	0.8038	0.0000	0.0000
150	β	8.2203	0.3247	0.3730	8.0455	0.2329	0.2347
	λ	0.9716	0.0124	0.0132	0.9985	0.0073	0.0073
	p	0.5552	0.2256	0.2852	0.8036	0.0000	0.0000
200	β	8.1724	0.2816	0.3110	8.0112	0.1919	0.1918
	λ	0.9731	0.0102	0.0109	1.0005	0.0052	0.0051
	p	0.5664	0.2121	0.2665	0.8038	0.0000	0.0000
300	β	8.1655	0.2193	0.2465	8.0103	0.1247	0.1247
	λ	0.9717	0.0084	0.0092	1.0014	0.0035	0.0035
	p	0.5635	0.2163	0.2720	0.8038	0.0000	0.0000
500	β	8.1711	0.1624	0.1915	8.0165	0.0773	0.0775
	λ	0.9665	0.0067	0.0078	0.9992	0.0022	0.0022
	p	0.5556	0.2104	0.2700	0.8038	0.0000	0.0000

Appendix Table B22 Simulation study of the MEIW distribution (True parameters: $\beta = 8$, $\lambda = 5$, $p = 0.2$)

n	Parameter	MLE			BE		
		Estimate	Var	MSE	Estimate	Var	MSE
15	β	8.5747	3.3423	3.6693	7.7386	2.2051	2.2712
	λ	6.6452	19.7423	22.4291	5.0756	7.6853	7.6834
	p	0.3927	0.2355	0.2724	0.2025	0.0001	0.0002
30	β	8.2728	1.5464	1.6193	7.9782	1.2885	1.2877
	λ	5.7698	3.8396	4.4284	5.1511	3.0047	3.0245
	p	0.4161	0.2348	0.2812	0.2024	0.0001	0.0002
50	β	8.0781	0.8807	0.8859	7.9718	0.7320	0.7320
	λ	5.4233	1.5491	1.7268	5.0587	1.3176	1.3198
	p	0.4332	0.2311	0.2852	0.2027	0.0002	0.0002
100	β	7.9597	0.4581	0.4593	7.9689	0.3416	0.3422
	λ	5.2203	0.6335	0.6814	5.0346	0.5882	0.5888
	p	0.4013	0.2208	0.2611	0.2024	0.0001	0.0002
150	β	7.8960	0.3323	0.3428	7.9648	0.2262	0.2272
	λ	5.1379	0.3447	0.3634	5.0024	0.3303	0.3300
	p	0.4283	0.2235	0.2754	0.2023	0.0001	0.0002
200	β	7.9100	0.2710	0.2789	7.9989	0.1754	0.1752
	λ	5.1320	0.2873	0.3044	5.0245	0.2744	0.2747
	p	0.4279	0.2185	0.2702	0.2029	0.0001	0.0002
300	β	7.8707	0.1912	0.2078	7.9810	0.1068	0.1070
	λ	5.0709	0.1851	0.1900	4.9875	0.1778	0.1778
	p	0.4304	0.2138	0.2667	0.2030	0.0002	0.0002
500	β	7.8628	0.1395	0.1582	7.9830	0.0667	0.0669
	λ	5.0499	0.1089	0.1113	4.9897	0.1034	0.1034
	p	0.4198	0.2003	0.2484	0.2029	0.0002	0.0002

Appendix Table B23 Simulation study of the MEIW distribution (True parameters: $\beta = 8$, $\lambda = 5$, $p = 0.5$)

n	Parameter	MLE			BE		
		Estimate	Var	MSE	Estimate	Var	MSE
15	β	8.8837	4.0203	4.8000	7.7584	2.5713	2.6271
	λ	6.9013	27.7023	31.3087	5.2257	10.2640	10.3047
	p	0.4470	0.2435	0.2462	0.4961	0.0001	0.0001
30	β	8.4426	1.3962	1.5906	7.9773	1.1454	1.1447
	λ	5.6978	3.1444	3.6281	5.1237	2.2585	2.2715
	p	0.4540	0.2411	0.2430	0.4966	0.0001	0.0001
50	β	8.2164	0.9035	0.9494	7.9405	0.7491	0.7519
	λ	5.3954	1.5599	1.7146	5.0754	1.2797	1.2841
	p	0.4540	0.2357	0.2376	0.4968	0.0001	0.0001
100	β	8.1412	0.4914	0.5108	8.0151	0.3767	0.3766
	λ	5.1789	0.6417	0.6731	5.0395	0.5806	0.5816
	p	0.4688	0.2278	0.2286	0.4965	0.0001	0.0001
150	β	8.0779	0.3204	0.3261	7.9880	0.2318	0.2318
	λ	5.0831	0.3790	0.3855	5.0018	0.3497	0.3493
	p	0.4575	0.2241	0.2257	0.4965	0.0001	0.0001
200	β	8.0360	0.2671	0.2681	7.9804	0.1856	0.1858
	λ	5.0446	0.2736	0.2753	4.9849	0.2583	0.2583
	p	0.4797	0.2220	0.2222	0.4966	0.0001	0.0001
300	β	8.0094	0.2184	0.2183	7.9809	0.1208	0.1211
	λ	5.0278	0.1771	0.1777	4.9905	0.1707	0.1707
	p	0.4886	0.2194	0.2193	0.4970	0.0001	0.0001
500	β	8.0264	0.1541	0.1546	7.9967	0.0711	0.0710
	λ	5.0095	0.1150	0.1149	4.9985	0.1062	0.1061
	p	0.4588	0.2132	0.2147	0.4966	0.0001	0.0001

Appendix Table B24 Simulation study of the MEIW distribution (True parameters: $\beta = 8$, $\lambda = 5$, $p = 0.8$)

n	**Parameter**	**MLE**			**BE**		
		Estimate	**Var**	**MSE**	**Estimate**	**Var**	**MSE**
15	β	9.0827	4.1835	5.3517	7.8335	2.6730	2.6980
	λ	7.0637	125.4833	129.6165	5.2302	6.4492	6.4957
	p	0.4748	0.2450	0.3505	0.8042	0.0000	0.0000
30	β	8.5244	1.5782	1.8517	7.8753	1.3353	1.3495
	λ	5.6171	3.1348	3.5124	5.0911	2.2234	2.2295
	p	0.4969	0.2419	0.3336	0.8042	0.0000	0.0000
50	β	8.4358	0.9664	1.1553	7.9919	0.7981	0.7974
	λ	5.4145	1.6500	1.8201	5.1321	1.3178	1.3339
	p	0.5169	0.2371	0.3170	0.8039	0.0000	0.0000
100	β	8.2979	0.4776	0.5659	8.0034	0.3676	0.3672
	λ	5.1453	0.5523	0.5729	5.0377	0.4914	0.4923
	p	0.5318	0.2321	0.3038	0.8041	0.0000	0.0000
150	β	8.2486	0.3657	0.4271	7.9852	0.2511	0.2510
	λ	5.0474	0.3379	0.3398	4.9947	0.3066	0.3063
	p	0.5116	0.2290	0.3119	0.8040	0.0000	0.0000
200	β	8.2043	0.2890	0.3305	7.9784	0.1893	0.1896
	λ	5.0120	0.2624	0.2623	4.9845	0.2397	0.2397
	p	0.5346	0.2234	0.2936	0.8041	0.0000	0.0000
300	β	8.1709	0.2236	0.2526	7.9886	0.1287	0.1287
	λ	5.0060	0.1653	0.1652	4.9973	0.1510	0.1508
	p	0.5691	0.2205	0.2736	0.8042	0.0000	0.0000
500	β	8.1627	0.1622	0.1885	7.9875	0.0724	0.0725
	λ	4.9829	0.1011	0.1013	4.9954	0.0932	0.0931
	p	0.5540	0.2138	0.2741	0.8041	0.0000	0.0000

Appendix Table B25 Simulation study of the MEIW distribution (True parameters: $\beta = 8$, $\lambda = 10$, $p = 0.2$)

n	Parameter	MLE			BE		
		Estimate	Var	MSE	Estimate	Var	MSE
15	β	8.5964	3.5137	3.8670	7.4841	1.9635	2.2277
	λ	16.2764	454.6843	493.7697	9.9326	35.6629	35.6317
	p	0.4558	0.2431	0.3084	0.2025	0.0001	0.0002
30	β	8.1692	1.4484	1.4756	7.7593	1.1522	1.2089
	λ	12.0310	31.3758	35.4693	10.2027	18.7775	18.7998
	p	0.4519	0.2388	0.3020	0.2018	0.0002	0.0002
50	β	8.0515	0.8437	0.8455	7.8759	0.7100	0.7247
	λ	10.9948	11.2557	12.2341	10.0816	8.9829	8.9806
	p	0.4337	0.2316	0.2859	0.2029	0.0002	0.0002
100	β	7.9739	0.4397	0.4400	7.9551	0.3496	0.3512
	λ	10.5667	4.4115	4.7282	10.1629	3.9598	3.9823
	p	0.4228	0.2271	0.2765	0.2032	0.0002	0.0002
150	β	7.8908	0.2908	0.3024	7.9273	0.2049	0.2100
	λ	10.1564	2.3925	2.4146	9.9099	2.2515	2.2574
	p	0.4220	0.2218	0.2709	0.2017	0.0002	0.0002
200	β	7.8966	0.2424	0.2529	7.9730	0.1620	0.1626
	λ	10.1684	1.7733	1.7999	10.0175	1.7586	1.7571
	p	0.4395	0.2205	0.2777	0.2030	0.0002	0.0002
300	β	7.8869	0.1844	0.1970	7.9779	0.1158	0.1162
	λ	10.1298	1.2380	1.2536	10.0503	1.2567	1.2580
	p	0.4191	0.2146	0.2624	0.2025	0.0002	0.0002
500	β	7.8917	0.1345	0.1461	8.0051	0.0638	0.0638
	λ	10.0627	0.6706	0.6739	10.0468	0.6883	0.6898
	p	0.4193	0.2062	0.2541	0.2028	0.0002	0.0002

Appendix Table B26 Simulation study of the MEIW distribution (True parameters: $\beta = 8$, $\lambda = 10$, $p = 0.5$)

n	Parameter	MLE			BE		
		Estimate	Var	MSE	Estimate	Var	MSE
15	β	8.8157	3.7251	4.3867	7.4459	2.1038	2.4087
	λ	16.1036	325.3951	362.3239	9.8945	42.7346	42.7029
	p	0.4318	0.2428	0.2472	0.4970	0.0001	0.0001
30	β	8.3919	1.4948	1.6469	7.7825	1.2121	1.2582
	λ	12.1398	25.0363	29.5902	10.2440	18.1219	18.1634
	p	0.4509	0.2384	0.2406	0.4966	0.0001	0.0001
50	β	8.1992	0.8634	0.9022	7.8626	0.6850	0.7032
	λ	11.1436	10.5890	11.8862	10.1288	8.1054	8.1139
	p	0.4972	0.2376	0.2373	0.4972	0.0001	0.0001
100	β	8.0949	0.4368	0.4454	7.9238	0.3406	0.3461
	λ	10.4571	4.0314	4.2363	10.0399	3.6460	3.6439
	p	0.4548	0.2268	0.2286	0.4972	0.0001	0.0001
150	β	8.0674	0.3311	0.3353	7.9605	0.2489	0.2502
	λ	10.2597	2.6783	2.7430	9.9978	2.4811	2.4786
	p	0.4710	0.2277	0.2283	0.4964	0.0001	0.0001
200	β	8.0391	0.2413	0.2426	7.9666	0.1740	0.1749
	λ	10.1675	1.8378	1.8640	9.9874	1.7385	1.7370
	p	0.4823	0.2220	0.2221	0.4971	0.0001	0.0001
300	β	8.0223	0.2000	0.2003	7.9832	0.1284	0.1286
	λ	10.1343	1.2402	1.2571	10.0305	1.1955	1.1953
	p	0.4914	0.2214	0.2213	0.4967	0.0001	0.0001
500	β	8.0170	0.1458	0.1459	7.9829	0.0660	0.0662
	λ	10.0248	0.6105	0.6105	9.9734	0.5938	0.5939
	p	0.4649	0.2145	0.2155	0.4967	0.0001	0.0001

Appendix Table B27 Simulation study of the MEIW distribution (True parameters: $\beta = 8$, $\lambda = 10$, $p = 0.8$)

n	**Parameter**	**MLE**			**BE**		
		Estimate	**Var**	**MSE**	**Estimate**	**Var**	**MSE**
15	β	9.0322	3.7077	4.7694	7.4414	2.1679	2.4778
	λ	15.4157	146.2006	175.3839	9.5950	27.3405	27.4772
	p	0.4611	0.2454	0.3600	0.8043	0.0000	0.0000
30	β	8.5616	1.5964	1.9102	7.7646	1.2115	1.2657
	λ	12.0490	26.6673	30.8392	10.0321	15.3583	15.3440
	p	0.5080	0.2421	0.3272	0.8044	0.0000	0.0000
50	β	8.3918	0.9472	1.0998	7.8841	0.7781	0.7907
	λ	11.3538	12.3262	14.1468	10.2307	8.2831	8.3281
	p	0.5535	0.2362	0.2968	0.8041	0.0000	0.0000
100	β	8.2703	0.4908	0.5634	7.9220	0.3823	0.3880
	λ	10.5706	4.3330	4.6543	10.0408	3.5703	3.5684
	p	0.5123	0.2330	0.3156	0.8044	0.0000	0.0000
150	β	8.2252	0.3422	0.3926	7.9438	0.2510	0.2539
	λ	10.3847	2.6659	2.8112	10.0271	2.3131	2.3115
	p	0.5243	0.2282	0.3040	0.8044	0.0000	0.0000
200	β	8.2181	0.2849	0.3322	7.9707	0.1921	0.1928
	λ	10.3169	2.1363	2.2346	10.0310	1.8906	1.8897
	p	0.5385	0.2218	0.2900	0.8046	0.0000	0.0000
300	β	8.1700	0.2291	0.2578	7.9660	0.1324	0.1334
	λ	10.1991	1.2359	1.2743	10.0049	1.1166	1.1155
	p	0.5574	0.2205	0.2791	0.8045	0.0000	0.0000
500	β	8.1629	0.1650	0.1914	7.9843	0.0752	0.0753
	λ	10.1267	0.6990	0.7144	9.9964	0.6228	0.6222
	p	0.5603	0.2125	0.2697	0.8044	0.0000	0.0000

Appendix C

R code for this thesis

****

R code of f(x), F(x), S(x) and h(x) for MEIW distribution

****

```
fx <- (p*(lambda*beta*y^(-(beta+1))*(exp(-y^(-beta)))^lambda))
       +((1-p)*(beta*lambda^(1-(1/beta))
       *y^(-beta)*(exp(-y^(-beta)))^lambda/gamma(1-(1/beta))))
Fx <- p*((exp(-y^(-beta)))^lambda)+(1-p)
       *pgamma(lambda/(y^(beta)),1-(1/beta),lower=FALSE)
       *gamma(1-(1/beta))/gamma(1-(1/beta))
Sx <- 1 - Fx
hx <- fx / Sx
```

****

R code of Random generation for EIW, LBEIW and MEIW distributions

****

```
rEIW <- function(n,beta,lambda)
{
x <- rep(0,n)
u <- runif(n)
x <- ( -log( u^(1/lambda) ) )^(-1/beta)
return(x)
}
rLBEIW <- function(n,beta,lambda)
{
v <- rep(0,n)
a <- rep(0,n)
x <- rep(0,n)
u <- runif(n)
```

****

R code of Random generation for EIW, LBEIW and MEIW distributions (Continued)

****

```
for(i in 1:n)
{
v[i] <- u[i] * gamma(1-(1/beta))
a[i] <- Igamma.inv(1-(1/beta), v[i], lower=FALSE)
x[i] <- (lambda/a[i])^(1/beta)
}
return(x)
}
rMEIW <- function(n,beta,lambda,p)
{
x1 <- rEIW(n,beta,lambda)
x2 <- rLBEIW(n,beta,lambda)
x <- rep(0,n)
u <- runif(n)
for(i in 1:n)
{
if(u[i] <= p)
{
x[i] <- x1[i]
}
if(u[i] > p)
{
x[i] <- x2[i]
}
}
return(x)
}
```

****

R code of MLE for MEIW distribution

****

```
log_like_MEIW <- function(beta, lambda, p)
{
log_like <- sum(log((p*(lambda*beta*y^(-(beta+1))
                  *(exp(-y^(-beta)))^lambda))
                  +((1-p)*(beta*lambda^(1-(1/beta))*y^(-beta)
                  *(exp(-y^(-beta)))^lambda/gamma(1-(1/beta))))))
return(-log_like)
}
library(stats4)
est <- mle(log_like_MEIW,start=list(beta=beta0,lambda=lambda0,p=p0),
        method="L-BFGS-B",lower=c(0, 0, 0),upper = c(Inf, Inf, 1))
mle_beta   <-       coef(est)[1]
mle_lambda <-        coef(est)[2]
mle_p      <-       coef(est)[3]
pMEIW <- function(y,beta,lambda,p)
{
     (p*((exp(-y^(-beta)))^lambda) )+((1-p)
     *Igamma(1-(1/beta),lambda/(y^(beta)),lower=FALSE)
     / gamma(1-(1/beta)) )
}
ks.test(y,pMEIW,mle_beta,mle_lambda,mle_p)
library(ADGofTest)
ad.test(y,pMEIW,mle_beta,mle_lambda,mle_p)
```

****

R code of Bayesian approach for MEIW distribution

****

```
dMEIW <- function(y, beta, lambda, p)
{
     (p*(lambda*beta*y^(-(beta+1))*(exp(-y^(-beta)))^lambda))
      +((1-p)*(beta lambda^(1-(1/beta))*y^(-beta)
      *(exp(-y^(-beta)))^lambda/gamma(1-(1/beta))))
}
burn_in <- 1500
iteration_bayes <- 10000
#beta is Gamma(a,b)
a_beta <- 0.01
b_beta <- 0.01
#lambda is Gamma(c,d)
c_lambda <- 0.01
d_lambda <- 0.01
#p is Beta(e,f)
e_p <- 5
f_p <- 5
#Initial Parameters of Proposal Distribution
proposal_b_beta  <- 1/b_beta
proposal_d_lambda <- 1/d_lambda
proposal_f_p     <- 1/f_p
#Initial Parameters
current_beta    <- beta0
current_lambda   <- lambda0
current_p        <- p0
beta_hat   <- numeric(iteration_bayes)
lambda_hat <- numeric(iteration_bayes)
p_hat      <- numeric(iteration_bayes)
```

****

R code of Bayesian approach for MEIW distribution (Continued)

****

```
for (i in 1:iteration_bayes)
{
proposal_a_beta <- current_beta / b_beta
proposal_beta <- rgamma(1,proposal_a_beta,proposal_b_beta)
loga_beta<-(sum(log(dMEIW(y,proposal_beta,current_lambda,current_p)))
-sum(log(dMEIW(y,current_beta,current_lambda,current_p)))
+sum(dgamma(proposal_beta,a_beta,b_beta,log=TRUE))
-sum(dgamma(current_beta,a_beta,b_beta,log=TRUE)))
u <- runif(1)
u <- log(u)
if( u < loga_beta )
{
current_beta <- proposal_beta
}
proposal_c_lambda <- current_lambda / d_lambda
proposal_lambda <- rgamma(1,proposal_c_lambda,proposal_d_lambda)
loga_lambda<-(sum(log(dMEIW(y,current_beta,proposal_lambda,current_p)))
-sum(log(dMEIW(y,current_beta,current_lambda,current_p)))
+sum(dgamma(proposal_lambda,c_lambda,d_lambda,log=TRUE))
-sum(dgamma(current_lambda,c_lambda,d_lambda,log=TRUE)))
u <- runif(1)
u <- log(u)
if( u < loga_lambda )
{
current_lambda <- proposal_lambda
}
```

****

R code of Bayesian approach for MEIW distribution (Continued)

****

```
proposal_e_p <- current_p / f_p
proposal_p <- rbeta(1,proposal_e_p,proposal_f_p)
loga_p<-(sum(log(dMEIW(y,current_beta,current_lambda,proposal_p)))
-sum(log(dMEIW(y,current_beta,current_lambda,current_p)))
+sum(dbeta(proposal_p,e_p,f_p,log=TRUE))
-sum(dbeta(current_p,e_p,f_p,log=TRUE)))
u <- runif(1)
u <- log(u)
if( u < loga_p )
{
current_p <- proposal_p
}
beta_hat[i]   <- current_beta
lambda_hat[i] <- current_lambda
p_hat[i]      <- current_p
}
bayes_beta   <- mean(beta_hat[(burn_in+1):iteration_bayes])
bayes_lambda <- mean(lambda_hat[(burn_in+1):iteration_bayes])
bayes_p      <- mean(p_hat[(burn_in+1):iteration_bayes])
```

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P. Seenoi ,W. Bodhisuwan and T. Supapakorn, The Length-biased Exponentiated Inverted Weibull Distribution, *International Journal of Pure and Applied Mathematics,* Vol. 92, No. 2, 2014, pp. 191-206.

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