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Maximizing the Retention Level for Proportional Reinsurance under α-regulation of the Finite-Time Surplus Process with Unit-equalized Interarrival Time

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Abstract: The research focuses on an insurance model controlled by proportional reinsurance in the finite-time surplus process with a unit-equalized time interval. We prove the existence of the maximal retention level for independent and identically distributed claim processes under α-regulation, i.e., a model where the insurance company has to manage the probability of insolvency to be at most α. In addition, we illustrate the maximal retention level for exponential claims by applying the bisection technique.

Keywords: probability of insolvency, proportional reinsurance, maximal retention level.

1. Introduction

Recently, an insurance business is a widely popular and highly interesting investment. An investment has risk as also insurance business. Moreover, the main interested in the view of an insurer is the occurring quantity of risk based upon capital which is affected by claim severities (the outflow of capital) and the premium rate (the inflow of capital).

In this paper, we assume that all of the random variables are defined in a probability space $(\Omega, \mathcal{F}, \Pr)$. The considered model is assumed that claim happens at the time n, for all $n = 1, 2, 3, \ldots$, called *claim arrivals*, and Y_n represents a claim size at the time n. In addition, the constant c_0 is assumed to be the premium rate for one unit of time. Therefore, the quantity is the insurer's balance (or surplus) at the time n with the constant $u \geq 0$ as the initial capital is define by

$$U_n(u, c_0) = u + c_0 n - \sum_{k=1}^{n} Y_k \qquad (1)$$

Furthermore, we assume that the claims are controlled by choosing the retention level $b \in [L, 1]$ of proportional reinsurance, that is the claim amount y, the insurer pay $b \cdot y$ and reinsurer pay $(1-b) \cdot y$. Moreover, level $b = 1$ is the action without reinsurance, and the level $b = L$ is the smallest retention level that the insurer can be chosen or accepted. The premium rate of one unit of time for the insurer and reinsurer are calculated by the expected value principle as follows:

$$c_0 = (1+\theta_0)\mathrm{E}[Y],$$

and

$$\begin{aligned} c(b) &= (1+\theta_0)\mathrm{E}[Y] - (1+\theta_1)(1-b)E[Y], \\ &= \{b(1+\theta_1) - (\theta_1 - \theta_0)\} E[Y], \end{aligned}$$

where θ_0, θ_1 are the safety loadings of the insurer and the reinsurer, respectively such that $\theta_1 \geq \theta_0 \geq 0.$ We note that $c(1) = c_0$ and assume that $c(L) > 0.$ Therefore, the surplus process for the insurance model with proportional reinsurance is given by

$$U_n(u, c_0, b) = u + nc(b) - b\sum_{k=1}^{n} Y_k \qquad (2)$$

for all $n = 1, 2, 3, ...,$ where $u \geq 0$ is the initial capital, $c(b)$ is a net premium rate of insurer with the retention level $b \in [L, 1]$ and $\{Y_n : n \in \mathbb{N}\}$ is a claim size process. The probability of insolvency at one of the time $1, 2, 3, \ldots, n$ with respect to the retention level b is defined by

$$\Phi_n(u, c_0, b) = \Pr(U_k(u, c_0, b) < 0 \text{ for some } k = 1, 2, 3, \ldots n).$$

The survival probability for one time unit $1, 2, 3, \ldots, n$ is denoted by

$$\varphi_n(u, c_0, b) = 1 - \Phi_n(u, c_0, b) \qquad (3)$$

Commonly, an application of the process (1) is

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used to describe the behavior of the surplus in the case of motor insurance, because traffic accidents are occurred every day. The process (1) is uncomplicated; so many researchers have studied the subject and the results. For example, Chan, W.S. and Zhang, L. (2006) considered a case of discrete time surplus process and proposed another approach to deriving recursive and explicit formulas for the ruin probabilities with exponential claim severity Y_n. Sattayatham, P., et. al.(2013) generalized the recursive formula of the probability of insolvency, introduced the minimum initial problem, and controlled the probability of insolvency so that it was not greater than a given quantity. Klongdee, W.(2013) studied the model (2) in the case $n=1$ under the α-regulation, introduced the maximum retention level for proportional reinsurance and gave an example of the existence of a maximal retention level by using exponential claims.

In this research, we study the model (2) and propose the existence of a maximal retention level for proportional reinsurance in the finite-time surplus process with unit-equalized inter-arrival time, then we maximize it for exponential claims under α-regulation. Moreover, we define the maximal retention level, denoted by

$$\mathrm{MaR}_\alpha(\Lambda_n,u,c_0)=\max\{b\in[L,1]:\Phi_n(u,c_0,b)\le\alpha\} \quad (4)$$

2. Main Results

In this section, we consider the finite time surplus process with an unit-equalized inter-arrival time in the equation (2), the claim process $\{Y_n : n\in\mathbb{N}\}$ constitutes a sequence of nonnegative random variables which are assumed to be independent and identically distributed (i.i.d), i.e., $\Pr(Y_1\le y)=\Pr(Y_k\le y),\ k=2,3,4,...,$ and Y_i,Y_j are mutually independent where $i\ne j$.

We define the total claim size process $\{\Lambda_n : n\in\mathbb{N}\}$ by

$$\Lambda_n := Y_1+Y_2+Y_3+\cdots+Y_n,$$

for all $n\in\mathbb{N}$. The survival probability at the time N as mentioned in (3) can be expressed as follows

$$\varphi_N(u,c_0,b)=\Pr\left(\Lambda_k\le\frac{u+kc(b)}{b},k=1,2,3,\dots,N\right)$$

$$=\Pr\left(\bigcap_{k=1}^{N}(b\Lambda_k-u-kc(b)\le 0)\right)$$

$$=\mathrm{E}\left[\prod_{k=1}^{N}\mathrm{I}_{(b\Lambda_k-u-kc(b)\le 0)}\right],$$

where I_A is an indicator in set A.

Lemma 2.1 Let N be a positive integer, $b\in[L,1]$ and c_0 be a non-negative real number .If $\{Y_n : n\in\mathbb{N}\}$ is an i.i.d claim size process, then $\varphi_N(u,c_0,b)$ is left continuous in b. In addition, $\Phi_n(u,c_0,b)$ is also left continuous in b.

Proof Sine $\mathrm{I}_{\{\omega:b\Lambda_n(\omega)-u-nc(b)\le 0\}}=\mathrm{I}_{(-\infty,0]}\left(b\Lambda_n-u-nc(b)\right)$, $n=1,2,3,\dots,N$,

$$\varphi_N(u,c_0,b)=\mathrm{E}\left[\prod_{n=1}^{N}\mathrm{I}_{(-\infty,0]}\left(b\Lambda_n-u-nc(b)\right)\right].$$

By the monotone convergence theorem, we have

$$\lim_{\delta\to b^-}\varphi_N(u,c_0,\delta)=\lim_{\delta\to b^-}\mathrm{E}\left[\prod_{n=1}^{N}\mathrm{I}_{(-\infty,0]}\left(\delta\Lambda_n-u-nc(\delta)\right)\right]$$

$$=\lim_{\delta\to b^-}\mathrm{E}\left[\prod_{n=1}^{N}\mathrm{I}_{(-\infty,0]}\left(\Lambda_n-\frac{u-nc(\delta)}{\delta}\right)\right]$$

$$=\mathrm{E}\left[\lim_{\delta\to b^-}\prod_{n=1}^{N}\mathrm{I}_{(-\infty,0]}\left(\Lambda_n-\frac{u-nc(\delta)}{\delta}\right)\right]$$

Sine $\mathrm{I}_{(-\infty,0]}\left(\Lambda_n-\dfrac{u-nc(\delta)}{\delta}\right)$ is left continuous on δ, $n=1,2,3,\dots,N$, so that

$$\lim_{\delta\to b^-}\varphi_N(u,c_0,\delta)=\mathrm{E}\left[\prod_{n=1}^{N}\lim_{\delta\to b^-}\mathrm{I}_{(-\infty,0]}\left(\Lambda_n-\frac{u-nc(\delta)}{\delta}\right)\right]$$

$$=\mathrm{E}\left[\prod_{n=1}^{N}\mathrm{I}_{(-\infty,0]}\left(\Lambda_n-\frac{u-nc(b)}{b}\right)\right]$$

$$=\mathrm{E}\left[\prod_{n=1}^{N}\mathrm{I}_{(-\infty,0]}\left(b\Lambda_n-u-nc(b)\right)\right]$$

$$=\varphi_N(u,c_0,b).$$

Therefore, $\varphi_N(u,c_0,b)$ is left continuous in b. Moreover, we can conclude that $\Phi_N(u)=1-\varphi(u)$ is also left continuous.

Lemma 2.2 Let a,b and α be real numbers such that $a\le b$. If f is increasing and left continuous on $[a,b]$ and $\alpha \in [f(a), f(b)]$, then there exists $d \in [a,b]$ such that

$$d = \max\{x \in [a,b] : f(x) \le \alpha\}.$$

Proof The proof is similarly to Lemma 2.1 in Sattayatham, P., et.al. (2013).

The maximum retention level for proportional reinsurance is defined by (4) and the proof of $\mathrm{MaR}_\alpha(\Lambda_n, u, c_0)$ was then done using the following Theorem 2.1.

Theorem 2.1 Let N be a positive integer and c_0 be a non-negative real number. If $\Phi_n(u, c_0, b)$ is increasing in b and $\alpha \in [\Phi_N(u, c_0, L), \Phi_N(u, c_0, 1)]$, then there exists $b^* \in [L,1]$ such that

$$b^* = \mathrm{MaR}_\alpha(\Lambda_n, u, c_0).$$

Proof We consider quantity $\Phi_n(u, c_0, 1)$ in the following cases:
Case 1 $\Phi_n(u, c_0, 1) \le \alpha$, $\mathrm{MaR}_\alpha(\Lambda_n, u, c_0) = 1$.
Case 2. $\Phi_n(u, c_0, 1) > \alpha$. By Lemma 2.1, we have $\Phi_n(u, c_0, b)$ is left continuous. Using Lemma 2.2, there exists $b^* \in [L,1]$ such that

$$b^* = \max\{b \in [L,1] : \Phi_N(u, c_0, b) \le \alpha\}.$$

This is, $b^* = \mathrm{MaR}_\alpha(\Lambda_n, u, c_0)$.

Lemma 2.3 [Supawan, K.,(2015)**]** Let n be positive integer and $\{Y_n : n \in \mathbb{N}\}$ be an i.i.d exponential claim process. If

$$u \ge \max\{n(\theta_1 - \theta_0)E[Y_1], 0\},$$

then $\Phi_n(u, c_0, b)$ is an increasing function in $b \in (0,1]$.

From the results of Theorem 2.1 and Lemma 2.3, Theorem 2.3 follows:

Theorem 2.3 Let N be a positive integer and c_0 be a non-negative real number. If $\Phi_n(u, c_0, L) \le \alpha$ and $u \ge \max\{n(\theta_1 - \theta_0)\mathbf{E}[Y_1], 0\}$, then $\mathrm{MaR}_\alpha(\Lambda_n, u, c_0) = \max\{b \in [L,1] : \Phi_n(u, c_0, b) \le \alpha\}$ exists.

Proof If $\Phi_n(u, c_0, 1) \le \alpha$, we have $\mathrm{MaR}_\alpha(\Lambda_n, u, c_0) = 1$. On the other hand, if $\Phi_n(u, c_0, 1) > \alpha$, we have $\alpha \in [\Phi_n(u, c_0, L), \Phi_n(u, c_0, 1)]$. By Theorem 2.1 and Lemma 2.3, we have

$$\Phi_n(u, c_0, K) = \lim_{b \to K^-} \Phi_n(u, c_0, b) \le \alpha,$$

where $K = \sup\{b \in [L,1] : \Phi_n(u, c_0, b) \le \alpha\}$. It follows that $K = \mathrm{MaR}_\alpha(\Lambda_n, u, c_0)$.

3. Numerical example

In this section, we compute the maximal retention level, $\mathrm{MaR}_\alpha(\Lambda_n, u, c_0)$, by applying the bisection technique for the decreasing and left continuous function as mentioned in next theorem which can be proved similarly to Theorem 2.8 in Sattayatham, P., (2013).

Theorem 3.1 Let N be a positive integer, $\alpha \in (0,1)$ and $L \le a_0 < b_0 \le 1$. Let $\{a_n : n \in \mathbb{N}\}$ and $\{b_n : n \in \mathbb{N}\}$ be a real sequence such that for each $n \in \mathbb{N}$,

$$a_n = a_{n-1} \text{ and } b_n = \frac{b_{n-1} + a_{n-1}}{2}, \text{ if } \Phi_N\left(u, c_0, \frac{a_{n-1} + b_{n-1}}{2}\right) > \alpha$$

$$b_n = b_{n-1} \text{ and } a_n = \frac{a_{n-1} + b_{n-1}}{2}, \text{ if } \Phi_N\left(u, c_0, \frac{a_{n-1} + b_{n-1}}{2}\right) \le \alpha.$$

If $\alpha \in [\Phi_N(u, c_0, a), \Phi_N(u, c_0, b)]$, then

$$\lim_{n \to \infty} a_n = \max\{a \in [L,1] : \Phi_n(u, c_0, a) \le \alpha\} \text{ and}$$

$$0 \le b_n - \mathrm{MaR}_\alpha(\Lambda_n, u) \le \frac{b_0 - a_0}{2^n}.$$

Next, we give an example to illustrate applications of the maximal retention level for exponential claims under α-regulation i.e., a model where the insurance company has to manage the probability of insolvency to be at most α. We assume that $\{Y_n : n \in \mathbb{N}\}$ is an i.i.d exponential claim size process such that Y_1 has an exponential with intensity $\lambda > 0$, i.e., $E[Y_1] = \frac{1}{\lambda}$. Thus, the recursive formula of probability of insolvency of the discrete time surplus process with a retention level b of proportional reinsurance is the form

$$\Phi_n(u, c_0, b) = \Phi_{n-1}(u, c(b), b) + \frac{[\frac{\lambda}{b}(u + nc(b))]^{n-1}}{(n-1)!} \cdot e^{-\frac{\lambda}{b}(u+nc(b))} \cdot \frac{u + c(b)}{u + nc(b)} \quad (5)$$

for all $n = 1, 2, 3, \ldots$ (Supawan, K., et.al. (2015)).

We estimate the numerical illustrations of the main results and approximate the Maximal retention level $\mathrm{MaR}_\alpha(\Lambda_n, u, c_0)$ in the finite-time surplus process with unit-equalized inter-arrival time (2) by using the Theorem 3.1 in the case of $\{Y_n : n \ge 1\}$ a sequence of i.i.d. exponential distribution with the intensity $\lambda = 1$. We choose parameter

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combinations $\theta = \theta_0 = \theta_1 = 0.1$ in which $c(b) = b(1.1)\text{E[Y]}$ and $\theta = \theta_0 = \theta_1 = 0.2$ in which $c(b) = b(1.2)\text{E[Y]}$ and setting $u = 2,4$ by using $\alpha = 0.05$ and $\alpha = 0.01$ respectively. As a result, we get the table of the maximal retention level b as below:

Table 1. Maximal retention levels $\text{MaR}_\alpha(\Lambda_n, u, c_0)$ in the finite-time surplus process with unit-equalized inter-arrival times with exponential claim ($\lambda = 1$)

	$u=2$ and $\alpha = 0.05$		$u=4$ and $\alpha = 0.01$	
N	$\theta = 0.1$	$\theta = 0.2$	$\theta = 0.1$	$\theta = 0.2$
5	0.899835	1.000000	0.819065	0.893983
10	0.764282	0.946663	0.716041	0.800137
20	0.698472	0.882411	0.645468	0.745491
30	0.678359	0.866225	0.617395	0.727946
40	0.669302	0.860026	0.602708	0.720351
50	0.664413	0.857143	0.593949	0.716547
60	0.661471	0.855640	0.588291	0.714460
100	0.656667	0.853776	0.578148	0.711719
200	0.654469	0.853361	0.572905	0.711063
300	0.654120	0.853345	0.572002	0.711035
400	0.654028	0.853344	0.571756	0.711033
500	0.653997	0.853344	0.571674	0.711033
1000	0.653978	0.853343	0.571620	0.711033

Table 1 shows the approximation of the maximal retention level $\text{MaR}_\alpha(\Lambda_n, u, c_0)$ with b as mentioned in Theorem 3.1, choosing $a_0 = 0$ and $b_0 = 20$, and $\Phi_n(u, c_0, b)$ computed by applying the bisection technique and using equation (5). The numerical results in Table 1 show maximal retention level $b = 0.899835$ for $u = 2$, $\alpha = 0.05$, $N = 5$ and $\theta = 0.1$ etc.

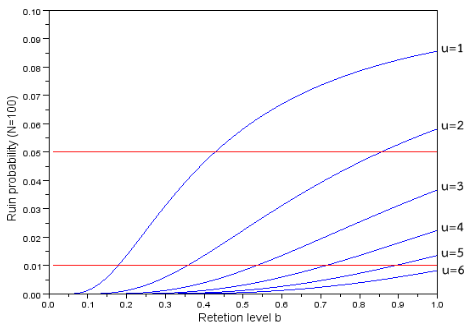

Figure 1. The relationship between proportional reinsurance b and $\Phi_n(u, c_0, b)$ with exponential claim ($\lambda = 1$)

Figure 1 we run numerical experiments to compare the retention level b and probability of insolvency for the finite-time surplus process with unit-equalized inter-arrival times. Here, we choose parameter combinations $u = \{1, 2, 3, ..., 6\}$, $N = 100$, $\alpha = 0.01$, $\alpha = 0.05$ and $\theta = 0.2$. The numerical results in Figure 1 show that the intersection of various values of α and u are the maximal retention level b.

4. Conclusions

In this paper, we consider a reinsurance problem for an insurer under the surplus process for a reinsurance model (4) in which we focus on the proportional reinsurance. We give the claim process $\{Y_n : n \geq 1\}$ which constitutes a sequence of nonnegative random variables which are assumed to be independent and identically distributed (i.i.d). The main result of the paper is the argument for the existence of the maximal retention level for independent and identically distributed claim processes under α-regulation, i.e., a model where the insurance company has to manage the probability of insolvency to be at most α. Finally, numerical examples of the maximal retention level, $\text{MaR}_\alpha(\Lambda_n, u, c_0)$, for exponential claims by applying the bisection technique to support the main result. In future works, we will study the maximal retention level, $\text{MaR}_\alpha(\Lambda_n, u, c_0)$, for this excess-of-loss reinsurance and alternate case.

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