APPENDIX B13

SUBGAME PERFECT NASH EQUILIBRIUM FINDING FOR THE CASE V

Objective function

Player 1:

$$\text{Max}_{p_1} \left(p_1 - \frac{1/2^2 + n_1^2}{2}\right)\left(\frac{1}{2} - \frac{p_1 - p_2}{2}\right)$$

$$\text{s.t.} \quad q_1 \geq 0 \text{ or } x_c \geq 0 \text{ or } \frac{1}{2} - \frac{p_1 - p_2}{2} \geq 0$$

$$\text{and} \quad q_1 \leq 1 \text{ or } x_c \leq 1 \text{ or } \frac{1}{2} - \frac{p_1 - p_2}{2} \leq 1$$

$$\text{and} \quad x_{c_1} \leq x_c \text{ or } \frac{m_1 + 1}{2} - \frac{(n_1 - 1/2)}{2(m_1 - 1)} + \frac{p_1 - p_2}{2(m_1 - 1)} - \left(\frac{1}{2} - \frac{p_1 - p_2}{2}\right) \leq 0$$

$$\text{and} \quad x_c \leq x_{c_2} \text{ or } \frac{1}{2} - \frac{p_1 - p_2}{2} - \left(\frac{m_2}{2} - \frac{(1/2 - n_2)}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)}\right) \leq 0$$

Player 2:

$$\text{Max}_{p_2} \left(p_2 - \frac{1/2^2 + n_2^2}{2}\right)\left(\frac{1}{2} + \frac{p_1 - p_2}{2}\right)$$

$$\text{s.t.} \quad q_1 \geq 0 \text{ or } x_c \geq 0 \text{ or } \frac{1}{2} - \frac{p_1 - p_2}{2} \geq 0$$

$$\text{and} \quad q_1 \leq 1 \text{ or } x_c \leq 1 \text{ or } \frac{1}{2} - \frac{p_1 - p_2}{2} \leq 1$$

$$\text{and} \quad x_{c_1} \leq x_c \text{ or } \frac{m_1 + 1}{2} - \frac{(n_1 - 1/2)}{2(m_1 - 1)} + \frac{p_1 - p_2}{2(m_1 - 1)} - \left(\frac{1}{2} - \frac{p_1 - p_2}{2}\right) \leq 0$$

$$\text{and} \quad x_c \leq x_{c_2} \text{ or } \frac{1}{2} - \frac{p_1 - p_2}{2} - \left(\frac{m_2}{2} - \frac{(1/2 - n_2)}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)}\right) \leq 0$$

Lagrange function:

$$L_1 = \left(p_1 - \frac{1/2^2 + n_1^2}{2}\right)\left(\frac{1}{2} - \frac{p_1 - p_2}{2}\right)$$

$$+\mu_1\left(\frac{1}{2}-\frac{p_1-p_2}{2}\right)-\mu_2\left(\frac{1}{2}-\frac{p_1-p_2}{2}-1\right)$$

$$-\mu_3\left(\frac{m_1+1}{2}-\frac{(n_1-1/2)}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}-\left(\frac{1}{2}-\frac{p_1-p_2}{2}\right)\right)$$

$$-\mu_4\left(\frac{1}{2}-\frac{p_1-p_2}{2}-\left(\frac{m_2}{2}-\frac{(1/2-n_2)}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}\right)\right) \quad \text{(B.237)}$$

$$L_2 = \left(p_2-\frac{1/2^2+n_2^2}{2}\right)\left(\frac{1}{2}+\frac{p_1-p_2}{2}\right)$$

$$+\lambda_1\left(\frac{1}{2}-\frac{p_1-p_2}{2}\right)-\lambda_2\left(\frac{1}{2}-\frac{p_1-p_2}{2}-1\right)$$

$$-\lambda_3\left(\frac{m_1+1}{2}-\frac{(n_1-1/2)}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}-\left(\frac{1}{2}-\frac{p_1-p_2}{2}\right)\right)$$

$$-\lambda_4\left(\frac{1}{2}-\frac{p_1-p_2}{2}-\left(\frac{m_2}{2}-\frac{(1/2-n_2)}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}\right)\right) \quad \text{(B.238)}$$

First order condition and complementary slackness are as follow.

Player 1:

$$\frac{\partial L_1}{\partial p_1} = \frac{1}{16}\left(9-8\mu_3+8\mu_4-\frac{8\mu_1 m_1}{m_1-1}+\frac{8\mu_2(m_2-1)}{m_2}\right.$$
$$\left.+4n_1^2-16p_1+8p_2\right) = 0 \quad \text{(B.239)}$$

$$\frac{\partial L_1}{\partial \mu_1} = \frac{1}{2}-\frac{p_1-p_2}{2} \geq 0 \quad \text{(B.240)}$$

$$\mu_1 \geq 0 \quad \text{(B.241)}$$

$$\mu_1\frac{\partial L_1}{\partial \mu_1} = \mu_1\left(\frac{1}{2}-\frac{p_1-p_2}{2}\right) = 0 \quad \text{(B.242)}$$

$$\frac{\partial L_1}{\partial \mu_2} = -\left(\frac{1}{2}-\frac{p_1-p_2}{2}-1\right) \geq 0 \quad \text{(B.243)}$$

$$\mu_2 \geq 0 \quad \text{(B.244)}$$

$$\mu_2\frac{\partial L_1}{\partial \mu_2} = -\mu_2\left(\frac{1}{2}-\frac{p_1-p_2}{2}-1\right) = 0 \quad \text{(B.245)}$$

$$\frac{\partial L_1}{\partial \mu_3} = -\left(\frac{m_1+1}{2} - \frac{(n_1-1/2)}{2(m_1-1)} + \frac{p_1-p_2}{2(m_1-1)} - \left(\frac{1}{2} - \frac{p_1-p_2}{2}\right)\right) \geq 0 \quad \text{(B.246)}$$

$$\mu_3 \geq 0 \quad \text{(B.247)}$$

$$\mu_3 \frac{\partial L_1}{\partial \mu_3} = -\mu_3\left(\frac{m_1+1}{2} - \frac{(n_1-1/2)}{2(m_1-1)} + \frac{p_1-p_2}{2(m_1-1)} - \left(\frac{1}{2} - \frac{p_1-p_2}{2}\right)\right) = 0 \quad \text{(B.248)}$$

$$\frac{\partial L_1}{\partial \mu_4} = -\left(\frac{1}{2} - \frac{p_1-p_2}{2} - \left(\frac{m_2}{2} - \frac{(1/2-n_2)}{2(-m_2)} + \frac{p_1-p_2}{2(-m_2)}\right)\right) \geq 0 \quad \text{(B.249)}$$

$$\mu_4 \geq 0 \quad \text{(B.250)}$$

$$\mu_4 \frac{\partial L_1}{\partial \mu_4} = -\mu_4\left(\frac{1}{2} - \frac{p_1-p_2}{2} - \left(\frac{m_2}{2} - \frac{(1/2-n_2)}{2(-m_2)} + \frac{p_1-p_2}{2(-m_2)}\right)\right) = 0 \quad \text{(B.251)}$$

Player 2:

$$\frac{\partial L_2}{\partial p_2} = \frac{1}{16}\left(9 + 8\lambda_3 - 8\lambda_4 + \frac{8\lambda_1 m_1}{m_1-1} - \frac{8\lambda_2(m_2-1)}{m_2}\right.$$

$$\left. + 4n_2^2 + 8p_1 - 16p_2\right) = 0 \quad \text{(B.252)}$$

$$\frac{\partial L_2}{\partial \lambda_1} = \frac{1}{2} - \frac{p_1-p_2}{2} \geq 0 \quad \text{(B.253)}$$

$$\lambda_1 \geq 0 \quad \text{(B.254)}$$

$$\lambda_1 \frac{\partial L_2}{\partial \lambda_1} = \lambda_1\left(\frac{1}{2} - \frac{p_1-p_2}{2}\right) = 0 \quad \text{(B.255)}$$

$$\frac{\partial L_2}{\partial \lambda_2} = -\left(\frac{1}{2} - \frac{p_1-p_2}{2} - 1\right) \geq 0 \quad \text{(B.256)}$$

$$\lambda_2 \geq 0 \quad \text{(B.257)}$$

$$\lambda_2 \frac{\partial L_2}{\partial \lambda_2} = -\lambda_2\left(\frac{1}{2} - \frac{p_1-p_2}{2} - 1\right) = 0 \quad \text{(B.258)}$$

$$\frac{\partial L_2}{\partial \lambda_3} = -\left(\frac{m_1+1}{2} - \frac{(n_1-1/2)}{2(m_1-1)} + \frac{p_1-p_2}{2(m_1-1)} - \left(\frac{1}{2} - \frac{p_1-p_2}{2}\right)\right) \geq 0 \quad \text{(B.259)}$$

$$\lambda_3 \geq 0 \quad \text{(B.260)}$$

$$\lambda_3 \frac{\partial L_2}{\partial \lambda_3} = -\lambda_3\left(\frac{m_1+1}{2} - \frac{(n_1-1/2)}{2(m_1-1)} + \frac{p_1-p_2}{2(m_1-1)} - \left(\frac{1}{2} - \frac{p_1-p_2}{2}\right)\right) = 0 \quad \text{(B.261)}$$

$$\frac{\partial L_2}{\partial \lambda_4} = -\left(\frac{1}{2}-\frac{p_1-p_2}{2}-\left(\frac{m_2}{2}-\frac{(1/2-n_2)}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}\right)\right) \geq 0 \quad \text{(B.262)}$$

$$\lambda_4 \geq 0 \quad \text{(B.263)}$$

$$\lambda_4 \frac{\partial L_2}{\partial \lambda_4} = -\lambda_4\left(\frac{1}{2}-\frac{p_1-p_2}{2}-\left(\frac{m_2}{2}-\frac{(1/2-n_2)}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}\right)\right) = 0 \quad \text{(B.264)}$$

By applying market clear condition $q_1+q_2=1$ and $0 \leq q_1 \leq 1$, $0 \leq q_2 \leq 1$, the only solution of $p_1, p_2, \mu_1, \mu_2, \mu_3, \mu_4, \lambda_1, \lambda_2, \lambda_3$ and λ_4 are as follow.

$$p_1 = \frac{1}{24}\left(27+8n_1^2+4n_2^2\right) \quad \text{(B.265)}$$

$$p_2 = \frac{1}{24}\left(27+4n_1^2+8n_2^2\right) \quad \text{(B.266)}$$

$$\mu_1 = 0$$

$$\mu_2 = 0$$

$$\mu_3 = 0$$

$$\mu_4 = 0$$

$$\lambda_1 = 0$$

$$\lambda_2 = 0$$

$$\lambda_3 = 0$$

$$\lambda_4 = 0$$

In order to check that these prices are the optimal prices that make each player has highest profit, second order condition is employed.

$$\frac{\partial^2 \pi_1}{\partial p_1^2} = -1 < 0 \quad , \quad \frac{\partial^2 \pi_2}{\partial p_2^2} = -1 < 0$$

Since the value of $\partial^2 \pi_1 / \partial p_1^2$ and $\partial^2 \pi_2 / \partial p_2^2$ are negative, we can ensure that these prices are the optimal prices for each player. By substitute optimal price of each player, equation (B.265) and (B.266), back into the profit function of each firm and applying first order condition again, we obtain:

$$\pi_1^* = \left(p_1^* - \frac{1/2^2 + n_1^2}{2}\right)\left(\frac{1}{2} - \frac{p_1^* - p_2^*}{2}\right)$$

$$= \frac{1}{72}(6 - n_1^2 + n_2^2)^2 \quad \text{(B.267)}$$

$$\frac{\partial \pi_1^*}{\partial n_1} = \frac{n_1(-6 + n_1^2 - n_2^2)}{18} \quad \text{(B.268)}$$

$$\pi_2^* = \left(p_1^* - \frac{1/2^2 + n_1^2}{2}\right)\left(\frac{1}{2} + \frac{p_1^* - p_2^*}{2}\right)$$

$$= \frac{1}{72}(6 + n_1^2 - n_2^2)^2 \quad \text{(B.269)}$$

$$\frac{\partial \pi_2^*}{\partial n_2} = \frac{n_2(-6 - n_1^2 + n_2^2)}{18} \quad \text{(B.270)}$$

Since the value of n_1, n_2 are not greater than 1, the value of $\partial\pi_1/\partial n_1$ and $\partial\pi_2/\partial n_2$ must be less than 0 except when n_1, n_2 equal to 0. Thus, the optimal product quality level of player 1 is 0 and player 2 is 0. Again, we will apply second order condition to check whether these are the optimal product quality level for both players. The second orders are as follow.

$$\frac{\partial^2 \pi_1^*}{\partial n_1^2} = \frac{1}{18}(-6 + 3n_1^2 - n_2^2) < 0$$

$$\frac{\partial^2 \pi_2^*}{\partial n_2^2} = \frac{1}{18}(-6 - n_1^2 + 3n_2^2) < 0$$

We can see that the value of $\partial^2\pi_1^*/\partial n_1^2$ and $\partial^2\pi_2^*/\partial n_2^2$ are always negative. Thus the optimal product quality level for each player is $n_1 = 0$ and $n_2 = 0$. Substitute the optimal quality back into profit function, equation (B.273) and (B.275), and do the partial differentiation with respect to m_i, we obtain:

$$\pi_1^* = \frac{1}{2}$$

$$\frac{\partial \pi_1^*}{\partial m_1} = 0$$

$$\pi_2^* = \frac{1}{2}$$

$$\frac{\partial \pi_2^*}{\partial m_2} = 0$$

Since $\partial\pi_1/\partial m_1$ and $\partial\pi_2/\partial m_2$ equals to 0, we can conclude that regardless the value of m_1, the profit of player 1 and player 2 are still the same. Therefore, the conclusion of this case is that there is subgame perfect Nash Equilibrium, which is when $m_1 \in [0,1]$, $m_2 \in [0,1]$, $n_1 = 0$, $n_2 = 0$, $p_1 = p_2 = 9/8$, and $\pi_1 = \pi_2 = 1/2$.