APPENDIX B11

SUBGAME PERFECT NASH EQUILIBRIUM FINDING FOR THE CASE IV.2

Objective functions

Player 1:

$$\text{Max}_{p_1}\left(p_1-\frac{1/2^2+n_1^2}{2}\right)\left(\frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}\right)$$

s.t. $x_{c_1}\le 1$ or $\frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}\le 1$

and $x_{c_1}\ge 0$ or $\frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}\ge 0$

and $p_1-n_1\le p_2-n_2$

Player 2:

$$\text{Max}_{p_2}\left(p_1-\frac{1/2^2+n_1^2}{2}\right)\left(1-\left(\frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}\right)\right)$$

s.t. $x_{c_1}\le 1$ or $\frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}\le 1$

and $x_{c_1}\ge 0$ or $\frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}\ge 0$

and $p_1-n_1\le p_2-n_2$

Lagrange function:

$$\begin{aligned}L_1 = {} & \left(p_1-\frac{1/2^2+n_1^2}{2}\right)\left(\frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}\right)\\ & -\mu_1\left(\frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}-1\right)\\ & +\mu_2\left(\frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}\right)\\ & -\mu_3\left(p_1-n_1-(p_2-n_2)\right)\end{aligned} \tag{B.199}$$

$$L_2 = \left(p_1 - \frac{1/2^2 + n_1^2}{2}\right)\left(1 - \left(\frac{m_1+1}{2} - \frac{n_1 - 1/2}{2(m_1-1)} + \frac{p_1 - p_2}{2(m_1-1)}\right)\right)$$
$$- \lambda_1\left(\frac{m_1+1}{2} - \frac{n_1 - 1/2}{2(m_1-1)} + \frac{p_1 - p_2}{2(m_1-1)} - 1\right)$$
$$+ \lambda_2\left(\frac{m_1+1}{2} - \frac{n_1 - 1/2}{2(m_1-1)} + \frac{p_1 - p_2}{2(m_1-1)}\right)$$
$$- \lambda_3\left(p_1 - n_1 - (p_2 - n_2)\right) \quad \text{(B.200)}$$

First order condition and complementary slackness are as follow.

Player 1:

$$\frac{\partial L_1}{\partial p_1} = \frac{1}{16(m_1-1)}\left(-5 - 8\mu_1 + 8\mu_2 + 8(2\mu_3 - 2\mu_3 m_1 + m_1^2)\right.$$
$$\left.- 4n_1(2+n_1) + 16p_1 - 8p_2\right) = 0 \quad \text{(B.201)}$$

$$\frac{\partial L_1}{\partial \mu_1} = -\left(\frac{m_1+1}{2} - \frac{n_1 - 1/2}{2(m_1-1)} + \frac{p_1 - p_2}{2(m_1-1)} - 1\right) \geq 0 \quad \text{(B.202)}$$

$$\mu_1 \geq 0 \quad \text{(B.203)}$$

$$\mu_1 \frac{\partial L_1}{\partial \mu_1} = -\mu_1\left(\frac{m_1+1}{2} - \frac{n_1 - 1/2}{2(m_1-1)} + \frac{p_1 - p_2}{2(m_1-1)} - 1\right) = 0 \quad \text{(B.204)}$$

$$\frac{\partial L_1}{\partial \mu_2} = \frac{m_1+1}{2} - \frac{n_1 - 1/2}{2(m_1-1)} + \frac{p_1 - p_2}{2(m_1-1)} \geq 0 \quad \text{(B.205)}$$

$$\mu_2 \geq 0 \quad \text{(B.206)}$$

$$\mu_2 \frac{\partial L_1}{\partial \mu_2} = \mu_2\left(\frac{m_1+1}{2} - \frac{n_1 - 1/2}{2(m_1-1)} + \frac{p_1 - p_2}{2(m_1-1)}\right) = 0 \quad \text{(B.207)}$$

$$\frac{\partial L_1}{\partial \mu_3} = -\left(p_1 - n_1 - (p_2 - n_2)\right) \geq 0 \quad \text{(B.208)}$$

$$\mu_3 \geq 0 \quad \text{(B.209)}$$

$$\mu_3 \frac{\partial L_1}{\partial \mu_3} = -\mu_3\left(p_1 - n_1 - (p_2 - n_2)\right) = 0 \quad \text{(B.210)}$$

Player 2:

$$\frac{\partial L_2}{\partial p_2} = \frac{1}{16(m_1-1)}\left(-13+8\lambda_1-8\lambda_2+16\lambda_3(m_1-1)-8(m_1-2)m_1\right.$$

$$\left.+8n_1-4n_2^2-8p_1+16p_2\right) = 0 \quad \text{(B.211)}$$

$$\frac{\partial L_2}{\partial \lambda_1} = -\left(\frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}-1\right) \geq 0 \quad \text{(B.212)}$$

$$\lambda_1 \geq 0 \quad \text{(B.213)}$$

$$\lambda_1\frac{\partial L_2}{\partial \lambda_1} = -\lambda_1\left(\frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}-1\right) = 0 \quad \text{(B.214)}$$

$$\frac{\partial L_2}{\partial \lambda_2} = \frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)} \geq 0 \quad \text{(B.215)}$$

$$\lambda_2 \geq 0 \quad \text{(B.216)}$$

$$\lambda_2\frac{\partial L_2}{\partial \lambda_2} = \lambda_2\left(\frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}\right) = 0 \quad \text{(B.217)}$$

$$\frac{\partial L_2}{\partial \lambda_3} = -\left(p_1-n_1-(p_2-n_2)\right) \geq 0 \quad \text{(B.218)}$$

$$\lambda_3 \geq 0 \quad \text{(B.219)}$$

$$\lambda_3\frac{\partial L_2}{\partial \lambda_3} = -\lambda_3\left(p_1-n_1-(p_2-n_2)\right) = 0 \quad \text{(B.220)}$$

By applying market clear condition $q_1+q_2=1$ and $0 \leq q_1 \leq 1$, $0 \leq q_2 \leq 1$, the solution of $p_1, p_2, \mu_1, \mu_2, \mu_3, \mu_4, \lambda_1, \lambda_2, \lambda_3, \text{and} \lambda_4$ are as follow.

$$p_1 = \frac{1}{24}\left(23-16m_1-8m_1^2+8n_1+8n_1^2+4n_2^2\right) \quad \text{(B.221)}$$

$$p_2 = \frac{1}{24}\left(31-32m_1+8m_1^2-8n_1+4n_1^2+8n_2^2\right) \quad \text{(B.222)}$$

$$\mu_1 = 0$$

$$\mu_2 = 0$$

$$\mu_3 = 0$$

$$\lambda_1 = 0$$

$$\lambda_2 \quad = \quad 0$$

$$\lambda_3 \quad = \quad 0$$

In order to check that these prices are the optimal prices that make each player have highest profit, second order condition, evaluate at p_1, p_2 in (B.221) and (B.222) is employed.

$$\frac{\partial^2 \pi_1}{\partial p_1^2} = \frac{1}{m_1 - 1} < 0, \qquad \frac{\partial^2 \pi_2}{\partial p_2^2} = \frac{1}{m_2 - 1} < 0$$

Since the value of $\partial^2 \pi_1 / \partial p_1^2$ and $\partial^2 \pi_2 / \partial p_2^2$ are negative, we can ensure that these prices are the optimal prices for each player. From these optimal pricing for each firm, we would be able to find the market share function under this case $m_1 = m_2$.

$$q_1 = \frac{-5 + 2m_2(2 + m_2) + (n_1 - 2)n_1 - n_2^2}{12(m_2 - 1)} \tag{B.223}$$

$$q_2 = \frac{-7 - 2(m_2 - 4)m_2 - (n_1 - 2)n_2 - n_2^2}{12(m_2 - 1)} \tag{B.224}$$

Since all Lagrange multiplier are zero, we can see that the constraints are not binding. We can conclude that the market demands of both players are positive. Thus, we can conclude that following results.

$$-5 + 2m_2(2 + m_2) + (n_1 - 2)n_1 - n_2^2 < 0 \tag{B.225}$$

$$-7 - 2(m_2 - 4)m_2 - (n_1 - 2)n_2 - n_2^2 < 0\,. \tag{B.226}$$

By substitute optimal price of each player, equation (B.221) and (B.222), back into the profit function of each firm and applying first order condition again, we obtain:

$$\pi_1^* = \left(p_1^* - \frac{1/2^2 + n_1^2}{2}\right)\left(\frac{m_1+1}{2} - \frac{n_1 - 1/2}{2(m_1-1)} + \frac{p_1^* - p_2^*}{2(m_1-1)}\right)$$

$$= -\frac{\left(5 - 2m_2(2+m_2) - (n_1-2)n_1 + n_2^2\right)^2}{72(m_2-1)} \qquad \text{(B.227)}$$

$$\frac{\partial \pi_1^*}{\partial n_1} = -\frac{(n_1-1)\left(-5 + 2m_2(2+m_2) + (n_1-2)n_1 - n_2^2\right)}{18(m_2-1)} \qquad \text{(B.228)}$$

$$\pi_2^* = \left(p_2^* - \frac{(1/2)^2 + n_2^2}{2}\right)\left(1 - \left(\frac{m_1+1}{2} - \frac{n_1 - 1/2}{2(m_1-1)} + \frac{p_1^* - p_2^*}{2(m_1-1)}\right)\right)$$

$$= \frac{\left(7 + 2(m_2-4)m_2 + (n_1-2)n_2 + n_2^2\right)^2}{72(m_2-1)} \qquad \text{(B.229)}$$

$$\frac{\partial \pi_2^*}{\partial n_2} = -\frac{n_2\left(-7 - 2(m_2-4)m_2 - (n_1-2)n_2 - n_2^2\right)}{18(m_2-1)} \qquad \text{(B.230)}$$

Considering equation (B.218) and (B.220), since we know from inequality (B.225) and (B.226) that the value of $-5 + 2m_2(2+m_2) + (n_1-2)n_1 - n_2^2$ and $-7 - 2(m_2-4)m_2 - (n_1-2)n_2 - n_2^2$ must be less than 0, the optimal product quality level of player 1 is 1 and player 2 is 0. Substitute the optimal product quality of both player back into profit function and do the partial differentiation with respect to m_i, we obtain:

$$\frac{\partial \pi_1^*}{\partial m_1} = -\frac{1}{18}(3+m_1)(1+3m_1)$$

$$\frac{\partial \pi_2^*}{\partial m_2} = -\frac{1}{18}(-3+m_2)(-5+3m_2)$$

As the value of $m_1 = m_2$ cannot be greater than 1, the value of $\partial\pi_1^*/\partial m_1$ and $\partial\pi_2^*/\partial m_2$ always be negative. Thus the value of m_1 and m_2 will move approach zero and the profit of both player will move toward $1/2$. Thus, for this case, we can conclude the subgame perfect Nash Equilibrium as follow. $m_1 = m_2 = 0$, $n_1 = 1$, $n_2 = 0$, $p_1 = 13/8$, $p_2 = 9/8$, and $\pi_1 = \pi_2 = 1/2$.

$$m_1 = m_2 = 0 \tag{B.231}$$

$$n_1 = 1 \tag{B.232}$$

$$n_2 = 0 \tag{B.233}$$

$$p_1 = \frac{13}{8} \tag{B.234}$$

$$p_2 = \frac{9}{8} \tag{B.235}$$

$$\pi_1 = \pi_2 = \frac{1}{2} \tag{B.236}$$

In conclusion of the case of $m_1 = m_2$, there is subgame perfect Nash Equilibrium occur under this case, which is when $m_1 = m_2 = 0$, $n_1 = 1$, $n_2 = 0$, $p_1 = 13/8$, and $p_2 = 9/8$. The profit of each player will be $1/2$. Also, since this game is symmetric, we can conclude that there is another subgame perfect Nash Equilibrium, which is when $m_1 = m_2 = 1$, $n_1 = 0$, $n_2 = 1$, $p_1 = 9/8$, and $p_2 = 13/8$. The profit of each player will also be $1/2$.