

APPENDIX B10

OPTIMAL PRICE FINDING FOR THE CASE IV.1

Objective function

Player 1:

$$\text{Max}_{p_1} \left(p_1 - \frac{1/2^2 + n_1^2}{2} \right) \left(\frac{1}{2} \left(\frac{m_1 + 1}{2} - \frac{n_1 - 1/2}{2(m_1 - 1)} + \frac{p_1 - p_2}{2(m_1 - 1)} + \frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)} \right) \right)$$

$$\text{s.t. } x_{c_1} \leq 1 \text{ or } \frac{m_1 + 1}{2} - \frac{n_1 - 1/2}{2(m_1 - 1)} + \frac{p_1 - p_2}{2(m_1 - 1)} \leq 1$$

$$\text{and } x_{c_2} \geq 0 \text{ or } \frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)} \geq 0$$

$$\text{and } x_{c_2} \leq x_{c_1} \text{ or } \frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)} \leq \frac{m_1 + 1}{2} - \frac{n_1 - 1/2}{2(m_1 - 1)} + \frac{p_1 - p_2}{2(m_1 - 1)}$$

$$\text{and } V - p_1 + n_1 = V - p_2 + n_2$$

Player 2:

$$\text{Max}_{p_2} \left(p_1 - \frac{1/2^2 + n_1^2}{2} \right) \left(1 - \frac{1}{2} \left(\frac{m_1 + 1}{2} - \frac{n_1 - 1/2}{2(m_1 - 1)} + \frac{p_1 - p_2}{2(m_1 - 1)} + \frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)} \right) \right)$$

$$\text{s.t. } x_{c_1} \leq 1 \text{ or } \frac{m_1 + 1}{2} - \frac{n_1 - 1/2}{2(m_1 - 1)} + \frac{p_1 - p_2}{2(m_1 - 1)} \leq 1$$

$$\text{and } x_{c_2} \geq 0 \text{ or } \frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)} \geq 0$$

$$\text{and } x_{c_2} \leq x_{c_1} \text{ or } \frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)} \leq \frac{m_1 + 1}{2} - \frac{n_1 - 1/2}{2(m_1 - 1)} + \frac{p_1 - p_2}{2(m_1 - 1)}$$

$$\text{and } V - p_1 + n_1 = V - p_2 + n_2$$

Lagrange function:

$$L_1 = \left(p_1 - \frac{1/2^2 + n_1^2}{2} \right) \left(\frac{1}{2} \left(\frac{m_1 + 1}{2} - \frac{n_1 - 1/2}{2(m_1 - 1)} + \frac{p_1 - p_2}{2(m_1 - 1)} + \frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)} \right) \right)$$

$$\begin{aligned}
& -\mu_1 \left(\frac{m_1+1}{2} - \frac{n_1-1/2}{2(m_1-1)} + \frac{p_1-p_2}{2(m_1-1)} - 1 \right) + \mu_2 \left(\frac{m_2}{2} - \frac{1/2-n_2}{2(-m_2)} + \frac{p_1-p_2}{2(-m_2)} \right) \\
& -\mu_3 \left(\frac{m_2}{2} - \frac{1/2-n_2}{2(-m_2)} + \frac{p_1-p_2}{2(-m_2)} - \left(\frac{m_1+1}{2} - \frac{n_1-1/2}{2(m_1-1)} + \frac{p_1-p_2}{2(m_1-1)} \right) \right) \\
& -\mu_4 (p_1 - n_1 - (p_2 - n_2)) \tag{B.173}
\end{aligned}$$

$$\begin{aligned}
L_2 &= \left(p_1 - \frac{1/2^2 + n_1^2}{2} \right) \left(1 - \frac{1}{2} \left(\frac{m_1+1}{2} - \frac{n_1-1/2}{2(m_1-1)} + \frac{p_1-p_2}{2(m_1-1)} + \frac{m_2}{2} - \frac{1/2-n_2}{2(-m_2)} + \frac{p_1-p_2}{2(-m_2)} \right) \right) \\
& -\mu_1 \left(\frac{m_1+1}{2} - \frac{n_1-1/2}{2(m_1-1)} + \frac{p_1-p_2}{2(m_1-1)} - 1 \right) + \mu_2 \left(\frac{m_2}{2} - \frac{1/2-n_2}{2(-m_2)} + \frac{p_1-p_2}{2(-m_2)} \right) \\
& -\mu_3 \left(\frac{m_2}{2} - \frac{1/2-n_2}{2(-m_2)} + \frac{p_1-p_2}{2(-m_2)} - \left(\frac{m_1+1}{2} - \frac{n_1-1/2}{2(m_1-1)} + \frac{p_1-p_2}{2(m_1-1)} \right) \right) \\
& -\mu_4 (p_1 - n_1 - (p_2 - n_2)) \tag{B.174}
\end{aligned}$$

First order condition and complementary slackness are as follow.

Player 1:

$$\begin{aligned}
\frac{\partial L_1}{\partial p_1} &= \frac{1}{32(m_1-1)m_2} (-5 - 16\mu_1(m_1-1) + 5m_1 - 5m_2 - 16\mu_2m_2 \\
& + 8n_2 + 16p_1 - 8p_2 + 4(-8\mu_4(m_1-1)m_2 + 2m_1^2m_2 - 2m_2^2 \\
& + 2m_1m_2^2 + 4\mu_3(m_1+m_2-1) - 2m_2n_1 - n_1^2 + (m_1-m_2)n_1^2 \\
& - 2m_1n_2 - 4(m_1-m_2)p_1 + 2(m_1-m_2)p_2) = 0 \tag{B.175}
\end{aligned}$$

$$\frac{\partial L_1}{\partial \mu_1} = - \left(\frac{m_1+1}{2} - \frac{n_1-1/2}{2(m_1-1)} + \frac{p_1-p_2}{2(m_1-1)} - 1 \right) \geq 0 \tag{B.176}$$

$$\mu_1 \geq 0 \tag{B.177}$$

$$\mu_1 \frac{\partial L_1}{\partial \mu_1} = -\mu_1 \left(\frac{m_1+1}{2} - \frac{n_1-1/2}{2(m_1-1)} + \frac{p_1-p_2}{2(m_1-1)} - 1 \right) = 0 \tag{B.178}$$

$$\frac{\partial L_1}{\partial \mu_2} = \frac{m_2}{2} - \frac{1/2-n_2}{2(-m_2)} + \frac{p_1-p_2}{2(-m_2)} \geq 0 \tag{B.179}$$

$$\mu_2 \geq 0 \tag{B.180}$$

$$\mu_2 \frac{\partial L_1}{\partial \mu_2} = \mu_2 \left(\frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)} \right) = 0 \quad (\text{B.181})$$

$$\begin{aligned} \frac{\partial L_1}{\partial \mu_3} = & - \left(\frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)} - \left(\frac{m_1 + 1}{2} \right. \right. \\ & \left. \left. - \frac{n_1 - 1/2}{2(m_1 - 1)} + \frac{p_1 - p_2}{2(m_1 - 1)} \right) \right) \geq 0 \end{aligned} \quad (\text{B.182})$$

$$\mu_3 \geq 0 \quad (\text{B.183})$$

$$\begin{aligned} \mu_3 \frac{\partial L_1}{\partial \mu_3} = & - \mu_3 \left(\frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)} - \left(\frac{m_1 + 1}{2} \right. \right. \\ & \left. \left. - \frac{n_1 - 1/2}{2(m_1 - 1)} + \frac{p_1 - p_2}{2(m_1 - 1)} \right) \right) = 0 \end{aligned} \quad (\text{B.184})$$

$$\frac{\partial L_1}{\partial \mu_4} = -(p_1 - n_1 - (p_2 - n_2)) = 0 \quad (\text{B.185})$$

Player 2:

$$\begin{aligned} \frac{\partial L_2}{\partial p_2} = & \frac{1}{32(m_1 - 1)m_2} (3 + 16\lambda_1(m_1 - 1) - 3m_1 - 29m_2 + 16\lambda_2 m_2 \\ & - 8n_2 - 8p_1 + 16p_2 - 4(-8\lambda_4(m_1 - 1)m_2 - 8m_1 m_2 + 2m_1^2 m_2 \\ & - 2m_2^2 + 2m_1 m_2^2 + 4\lambda_3(m_1 + m_2 - 1) + 2m_2 n_1 - 2m_1 n_2 \\ & + (1 - m_1 + m_2)n_2^2 - 2(m_1 - m_2)p_1 + 4(m_1 - m_2)p_2)) = 0 \end{aligned} \quad (\text{B.186})$$

$$\frac{\partial L_2}{\partial \lambda_1} = - \left(\frac{m_1 + 1}{2} - \frac{n_1 - 1/2}{2(m_1 - 1)} + \frac{p_1 - p_2}{2(m_1 - 1)} - 1 \right) \geq 0 \quad (\text{B.187})$$

$$\lambda_1 \geq 0 \quad (\text{B.188})$$

$$\lambda_1 \frac{\partial L_2}{\partial \lambda_1} = - \lambda_1 \left(\frac{m_1 + 1}{2} - \frac{n_1 - 1/2}{2(m_1 - 1)} + \frac{p_1 - p_2}{2(m_1 - 1)} - 1 \right) = 0 \quad (\text{B.189})$$

$$\frac{\partial L_2}{\partial \lambda_2} = \frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)} \geq 0 \quad (\text{B.190})$$

$$\lambda_2 \geq 0 \quad (\text{B.191})$$

$$\lambda_2 \frac{\partial L_2}{\partial \lambda_2} = \lambda_2 \left(\frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)} \right) = 0 \quad (\text{B.192})$$

$$\frac{\partial L_2}{\partial \lambda_3} = - \left(\frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)} - \left(\frac{m_1 + 1}{2} - \frac{n_1 - 1/2}{2(m_1 - 1)} + \frac{p_1 - p_2}{2(m_1 - 1)} \right) \right) \geq 0 \quad (\text{B.193})$$

$$\lambda_3 \geq 0 \quad (\text{B.194})$$

$$\lambda_3 \frac{\partial L_2}{\partial \lambda_3} = - \lambda_3 \left(\frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)} - \left(\frac{m_1 + 1}{2} - \frac{n_1 - 1/2}{2(m_1 - 1)} + \frac{p_1 - p_2}{2(m_1 - 1)} \right) \right) = 0 \quad (\text{B.195})$$

$$\frac{\partial L_2}{\partial \lambda_4} = -(p_1 - n_1 - (p_2 - n_2)) = 0 \quad (\text{B.196})$$

By applying market clear condition $q_1 + q_2 = 1$ and $0 \leq q_1 \leq 1$, $0 \leq q_2 \leq 1$, the solution of $p_1, p_2, \mu_1, \mu_2, \mu_3, \mu_4, \lambda_1, \lambda_2, \lambda_3$, and λ_4 are as follow.

$$p_1 = \frac{1}{8(m_1 - m_2 - 1)} \left(1 - m_1 + 17m_2 - 16m_1m_2 + 4n_1 - 4m_1n_1 + 4m_2n_1 + 2n_1^2 - 2m_1n_1^2 + 2m_2n_1^2 - 4n_2 + 4m_1n_2 - 4m_2n_2 + 2n_2^2 - 2m_1n_2^2 + 2m_2n_2^2 \right) \quad (\text{B.197})$$

$$p_2 = \frac{1}{8(m_1 - m_2 - 1)} \left(-1 + m_1 - 17m_2 + 16m_1m_2 + 4n_1 - 4m_1n_1 + 4m_2n_1 - 2n_1^2 + 2m_1n_1^2 - 2m_2n_1^2 - 4n_2 + 4m_1n_2 - 4m_2n_2 - 2n_2^2 + 2m_1n_2^2 - 2m_2n_2^2 \right) \quad (\text{B.198})$$

$$\mu_1 = 0$$

$$\mu_2 = 0$$

$$\mu_3 = 0$$

$$\begin{aligned}
\mu_4 &= \frac{1}{16(1-m_1)m_2} \left(2 - 2m_1 - 6m_2 + 8m_1m_2 - 4m_1^2m_2 + 4m_2^2 \right. \\
&\quad \left. - 4m_1m_2^2 - 6n_1 + 6m_1n_1 - 2m_2n_1 + n_1^2 - m_1n_1^2 + m_2n_1^2 + 2n_2 \right. \\
&\quad \left. - 2m_1n_2 + 6m_2n_2 - n_2^2 + m_1n_2^2 - m_2n_2^2 \right) \\
\lambda_1 &= 0 \\
\lambda_2 &= 0 \\
\lambda_3 &= 0 \\
\lambda_4 &= \frac{1}{16(1-m_1)m_2} \left(2 - 2m_1 - 6m_2 + 8m_1m_2 - 4m_1^2m_2 + 4m_2^2 \right. \\
&\quad \left. - 4m_1m_2^2 - 6n_1 + 6m_1n_1 - 2m_2n_1 + n_1^2 - m_1n_1^2 + m_2n_1^2 + 2n_2 \right. \\
&\quad \left. - 2m_1n_2 + 6m_2n_2 - n_2^2 + m_1n_2^2 - m_2n_2^2 \right)
\end{aligned}$$

Since this is the only result from this system of equations, we can interpret the result $\mu_1 = \mu_2 = \mu_3 = \lambda_1 = \lambda_2 = \lambda_3 = 0$ that the case when the market separating line, $x_{c_1} = 0$ or $x_{c_2} = 1$ will not occur because the constraints are not binding.