APPENDIX B9

OPTIMAL PRICE FINDING FOR THE CASE III

Objective function

Player 1:

$$\text{Max}_{p_1}\left(p_1-\frac{1/2^2+n_1^2}{2}\right)\left(\frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}-\left(\frac{m_1+m_2}{2}\right.\right.$$
$$\left.\left.-\frac{(n_1-n_2)}{2(m_1-m_2)}+\frac{p_1-p_2}{2(m_1-m_2)}\right)+\frac{m_2}{2}-\frac{\frac{1}{2}-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}\right)$$

s.t. $\quad x_{c_1}\le 1$ or $\frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}\le 1$

and $\quad x_{c_2}\ge 0$ or $\frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}\ge 0$

and $\quad x_{c_2}\le x_{c_3}$ or $\frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}\le\frac{m_1+m_2}{2}-\frac{(n_1-n_2)}{2(m_1-m_2)}+\frac{p_1-p_2}{2(m_1-m_2)}$

and $\quad x_{c_3}\le x_{c_1}$ or $\frac{m_1+m_2}{2}-\frac{(n_1-n_2)}{2(m_1-m_2)}+\frac{p_1-p_2}{2(m_1-m_2)}\le\frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}$

Player 2:

$$\text{Max}_{p_2}\left(p_1-\frac{1/2^2+n_1^2}{2}\right)\left(1-\left(\frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}-\left(\frac{m_1+m_2}{2}\right.\right.\right.$$
$$\left.\left.\left.-\frac{(n_1-n_2)}{2(m_1-m_2)}+\frac{p_1-p_2}{2(m_1-m_2)}\right)+\frac{m_2}{2}-\frac{\frac{1}{2}-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}\right)\right)$$

s.t. $\quad x_{c_1}\le 1$ or $\frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}\le 1$

and $\quad x_{c_2}\ge 0$ or $\frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}\ge 0$

and $\quad x_{c_2}\le x_{c_3}$ or $\frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}\le\frac{m_1+m_2}{2}-\frac{(n_1-n_2)}{2(m_1-m_2)}+\frac{p_1-p_2}{2(m_1-m_2)}$

and $x_{c_3} \le x_{c_1}$ or $\frac{m_1+m_2}{2} - \frac{(n_1-n_2)}{2(m_1-m_2)} + \frac{p_1-p_2}{2(m_1-m_2)} \le \frac{m_1+1}{2} - \frac{n_1-1/2}{2(m_1-1)} + \frac{p_1-p_2}{2(m_1-1)}$

Lagrange function:

$$
\begin{aligned}
L_1 = {} & \left(p_1 - \frac{1/2^2 + n_1^2}{2}\right)\left(\frac{m_1+1}{2} - \frac{n_1-1/2}{2(m_1-1)} + \frac{p_1-p_2}{2(m_1-1)} - \left(\frac{m_1+m_2}{2}\right.\right. \\
& \left.\left. - \frac{(n_1-n_2)}{2(m_1-m_2)} + \frac{p_1-p_2}{2(m_1-m_2)}\right) + \frac{m_2}{2} - \frac{1/2-n_2}{2(-m_2)} + \frac{p_1-p_2}{2(-m_2)}\right) \\
& - \mu_1\left(\frac{m_1+1}{2} - \frac{n_1-1/2}{2(m_1-1)} + \frac{p_1-p_2}{2(m_1-1)} - 1\right) + \mu_2\left(\frac{m_2}{2} - \frac{1/2-n_2}{2(-m_2)} + \frac{p_1-p_2}{2(-m_2)}\right) \\
& - \mu_3\left(\frac{m_1+m_2}{2} - \frac{(n_1-n_2)}{2(m_1-m_2)} + \frac{p_1-p_2}{2(m_1-m_2)} - \left(\frac{m_1+1}{2} - \frac{n_1-1/2}{2(m_1-1)} + \frac{p_1-p_2}{2(m_1-1)}\right)\right) \\
& - \mu_4\left(\frac{m_2}{2} - \frac{1/2-n_2}{2(-m_2)} + \frac{p_1-p_2}{2(-m_2)} - \left(\frac{m_1+m_2}{2}\right.\right. \\
& \left.\left. - \frac{(n_1-n_2)}{2(m_1-m_2)} + \frac{p_1-p_2}{2(m_1-m_2)}\right)\right)
\end{aligned}
\tag{B.143}
$$

$$
\begin{aligned}
L_2 = {} & \left(p_2 - \frac{1/2^2 + n_2^2}{2}\right)\left(1 - \left(\frac{m_1+1}{2} - \frac{n_1-1/2}{2(m_1-1)} + \frac{p_1-p_2}{2(m_1-1)} - \left(\frac{m_1+m_2}{2}\right.\right.\right. \\
& \left.\left.\left. - \frac{(n_1-n_2)}{2(m_1-m_2)} + \frac{p_1-p_2}{2(m_1-m_2)}\right) + \frac{m_2}{2} - \frac{1/2-n_2}{2(-m_2)} + \frac{p_1-p_2}{2(-m_2)}\right)\right) \\
& - \lambda_1\left(\frac{m_1+1}{2} - \frac{n_1-1/2}{2(m_1-1)} + \frac{p_1-p_2}{2(m_1-1)} - 1\right) + \lambda_2\left(\frac{m_2}{2} - \frac{1/2-n_2}{2(-m_2)} + \frac{p_1-p_2}{2(-m_2)}\right) \\
& - \lambda_3\left(\frac{m_1+m_2}{2} - \frac{(n_1-n_2)}{2(m_1-m_2)} + \frac{p_1-p_2}{2(m_1-m_2)} - \left(\frac{m_1+1}{2} - \frac{n_1-1/2}{2(m_1-1)} + \frac{p_1-p_2}{2(m_1-1)}\right)\right) \\
& - \lambda_4\left(\frac{m_2}{2} - \frac{1/2-n_2}{2(-m_2)} + \frac{p_1-p_2}{2(-m_2)} - \left(\frac{m_1+m_2}{2}\right.\right. \\
& \left.\left. - \frac{(n_1-n_2)}{2(m_1-m_2)} + \frac{p_1-p_2}{2(m_1-m_2)}\right)\right)
\end{aligned}
\tag{B.144}
$$

First order condition and complementary slackness are as follow.

Player 1:

$$\frac{\partial L_1}{\partial p_1} = \frac{1}{16(m_1-1)(m_1-m_2)m_2}(m_1(5-8\mu_2+8\mu_4+8m_2+4n_1^2-8n_2$$

$$-16p_1+8p_2)+m_1(-5-8\mu_4+8\mu_2(1+m_2)-4n_1^2+8n_2+16p_1$$

$$-8p_2-m_2(9+8\mu_1+8m_2+4n_1^2-16p_1+8p_2))+m_2(4-8\mu_2$$

$$-8\mu_3(m_2-1)-8n_1+m_2(5+8\mu_1+4n_1(2+n_1)$$

$$-16p_1+8p_2))) = 0 \qquad \text{(B.145)}$$

$$\frac{\partial L_1}{\partial \mu_1} = -\left(\frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}-1\right) \geq 0 \qquad \text{(B.146)}$$

$$\mu_1 = 0 \qquad \text{(B.147)}$$

$$\mu_1\frac{\partial L_1}{\partial \mu_1} = -\mu_1\left(\frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}-1\right) = 0 \qquad \text{(B.148)}$$

$$\frac{\partial L_1}{\partial \mu_2} = \frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)} \geq 0 \qquad \text{(B.149)}$$

$$\mu_2 = 0 \qquad \text{(B.150)}$$

$$\mu_2\frac{\partial L_1}{\partial \mu_2} = \mu_2\left(\frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}\right) = 0 \qquad \text{(B.151)}$$

$$\frac{\partial L_1}{\partial \mu_3} = \frac{m_1+m_2}{2}-\frac{(n_1-n_2)}{2(m_1-m_2)}+\frac{p_1-p_2}{2(m_1-m_2)}-\left(\frac{m_1+1}{2}\right.$$

$$\left.-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}\right) \geq 0 \qquad \text{(B.152)}$$

$$\mu_3 = 0 \qquad \text{(B.153)}$$

$$\mu_3\frac{\partial L_1}{\partial \mu_3} = -\mu_3\left(\frac{m_1+m_2}{2}-\frac{(n_1-n_2)}{2(m_1-m_2)}+\frac{p_1-p_2}{2(m_1-m_2)}-\left(\frac{m_1+1}{2}\right.\right.$$

$$\left.\left.-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}\right)\right) = 0 \qquad \text{(B.154)}$$

$$\frac{\partial L_1}{\partial \mu_4} = \frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}-\left(\frac{m_1+m_2}{2}\right.$$

$$-\frac{(n_1-n_2)}{2(m_1-m_2)}+\frac{p_1-p_2}{2(m_1-m_2)}\Bigg) \quad \geq \quad 0 \tag{B.155}$$

$$\mu_4 \quad = \quad 0 \tag{B.156}$$

$$\mu_4 \frac{\partial L_1}{\partial \mu_4} = \quad -\mu_4\left(\frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}-\left(\frac{m_1+m_2}{2}\right.\right.$$

$$\left.\left.-\frac{(n_1-n_2)}{2(m_1-m_2)}+\frac{p_1-p_2}{2(m_1-m_2)}\right)\right) \quad = \quad 0 \tag{B.157}$$

Player 2:

$$\frac{\partial L_2}{\partial p_2} \quad = \quad \frac{1}{16(m_1-1)(m_1-m_2)m_2}\Big(m_2(-4+8\lambda_2+8\lambda_3(m_2-1)+8n_1+m_2(13$$

$$-8\lambda_1-8n_1+4n_2^2+8p_1-16p_2))+m_1^2(-3+8\lambda_2-8\lambda_4+8m_2$$

$$+4n_2(2+n_2)+8p_1-16p_2)+m_1(3+8\lambda_4-8\lambda_2(1+m_2)$$

$$-4n_2(2+n_2)-8p_1-m_2(9-8\lambda_1+8m_2+4n_2^2+8p_1)$$

$$+16(1+m_2)p_2\Big) \quad = \quad 0 \tag{B.158}$$

$$\frac{\partial L_2}{\partial \lambda_1} \quad = \quad -\left(\frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}-1\right) \quad \geq \quad 0 \tag{B.159}$$

$$\lambda_1 \quad = \quad 0 \tag{B.160}$$

$$\lambda_1 \frac{\partial L_2}{\partial \lambda_1} = \quad -\lambda_1\left(\frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}-1\right) \quad = \quad 0 \tag{B.161}$$

$$\frac{\partial L_2}{\partial \lambda_2} \quad = \quad \frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)} \quad \geq \quad 0 \tag{B.162}$$

$$\lambda_2 \quad = \quad 0 \tag{B.163}$$

$$\lambda_2 \frac{\partial L_2}{\partial \lambda_2} = \quad \lambda_2\left(\frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}\right) \quad = \quad 0 \tag{B.164}$$

$$\frac{\partial L_2}{\partial \lambda_3} \quad = \quad \frac{m_1+m_2}{2}-\frac{(n_1-n_2)}{2(m_1-m_2)}+\frac{p_1-p_2}{2(m_1-m_2)}-\left(\frac{m_1+1}{2}\right.$$

$$\left.-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}\right) \quad \geq \quad 0 \tag{B.165}$$

$$\lambda_3 \quad = \quad 0 \tag{B.166}$$

$$\lambda_3 \frac{\partial L_2}{\partial \lambda_3} = \quad -\lambda_3\left(\frac{m_1+m_2}{2} - \frac{(n_1-n_2)}{2(m_1-m_2)} + \frac{p_1-p_2}{2(m_1-m_2)} - \left(\frac{m_1+1}{2}\right.\right.$$

$$\left.\left. - \frac{n_1 - 1/2}{2(m_1-1)} + \frac{p_1-p_2}{2(m_1-1)}\right)\right) \quad = \quad 0 \tag{B.167}$$

$$\frac{\partial L_2}{\partial \lambda_4} \quad = \quad \frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1-p_2}{2(-m_2)} - \left(\frac{m_1+m_2}{2}\right.$$

$$\left. - \frac{(n_1-n_2)}{2(m_1-m_2)} + \frac{p_1-p_2}{2(m_1-m_2)}\right) \quad \geq \quad 0 \tag{B.168}$$

$$\lambda_4 \quad = \quad 0 \tag{B.169}$$

$$\lambda_4 \frac{\partial L_2}{\partial \lambda_4} = \quad -\lambda_4\left(\frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1-p_2}{2(-m_2)} - \left(\frac{m_1+m_2}{2}\right.\right.$$

$$\left.\left. - \frac{(n_1-n_2)}{2(m_1-m_2)} + \frac{p_1-p_2}{2(m_1-m_2)}\right)\right) \quad = \quad 0 \tag{B.170}$$

By applying market clear condition $q_1 + q_2 = 1$ and $0 \le q_1 \le 1$, $0 \le q_2 \le 1$, the solution of $p_1, p_2, \mu_1, \mu_2, \mu_3, \mu_4, \lambda_1, \lambda_2, \lambda_3, \text{and} \lambda_4$ are as follow.

$$\begin{aligned} p_1 \quad = \quad & -\frac{1}{24(-m_1+m_1^2-m_1m_2+m_2^2)}\left(7m_1 - 7m_1^2 - 4m_2 + 27m_1m_2\right. \\ & - 24m_1^2m_2 - 23m_2^2 + 24m_1m_{21}^2 + 8m_2n_1 - 8m_2^2n_1 + 8m_1n_1^2 \\ & - 8m_1^2n_1^2 + 8m_1m_2n_1^2 - 8m_2^2n_1^2 - 8m_1n_2 + 8m_1^2n_2 \\ & \left. + 4m_1n_2^2 - 4m_1^2n_2^2 + 4m_1m_2n_2^2 - 4m_2^2n_2^2\right) \end{aligned} \tag{B.171}$$

$$\begin{aligned} p_2 \quad = \quad & -\frac{1}{24(-m_1+m_1^2-m_1m_2+m_2^2)}\left(-m_1 + m_1^2 + 4m_2 + 27m_1m_2\right. \\ & - 24m_1^2m_2 - 31m_2^2 + 24m_1m_2^2 - 8m_2n_1 + 8m_2^2n_1 + 4m_1n_1^2 \\ & - 4m_1^2n_1^2 + 4m_1m_2n_1^2 - 4m_2^2n_1^2 + 8m_1n_2 - 8m_1^2n_2 \end{aligned}$$

$$+8m_1n_2^2 - 8m_1^2n_2^2 + 8m_1m_2n_2^2 - 8m_2^2n_2^2\Big) \tag{B.172}$$

$$\mu_1 = 0$$

$$\mu_2 = 0$$

$$\mu_3 = 0$$

$$\mu_4 = 0$$

$$\lambda_1 = 0$$

$$\lambda_2 = 0$$

$$\lambda_3 = 0$$

$$\lambda_4 = 0$$