APPENDIX B8

OPTIMAL PRICE FINDING FOR THE CASE II.1

Objective functions

Player 1:

$$\text{Max}_{p_1}\left(p_1 - -\frac{1/2^2+n_1^2}{2}\right)\left(1-\left(\frac{m_1+m_2}{2}-\frac{(n_1-n_2)}{2(m_1-m_2)}+\frac{p_1-p_2}{2(m_1-m_2)}\right)+\frac{m_2}{2}-\frac{\frac{1}{2}-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}\right)$$

s.t. $\quad x_{c_1} \geq 1$ or $\dfrac{m_1+1}{2}-\dfrac{n_1-1/2}{2(m_1-1)}+\dfrac{p_1-p_2}{2(m_1-1)} \geq 1$

and $\quad x_{c_2} \geq 0$ or $\dfrac{m_2}{2}-\dfrac{1/2-n_2}{2(-m_2)}+\dfrac{p_1-p_2}{2(-m_2)} \geq 0$

and $\quad x_{c_3} \leq 1$ or $\dfrac{m_1+m_2}{2}-\dfrac{(n_1-n_2)}{2(m_1-m_2)}+\dfrac{p_1-p_2}{2(m_1-m_2)} \leq 1$

and $\quad x_{c_2} \leq x_{c_3}$ or $\dfrac{m_2}{2}-\dfrac{1/2-n_2}{2(-m_2)}+\dfrac{p_1-p_2}{2(-m_2)} \leq \dfrac{m_1+m_2}{2}-\dfrac{(n_1-n_2)}{2(m_1-m_2)}+\dfrac{p_1-p_2}{2(m_1-m_2)}$

Player 2:

$$\text{Max}_{p_2}\left(p_2 - \frac{1/2^2+n_2^2}{2}\right)\left(\frac{m_1+m_2}{2}-\frac{(n_1-n_2)}{2(m_1-m_2)}+\frac{p_1-p_2}{2(m_1-m_2)}-\left(\frac{m_2}{2}-\frac{\frac{1}{2}-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}\right)\right)$$

s.t. $\quad x_{c_1} \geq 1$ or $\dfrac{m_1+1}{2}-\dfrac{n_1-1/2}{2(m_1-1)}+\dfrac{p_1-p_2}{2(m_1-1)} \geq 1$

and $\quad x_{c_2} \geq 0$ or $\dfrac{m_2}{2}-\dfrac{1/2-n_2}{2(-m_2)}+\dfrac{p_1-p_2}{2(-m_2)} \geq 0$

and $\quad x_{c_3} \leq 1$ or $\dfrac{m_1+m_2}{2}-\dfrac{(n_1-n_2)}{2(m_1-m_2)}+\dfrac{p_1-p_2}{2(m_1-m_2)} \leq 1$

and $\quad x_{c_2} \leq x_{c_3}$ or $\dfrac{m_2}{2}-\dfrac{1/2-n_2}{2(-m_2)}+\dfrac{p_1-p_2}{2(-m_2)} \leq \dfrac{m_1+m_2}{2}-\dfrac{(n_1-n_2)}{2(m_1-m_2)}+\dfrac{p_1-p_2}{2(m_1-m_2)}$

Lagrange function:

$$L_1 = \left(p_1 - \frac{1/2^2 + n_1^2}{2}\right)\left(1 - \left(\frac{m_1+m_2}{2} - \frac{(n_1-n_2)}{2(m_1-m_2)} + \frac{p_1-p_2}{2(m_1-m_2)}\right) + \frac{m_2}{2} - \frac{\frac{1}{2}-n_2}{2(-m_2)} + \frac{p_1-p_2}{2(-m_2)}\right)$$

$$+ \mu_1\left(\frac{m_1+1}{2} - \frac{n_1-1/2}{2(m_1-1)} \quad \frac{p_2}{2(m_1-1)} - 1\right) + \mu_2\left(\frac{m_2}{2} - \frac{1/2-n_2}{2(-m_2)} + \frac{p_1-p_2}{2(-m_2)}\right)$$

$$- \mu_3\left(\frac{m_1+m_2}{2} - \frac{(n_1-n_2)}{2(m_1-m_2)} + \frac{p_1-p_2}{2(m_1-m_2)} - 1\right)$$

$$- \mu_4\left(\frac{m_2}{2} - \frac{1/2-n_2}{2(-m_2)} + \frac{p_1-p_2}{2(-m_2)} - \left(\frac{m_1+m_2}{2}\right.\right.$$

$$\left.\left. - \frac{(n_1-n_2)}{2(m_1-m_2)} + \frac{p_1-p_2}{2(m_1-m_2)}\right)\right) \qquad \text{(B.113)}$$

$$L_2 = \left(p_2 - \frac{1/2^2 + n_2^2}{2}\right)\left(\frac{m_1+m_2}{2} - \frac{(n_1-n_2)}{2(m_1-m_2)} + \frac{p_1-p_2}{2(m_1-m_2)} - \left(\frac{m_2}{2} - \frac{\frac{1}{2}-n_2}{2(-m_2)} + \frac{p_1-p_2}{2(-m_2)}\right)\right)$$

$$+ \lambda_1\left(\frac{m_1+1}{2} - \frac{n_1-1/2}{2(m_1-1)} + \frac{p_1-p_2}{2(m_1-1)} - 1\right) + \lambda_2\left(\frac{m_2}{2} - \frac{1/2-n_2}{2(-m_2)} + \frac{p_1-p_2}{2(-m_2)}\right)$$

$$- \lambda_3\left(\frac{m_1+m_2}{2} - \frac{(n_1-n_2)}{2(m_1-m_2)} + \frac{p_1-p_2}{2(m_1-m_2)} - 1\right)$$

$$- \lambda_4\left(\frac{m_2}{2} - \frac{1/2-n_2}{2(-m_2)} + \frac{p_1-p_2}{2(-m_2)} - \left(\frac{m_1+m_2}{2}\right.\right.$$

$$\left.\left. - \frac{(n_1-n_2)}{2(m_1-m_2)} + \frac{p_1-p_2}{2(m_1-m_2)}\right)\right) \qquad \text{(B.114)}$$

First order condition and complementary slackness are as follow.

Player 1:

$$\frac{\partial L_1}{\partial p_1} = \frac{1}{16(m_1-1)(m_1-m_2)m_2}\left(4m_2 - 8m_1^3 m_2 - 8m_2\left(\mu_2 - \mu_3 - (\mu_1 - 2)m_2 - n_1\right)\right.$$

$$+ m_1\left(-5 - 8\mu_4 + 8\mu_2(1+m_2) - 4m_2(5 - 2\mu_1 + 2\mu_3 + 6m_2)\right.$$

$$+ 8m_2 n_1 - 4n_1^2 + 8n_2 + 16p_1 - 8p_2 + m_1^2\left(5 - 8\mu_2 + 8\mu_4\right.$$

$$+8m_2(m_2+3)+4n_1^2-8n_2-16p_1+8p_2)))\quad = \quad 0 \qquad \text{(B.115)}$$

$$\frac{\partial L_1}{\partial \mu_1} = \frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(n}-1 \quad \geq \quad 0 \qquad \text{(B.116)}$$

$$\mu_1 \quad \geq \quad 0 \qquad \text{(B.117)}$$

$$\mu_1\frac{\partial L_1}{\partial \mu_1} = \mu_1\left(\frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}-1\right) \quad = \quad 0 \qquad \text{(B.118)}$$

$$\frac{\partial L_1}{\partial \mu_2} = \frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)} \quad \geq \quad 0 \qquad \text{(B.119)}$$

$$\mu_2 \quad = \quad 0 \qquad \text{(B.120)}$$

$$\mu_2\frac{\partial L_1}{\partial \mu_2} = \mu_2\left(\frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}\right) \quad = \quad 0 \qquad \text{(B.121)}$$

$$\frac{\partial L_1}{\partial \mu_3} = -\left(\frac{m_1+m_2}{2}-\frac{(n_1-n_2)}{2(m_1-m_2)}+\frac{p_1-p_2}{2(m_1-m_2)}-1\right) \quad \geq \quad 0 \qquad \text{(B.122)}$$

$$\mu_3 \quad = \quad 0 \qquad \text{(B.123)}$$

$$\mu_3\frac{\partial L_1}{\partial \mu_3} = -\mu_3\left(\frac{m_1+m_2}{2}-\frac{(n_1-n_2)}{2(m_1-m_2)}+\frac{p_1-p_2}{2(m_1-m_2)}-1\right) \quad = \quad 0 \qquad \text{(B.124)}$$

$$\frac{\partial L_1}{\partial \mu_4} = -\left(\frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}-\left(\frac{m_1+m_2}{2}\right.\right.$$

$$\left.\left.-\frac{(n_1-n_2)}{2(m_1-m_2)}+\frac{p_1-p_2}{2(m_1-m_2)}\right)\right) \quad \geq \quad 0 \qquad \text{(B.125)}$$

$$\mu_4 \quad = \quad 0 \qquad \text{(B.126)}$$

$$\mu_4\frac{\partial L_1}{\partial \mu_4} = -\mu_4\left(\frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}-\left(\frac{m_1+m_2}{2}\right.\right.$$

$$\left.\left.-\frac{(n_1-n_2)}{2(m_1-m_2)}+\frac{p_1-p_2}{2(m_1-m_2)}\right)\right) \quad = \quad 0 \qquad \text{(B.127)}$$

Player 2:

$$\frac{\partial L_2}{\partial p_2} = \frac{1}{16(m_1-1)(m_1-m_2)m_2}\left(-4m_2+8m_1^3m_2+8m_2(\lambda_2-\lambda_3+\lambda_1m_2+n_1)\right.$$

$$+m_1^2\left(-3+8\lambda_2-8\lambda_4-8m_2(1+m_2)+4n_2(n_2+2)+8p_1-16p_2\right)$$

$$+m_1(3+8\lambda_4-8\lambda_2(m_2+1)+4m_2(1-2\lambda_1+2\lambda_3+2m_2-2n_1)$$

$$-4n_2(n_2+2)-8p_1+16p_2)) \quad = \quad 0 \qquad \text{(B.128)}$$

$$\frac{\partial L_2}{\partial \lambda_1} = \frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1}{2(m_1-1)} \geq 0 \qquad \text{(B.129)}$$

$$\lambda_1 \geq 0 \qquad \text{(B.130)}$$

$$\lambda_1\frac{\partial L_2}{\partial \lambda_1} = \lambda_1\left(\frac{m_1+1}{2}-\frac{n_1-1/2}{2(m_1-1)}+\frac{p_1-p_2}{2(m_1-1)}-1\right) = 0 \qquad \text{(B.131)}$$

$$\frac{\partial L_2}{\partial \mu_2} = \frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)} \geq 0 \qquad \text{(B.132)}$$

$$\lambda_2 = 0 \qquad \text{(B.133)}$$

$$\lambda_2\frac{\partial L_2}{\partial \mu_2} = \lambda_2\left(\frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}\right) = 0 \qquad \text{(B.134)}$$

$$\frac{\partial L_2}{\partial \lambda_3} = -\left(\frac{m_1+m_2}{2}-\frac{(n_1-n_2)}{2(m_1-m_2)}+\frac{p_1-p_2}{2(m_1-m_2)}-1\right) \geq 0 \qquad \text{(B.135)}$$

$$\lambda_3 = 0 \qquad \text{(B.136)}$$

$$\lambda_3\frac{\partial L_2}{\partial \lambda_3} = -\lambda_3\left(\frac{m_1+m_2}{2}-\frac{(n_1-n_2)}{2(m_1-m_2)}+\frac{p_1-p_2}{2(m_1-m_2)}-1\right) = 0 \qquad \text{(B.137)}$$

$$\frac{\partial L_2}{\partial \lambda_4} = -\left(\frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}-\left(\frac{m_1+m_2}{2}\right.\right.$$

$$\left.\left.-\frac{(n_1-n_2)}{2(m_1-m_2)}+\frac{p_1-p_2}{2(m_1-m_2)}\right)\right) \geq 0 \qquad \text{(B.138)}$$

$$\lambda_4 = 0 \qquad \text{(B.139)}$$

$$\lambda_4\frac{\partial L_2}{\partial \lambda_4} = -\lambda_4\left(\frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}-\left(\frac{m_1+m_2}{2}\right.\right.$$

$$\left.\left.-\frac{(n_1-n_2)}{2(m_1-m_2)}+\frac{p_1-p_2}{2(m_1-m_2)}\right)\right) = 0 \qquad \text{(B.140)}$$

By applying market clear condition $q_1+q_2=1$ and $0\leq q_1\leq 1$, $0\leq q_2\leq 1$, the only solution of $p_1, p_2, \mu_1, \mu_2, \mu_3, \mu_4, \lambda_1, \lambda_2, \lambda_3, \text{and} \lambda_4$ are as follow.

$$p_1 = -\frac{5m_1 - 8m_1^2 + 4m_1^3 - 8m_1m_2 + 8m_2^2 - 4m_1n_1 - 2m_1n_1^2 - 2m_1n_2^2}{8m_1} \quad \text{(B.141)}$$

$$p_2 = \frac{7m_1 - 8m_1^2 + 4m_1^3 + 8m_1m_2 - 8m_2^2 - 4m_1n_1 + 2m_1n_1^2 + 2m_1n_2^2}{8m_1} \quad \text{(B.142)}$$

$$\mu_1 = 0$$

$$\mu_2 = 0$$

$$\mu_3 = 0$$

$$\mu_4 = 0$$

$$\lambda_1 = 0$$

$$\lambda_2 = 0$$

$$\lambda_3 = 0$$

$$\lambda_4 = 0$$