APPENDIX B5

SUBGAME PERFECT NASH EQUILIBRIUM FINDING FOR THE CASE I.3

Objective function

Player 1:

$$\text{Max}_{p_1} \left(p_1 - \frac{1/2^2 + n_1^2}{2} \right)\left(\frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)} \right)$$

$$\text{s.t.} \qquad q_1 \geq 0 \quad \text{or} \quad x_{c_2} \geq 0 \quad \text{or} \quad \frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)} \geq 0$$

$$\text{and} \qquad q_1 \leq 1 \quad \text{or} \quad x_{c_2} \leq 1 \quad \text{or} \quad \frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)} \leq 1$$

Player 2:

$$\text{Max}_{p_2} \left(p_2 - \frac{1/2^2 + n_2^2}{2} \right)\left(1 - \frac{m_2}{2} + \frac{1/2 - n_2}{2(-m_2)} - \frac{p_1 - p_2}{2(-m_2)} \right)$$

$$\text{s.t.} \qquad q_2 \leq 1 \quad \text{or} \quad 1 - x_{c_2} \leq 1 \quad \text{or} \quad \frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)} \geq 0$$

$$\text{and} \qquad q_2 \geq 0 \quad \text{or} \quad 1 - x_{c_2} \geq 0 \quad \text{or} \quad \frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)} \leq 1$$

Lagrange function:

$$\begin{aligned} L_1 \quad = \quad & \left(p_1 - \frac{1/2^2 + n_1^2}{2} \right)\left(\frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)} \right) \\ & + \mu_1 \left(\frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)} \right) \\ & - \mu_2 \left(\frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)} - 1 \right) \qquad \text{(B.61)} \\ L_2 \quad = \quad & \left(p_2 - \frac{1/2^2 + n_2^2}{2} \right)\left(1 - \frac{m_2}{2} + \frac{1/2 - n_2}{2(-m_2)} - \frac{p_1 - p_2}{2(-m_2)} \right) \\ & + \lambda_1 \left(\frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)} \right) \end{aligned}$$

$$-\lambda_2\left(\frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}-1\right) \quad \text{(B.62)}$$

First order condition and complementary slackness are as follow.

Player 1:

$$\frac{\partial L_1}{\partial p_1} = \frac{5-8\mu_1+8\mu_2+8m_2^2+4n_1^2-8n_2-16p_1+8p_2}{16m_2} = 0 \quad \text{(B.63)}$$

$$\frac{\partial L_1}{\partial \mu_1} = \frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)} \geq 0 \quad \text{(B.64)}$$

$$\mu_1 \geq 0 \quad \text{(B.65)}$$

$$\mu_1\frac{\partial L_1}{\partial \mu_1} = \mu_1\left(\frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}\right) = 0 \quad \text{(B.66)}$$

$$\frac{\partial L_1}{\partial \mu_2} = -\left(\frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}-1\right) \geq 0 \quad \text{(B.67)}$$

$$\mu_2 \geq 0 \quad \text{(B.68)}$$

$$\mu_2\frac{\partial L_1}{\partial \mu_2} = -\mu_2\left(\frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}-1\right) = 0 \quad \text{(B.69)}$$

Player 2:

$$\frac{\partial L_2}{\partial p_2} = \frac{-3+8\lambda_1-8\lambda_2-8m_2(m_2-2)+4n_2(n_2+2)+8p_1 16p_2}{16m_2} = 0 \quad \text{(B.70)}$$

$$\frac{\partial L_2}{\partial \lambda_1} = \frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)} \geq 0 \quad \text{(B.71)}$$

$$\lambda_1 \geq 0 \quad \text{(B.72)}$$

$$\lambda_1\frac{\partial L_2}{\partial \lambda_1} = \lambda_1\left(\frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}\right) = 0 \quad \text{(B.73)}$$

$$\frac{\partial L_2}{\partial \lambda_2} = -\left(\frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}-1\right) \geq 0 \quad \text{(B.74)}$$

$$\lambda_2 \geq 0 \quad \text{(B.75)}$$

$$\lambda_2\frac{\partial L_2}{\partial \lambda_2} = -\lambda_2\left(\frac{m_2}{2}-\frac{1/2-n_2}{2(-m_2)}+\frac{p_1-p_2}{2(-m_2)}-1\right) = 0 \quad \text{(B.76)}$$

By applying market clear condition $q_1 + q_2 = 1$ and $0 \le q_1 \le 1$, $0 \le q_2 \le 1$, the only solution of $p_1, p_2, \mu_1, \mu_2, \lambda_1, \text{and} \lambda_2$ are as follow.

$$p_1 = \frac{1}{24}\left(7 + 16m_2 + 8m_2^2 + 8n_1^2 - 8n_2 + 4n_2^2\right) \quad \text{(B.77)}$$

$$p_2 = \frac{1}{24}\left(-1 + 32m_2 - 8m_2^2 + 4n_1^2 + 8n_2 + 8n_2^2\right) \quad \text{(B.78)}$$

$$\mu_1 = 0$$

$$\mu_2 = 0$$

$$\lambda_1 = 0$$

$$\lambda_2 = 0$$

In order to check that these prices are the optimal prices that make each player has highest profit, second order condition is employed.

$$\frac{\partial^2 \pi_1}{\partial p_1^2} = -\frac{1}{m_2} < 0 \quad , \quad \frac{\partial^2 \pi_2}{\partial p_2^2} = -\frac{1}{m_2} < 0$$

Since the value of $\partial^2 \pi_1 / \partial p_1^2$ and $\partial^2 \pi_2 / \partial p_2^2$ are negative, we can ensure that these prices are the optimal prices for each player. By substitute the optimal price, equation (B.77) and (B.78), into market share function we get:

$$q_1 = \frac{4m_2 + 2m_2^2 - n_1^2 + (n_2 - 1)^2}{12m_2} \quad \text{(B.79)}$$

$$q_2 = \frac{8m_2 - 2m_2^2 + n_1^2 - (n_2 - 1)^2}{12m_2} \quad \text{(B.80)}$$

Since all Lagrange multiplier are zero, we can see that the constraints are not binding. We can conclude that the market demands of both players are positive. Thus, we can conclude the following results.

$$4m_2 + 2m_2^2 - n_1^2 + (n_2 - 1)^2 > 0 \quad \text{(B.81)}$$

$$8m_2 - 2m_2^2 + n_1^2 - (n_2 - 1)^2 > 0 \quad \text{(B.82)}$$

By substitute optimal price of each player, equation (B.77) and (B.78), back into the profit function of each firm and applying first order condition again, we obtain:

$$\pi_1^* = \left(p_1^* - \frac{1/2^2 + n_1^2}{2}\right)\left(\frac{m_2}{2} - \frac{1/2 - n_2}{2(-m_2)} + \frac{p_1^* - p_2^*}{2(-m_2)}\right)$$

$$= \frac{\left(4m_2 + 2m_2^2 - n_1^2 + (n_2 - 1)^2\right)^2}{72m_2} \quad \text{(B.83)}$$

$$\frac{\partial \pi_1^*}{\partial n_1} = -\frac{n_1\left(4m_2 + 2m_2^2 - n_1^2 + (n_2 - 1)^2\right)}{18m_2} \quad \text{(B.84)}$$

$$\pi_2^* = \left(p_2^* - \frac{(1/2)^2 + n_2^2}{2}\right)\left(1 - \frac{m_2}{2} + \frac{1/2 - n_2}{2(-m_2)} - \frac{p_1^* - p_2^*}{2(-m_2)}\right)$$

$$= \frac{\left(8m_2 - 2m_2^2 + n_1^2 - (n_2 - 1)^2\right)^2}{72m_2} \quad \text{(B.85)}$$

$$\frac{\partial \pi_2^*}{\partial n_2} = -\frac{(n_2 - 1)\left(8m_2 - 2m_2^2 + n_1^2 - (n_2 - 1)^2\right)}{18m_2} \quad \text{(B.86)}$$

Considering equation (B.84) and (B.86), since we know from inequality (B.81) and (B.82) that the value $4m_2 + 2m_2^2 - n_1^2 + (n_2 - 1)^2$ and $8m_2 - 2m_2^2 + n_1^2 - (n_2 - 1)^2$ must be greater than 0, the optimal product quality level of player 1 is 0 and player 2 is 1. Again, we will apply second order condition to check whether these are the optimal product quality level for both players. The second order conditions are as follow.

$$\frac{\partial^2 \pi_1^*}{\partial n_1^2} = -\frac{2m_2(2+m_2) - 3n_1^2 + (n_2-1)^2}{18m_2} < 0$$

$$\frac{\partial^2 \pi_2^*}{\partial n_2^2} = \frac{2m_2(-4+m_2) - n_1^2 + 3(n_2-1)^2}{18m_2} < 0$$

We can see that when $n_1 = 0$ the value of $\partial^2 \pi_1^* / \partial n_1^2$ will always be negative and also when $n_2 = 1$ the value of $\partial^2 \pi_2^* / \partial n_2^2$ is always negative. Thus the optimal product quality level for each player is $n_1 = 0$ and $n_2 = 1$. Substitute the optimal quality back into profit function, equation (B.83) and (B.85), and do the partial differentiation with respect to m_i, we obtain:

$$\pi_1^* = \frac{m_2(2+m_2)^2}{18}$$

$$\frac{\partial \pi_1^*}{\partial m_1} = 0$$

$$\pi_2^* = \frac{m_2(-4+m_2)^2}{18}$$

$$\frac{\partial \pi_2^*}{\partial m_2} = \frac{(-4+3m_2)(-4+m_2)}{18}$$

Since $\partial \pi_2 / \partial m_2$ is always positive regardless the value of m_1, the optimal value of m_2 is 1. Since $\partial \pi_1 / \partial m_1$ equals to 0, we can conclude that regardless the value of m_1, the profit of player 1 is still the same. Therefore, by substitute these optimal values of m_i back into profit function, the optimal profit for each player is $1/2$.

Therefore, under the assumption $m_1 < m_2$ and the market separating line is x_{c_2}, the optimal locations are $m_1^* \in [0,1)$ and $m_2^* = 1$, the optimal product quality levels are $n_1^* = 0, n_2^* = 1$, the optimal pricing are $p_1^* = 9/8$, $p_2^* = 13/8$, optimal market share are $q_1^* = q_2^* = 1/2$, and optimal profit are $\pi_1^* = \pi_2^* = 1/2$.

Since this is a symmetric game, we can also conclude that under the assumption $m_1 < m_2$ and the market separating line is x_{c_1}, the optimal locations are $m_1^* = 0$ and $m_2^* \in (0,1]$, the optimal product quality levels are $n_1^* = 1, n_2^* = 0$, the optimal pricing are $p_1^* = 13/8$, $p_2^* = 9/8$, optimal market share are $q_1^* = q_2^* = 1/2$, and optimal profit are $\pi_1^* = \pi_2^* = 1/2$.