

APPENDIX B2

SUBGAME PERFECT NASH EQUILIBRIUM FINDING FOR THE CASE I.1

Objective functions

Player 1:

$$\begin{aligned} \text{Max}_{p_1} & \left(p_1 - \frac{1/2^2 + n_1^2}{2} \right) \left(\frac{m_1 + m_2}{2} - \frac{(n_1 - n_2)}{2(m_1 - m_2)} + \frac{p_1 - p_2}{2(m_1 - m_2)} \right) \\ \text{s.t.} & \quad q_1 \geq 0 \quad \text{or} \quad x_{c_3} \geq 0 \quad \text{or} \quad \frac{m_1 + m_2}{2} - \frac{(n_1 - n_2)}{2(m_1 - m_2)} + \frac{p_1 - p_2}{2(m_1 - m_2)} \geq 0 \\ \text{and} & \quad q_1 \leq 1 \quad \text{or} \quad x_{c_3} \leq 1 \quad \text{or} \quad \frac{m_1 + m_2}{2} - \frac{(n_1 - n_2)}{2(m_1 - m_2)} + \frac{p_1 - p_2}{2(m_1 - m_2)} \leq 1 \end{aligned}$$

Player 2:

$$\begin{aligned} \text{Max}_{p_2} & \left(p_2 - \frac{1/2^2 + n_2^2}{2} \right) \left(1 - \frac{m_1 + m_2}{2} + \frac{(n_1 - n_2)}{2(m_1 - m_2)} - \frac{p_1 - p_2}{2(m_1 - m_2)} \right) \\ \text{s.t.} & \quad q_2 \leq 1 \quad \text{or} \quad 1 - x_{c_3} \leq 1 \quad \text{or} \quad \frac{m_1 + m_2}{2} - \frac{(n_1 - n_2)}{2(m_1 - m_2)} + \frac{p_1 - p_2}{2(m_1 - m_2)} \geq 0 \\ \text{and} & \quad q_2 \geq 0 \quad \text{or} \quad 1 - x_{c_3} \geq 0 \quad \text{or} \quad \frac{m_1 + m_2}{2} - \frac{(n_1 - n_2)}{2(m_1 - m_2)} + \frac{p_1 - p_2}{2(m_1 - m_2)} \leq 1 \end{aligned}$$

The Lagrange functions:

Player 1:

$$\begin{aligned} L_1 & = \left(p_1 - \frac{1/2^2 + n_1^2}{2} \right) \left(\frac{m_1 + m_2}{2} - \frac{(n_1 - n_2)}{2(m_1 - m_2)} + \frac{p_1 - p_2}{2(m_1 - m_2)} \right) \\ & \quad + \mu_1 \left(\frac{m_1 + m_2}{2} - \frac{(n_1 - n_2)}{2(m_1 - m_2)} + \frac{p_1 - p_2}{2(m_1 - m_2)} \right) - \mu_2 \left(\frac{m_1 + m_2}{2} - \frac{(n_1 - n_2)}{2(m_1 - m_2)} + \frac{p_1 - p_2}{2(m_1 - m_2)} - 1 \right) \end{aligned}$$

Player 2:

$$\begin{aligned} L_2 & = \left(p_2 - \frac{1/2^2 + n_2^2}{2} \right) \left(1 - \frac{m_1 + m_2}{2} + \frac{(n_1 - n_2)}{2(m_1 - m_2)} - \frac{p_1 - p_2}{2(m_1 - m_2)} \right) \\ & \quad + \lambda_1 \left(\frac{m_1 + m_2}{2} - \frac{(n_1 - n_2)}{2(m_1 - m_2)} + \frac{p_1 - p_2}{2(m_1 - m_2)} \right) - \lambda_2 \left(\frac{m_1 + m_2}{2} - \frac{(n_1 - n_2)}{2(m_1 - m_2)} + \frac{p_1 - p_2}{2(m_1 - m_2)} - 1 \right) \end{aligned}$$

First order condition and complementary slackness are as follow.

Player 1:

$$\begin{aligned} \frac{\partial L_1}{\partial p_1} &= \frac{1}{2(m_1 - m_2)} \left(p_1 - (1/4 - n_1^2)/2 + (m_1^2 - m_2^2) - (n_1 - n_2) \right. \\ &\quad \left. + (p_1 - p_2) + \mu_1 - \mu_2 \right) = 0 \end{aligned} \quad (\text{B.33})$$

$$\frac{\partial L_1}{\partial \mu_1} = \frac{m_1 + m_2}{2} - \frac{(n_1 - n_2)}{2(m_1 - m_2)} + \frac{p_1 - p_2}{2(m_1 - m_2)} \geq 0 \quad (\text{B.34})$$

$$\mu_1 \geq 0 \quad (\text{B.35})$$

$$\mu_1 \frac{\partial L_1}{\partial \mu_1} = \mu_1 \left(\frac{m_1 + m_2}{2} - \frac{(n_1 - n_2)}{2(m_1 - m_2)} + \frac{p_1 - p_2}{2(m_1 - m_2)} \right) = 0 \quad (\text{B.36})$$

$$\frac{\partial L_1}{\partial \mu_2} = - \left(\frac{m_1 + m_2}{2} - \frac{(n_1 - n_2)}{2(m_1 - m_2)} + \frac{p_1 - p_2}{2(m_1 - m_2)} - 1 \right) \geq 0 \quad (\text{B.37})$$

$$\mu_2 \geq 0 \quad (\text{B.38})$$

$$\mu_2 \frac{\partial L_1}{\partial \mu_2} = - \mu_2 \left(\frac{m_1 + m_2}{2} - \frac{(n_1 - n_2)}{2(m_1 - m_2)} + \frac{p_1 - p_2}{2(m_1 - m_2)} - 1 \right) = 0 \quad (\text{B.39})$$

Player 2:

$$\begin{aligned} \frac{\partial L_2}{\partial p_2} &= \frac{1}{2(m_1 - m_2)} \left(p_2 - (1/4 - n_2^2)/2 - (m_1^2 - m_2^2) + (n_1 - n_2) \right. \\ &\quad \left. - (p_1 - p_2) - \lambda_1 - \lambda_2 \right) = 0 \end{aligned} \quad (\text{B.40})$$

$$\frac{\partial L_2}{\partial \lambda_1} = \frac{m_1 + m_2}{2} - \frac{(n_1 - n_2)}{2(m_1 - m_2)} + \frac{p_1 - p_2}{2(m_1 - m_2)} \geq 0 \quad (\text{B.41})$$

$$\lambda_1 \geq 0 \quad (\text{B.42})$$

$$\lambda_1 \frac{\partial L_2}{\partial \lambda_1} = \lambda_1 \left(\frac{m_1 + m_2}{2} - \frac{(n_1 - n_2)}{2(m_1 - m_2)} + \frac{p_1 - p_2}{2(m_1 - m_2)} \right) = 0 \quad (\text{B.43})$$

$$\frac{\partial L_2}{\partial \lambda_2} = - \left(\frac{m_1 + m_2}{2} - \frac{(n_1 - n_2)}{2(m_1 - m_2)} + \frac{p_1 - p_2}{2(m_1 - m_2)} - 1 \right) \geq 0 \quad (\text{B.44})$$

$$\lambda_2 \geq 0 \quad (\text{B.45})$$

$$\lambda_2 \frac{\partial L_2}{\partial \lambda_2} = - \lambda_2 \left(\frac{m_1 + m_2}{2} - \frac{(n_1 - n_2)}{2(m_1 - m_2)} + \frac{p_1 - p_2}{2(m_1 - m_2)} - 1 \right) = 0 \quad (\text{B.46})$$

By applying market clear condition $q_1 + q_2 = 1$ and $0 \leq q_1 \leq 1$, $0 \leq q_2 \leq 1$, the only solution of $p_1, p_2, \mu_1, \mu_2, \lambda_1$, and λ_2 are as follow.

$$p_1 = \frac{1}{24} (3 - 16m_1 - 8m_1^2 + 16m_2 + 8m_2^2 + 8n_1 + 8n_1^2 - 8n_2 + 4n_2^2) \quad (\text{B.47})$$

$$p_2 = \frac{1}{24} (3 - 32m_1 + 8m_1^2 + 32m_2 - 8m_2^2 - 8n_1 + 4n_1^2 + 8n_2 + 8n_2^2) \quad (\text{B.48})$$

$$\mu_1 = 0$$

$$\mu_2 = 0$$

In order to check that these prices are the optimal prices that make each player has highest profit, second order condition, evaluate at p_1, p_2 in (B.47) and (B.48), is employed.

$$\frac{\partial^2 \pi_1}{\partial p_1^2} = \frac{1}{m_1 - m_2} < 0 \quad , \quad \frac{\partial^2 \pi_2}{\partial p_2^2} = \frac{1}{m_1 - m_2} < 0$$

Since the value of $\partial^2 \pi_1 / \partial p_1^2$ and $\partial^2 \pi_2 / \partial p_2^2$ are negative, we can ensure that these prices are the optimal prices for each player. From these optimal pricing for each firm, we would be able to find the market share function under the assumption that $m_1 < m_2$ by substitute the optimal price, equation (B.47) and (B.48) into the market share function. The result is as follow.

$$q_1 = \frac{2(m_1 + 1)^2 - 2(m_2 + 1)^2 + (n_1 - 1)^2 - (n_2 - 1)^2}{12(m_1 - m_2)} \quad (\text{B.49})$$

$$q_2 = -\frac{2(m_1 - 2)^2 - 2(m_2 - 2)^2 + (n_1 - 1)^2 - (n_2 - 1)^2}{12(m_1 - m_2)} \quad (\text{B.50})$$

Since all Lagrange multiplier are zero, we can see that the constraints are not binding. We can conclude that the market demands of both players are positive. Thus, we can conclude the following results

$$2(m_1 + 1)^2 - 2(m_2 + 1)^2 + (n_1 - 1)^2 - (n_2 - 1)^2 < 0 \quad (\text{B.51})$$

$$2(m_1 - 2)^2 - 2(m_2 - 2)^2 + (n_1 - 1)^2 - (n_2 - 1)^2 > 0 \quad (\text{B.52})$$

By substitute optimal prices of both players, equation (B.47) and (B.48), back into the profit function of each firm and applying first order condition, we obtain:

$$\begin{aligned} \pi_1^* &= \left(p_1^* - \frac{(1/2)^2 + n_1^2}{2} \right) \left(\frac{m_1 + m_2}{2} - \frac{(n_1 - n_2)}{2(m_1 - m_2)} + \frac{p_1^* - p_2^*}{2(m_1 - m_2)} \right) \\ &= - \frac{\left(2(m_1 + 1)^2 - 2(m_2 + 1)^2 + (n_1 - 1)^2 - (n_2 - 1)^2 \right)^2}{72(m_1 - m_2)} \end{aligned} \quad (\text{B.53})$$

$$\frac{\partial \pi_1^*}{\partial n_1} = - \frac{(-1 + n_1) \left(2(m_1 + 1)^2 - 2(m_2 + 1)^2 + (n_1 - 1)^2 - (n_2 - 1)^2 \right)}{18(m_1 - m_2)} \quad (\text{B.54})$$

$$\begin{aligned} \pi_2^* &= \left(p_2^* - \frac{(1/2)^2 + n_2^2}{2} \right) \left(1 - \frac{m_1 + m_2}{2} + \frac{(n_1 - n_2)}{2(m_1 - m_2)} - \frac{p_1^* - p_2^*}{2(m_1 - m_2)} \right) \\ &= - \frac{\left(2(m_1 - 2)^2 - 2(m_2 - 2)^2 + (n_1 - 1)^2 - (n_2 - 1)^2 \right)^2}{72(m_1 - m_2)} \end{aligned} \quad (\text{B.55})$$

$$\frac{\partial \pi_2^*}{\partial n_2} = \frac{(-1 + n_2) \left(2(m_1 - 2)^2 - 2(m_2 - 2)^2 + (n_1 - 1)^2 - (n_2 - 1)^2 \right)}{72(m_1 - m_2)} \quad (\text{B.56})$$

Considering equation (B.54) and (B.56), since we know from inequality (B.51) and (B.52) that the value of $2(m_1 + 1)^2 - 2(m_2 + 1)^2 + (n_1 - 1)^2 - (n_2 - 1)^2$ must be less than 0 and the value of $2(m_1 - 2)^2 - 2(m_2 - 2)^2 + (n_1 - 1)^2 - (n_2 - 1)^2$ must be greater than 0, the optimal product quality level for both players are 1. Again, we will apply second order condition to check whether these product quality levels are the optimal product quality level for both players. At the point where $n_1 = n_2 = 1$, we have:

$$\frac{\partial^2 \pi_1^*}{\partial n_1^2} = \frac{-2m_1(2+m_1) + 2m_2(2+m_2)}{9(m_1 - m_2)} = -\frac{2}{9}(m_1 + m_2 + 2) < 0$$

$$\frac{\partial^2 \pi_2^*}{\partial n_2^2} = \frac{2m_1(-4+m_1) - 2m_2(-4+m_2)}{9(m_1 - m_2)} = \frac{2}{9}(m_1 + m_2 - 8) < 0$$

Since the value of $\partial^2 \pi_1^*/\partial n_1^2$ and $\partial^2 \pi_2^*/\partial n_2^2$ are negative. Thus the optimal product quality level for both players are $n_1 = n_2 = 1$. Substitute the optimal quality of both players back into profit function, equation (B.53) and (B.55), and do the partial differentiation with respect to m_i , we obtain:

$$\pi_1^* = -\frac{1}{18}(m_1 - m_2)(2 + m_1 + m_2)^2 \quad (\text{B.57})$$

$$\frac{\partial \pi_1^*}{\partial m_1} = -\frac{1}{18}(2 + 3m_1 - m_2)(2 + m_1 + m_2) \quad (\text{B.58})$$

$$\pi_2^* = -\frac{1}{18}(m_1 - m_2)(-4 + m_1 + m_2)^2 \quad (\text{B.59})$$

$$\frac{\partial \pi_2^*}{\partial m_2} = -\frac{1}{18}(4 + m_1 - 3m_2)(-4 + m_1 + m_2) \quad (\text{B.60})$$

Since $\partial \pi_1^*/\partial m_1$ is always negative and $\partial \pi_2^*/\partial m_2$ is always positive, we can conclude the maximal differentiation or the optimal value of m_1 and m_2 are 0 and 1, respectively. Therefore, by substitute these optimal values of m_i back into profit function, the optimal profit for each player is $1/2$.

Therefore, under the assumption $m_1 < m_2$, the optimal locations are $m_1^* = 0$ and $m_2^* = 1$, the optimal product quality levels are $n_1^* = n_2^* = 1$, the optimal pricing are $p_1^* = p_2^* = 13/8$, optimal market share are $q_1^* = q_2^* = 1/2$, and optimal profit are $\pi_1^* = \pi_2^* = 1/2$.