APPENDIX B1

SUBGAME PERFECT NASH EQUILIBRIUM FINDING IN THE FIRST COMEPTITION

Objective functions

Player 1:

$$\text{Max}_{p_1}\ (p_1 - y_1^2)(q_1) \ \text{ or } \ \text{Max}_{p_1}\ (p_1 - y_1^2)\left(\frac{x_1 + x_2}{2} - \frac{(y_1 - y_2)}{2(x_1 - x_2)} + \frac{p_1 - p_2}{2(x_1 - x_2)}\right)$$

s.t. $\quad q_1 \geq 0$ or $x_c \geq 0$ or $\dfrac{x_1 + x_2}{2} - \dfrac{(y_1 - y_2)}{2(x_1 - x_2)} + \dfrac{p_1 - p_2}{2(x_1 - x_2)} \geq 0$

and $\quad q_1 \leq 1$ or $x_c \leq 1$ or $\dfrac{x_1 + x_2}{2} - \dfrac{(y_1 - y_2)}{2(x_1 - x_2)} + \dfrac{p_1 - p_2}{2(x_1 - x_2)} \leq 1$

Player 2:

$$\text{Max}_{p_2}\ (p_2 - y_2^2)(q_2) \ \text{ or } \ \text{Max}_{p_2}\ (p_2 - y_2^2)\left(1 - \frac{x_1 + x_2}{2} + \frac{(y_1 - y_2)}{2(x_1 - x_2)} - \frac{p_1 - p_2}{2(x_1 - x_2)}\right)$$

s.t. $\quad q_2 \leq 1$ or $1 - x_c \leq 1$ or $\dfrac{x_1 + x_2}{2} - \dfrac{(y_1 - y_2)}{2(x_1 - x_2)} + \dfrac{p_1 - p_2}{2(x_1 - x_2)} \geq 0$

and $\quad q_2 \geq 0$ or $1 - x_c \geq 0$ or $\dfrac{x_1 + x_2}{2} - \dfrac{(y_1 - y_2)}{2(x_1 - x_2)} + \dfrac{p_1 - p_2}{2(x_1 - x_2)} \leq 1$

The Lagrange functions:

Player 1:

$$L_1 = (p_1 - y_1^2)\left(\frac{x_1 + x_2}{2} - \frac{(y_1 - y_2)}{2(x_1 - x_2)} + \frac{p_1 - p_2}{2(x_1 - x_2)}\right) + \mu_1\left(\frac{x_1 + x_2}{2} - \frac{(y_1 - y_2)}{2(x_1 - x_2)} + \frac{p_1 - p_2}{2(x_1 - x_2)}\right)$$
$$- \mu_2\left(\frac{x_1 + x_2}{2} - \frac{(y_1 - y_2)}{2(x_1 - x_2)} + \frac{p_1 - p_2}{2(x_1 - x_2)} - 1\right)$$

Player 2:

$$L_2 = (p_2 - y_2^2)\left(1 - \frac{x_1 + x_2}{2} + \frac{(y_1 - y_2)}{2(x_1 - x_2)} - \frac{p_1 - p_2}{2(x_1 - x_2)}\right) + \lambda_1\left(\frac{x_1 + x_2}{2} - \frac{(y_1 - y_2)}{2(x_1 - x_2)} + \frac{p_1 - p_2}{2(x_1 - x_2)}\right)$$
$$- \lambda_2\left(\frac{x_1 + x_2}{2} - \frac{(y_1 - y_2)}{2(x_1 - x_2)} + \frac{p_1 - p_2}{2(x_1 - x_2)} - 1\right)$$

First order condition and complementary slackness are as follow.

Player 1:

$$\frac{\partial L_1}{\partial p_1} = \frac{p_1 - y_1^2}{2(x_1 - x_2)} + \frac{x_1 + x_2}{2} - \frac{y_1 - y_2}{2(x_1 - x_2)} + \frac{p_1 - p_2}{2(x_1 - x_2)}$$

$$+\frac{\mu_1}{2(x_1 - x_2)} - \frac{\mu_2}{2(x_1 - x_2)} = 0 \qquad \text{(B.1)}$$

$$\frac{\partial L_1}{\partial \mu_1} = \frac{x_1 + x_2}{2} - \frac{(y_1 - y_2)}{2(x_1 - x_2)} + \frac{p_1 - p_2}{2(x_1 - x_2)} \geq 0 \qquad \text{(B.2)}$$

$$\mu_1 \geq 0 \qquad \text{(B.3)}$$

$$\mu_1 \frac{\partial L_1}{\partial \mu_1} = \mu_1\left(\frac{x_1 + x_2}{2} - \frac{(y_1 - y_2)}{2(x_1 - x_2)} + \frac{p_1 - p_2}{2(x_1 - x_2)}\right) = 0 \qquad \text{(B.4)}$$

$$\frac{\partial L_1}{\partial \mu_2} = -\left(\frac{x_1 + x_2}{2} - \frac{(y_1 - y_2)}{2(x_1 - x_2)} + \frac{p_1 - p_2}{2(x_1 - x_2)} - 1\right) \geq 0 \qquad \text{(B.5)}$$

$$\mu_2 \geq 0 \qquad \text{(B.6)}$$

$$\mu_2 \frac{\partial L_1}{\partial \mu_2} = -\mu_2\left(\frac{x_1 + x_2}{2} - \frac{(y_1 - y_2)}{2(x_1 - x_2)} + \frac{p_1 - p_2}{2(x_1 - x_2)} - 1\right) = 0 \qquad \text{(B.7)}$$

Player 2:

$$\frac{\partial L_2}{\partial p_2} = \frac{p_2 - y_2^2}{2(x_1 - x_2)} + 1 - \frac{x_1 + x_2}{2} + \frac{y_1 - y_2}{2(x_1 - x_2)} - \frac{p_1 - p_2}{2(x_1 - x_2)}$$

$$-\frac{\lambda_1}{2(x_1 - x_2)} + \frac{\lambda_2}{2(x_1 - x_2)} = 0 \qquad \text{(B.8)}$$

$$\frac{\partial L_2}{\partial \lambda_1} = \frac{x_1 + x_2}{2} - \frac{(y_1 - y_2)}{2(x_1 - x_2)} + \frac{p_1 - p_2}{2(x_1 - x_2)} \geq 0 \qquad \text{(B.9)}$$

$$\lambda_1 \geq 0 \qquad \text{(B.10)}$$

$$\lambda_1 \frac{\partial L_2}{\partial \lambda_1} = \lambda_1\left(\frac{x_1 + x_2}{2} - \frac{(y_1 - y_2)}{2(x_1 - x_2)} + \frac{p_1 - p_2}{2(x_1 - x_2)}\right) = 0 \qquad \text{(B.11)}$$

$$\frac{\partial L_2}{\partial \lambda_2} = -\left(\frac{x_1+x_2}{2} - \frac{(y_1-y_2)}{2(x_1-x_2)} + \frac{p_1-p_2}{2(x_1-x_2)} - 1\right) \geq 0 \quad \text{(B.12)}$$

$$\lambda_2 \geq 0 \quad \text{(B.13)}$$

$$\lambda_2 \frac{\partial L_2}{\partial \lambda_2} = -\lambda_2\left(\frac{x_1+x_2}{2} - \frac{(y_1-y_2)}{2(x_1-x_2)} + \frac{p_1-p_2}{2(x_1-x_2)} - 1\right) = 0 \quad \text{(B.14)}$$

By applying market clear condition $q_1 + q_2 = 1$ and $0 \leq q_1 \leq 1$, $0 \leq q_2 \leq 1$, the only solution of $p_1, p_2, \mu_1, \mu_2, \lambda_1, \text{and} \lambda_2$ are as follow.

$$p_1 = \frac{1}{3}\left(-(x_1^2 - x_2^2) - 2(x_1 - x_2) + (y_1 - y_2) + 2y_1^2 + y_2^2\right) \quad \text{(B.15)}$$

$$p_2 = \frac{1}{3}\left((x_1^2 - x_2^2) - 4(x_1 - x_2) - (y_1 - y_2) + y_1^2 + 2y_2^2\right) \quad \text{(B.16)}$$

$$\mu_1 = 0$$

$$\mu_2 = 0$$

$$\lambda_1 = 0$$

$$\lambda_2 = 0$$

In order to check that these prices are the optimal prices that make each player has highest profit, the second order condition, evaluate at $p_{1,} p_2$ in (B.15) and (B.16), is employed.

$$\frac{\partial^2 \pi_1}{\partial p_1^2} = \frac{1}{x_1 - x_2} < 0 \quad , \quad \frac{\partial^2 \pi_2}{\partial p_2^2} = \frac{1}{x_1 - x_2} < 0$$

Since the value of $\partial^2 \pi_1 / \partial p_1^2$ and $\partial^2 \pi_2 / \partial p_2^2$ are negative, we can ensure that these prices are the optimal prices for each player. From these optimal pricing for each firm, we would be able to find the market share function under the assumption that $x_1 < x_2$ by substitute the optimal price, equation (B.15) and (B.16) into the market share function q_1 and q_2 or x_c and $1 - x_c$. The result is as follow.

$$q_1 \quad = \quad \frac{(x_1+1)^2-(x_2+1)^2+(y_1-1/2)^2-(y_2-1/2)^2}{6(x_1-x_2)} \tag{B.17}$$

$$q_2 \quad = \quad -\frac{(x_1-2)^2-(x_2-2)^2+(y_1-1/2)^2-(y_2-1/2)^2}{6(x_1-x_2)} \tag{B.18}$$

Since all Lagrange multiplier are zero, we can see that the constraints are not binding. We can conclude that the market demands of both players are positive. Thus, we can conclude the following results.

$$(x_1+1)^2-(x_2+1)^2+(y_1-1/2)^2-(y_2-1/2)^2<0 \tag{B.19}$$

$$(x_1-2)^2-(x_2-2)^2+(y_1-1/2)^2-(y_2-1/2)^2>0 \tag{B.20}$$

By substitute optimal prices of both players, equation (B.15) and (B.16), back into the profit function of each firm and applying first order condition, we obtain:

$$\pi_1^* \quad = \quad (p_1^*-y_1^2)\left(\frac{x_1+x_2}{2}-\frac{(y_1-y_2)}{2(x_1-x_2)}+\frac{p_1^*-p_2^*}{2(x_1-x_2)}\right)$$

$$= \quad -\frac{\left((x_1-x_2)(2+x_1+x_2)-(y_1-y_2)+y_1^2-y_2^2\right)^2}{18(x_1-x_2)} \tag{B.21}$$

$$\frac{\partial \pi_1^*}{\partial y_1} \quad = \quad \frac{(1-2y_1)\left((x_1-x_2)(2+x_1+x_2)-(y_1-y_2)+y_1^2-y_2^2\right)}{9(x_1-x_2)}$$

$$= \quad \frac{(1-2y_1)\left((x_1+1)^2-(x_2+1)^2+(y_1-1/2)^2-(y_2-1/2)^2\right)}{9(x_1-x_2)}$$

$$\frac{(1-2y_1)\left((x_1+1)^2-(x_2+1)^2+(y_1-1/2)^2-(y_2-1/2)^2\right)}{9(x_1-x_2)} \quad = \quad 0 \tag{B.22}$$

$$\pi_2^* \quad = \quad (p_2^*-y_2^2)\left(1-\frac{x_1+x_2}{2}+\frac{(y_1-y_2)}{2(x_1-x_2)}-\frac{p_1^*-p_2^*}{2(x_1-x_2)}\right)$$

$$= \quad -\frac{\left((x_1 - x_2)(-4 + x_1 + x_2) - (y_1 - y_2) + y_1^2 - y_2^2\right)^2}{18(x_1 - x_2)} \qquad \text{(B.23)}$$

$$\frac{\partial \pi_2^*}{\partial y_2} \quad = \quad -\frac{(1 - 2y_2)\left((x_1 - x_2)(-4 + x_1 + x_2) - (y_1 - y_2) + y_1^2 - y_2^2\right)}{9(x_1 - x_2)}$$

$$= \quad -\frac{(1 - 2y_2)\left((x_1 - 2)^2 - (x_2 - 2)^2 + (y_1 - 1/2)^2 - (y_2 - 1/2)^2\right)}{9(x_1 - x_2)}$$

$$-\frac{(1 - 2y_2)\left((x_1 - 2)^2 - (x_2 - 2)^2 + (y_1 - 1/2)^2 - (y_2 - 1/2)^2\right)}{9(x_1 - x_2)} \quad = \quad 0 \qquad \text{(B.24)}$$

Considering equation (B.23) and (B.24), since we know from inequality (B.19) and (B.20) that the value of $(x_1 + 1)^2 - (x_2 + 1)^2 + (y_1 - 1/2)^2 - (y_2 - 1/2)^2$ must be less than 0 and the value of $(x_1 - 2)^2 - (x_2 - 2)^2 + (y_1 - 1/2)^2 - (y_2 - 1/2)^2$ must be greater than 0, the optimal product quality level for both players are $1/2$. Again, we will apply second order condition to check whether these are the optimal product quality level for both players. At the point where $y_1 = y_2 = 1/2$, we have:

$$\frac{\partial^2 \pi_1^*}{\partial y_1^2} \quad = \quad \frac{-2x_1(2 + x_1) + 2x_2(2 + x_2)}{9(x_1 - x_2)} \quad = \quad -\frac{2}{9}(x_1 + x_2 + 2) \quad < \quad 0$$

$$\frac{\partial^2 \pi_2^*}{\partial y_1^2} \quad = \quad \frac{2x_1(-4 + x_1) - 2x_2(-4 + x_2)}{9(x_1 - x_2)} \quad = \quad \frac{2}{9}(x_1 + x_2 - 8) \quad < \quad 0$$

Since the value of $\partial^2 \pi_1^* / \partial y_1^2$ and $\partial^2 \pi_2^* / \partial y_2^2$ are negative. Thus the optimal product quality level for both players are $y_1 = y_2 = 1/2$. Substitute the optimal quality of both players back into profit function, equation (B.21) and (B.23), and do the partial differentiation with respect to x_1 and x_2 respectively, we obtain:

$$\pi_1^* \quad = \quad -\frac{1}{18}(x_1 - x_2)(2 + x_1 + x_2)^2 \qquad \text{(B.25)}$$

$$\frac{\partial \pi_1^*}{\partial x_1} = -\frac{1}{18}(2+3x_1-x_2)(2+x_1+x_2) \quad \text{(B.26)}$$

$$\pi_2^* = -\frac{1}{18}(x_1-x_2)(-4+x_1+x_2)^2 \quad \text{(B.27)}$$

$$\frac{\partial \pi_2^*}{\partial x_2} = -\frac{1}{18}(4+x_1-3x_2)(-4+x_1+x_2) \quad \text{(B.28)}$$

Since $x_1 \in [0,1]$ and $x_2 \in [0,1]$, then $\partial\pi_1/\partial x_1$ is negative and $\partial\pi_2/\partial x_2$ is positive, we can conclude the maximal differentiation or the optimal value of x_1 and x_2 are 0 and 1, respectively. Therefore, by substitute these optimal values of x_i back into profit function, the optimal profit for each player is $1/2$. Thus, we can conclude the result of this case as follow.

$$x_1 = 0 \quad \text{(B.29)}$$

$$x_2 = 1 \quad \text{(B.30)}$$

$$y_1 = y_2 = \frac{1}{2} \quad \text{(B.31)}$$

$$\pi_1 = \pi_2 = \frac{1}{2} \quad \text{(B.32)}$$

For the second case, when $x_1 = x_2$, we will divide this case into two sub-cases. The first sub-case is when the game is tie or both players share the whole market share together. The objective functions are as follow.

Player 1:

$$\text{Max}_{p_1}\ (p_1 - y_1^2)(q_1) \ \text{ or } \ \text{Max}_{p_1}\ (p_1 - y_1^2)\left(\frac{1}{2}\right)$$

s.t. $y_1 - p_1 = y_2 - p_2$

Player 2:

$$\text{Max}_{p_2}\ (p_2 - y_2^2)(q_2) \ \text{ or } \ \text{Max}_{p_2}\ (p_2 - y_2^2)\left(\frac{1}{2}\right)$$

s.t. $\quad y_1 - p_1 = y_2 - p_2$

If firm *i* locates at $y_i \neq 1/2$, it can increase the profit by changing its location to $y_i = 1/2$ and changing its corresponds price $p_i(y_i)$ by the same amount in the same direction regardless the other player's strategy. This is because of the fact, in this case, $u_i = V - p_i(y_i) - (x - x_i)^2 + y_i$, which will stay the same if player change y_i and $p_i(y_i)$ by the same amount at the same direction. However, the change in profit will be

$$\Delta\pi_i \quad \approx \quad \frac{\partial \pi_i}{\partial y_i}\Delta y_i + \frac{\partial \pi_i}{\partial p_i}\Delta p_i \quad = \quad \left(\frac{\partial \pi_i}{\partial y_i} + \frac{\partial \pi_i}{\partial p_i}\right)\Delta p_i \quad = \quad \left(\frac{1}{2} - y_i\right)\Delta p_i$$

If y_i is less than $1/2$ and player increases y_i, $\Delta y_i = \Delta p_i > 0$, then $\Delta\pi_i > 0$. Also, If y_i is greater than $1/2$ and player decreases y_i, $\Delta y_i = \Delta p_i < 0$, then $\Delta\pi_i > 0$. Therefore, the optimal level of product quality is at $1/2$ for both players. When the location and product quality level of both players are the same, the competition will be the same as Bertrand competition, which will drive the price of both players to the production cost or $p_1 = p_2 = 1/4$ and the profit for both players will be 0.

The second sub-case is when there is a winner and loser or when one player obtain the whole market share. Regardless the loser's action, the winner's product quality will be at $1/2$ as describe in the first sub-case. For the loser strategy, choosing product quality other than $1/2$ will not help as it lead to the other player's undercutting strategy and the loser still does not gain any market share. As the loser choose product quality equal to $1/2$, the game will move back to the first sub-case. Thus the profit of both players will be 0.

Hence the solution $x_1 = x_2$ is not Nash Equilibrium in the first stage since both players can gain positive profit by choosing $x_1 = 0$ and $x_2 = 1$ to obtain equilibrium as discuss above. Therefore, the equilibrium in the first stage is when $x_1 = 0$, $x_2 = 1$, $y_1 = y_2 = 1/2$, and $p_1 = p_2 = 5/4$.