CHAPTER 4

EQUILIBRIUM

4.1 First Competition

Since the consumer has a unit demand and will buy the product from the player who give them the higher utility, consumer who locate at x will buy the product from player 1 instead of player 2 if

$$
\begin{aligned}
V - p_1 - (x - x_1)^2 + y_1 &> V - p_2 - (x - x_2)^2 + y_2 \\
-p_1 - x^2 + 2x_1 x - x_1^2 + y_1 &> -p_2 - x^2 + 2x_2 x - x_2^2 + y_2 \\
-p_1 + 2x_1 x - x_1^2 + y_1 &> -p_2 + 2x_2 x - x_2^2 + y_2 \\
2(x_1 - x_2)x &> x_1^2 - x_2^2 - (y_1 - y_2) + p_1 - p_2
\end{aligned}
$$

If $x_1 > x_2$:

$$x > \frac{x_1^2 - x_2^2}{2(x_1 - x_2)} - \frac{(y_1 - y_2)}{2(x_1 - x_2)} + \frac{p_1 - p_2}{2(x_1 - x_2)} = \frac{x_1 + x_2}{2} - \frac{(y_1 - y_2)}{2(x_1 - x_2)} + \frac{p_1 - p_2}{2(x_1 - x_2)} = x_c$$

If $x_1 < x_2$:

$$x < \frac{x_1^2 - x_2^2}{2(x_1 - x_2)} - \frac{(y_1 - y_2)}{2(x_1 - x_2)} + \frac{p_1 - p_2}{2(x_1 - x_2)} = \frac{x_1 + x_2}{2} - \frac{(y_1 - y_2)}{2(x_1 - x_2)} + \frac{p_1 - p_2}{2(x_1 - x_2)} = x_c$$

If $x_1 = x_2$:

$$
\begin{aligned}
0 &> -(y_1 - y_2) + p_1 - p_2 \\
y_1 - p_1 &> y_2 - p_2
\end{aligned}
$$

We can see that the case $x_1 > x_2$ and $x_1 < x_2$ are similar except that the sign is opposite. Therefore, without loss of generality, we may assume that $x_1 \le x_2$. Since the value of market of each player is restricted to be in the interval $[0,1]$, while the value of x_c can be outside the interval, the market share of each will be as follow.

If $x_1 < x_2$:
$$q_1 = \begin{cases} 0 & \text{if } x_c < 0 \\ x_c & \text{if } 0 \le x_c \le 1 \\ 1 & \text{if } x_c > 1 \end{cases}$$
$$q_2 = 1 - q_1$$

If $x_1 = x_2$:
$$q_1 = \begin{cases} 0 & \text{if } y_1 - p_1 < y_2 - p_2 \\ 1/2 & \text{if } y_1 - p_1 = y_2 - p_2 \\ 1 & \text{if } y_1 - p_1 > y_2 - p_2 \end{cases}$$
$$q_2 = 1 - q_1$$

We are going to consider these two cases, $x_1 < x_2$ and $x_1 = x_2$ separately. For the first case, the analysis is as following. Since this game is dynamic, backward induction will be used.

From the calculation, the optimal location, product quality level, and pricing strategy are solved. Please see appendix B1 for the calculation of this case. Players will try to maximize location differentiation or $x_1 = 0$ and $x_2 = 1$ in order to decrease the competitive environment in term of location. As they locate near to each other, the competition will be more intense. Thus, the profit will be lower. However, the product quality level differentiation is minimized as $y_1 = y_2 = 1/2$. The lower product quality level will lead to the lower utility provided to consumer. Also, the higher product quality level leads to the higher cost. Thus, the optimal product quality level strategy is to balance between the benefit and cost, which is to choose at half. For the pricing, as the profit function is the quadratic function of price. Thus, the optimal price of both players is equal to $5/4$.

Proposition 1: *The three-stage game of location in the first stage, product quality level in the second stage, and price in the third stage, with quadratic transportation cost has a unique subgame perfect equilibrium at maximal variety differentiation* $x_1^* = 0$, $x_2^* = 1$, *minimal quality differentiation* $y_1^* = y_2^* = 1/2$, *and equal price* $p_1^* = p_2^* = 5/4$. *Both players will share the same amount of market share* $q_1^* = q_2^* = 1/2$. *And, both player will receive equal profit* $\pi_1^* = \pi_2^* = 1/2$.

4.2 Second Competition

In this competition, there will be another two branches, one for each player, in the market. Therefore we need to consider three more market separating lines, which come from the following equations.

$$V - p_1 - (x_{c_1} - m_1)^2 - (y_{c_1} - n_1) \quad = \quad V - p_2 - (x_{c_1} - x_2)^2 - (y_{c_1} - y_2) \qquad (4.1)$$

$$V - p_1 - (x_{c_2} - x_1)^2 - (y_{c_2} - y_1) \quad = \quad V - p_2 - (x_{c_2} - m_2)^2 - (y_{c_2} - n_2) \qquad (4.2)$$

$$V - p_1 - (x_{c_3} - m_1)^2 - (y_{c_3} - n_1) \quad = \quad V - p_2 - (x_{c_3} - m_2)^2 - (y_{c_3} - n_2) \qquad (4.3)$$

Where $m_1, m_2, n_1,$ and n_2 are location of player1, location of player 2, product quality level of player 1, and product quality level of player 2, respectively. Equation (4.1), (4.2), and (4.3) represent the line where consumers are indifferent buying the product from new branch of player 1 and original branch of player 2, original branch of player 1 and new branch of player 2, and new branches of both players, respectively. By solving these equations and use the optimal location and product quality level from "first competition", we get:

$$x_{c_1} \quad = \quad \frac{m_1 + 1}{2} - \frac{n_1 - \frac{1}{2}}{2(m_1 - 1)} + \frac{p_1 - p_2}{2(m_1 - 1)} \qquad (4.4)$$

$$x_{c_2} \quad = \quad \frac{m_2}{2} - \frac{\frac{1}{2} - n_2}{2(-m_2)} + \frac{p_1 - p_2}{2(-m_2)} \qquad (4.5)$$

$$x_{c_3} \quad = \quad \frac{m_1 + m_2}{2} - \frac{(n_1 - n_2)}{2(m_1 - m_2)} + \frac{p_1 - p_2}{2(m_1 - m_2)} \qquad (4.6)$$

The analysis will be divided into five main cases and sub-cases as follow

- Case I: There is one market separating line in the market and $m_1 \neq m_2$.
  - Case I.1: The market separating line is x_{c_3} and $m_1 < m_2$

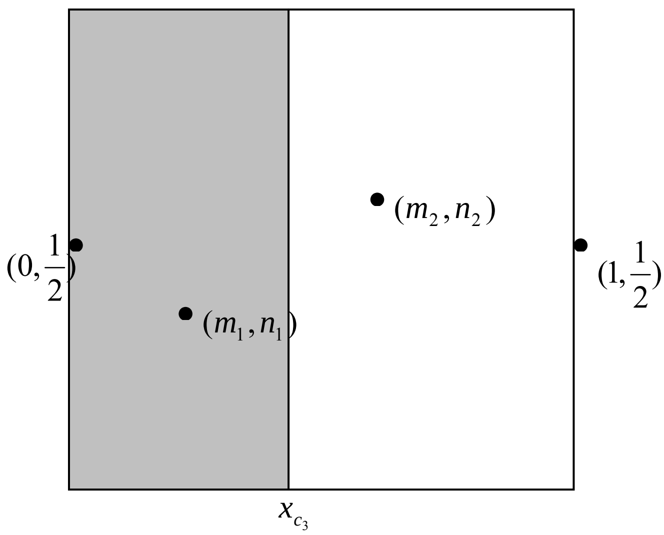
- Case I.2: The market separating line is x_{c_3} and $m_1 > m_2$
- Case I.3: The market separating line is x_{c_1} or x_{c_2} and $m_1 < m_2$
- Case I.4: The market separating line is x_{c_1} or x_{c_2} and $m_1 > m_2$

- Case II: There are two market separating lines in the market and $m_1 > m_2$.
  - Case II.1: The market separating lines are x_{c_2} and x_{c_3}
  - Case II.2: The market separating lines are x_{c_1} and x_{c_3}
- Case III: There are three market separating lines in the market, x_{c_1}, x_{c_2}, and x_{c_3}.
- Case IV: When $m_1 = m_2$
  - Case IV.1: There are two market separating lines x_{c_1}, x_{c_2} and both players share the middle area.
  - Case IV.2 There is one market separating lines x_{c_1} or x_{c_2}. In other word, one player obtain the whole middle area.
- Case V: When the market separating line is still the same one as in the first competition, x_c.

For case I we are going to analyze when there is only one market separating line in the market. Thus for caseI.1, the market separating line existed in the market is x_{c_3} and $m_1 < m_2$. The market shares of each player are as in the figure below.

Figure 4.1

The market share shape for case I.1

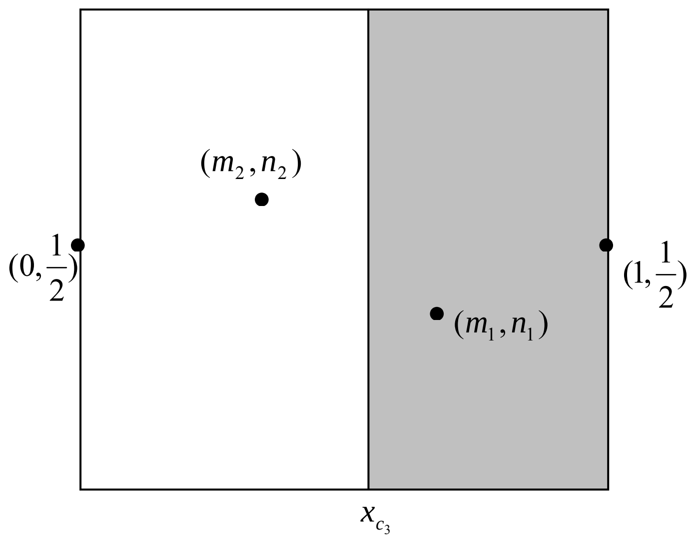


Since this game is dynamic, backward induction will be used. Please see appendix B2 and B3 for subgame perfect Nash Equilibrium finding and assumptions' cross checking. The In this case, players are still maximize their location, $m_1 = 0$ and $m_2 = 1$ due to the same reason as in the first competition, the intense pricing competition environment. For the product quality level, the minimize product quality level differentiation is still used. However, the value of product quality levels are changed from $1/2$ to be $n_1 = n_2 = 1/2$. The higher product quality level is still lead to the higher cost, but the optimal point is higher due to the assumption of average cost between original and new branch, $(y_i^2 + n_i^2)/2$ for $i = 1,2$. And, as usual, the higher product quality level leads to higher cost. Thus the optimal price is changed to $13/8$ from $5/4$. From all of these strategy, the result in term of profit of each player is that both players receive the same amount of profit, which is $1/2$.

The next case, case I.2, is when the market separating line existed in the market is x_{c_3} and $m_1 > m_2$. The market share will be as in the figure below.

Figure 4.2

The market share shape for case I.2



This is the same as the previous case except that players interchange their locations. Thus, under the assumption $m_1 > m_2$, the optimal locations are $m_1^* = 1$ and $m_2^* = 0$, the optimal product quality levels are $n_1^* = n_2^* = 1$, the optimal pricing are $p_1^* = p_2^* = 13/8$, optimal market share are $q_1^* = q_2^* = 1/2$, and optimal profit are $\pi_1^* = \pi_2^* = 1/2$. Please see appendix B4 for the assumptions' cross checking for this case.

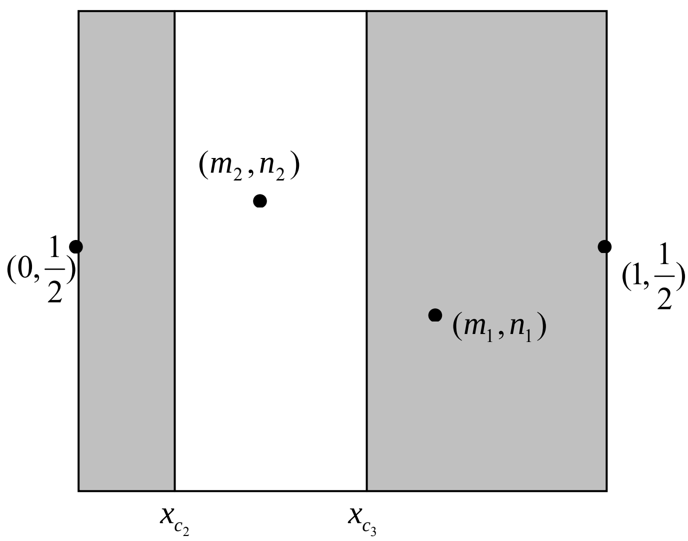
The next case, case I.3, we are going to consider is when $m_1 < m_2$ the market separating line existed in the market is x_{c_1} or x_{c_2}. Since this is a symmetric competition, without loss of generality, we will consider these as only one case, which is x_{c_2}. Please see appendix B5 and B6 for the equilibrium finding and assumptions' cross checking. In this case, the new strategies for location and product quality level are implemented for one player and the old strategies are implemented for another player. One player, player 1 for example, will not concern about the location as $m_1 \in [0.1)$ and trying to minimized the cost of production as $n_1 = 0$. This due to the fact that he does not use this new branch as a selling branch, but he uses it as a cost reduction branch. On the other hand, player 2 is still trying to maximize the location differentiation. Since the selling branch of player 1 in this case is the original one, player 2 optimal location, by knowing player 1's action, is the maximum differentiation or $m_2 = 1$.

The next case, case I.4, we are going to consider is when $m_1 > m_2$ the market separating line existed in the market is x_{c_1} or x_{c_2}. Since this is a symmetric competition, without loss of generality, we will consider these as only one case, which is x_{c_2}. Please see appendix B7 for the subgame perfect Nash Equilibrium finding. Although the setup and assumptions are almost the same as in case I.3, but there is no subgame perfect Nash Equilibrium in this case as one of the assumption is violated.

After we analyze case I, one market separating line, already, the next step is to analyze the case when there are two market separating line, case II. The first sub-case, case II.1, that we are going to consider is when $m_1 > m_2$ and there is two market separating line, which are x_{c_2} and x_{c_3}. This means that x_{c_1} is greater than or equal to 1. Thus, the shape of market share will be as follow.

Figure 4.3

The market share shape for case II.1



The gray area is the market share of player 1 and the white area is the market share of player 2. For this sub case, the numerical method is applied. The simulation using MATLAB programming is implemented. The concept is that we are going to simulate the value of $m_1, m_2, n_1,$ and n_2 in a matrix 101×101 form using MATLAB.

However, since we can calculate the optimal pricing of each player using Lagrange method, the optimal price of each player are not simulated, but calculated. Please see appendix B8 for the optimal price finding.

From the optimal prices calculated this way, the simulation is created. Please see Appendix A1 for the MATLAB code. By using this simulation, the result is that the checker of subgame perfect Nash Equilibrium has the value equal to 0. Thus, there is no subgame perfect Nash Equilibrium for this sub-case.

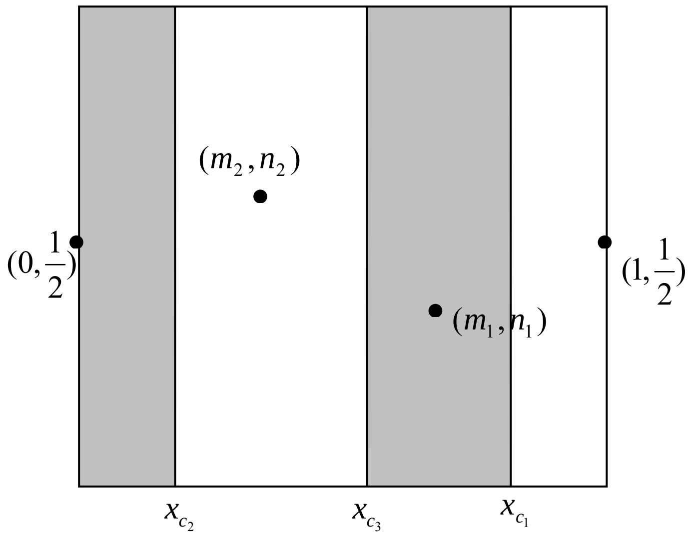
For the next sub-case, case II.2, is when $m_1 > m_2$ and two market separating line are x_{c_1} and x_{c_3} or when the line x_{c_2} is less than or equal to 0. However, for this sub-case, the result will be the same as the first sub-case as the input are the same except that the market share of player 1 is, now player 2's and the market share of player 2 is, now, player1's. Therefore, we can conclude, for this sub-case, that there is no subgame perfect Nash Equilibrium for this sub-case.

Thus, in conclusion for the case of two market separating lines, there is no subgame perfect Nash Equilibrium.

The next case we are going to consider is the case when there are three market separating line, case III. This will happen when $m_1 > m_2$ and all x_{c_1}, x_{c_2}, and x_{c_3} exist. The shape of market share of each player will be as the figure below.

Figure 4.4

The market share shape for case III



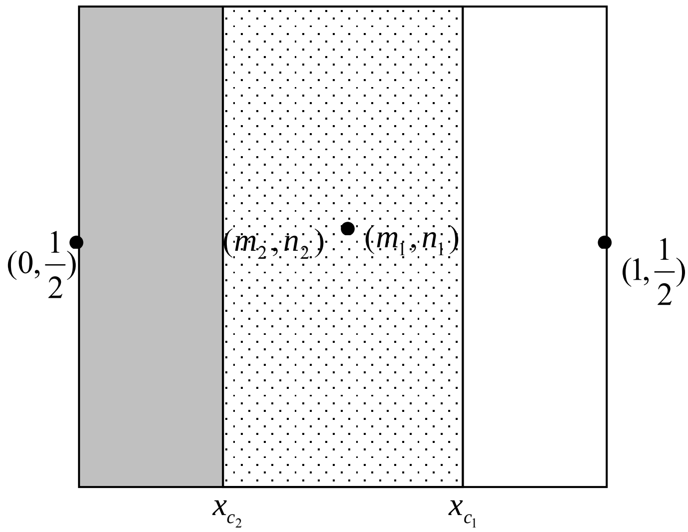
The gray area is the market share of player 1 and the white area is the market share of player 2. Also this case, the numerical method is applied. The concept is the same as before that we are going to simulate the value of m_1, m_2, n_1, and n_2 in a matrix 101×101 form using MATLAB. Thus, before doing the simulation, we are going to calculate to optimal price of this case by using Lagrange method. Please see appendix B9 for the optimal price finding of this case.

Therefore, by using the optimal price calculated this way, we can create the simulation code. Please see the Appendix A2 for the simulation code of this case. By using this simulation, the result is that the checker of subgame perfect Nash Equilibrium has the value equal to 0. Thus, we can conclude that there is no subgame perfect Nash Equilibrium in the case of three market separating line.

The next case, called case IV, that we are going to consider is the case when the position of new location of both players are at the same point. Thus this means that the value of m_1 and m_2 are equal. This brings us to two separate subcases. The first one is when they share the market of the middle area or when $V - p_1 + n_1 = V - p_2 + n_2$. And, the second subcase is when there is one player take the whole middle area or when $V - p_1 + n_1 < V - p_2 + n_2$ or $V - p_1 + n_1 > V - p_2 + n_2$. The market share shapes are as below.

Figure 4.5

The market share shape for case IV



The dots area is the area that will be shared by both players in the first sub-case, and, will be taken by one player in the second sub-case.

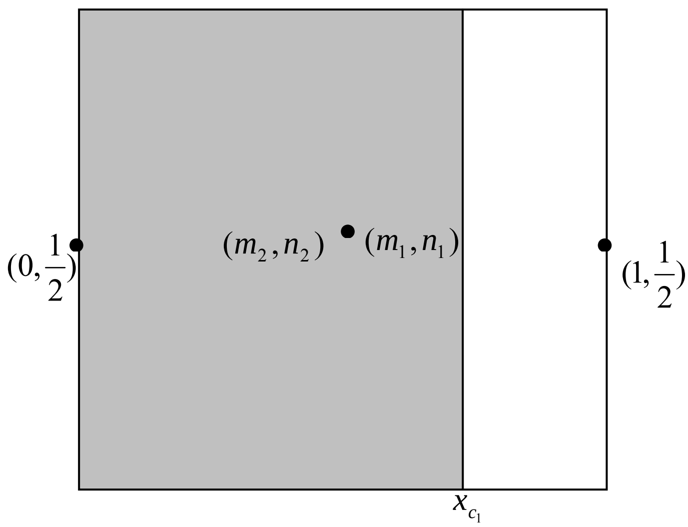
Firstly, we are going to consider the case when they share the middle area, called case IV.1. Please see appendix B10 for the optimal price finding of this case.

From the result of optimal price, the numerical method is applied. Please see the Appendix A3 for the simulation code. By using this simulation, the result is that the checker of subgame perfect Nash Equilibrium has the value equal to 0. Thus, we can conclude that there is no subgame perfect Nash Equilibrium in the case of sharing middle area or $p_1 - n_1 = p_2 - n_2$.

Next, case IV.2, it is the case when there is one player take the whole middle area. As mention before, this case will occur when $V - p_1 + n_1 < V - p_2 + n_2$ or $V - p_1 + n_1 > V - p_2 + n_2$. However, since this is the symmetric game, without loss of generality, we are going to consider only the case when $V - p_1 + n_1 > V - p_2 + n_2$, which means that the player who take the whole middle area is player 1. Thus, the shape of market share will be as the follow figure.

Figure 4.6

The market share shape for case IV.2



For this case, the subgame perfect Nash Equilibrium is calculated using backward induction as same as the case stated before. Please see appendix B11 and appendix B12 for the subgame perfect Nash Equilibrium finding and assumptions' cross checking of this case, respectively. The finding of this case confirms the finding in case I.3 as this case assumes that $m_1 = m_2$ while the case I.3 assumes $m_1 < m_2$ and the optimal strategies are the same. One player is going to use the new branch as the cost reduction branch while another player uses the new branch as the selling branch. Thus one player will choose to minimize the product quality level of the new branch while another player chooses the product quality level equal to 1.

The last case, case V, we are going to consider is when the market separating line existed in the market is the x_c or the original line from the first competition. This will be occurred whenever the line x_{c_1} is less than the line x_c and line x_{c_2} is greater than line x_c. Please see appendix B13 and B14 for subgame perfect Nash Equilibrium finding and assumption cross checking for this case, respectively. This case will occur when both player applying the cost cutting strategy. Both player will not concern about the location of the new branches, $m_1 \in [0,1]$ and $\mathrm{m}_2 \in [0,1]$ as they will not use this new branch as a selling branch anyway. Also, for the product quality level, both player will minimized the cost through the product quality level of their new branched or choosing the lowest product quality level as $n_1 = n_2 = 0$.

From the analysis of all cases discussed above, we can see that there are five (pure) subgame perfect Nash Equilibria in this game which are as follow.

1) $m_1^* = 0$, $m_2^* = 1$, $n_1^* = 1$, $n_2^* = 1$, $p_1^* = 13/8$, $p_2^* = 13/8$, $\pi_1^* = 1/2$, $\pi_2^* = 1/2$

2) $m_1^* = 1$, $m_2^* = 0$, $n_1^* = 1$, $n_2^* = 1$, $p_1^* = 13/8$, $p_2^* = 13/8$, $\pi_1^* = 1/2$, $\pi_2^* = 1/2$

3) $m_1^* = 0$, $m_2^* \in [0,1]$, $n_1^* = 1$, $n_2^* = 0$, $p_1^* = 13/8$, $p_2^* = 9/8$, $\pi_1^* = 1/2$, $\pi_2^* = 1/2$

4) $m_1^* \in [0,1]$, $m_2^* = 1$, $n_1^* = 0$, $n_2^* = 1$, $p_1^* = 9/8$, $p_2^* = 13/8$, $\pi_1^* = 1/2$, $\pi_2^* = 1/2$

5) $m_1^* \in [0,1]$, $m_2^* = [0,1]$, $n_1^* = 0$, $n_2^* = 0$, $p_1^* = 9/8$, $p_2^* = 9/8$, $\pi_1^* = 1/2$, $\pi_2^* = 1/2$

As we can see, from all of the cases, the profits of each player are all equal to $1/2$. Since, in the first stage, both players will choose the locations or the value of m_i simultaneously, all of the cases are Nash Equilibrium for this second location competition.

Proposition 2: *Given that both players already have one existing branch where maximal variety differentiation* $x_1^* = 0$, $x_2^* = 1$, *minimal quality differentiation* $y_1^* = y_2^* = 1/2$ *are applied, the addition three-stage game of location in the first stage, product quality level in the second stage, and price in the third stage, with quadratic transportation cost has subgame perfect Nash equilibria outcome as follow.*

i) *Locate at* $m_1^* = 0$, $m_2^* = 1$, *minimal quality differentiation* $n_1^* = n_2^* = 1$, *and equal price* $p_1^* = p_2^* = 13/8$.

ii) *Locate at* $m_1^* = 1$, $m_2^* = 0$, *minimal quality differentiation* $n_1^* = n_2^* = 1$, *and equal price* $p_1^* = p_2^* = 13/8$.

iii) *Locate at* $m_1^* = 0$, $m_2^* \in [0,1]$, *maximal quality differentiation* $n_1^* = 1$, $n_2^* = 0$, *and unequal price* $p_1^* = 13/8$, $p_2^* = 9/8$.

iv) *Locate at* $m_1^* \in [0,1]$, $m_2^* = 1$, *maximal quality differentiation* $n_1^* = 0$, $n_2^* = 1$, *and unequal price* $p_1^* = 9/8$, $p_2^* = 13/8$.

v) *Locate at* $m_1^* \in [0,1]$, $m_2^* \in [0,1]$, *minimal quality differentiation* $n_1^* = 0$, $n_2^* = 0$, *and equal price* $p_1^* = p_2^* = 9/8$.