1 Introduction

It is well known that the metric projection operator plays an important role in various fields such as nonlinear functional analysis, optimization theory, fixed point theory, nonlinear programming, game theory, variational inequality and complementarity problems. (see [3]). In 1994, Alber [2] introduced and studied the generalized projections from Hilbert spaces to uniformly convex and uniformly smooth Banach spaces. Moreover Alber [3] presented some applications of the generalized projections to approximately solving variational inequalities and von Neumann intersection problem in Banach spaces. In 2005, Li [50] extended the generalized projection operator from uniformly convex and uniformly smooth Banach spaces to reflexive Banach spaces and studied some properties of the generalized projection operator with applications to solve the variational inequality in Banach spaces. Later, Wu and Huang [85] introduced a new generalized f-projection operator in Banach spaces. They extended the definition of the generalized projection operators introduced by Abler [2] and proved some properties of the generalized f-projection operator. In 2009, Fan et al. [30] presented some basic results for the generalized f-projection operator, and discussed the existence of solutions and approximation of the solutions for generalized variational inequalities in noncompact subsets of Banach spaces.

The study of Ky Fan inequality [31], fixed points of nonlinear mappings, and their approximation algorithms constitutes a topic of intensive research efforts. Many well known problems arising in various branches of science can be studied by using algorithms, which are iterative in their nature. As an example, in computer tomography with limited data, each piece of information implies the existence of a convex set in which the required solution lies.

The problem of finding a point in the intersection of the convex sets is then of crucial interest, and it cannot be usually solved directly. Therefore, an iterative algorithm must be used to approximate such point. The well known convex feasibility problem which captures applications in various disciplines such as image restoration, computer tomography, and radiation therapy treatment planning is to find a point in the intersection of common fixed point sets of a family of nonexpansive mappings. A simple algorithmic solution to the problem of minimizing a quadratic function over the intersection of the convex sets is of extreme value in many applications including set theoretic signal estimation.

For solving convex feasibility problem of a system of generalized Ky Fan inequalities is very general in the sense that it includes, as special cases optimization problems, equilibrium problems, variational inequality problems, Min-Max problems. Moreover, the the generalized Ky Fan inequality was shown in [8] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational inequality problems, vector equilibrium problems, Nash equilibria in noncooperative games. In addition, there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of a generalized Ky Fan inequality. In other words, the generalized Ky Fan inequality and equilibrium problem are an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. In the last two decades, many papers have appeared in the literature on the existence of solutions of a generalized Ky Fan inequality (or equilibrium problem; see, for example [8, 22, 96] and references therein). Some solution methods have been proposed to solve the generalized Ky Fan inequality and equilibrium problems; see, for example, [8, 39, 41, 43, 55, 58, 63, 61] and references therein.

Let $f : C \times C \to \mathbb{R}$ be a bifunction, the *equilibrium problem*, is to find $x \in C$ such that

$$f(x,y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.2) is denoted by $EP(f)$. The equilibrium problem is very general in the sense that it includes, as special cases optimization problems, variational inequality problems, Min-Max problems, saddle point problem, fixed point problem, Nash EP. In 2008, Takahashi and Zembayashi [82, 83], introduced iterative sequences for finding a common solution of an equilibrium problem and fixed point problem. Some solution methods have been proposed to solve the equilibrium problem; see, for instance, [60, 18, 36, 73, 26, 92, 27, 28, 93, 94].

Let θ be a bifunction of $C \times C$ into $\mathbb{R}$ and $\varphi : C \to \mathbb{R}$ be a real-valued function. The *mixed equilibrium problem*, denoted by $MEP(\theta, \varphi)$, is to find $x \in C$ such that

$$\theta(x,y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.2}$$

If $\varphi \equiv 0$, the problem (1.2) reduce into the *equilibrium problem for θ*, denoted by $EP(\theta)$, is to find $x \in C$ such that

$$\theta(x,y) \geq 0, \quad \forall y \in C. \tag{1.3}$$

If $\theta \equiv 0$, the problem (1.2) reduce into the *minimize problem*, denoted by $Argmin(\varphi)$, is to find $x \in C$ such that

$$\varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.4}$$

The above formulation (1.3) was shown in [8] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational inequality problems, vector equilibrium problems, Nash equilibria in noncooperative games. In addition, there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an $EP(\theta)$. In other words, the $EP(\theta)$ is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. In the last two decades, many papers have appeared in the literature on the existence of solutions of $EP(\theta)$; see, for example [8, 42] and references therein. Some solution methods have been proposed to solve the $EP(\theta)$; see, for example, [8, 39, 41, 43, 55, 63, 61] and references therein.

Let E be a real Banach space with dual E^* and let C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be an operator. A is called *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C;$$

α–inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C;$$

L-Lipschitz continuous if there exists a constant $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

If A is α-inverse strongly monotone, then it is $\frac{1}{\alpha}$-Lipschitz continuous, i.e.,

$$\|Ax - Ay\| \leq \frac{1}{\alpha}\|x - y\|, \quad \forall x, y \in C.$$

A monotone operator A is said to be *maximal* if its graph $G(A) = \{(x, x^*) : x^* \in Ax\}$ is not properly contained in the graph of any other monotone operator.

Let A be a monotone operator. We consider the problem for finding $x \in E$ such that

$$0 \in Ax, \tag{1.5}$$

a point $x \in E$ is called a *zero point* of A. Denote by $A^{-1}0$ the set of all point $x \in E$ such that $0 \in Ax$. This problem is very important in optimization theory and related fields.

Let A be a monotone operator. *The classical variational inequality problem* for an operator A is to find $\hat{z} \in C$ such that

$$\langle A\hat{z}, y - \hat{z}\rangle \geq 0, \quad \forall y \in C. \tag{1.6}$$

The set of solution of (1.6) is denoted by $VI(A, C)$. This problem is connected with the convex minimization problem, the complementary problem, the problem of finding a point $x \in E$ satisfying $Ax = 0$.

The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^*\rangle$ or $x^*(x)$. For each $p > 1$, the *generalized duality mapping* $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\}$$

for all $x \in E$. In particular, $J = J_2$ is called the *normalized duality mapping*. If E is a Hilbert space, then $J = I$, where I is the identity mapping.

Consider the functional defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2, \quad \text{for } x, y \in E, \tag{1.7}$$

where J is the normalized duality mapping. It is obvious from the definition of ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2, \quad \forall x, y \in E. \tag{1.8}$$

Alber [3] introduced The *generalized projection* $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution of the minimization problem

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x), \tag{1.9}$$

existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J.

Iiduka and Takahashi [37] introduced the following iterative scheme for finding a solution of the variational inequality problem for an inverse-strongly monotone operator A in a 2-uniformly convex and uniformly smooth Banach space E: $x_1 = x \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \quad \forall n \geq 1, \tag{1.10}$$

where Π_C is the generalized projection from E onto C, J is the duality mapping from E into E^* and $\{\lambda_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (1.10) converges weakly to some element of $VI(A, C)$. In connection, Iiduka and Takahashi [34] studied the following iterative scheme for finding a zero point of a monotone operator A in a 2-uniformly convex and uniformly smooth Banach space E:

$$\begin{cases} x_1 = x \in E \text{ chosen arbitrarily,} \\ y_n = J^{-1}(Jx_n - \lambda_n Ax_n), \\ X_n = \{z \in E : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Y_{n+1} = \{z \in E : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{X_n \cap Y_n}(x). \end{cases} \tag{1.11}$$

where $\Pi_{X_n \cap Y_n}$ is the generalized projection from E onto $X_n \cap Y_n$, J is the duality mapping from E into E^* and $\{\lambda_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ converges strongly to an element of $A^{-1}0$. Moreover, under the additional suitable assumption they proved that the sequence $\{x_n\}$ converges strongly to some element of $VI(A, C)$. Some solution methods have been proposed to solve the variational inequality problem; see, for instance, [19, 47, 48].

Let E be a real Banach space with the dual space E^* and C be a nonempty closed and convex subset of E. A mapping $S : C \to C$ is said to be:

(1) *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|$$

for all $x, y \in C$;

(2) *quasi-nonexpansive* if $F(S) \neq \emptyset$ and

$$\|Sx - y\| \leq \|x - y\|$$

for all $x \in C$ and $y \in F(S)$;

(3) *quasi-ϕ-nonexpansive* if $F(S) \neq \emptyset$ and

$$\phi(p, Sx) \leq \phi(p, x), \forall x \in C, \ p \in F(S).$$

;

(4) *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|S^n x - S^n y\| \leq k_n \|x - y\|$$

for all $x, y \in C$;

(5) *asymptotically quasi-nonexpansive* if $F(S) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|S^n x - y\| \leq k_n \|x - y\|$$

for all $x \in C$ and $y \in F(S)$;

(6) *total asymptotically nonexpansive* if there exist the nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ with $\psi(0) = 0$ such that

$$\|S^n x - S^n y\| \le \|x - y\| + \mu_n \psi(\|x - y\|) + \nu_n$$

for all $x, y \in C$ and $n \ge 1$;

(7) *total quasi-ϕ-asymptotically nonexpansive*, if $F(S) \neq \emptyset$ and there exist nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$ such that

$$\phi(p, S^n x) \le \phi(p, x) + \nu_n \varphi(\phi(p, x)) + \mu_n, \ \forall n \ge 1, \ \forall x \in C, \ p \in F(S).$$

A mapping $S : C \to C$ is said to be *uniformly L-Lipschitz continuous* if there exists a constant $L > 0$ such that

$$\|S^n x - S^n y\| \le L\|x - y\| \tag{1.12}$$

for all $x, y \in C$. A mapping $S : C \to C$ is said to be *closed* if, for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Sx_n = y_0$, we have $Sx_0 = y_0$.

Let 2^C be the family of all nonempty subsets of C and let $S : C \to 2^C$ be a multi-valued mapping. For a point $q \in C$, $n \ge 1$ define an iterative sequence as follows:

$$Sq := \{q_1 : q_1 \in Sq\}$$

$$S^2 q = SSq := \bigcup_{q_1 \in Sq} Sq_1$$

$$S^3 q = SS^2 q := \bigcup_{q_2 \in T^2 q} Sq_2$$

$$\vdots$$

$$S^n q = SS^{n-1} q := \bigcup_{q_{n-1} \in S^{n-1} q} Sq_{n-1}.$$

A point $p \in C$ is said to be an *asymptotic fixed point* of S, if there exists a sequence $\{x_n\}$ in C such that $\{x_n\}$ converges weakly to p and

$$\lim_{n\to\infty} d(x_n, Sx_n) := \lim_{n\to\infty} \inf_{x \in Sx_n} \|x_n - x\| = 0.$$

The asymptotic fixed point set of S denoted by $\widehat{F}(S)$.

A multi-valued mapping S is said to be *total quasi-ϕ-asymptotically nonexpansive*, if $F(S) \neq \emptyset$ and there exist nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$ such that for all $x \in C$, $p \in F(S)$

$$\phi(p, w_n) \le \phi(p, x) + \nu_n \varphi(\phi(p, x)) + \mu_n, \ \forall n \ge 1, \ w_n \in S^n x.$$

S is said to be *closed* if for any sequence $\{x_n\}$ and $\{w_n\}$ in C with $w_n \in Sx_n$ if $x_n \to x$ and $w_n \to w$, then $w \in Sx$.

A multi-valued mapping S is said to be *uniformly asymptotically regular* on C if

$$\lim_{n\to\infty}\left(\sup_{x\in C}\|s_{n+1}-s_n\|\right)=0,\ \ s_n\in S^n x.$$

Every quasi-ϕ-asymptotically nonexpansive multi-valued mappings implies quasi-ϕ-asymptotically nonexpansive mappings but the converse is not true.

In 2012, Chang et al. [13] introduced the concept of total quasi-ϕ-asymptotically nonexpansive multi-valued mapping and then proved some strong convergence theorem by using the hybrid shrinking projection method.

For a mapping $A : C \to E^*$, let $f(x,y) = \langle Ax, y-x\rangle$ for all $x, y \in C$. Then $x \in EP(f)$ if and only if $\langle Tx, y-x\rangle \geq 0$ for all $y \in C$; i.e., x is a solution of the variational inequality.

Motivated and inspired by the work mentioned above, in this work, we introduce and prove strong convergence theorems of hybrid projection algorithm for a fixed point set. In the first theorem, we prove strong convergence theorem of a family of relatively quasi-nonexpansive mapping. Next, we extend a mapping to a total quasi-ϕ-asymopotically nonexpansive mapping and then we can prove strong convergence theorem. Finally, we extend from a single-value mapping to a multi-valued mapping. In the final theorem we show the fixed set of a total quasi-ϕ-asymopotically nonexpansive multi-valued mappings. Moreover, we prove strong convergence to solution of the equilibrium problem, zero points of monotone operators, the solution of the variation inequality in Banach space.

2 Preliminaries

A Banach space E with norm $\|\cdot\|$, is called *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. A Banach space E is called *smooth* if the limit $\lim\limits_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U$. It is also called *uniformly smooth* if the limit exists uniformly for all $x, y \in U$. The *modulus of convexity* of E is the function $\delta : [0,2] \to [0,1]$ defined by

$$\delta(\epsilon) = \inf\{1-\|\tfrac{x+y}{2}\| : x,y \in E, \|x\| = \|y\| = 1, \|x-y\| \geq \epsilon\}.$$

A Banach space E is a *uniformly convex* if and only if $\delta(\epsilon) > 0$ for all $\epsilon \in (0,2]$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be a *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in [0,2]$. Observe that every p-uniform convex is uniformly convex. Every uniformly convex Banach space E has the *Kadec-Klee property*, that is, for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.

Let E be a real Banach space with dual E^*, E is a uniformly smooth if and only if E^* is a uniformly convex Banach space. If E is a uniformly smooth Banach space, then E is a smooth and reflexive Banach space.

Remark 2.1. • If E is smooth, then J is single-valued;

- If E is strictly convex, then J is one-to-one and $\langle x-y, x^*-y^*\rangle > 0$ holds for all $(x,x^*),(y,y^*) \in J$ with $x \neq y$;
- If E is a strictly convex, then J is strictly monotone;
- If E is a smooth, then J is single valued and semi-continuous;
- If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E;
- If E is reflexive smooth and strictly convex, then the normalized duality mapping J is single valued, one-to-one and onto;
- If E be a reflexive strictly convex and smooth Banach space and J is the duality mapping from E into E^*, then J^{-1} is also single-value, bijective and is also the duality mapping from E^* into E and thus $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$;
- If E is uniformly smooth, then E is smooth and reflexive;
- E is uniformly smooth if and only if E^* is uniformly convex.

See [17] for more details.

Remark 2.2. If E is a reflexive, strictly convex and smooth Banach space, then $\phi(x,y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x,y) = 0$ then $x = y$. From (1.7), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$ (see [17, 68] for more details).

Lemma 2.3. (*Beauzamy* [10] and Xu[86]) *If E be a 2-uniformly convex Banach space. Then, for all $x, y \in E$ we have*

$$\|x-y\| \leq \frac{2}{c^2}\|Jx - Jy\|,$$

where J is the normalized duality mapping of E and $0 < c \leq 1$.

The best constant $\frac{1}{c}$ in Lemma is called the *p-uniformly convex constant* of E.

Lemma 2.4. (*Beauzamy* [10] and Zalinescu[89]) *If E be a p-uniformly convex Banach space and let p be a given real number with $p \geq 2$. Then for all $x, y \in E$, $J_x \in J_p(x)$ and $J_y \in J_p(y)$*

$$\langle x-y, J_x - J_y\rangle \geq \frac{c^p}{2^{p-2}p}\|x-y\|^p,$$

where J_p is the generalized duality mapping of E and $\frac{1}{c}$ is the p-uniformly convexity constant of E.

Lemma 2.5. (Kamimura and Takahashi [45]) *Let E be a uniformly convex and smooth Banach space and $\{x_n\}$, $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 2.6. (Alber [3]) *Let C be a nonempty closed convex subset of a smooth Banach space E and let $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.7. (Alber [3]) *Let E be a reflexive strictly convex and smooth Banach space, C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 2.8. (*Qin et al.* [63]). *Let E be a uniformly convex and smooth Banach space, let C be a closed convex subset of E, and let T be a closed relatively quasi-nonexpansive mapping from C into itself. Then $F(T)$ is a closed convex subset of C.*

Lemma 2.9. ([21]) *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. Let $S : C \to C$ be a closed and total quasi-ϕ-asymptotically nonexpansive mapping with the nonnegative real sequences $\{\nu_n\}$ and $\{\mu_n\}$ with $\nu_n \to 0$ and $\mu_n \to 0$ as $n \to \infty$, respectively, and a strictly increasing continuous function $\zeta : \mathbf{R}^+ \to \mathbf{R}^+$ with $\zeta(0) = 0$. If $\mu_1 = 0$, then the set $F(S)$ of fixed points of S is a closed convex subset of C.*

Lemma 2.10. (Change et al.[13]) *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. Let $S : C \to 2^C$ be a closed and total quasi-ϕ-asymptotically nonexpansive multi-valued mapping with nonnegtive real sequence ν_n and μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$. If $\mu_1 = 0$, then the fixed point set $F(S)$ is a closed convex subset of C.*

Let A be an inverse-strongly monotone mapping of C into E^* which is said to be *hemicontinuous* if for all $x, y \in C$, the mapping h of $[0, 1]$ into E^*, defined by $h(t) = A(tx + (1-t)y)$, is continuous with respect to the weak* topology of E^*. We define by $N_C(v)$ *the normal cone* for C at a point $v \in C$, that is,

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^* \rangle \geq 0, \ \forall y \in C\}. \tag{2.1}$$

Theorem 2.11. (*Rockafellar* [72]) *Let C be a nonempty, closed convex subset of a Banach space E and A a monotone, hemicontinuous operator of C into E^*. Let $B \subset E \times E^*$ be an operator defined as follows:*

$$Bv = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & otherwise. \end{cases} \tag{2.2}$$

Then B is maximal monotone and $B^{-1}0 = VI(A, C)$.

Theorem 2.12. (*Takahashi* [81]) *Let C be a nonempty subset of a Banach space E and A a monotone, hemicontinuous operator of C into E^* with $C = D(A)$. Then*

$$VI(A, C) = \{u \in C : \langle v - u, Av \rangle \geq 0, \ \forall v \in C\}. \tag{2.3}$$

It is obvious that the set $VI(A, C)$ is a closed and convex subset of C and the set $A^{-1}0 = VI(A, E)$ is closed and convex subset of E.

Theorem 2.13. (*Takahashi* [81]) *Let C be a nonempty compact convex subset of a Banach space E and A a monotone, hemicontinuous operator of C into E^* with $C = D(A)$. Then $VI(A, C)$ is nonempty.*

We make use of the following mapping V studied in Alber [3]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2, \quad \forall x \in E,\, x^* \in E^*, \tag{2.4}$$

that is, $V(x, x^*) = \phi(x, J^{-1}(x^*))$.

Lemma 2.14. (Alber [3]) *Let E be a reflexive strictly convex smooth Banach space and V be as in (2.4). Then we have*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^*\rangle \leq V(x, x^* + y^*), \quad \forall x \in E,\, x^*, y^* \in E^*.$$

Lemma 2.15. (Cho et al.[16]) *Let E be a uniformly convex Banach space and $B_r(0) = \{x \in E : \|x\| \leq r\}$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \|\lambda x\|^2 + \|\mu y\|^2 + \|\gamma z\|^2 - \lambda\mu g(\|x - y\|),$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

Lemma 2.16. (Pascali and Sburlan [59]) *Let E be a real smooth Banach spaces and $A : E \to 2^{E^*}$ be a maximal monotone mapping. Then $A^{-1}0$ is closed and convex subset of E and the graph $G(A)$ of A is demiclosed in the following sense: if $\{x_n\} \subset D(A)$ with $x_n \rightharpoonup x \in E$ and $y_n \in Ax_n$ with $y_n \to y \in E^*$, then $x \in D(A)$ and $y \in Ax$.*

In 2006, Wu and Huang [85] introduced a new generalized f-projection operator in Banach space. They extended the definition of the generalized projection operators introduced by Abler [2] and proved some properties of the generalized f-projection operator. Consider the functional $G : C \times E^* \longrightarrow \mathbf{R} \cup \{+\infty\}$ defined by

$$G(y, \varpi) = \|y\|^2 - 2\langle y, \varpi\rangle + \|\varpi\|^2 + 2\rho f(y) \tag{2.5}$$

for all $(y, \varpi) \in C \times E^*$, where ρ is a positive number and $f : C \to \mathbf{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous. From the definition of G, Wu and Huang [85] proved the following properties:

(1) $G(y, \varpi)$ is convex and continuous with respect to ϖ when y is fixed;

(2) $G(y, \varpi)$ is convex and lower semicontinuous with respect to y when ϖ is fixed.

Let E be a real Banach space with its dual space E^* and C be a nonempty closed and convex subset of E. We say that $\pi_C^f : E^* \to 2^C$ is a *generalized f-projection operator* if

$$\pi_C^f \varpi = \{u \in C : G(u, \varpi) = \inf_{y \in C} G(y, \varpi), \forall \varpi \in E^*\}.$$

Recall that a Banach space E has the Kadec-Klee property ([17, 33, 81]) if, for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, we have $\|x_n - x\| \to 0$ as $n \to \infty$. It is well-known that, if E is a uniformly convex Banach space, then E has the Kadec-Klee property.

Lemma 2.17. ([85]) *Let E be a real reflexive Banach space with its dual space E^* and C be a nonempty closed and convex subset of E. The following statement hold:*

(1) $\pi_C^f \varpi$ *is a nonempty, closed and convex subset of C for all $\varpi \in E^*$;*

(2) *If E is smooth, then for all $\varpi \in E^*$, $x \in \pi_C^f \varpi$ if and only if*

$$\langle x - y, \varpi - Jx \rangle + \rho f(y) - \rho f(x) \geq 0$$

for all $y \in C$;

(3) *If E is strictly convex and $f : C \to \mathbf{R} \cup \{+\infty\}$ is positive homogeneous (i.e., $f(tx) = tf(x)$ for all $t > 0$ such that $tx \in C$, where $x \in C$), then $\pi_C^f \varpi$ is single valued mapping.*

Recently, Fan et al. [30] showed that the condition, f is positive homogeneous, which appears in [30, Lemma 2.1 (iii)], can be removed.

Lemma 2.18. ([30]) *Let E be a real reflexive Banach space with its dual space E^* and let C be a nonempty closed and convex subset of E. If E is strictly convex, then $\pi_C^f \varpi$ is single-valued.*

Recall that J is single-value mapping when E is a smooth Banach space. There exists a unique element $\varpi \in E^*$ such that $\varpi = Jx$, where $x \in E$. This substitution in (2.5) gives the following:

$$G(y, Jx) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2 + 2\rho f(y). \tag{2.6}$$

Now, we consider the second generalized f projection operator in a Banach space (see [50]).

Let E be a real smooth Banach space and C be a nonempty closed and convex subset of E. We say that $\Pi_C^f : E \to 2^C$ is a *generalized f-projection operator* if

$$\Pi_C^f x = \{u \in C : G(u, Jx) = \inf_{y \in C} G(y, Jx), \forall x \in E\}.$$

Lemma 2.19. ([29]) *Let E be a Banach space and $f : E \to \mathbf{R} \cup \{+\infty\}$ be a lower semicontinuous and convex function. Then there exist $x^* \in E^*$ and $\alpha \in \mathbf{R}$ such that*

$$f(x) \geq \langle x, x^* \rangle + \alpha$$

for all $x \in E$.

Lemma 2.20. ([50]) *Let E be a reflexive smooth Banach space and C be a nonempty closed and convex subset of E. The following statements hold:*

(1) $\Pi_C^f x$ *is nonempty closed and convex subset of C for all $x \in E$;*

(2) *For all $x \in E$, $\hat{x} \in \Pi_C^f x$ if and only if*

$$\langle \hat{x} - y, Jx - J\hat{x} \rangle + \rho f(y) - \rho f(\hat{x}) \geq 0$$

for all $y \in C$;

(3) *If* E *is strictly convex, then* Π_C^f *is single-valued mapping.*

Lemma 2.21. ([50]) *Let* E *be a real reflexive smooth Banach space and* C *be a nonempty closed and convex subset of* E. *Then, for any* $x \in E$ *and* $\hat{x} \in \Pi_C^f x$,

$$\phi(y, \hat{x}) + G(\hat{x}, Jx) \leq G(y, Jx)$$

for all $y \in C$.

Lemma 2.22. ([50]). *Let* E *be a Banach space and* $f : E \to \mathbf{R} \cup \{+\infty\}$ *be a proper, convex and lower semicontinuous mapping with convex domain* $D(f)$. *If* $\{x_n\}$ *is a sequence in* $D(f)$ *such that* $x_n \rightharpoonup \hat{x} \in D(f)$ *and* $\lim_{n\to\infty} G(x_n, Jy) = G(\hat{x}, Jy)$, *then* $\lim_{n\to\infty} \|x_n\| = \|\hat{x}\|$.

Remark 2.23. Let E be a uniformly convex and uniformly smooth Banach space and $f(y) = 0$ for all $y \in E$. Then Lemma 2.21 reduces to the property of the generalized projection operator considered by Alber [3].

Let $f(y) \geq 0$ for all $y \in C$ and $f(0) = 0$, then the definition of a totally quasi-ϕ-asymptotically nonexpansive S is equivalent to the following:

If $F(S) \neq \emptyset$ and there exist the nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$, respectively, and a strictly increasing continuous function $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ with $\psi(0) = 0$ such that

$$G(p, S^n x) \leq G(p, x) + \nu_n \psi G(p, x) + \mu_n$$

for all $x \in C$, $p \in F(S)$ and $n \geq 1$.

Let θ be a bifunction from $C \times C$ to $\mathbf{R}$, where $\mathbf{R}$ denotes the set of real numbers. *The equilibrium problem* (for short, (EP)) is to find $\hat{x} \in C$ such that

$$\theta(\hat{x}, y) \geq 0 \tag{2.7}$$

for all $y \in C$. The set of solutions of the (EP) 2.7 is denoted by $EP(\theta)$.

An operator $A \subset E \times E^*$ is said to be *monotone* if

$$\langle x - y, x^* - y^* \rangle \geq 0$$

for all $(x, x^*), (y, y^*) \in A$. A point $z \in E$ is called a *zero point* of A if

$$0 \in Az. \tag{2.8}$$

We denote the set of zeroes of the operator A by $A^{-1}0$, that is,

$$A^{-1}0 = \{z \in E : 0 \in Az\}.$$

A monotone $A \subset E \times E^*$ is said to be *maximal* if its graph $G(A) = \{(x, y^*) : y^* \in Ax\}$ is not property contained in the graph of any other monotone operator. If A is maximal monotone, then the solution set $A^{-1}0$ is closed and convex.

Let E be a smooth strictly convex and reflexive Banach space, C be a nonempty closed convex subset of E and $A \subset E \times E^*$ be a monotone operator satisfying $D(A) \subset C \subset J^{-1}(\cap_{\lambda>0} R(J + \lambda A))$. Then the *resolvent* $J_\lambda : C \to D(A)$ of A is defined by

$$J_\lambda x = \{z \in D(A) : Jx \in Jz + \lambda Az, \forall x \in C\}.$$

J_λ is a single-valued mapping from E to $D(A)$. On the other words, $J_\lambda = (J + \lambda A)^{-1} J$ for all $\lambda > 0$.

For any $\lambda > 0$, the *Yosida approximation* $A_\lambda : C \to E^*$ of A is defined by $A_\lambda x = \frac{Jx - JJ_\lambda x}{\lambda}$ for all $x \in C$. We know that $A_\lambda x \in A(J_\lambda x)$ for all $\lambda > 0$ and $x \in E$. Since relatively quasi nonexpansive mappings and quasi-ϕ-nonexpansive mappings are same, we can see that J_λ is a quasi-ϕ-nonexpansive mapping (see [76, Theorem 4.7]).

Lemma 2.24. ([44]) *Let E be a smooth strictly convex and reflexive Banach space, C be a nonempty closed convex subset of E and $A \subset E \times E^*$ be a monotone operator satisfying $D(A) \subset C \subset J^{-1}(\cap_{\lambda>0} R(J + \lambda A))$. For any $\lambda > 0$, let J_λ and A_λ be the resolvent and the Yosida approximation of A, respectively. Then the following hold:*

(1) $\phi(p, J_\lambda x) + \phi(J_\lambda x, x) \le \phi(p, x)$ *for all* $x \in C$ *and* $p \in A^{-1}0$;

(2) (*b*) $(J_\lambda x, A_\lambda x) \in A$ *for all* $x \in C$;

(3) (*c*) $F(J_\lambda) = A^{-1}0$.

Lemma 2.25. ([72]) *Let E be a reflexive strictly convex and smooth Banach space. Then an operator $A \subset E \times E^*$ is maximal monotone if and only if $R(J + \lambda A) = E^*$ for all $\lambda > 0$.*

For solving the equilibrium problem for a bifunction $\theta : C \times C \to \mathbb{R}$, let us assume that θ satisfies the following conditions:

(A1) $\theta(x, x) = 0$ for all $x \in C$;

(A2) θ is monotone, i.e., $\theta(x, y) + \theta(y, x) \le 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} \theta(tz + (1 - t)x, y) \le \theta(x, y);$$

(A4) for each $x \in C$, $y \mapsto \theta(x, y)$ is convex and lower semi-continuous.

Lemma 2.26. (*Blum and Oettli* [8]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, let θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), and let $r > 0$ and $x \in E$. Then there exists $z \in C$ such that*

$$\theta(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \quad \forall y \in C.$$

The following lemma is a spacial case of Zhang [96].

Lemma 2.27. *(Zhang [96]). Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let $\varphi : C \to \mathbb{R}$ is convex and lower semi-continuous and θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). For $r > 0$ and $x \in E$, then there exists $u \in C$ such that*

$$\theta(u,y) + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \quad \forall y \in C.$$

Define a mapping $K_r : C \to C$ as follows:

$$K_r(x) = \{u \in C : \theta(u,y) + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \quad \forall y \in C\} \tag{2.9}$$

for all $x \in C$. Then the followings hold:

1. *K_r is single-valued;*
2. *K_r is firmly nonexpansive, i.e., for all $x, y \in E$, $\langle K_r x - K_r y, JK_r x - JK_r y\rangle \leq \langle K_r x - K_r y, Jx - Jy\rangle$;*
3. *$F(K_r) = MEP(\theta, \varphi)$;*
4. *$MEP(\theta, \varphi)$ is closed and convex;*
5. *$\phi(p, K_r z) + \phi(K_r z, z) \leq \phi(p, z)$, $\forall p \in F(K_r)$ and $z \in E$.*

3 Methodology

1. Studying and investigating on fixed point problems.
2. Studying and investigating on nonlinear problems.
3. Establishing new theorems for fixed point problems and nonlinear problems in Banach spaces.
4. Finding necessary and sufficient conditions which are established for mappings to have fixed points and common solutions of nonlinear mappings.

4 Main results

Let C be a closed subset of a Banach space E. Recall that a mapping $S : C \to C$ is closed if for each $\{x_n\}$ in C, if $x_n \to x$ and $Sx_n \to y$, then $Sx = y$. Let $\{S_n\}$ be a family of mappings of C in to itself with $F := \cap_{n=1}^{\infty} F(S_n) \neq \emptyset$, $\{S_n\}$ is said to satisfy the $(*)$-condition if for each bounded sequence $\{z_n\}$ in C,

$$\lim_{n\to\infty} \|z_n - S_n z_n\| = 0, \; z_n \to z \text{ imply } z \in F. \tag{4.1}$$

Remark 4.1. It follows directly from the definitions above that if $\{S_n\}$ satisfies NST-condition, then $\{S_n\}$ satisfies $(*)$-condition. If $S_n \equiv S$ and S is closed, then $\{S_n\}$ satisfies $(*)$-condition.

Using the (*)-condition, we prove the new convergence theorems for finding a common element of the set of solutions of an equilibrium problems, the common fixed point set of a family of relatively quasi-nonexpansive mappings, a zero of maximal monotone operators and the solution set of variational inequalities for an α-inverse strongly monotone mapping in a 2-uniformly convex and uniformly smooth Banach space.

Theorem 4.2. *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let $T \subset E \times E^*$ be a maximal monotone operator satisfying $D(T) \subset C$ and let $J_{r_n} = (J + r_nT)^{-1}J$ for all $r_n > 0$. Let θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $\|Ay\| \leq \|Ay - Au\|$, $\forall y \in C$ and $u \in VI(A, C) \neq \emptyset$. Let $S_n : C \to C$ be a family of relatively quasi-nonexpansive mappings such that satisfies the (*)-condition and $\Theta := (\cap_{n=1}^{\infty} F(S_n)) \cap T^{-1}0 \cap MEP(\theta, \varphi) \cap VI(A, C) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JS_nJ_{r_n}z_n), \\ u_n \in C \text{ such that } \theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \alpha_n\phi(z, x_n) + (1 - \alpha_n)\phi(z, z_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \tag{4.2}$$

where J is the duality mapping on E, $\{\alpha_n\}$ is sequence in $[0,1]$ and $\{r_n\} \subset [d, \infty)$ for some $d > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E. If $\liminf_{n\to\infty}(1 - \alpha_n) > 0$, then $\{x_n\}$ converges strongly to $p \in \Theta$, where $p = \Pi_\Theta x_0$.

When we extend mapping S to totally quasi-ϕ-asymptotically nonexpansive, we have the following Theorem.

Theorem 4.3. *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. For each $i = 1, 2, \cdots, m$, let θ_i be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying the conditions (A1)-(A4). Let $A_j \subset E \times E^*$ be a maximal monotone operator satisfying $D(A_j) \subset C$ and $J_{\lambda_{j,n}}^{A_j} = (J + \lambda_{j,n}A_j)^{-1}J$ for all $\lambda_{j,n} > 0$ and $j = 1, 2, \cdots, l$. Let $S : C \to C$ be a closed and totally quasi-ϕ-asymptotically nonexpansive mapping with the nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$, respectively, and a strictly increasing continuous function $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ with $\psi(0) = 0$. Let $f : E \to \mathbf{R}^+$ be a convex and lower semicontinuous function with $C \subset int(D(f))$ and $f(0) = 0$. Assume that S is uniformly L-Lipschitz continuous and $\mathcal{F} = F(S) \cap (\cap_{i=1}^{m} EP(\theta_i)) \cap (\cap_{j=1}^{l} A_j^{-1}0) \neq \emptyset$. For any initial point $x_1 \in E$, define $C_1 = C$ and the sequence $\{x_n\}$ in C by*

$$\begin{cases} z_n = J_{\lambda_{l,n}}^{A_l} \circ J_{\lambda_{l-1,n}}^{A_{l-1}} \circ \cdots \circ J_{\lambda_{1,n}}^{A_1} x_n, \\ u_n = T_{r_{m,n}}^{\theta_m} \circ T_{r_{m-1,n}}^{\theta_{m-1}} \circ \cdots \circ T_{r_{1,n}}^{\theta_1} z_n, \\ y_n = J^{-1}(\alpha_n Jx_1 + \beta_n JS^n x_n + \gamma_n Ju_n), \\ C_{n+1} = \{v \in C_n : G(v, Jy_n) \leq \alpha_n G(v, Jx_1) + (1 - \alpha_n)G(v, Jx_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}^f x_1 \end{cases} \tag{4.3}$$

for each $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\zeta_n = \nu_n \sup_{q\in\mathcal{F}} \psi(G(q, Jx_n)) + \mu_n$ and for each $i = 1, 2, \cdots, m$, $\{r_{i,n}\} \subset [d, \infty)$ for some $d > 0$. If, for each $j = 1, 2, \cdots, l$, $\liminf_{n\to\infty} \lambda_{j,n} > 0$, $\lim_{n\to\infty} \alpha_n = 0$ and $\liminf_{n\to\infty} \beta_n < 1$, then the sequence $\{x_n\}$ converges strongly to a point $\Pi_{\mathcal{F}}^f x_1$.

Theorem 4.4. *Let C be a nonempty closed and convex subset of a uniformly smooth and 2-uniformly convex Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions (A1)-(A4) and let A be an α-inverse-strongly monotone mapping of E into E^*. Let $S : C \to 2^C$ be a closed and total quasi-ϕ-asymptotically nonexpansive multi-valued mapping with nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$. Assume that S is uniformly asymptotically regular on C with $\mu_1 = 0$ and $F := F(S) \cap EF(f) \cap A^{-1}0 \neq \emptyset$. For arbitrary $x_1 \in C$, $C_1 = C$, generate a sequences $\{x_n\}$ by*

$$\begin{cases} z_n = J^{-1}(Jx_n - \lambda_n Ax_n), \\ u_n = T_{r_n} z_n, \\ y_n = J^{-1}(\alpha_n Jx_n + \beta_n Jw_n + \gamma_n Ju_n), \ w_n \in S^n x_n, \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, z_n) \leq \phi(v, x_n) + K_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \in \mathbb{N}, \end{cases} \tag{4.4}$$

where $K_n = \nu_n \sup_{q \in F} \psi(\phi(q, x_n)) + \mu_n$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ and $\{r_n\}$ satisfy the following conditions:

1. *$\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\liminf_{n\to\infty} \alpha_n \beta_n > 0$,*
2. *$\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < \frac{c^2 \alpha}{2}$ and $\frac{1}{c}$ is the 2-uniformly convexity constant of E,*
3. *$\{r_n\} \subset [d, \infty)$ for some $d > 0$,*

then $\{x_n\}$ converges strongly to $\Pi_F x_1$.

5 Conclusion and Suggestion

In this work, we introduce new hybrid iterations of the generalized projection operators and the generalized f-projection operators for finding a common element of the fixed point set and solution of nonlinear problems. This research has three theorems for fixed point problems and nonlinear problems. Each theorem compound by difference mapping and differences conditions for get the strong convergence. For other researcher should study other mappings or iterations and study the necessary conditions to get some new convergence.

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