SOME PROPERTIES OF 3 - (γ , 2) – CRITICAL GRAPHS

By
Wajananu Kulclung

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree
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สมบัติของกราฟ 3 - (γ , 2) - critical

โดย
นายวัจนานุค์ กุลคลัง

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
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The Graduate School, Silpakorn University has approved and accredited the Thesis title of “Some properties of 3 - (γ,2) – critical graphs” submitted by MR Wajananu Kulclung as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics

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A graph G is said to be k - γ - critical if $\gamma(G)=k$ but $\gamma(G+uv)<k$ for every pair of non-adjacent vertices u and v of G, where $\gamma(G)$ is the domination number of G. If $\gamma(G)=k$ but $\gamma(G+uv)<k$ for every pair of non-adjacent vertices u and v of G with $d(u,v)\leq t$ then G is called k - (γ,t) - critical. Henning et al. gave a complete characterization of 2 - $(\gamma,2)$ - critical graphs and a characterization of 3 - $(\gamma,2)$ - critical graphs of diameter 4.

In this thesis, we study 3 - $(\gamma,2)$ - critical graphs of diameter 3 which are not 3 - γ - critical. We establish some properties and some characterizations of 3 - $(\gamma,2)$ - critical graphs of diameter 3 which are not 3 - γ - critical. Further, we show that there are exactly 10 non-isomorphic such graphs of order at most 8.

Department of Mathematics Graduate School, Silpakorn University Academic Year 2011
Student's signature ....
Thesis Advisor's signature ....

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เรากล่าว G เป็นกราฟ k - γ - critical ถ้า $\gamma(G)=k$ และ $\gamma(G+uv)<k$ สำหรับทุกคู่ของจุด u และ v ใน G ที่ไม่ประชิดกัน โดยที่ $\gamma(G)$ คือ จำนวนควบคุม ถ้า $\gamma(G)=k$ สำหรับทุกคู่ของจุด u และ v ใน G ที่ไม่ประชิดกัน โดยที่ $d(u,v)\leq t$ แล้วเราจะกล่าวว่า G เป็นกราฟ k - (γ,t) - critical Henning et al.[6] ได้ให้ลักษณะเฉพาะ ของกราฟ 2 - $(\gamma,2)$ - critical และกราฟ 3 - $(\gamma,2)$ - critical ที่มีเส้นผ่านศูนย์กลางเท่ากับ 4

ในวิทยานิพนธ์นี้ เราศึกษากราฟ 3 - $(\gamma,2)$ - critical ที่มีเส้นผ่านศูนย์กลางเท่ากับ 3 ที่ไม่เป็นกราฟ 3 - γ - critical โดยได้พิสูจน์สมบัติบางประการ และได้ให้ลักษณะเฉพาะบางส่วนของกราฟ 3 - $(\gamma,2)$ - critical ที่ไม่เป็นกราฟ 3 - γ - critical ที่มีเส้นผ่านศูนย์กลางเท่ากับ 3 นอกจากนี้ ยังได้แสดงว่า ถ้าจำนวนจุดไม่เกิน 8 แล้วจะมี กราฟ 3 - $(\gamma,2)$ - critical ที่มีเส้นผ่านศูนย์กลางเท่ากับ 3 เพียง 10 แบบที่แตกต่างกัน

ภาควิชาคณิตศาสตร์ บัณฑิตวิทยาลัย มหาวิทยาลัยศิลปากร ปีการศึกษา 2554

ลายมือชื่อนักศึกษา..

ลายมือชื่ออาจารย์ที่ปรึกษาวิทยานิพนธ์ ....

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Finally, none of this would have been possible without love and encouragement of my family.

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Chapter 1

Introduction

In this chapter, we introduce some definitions and notations used in this thesis.

A **graph** is a triple $G = (V(G), E(G), \omega_G)$, where $V(G)$ is a finite (possibly empty) set of **vertices**, $E(G)$ is a set of **edges** and an **incidence** function ω_G that associates with each edge of G and unordered pairs of vertices of G. If e is an edge and x and y are vertices such that $\omega_G(e) = \{x, y\}$, then e is said to **incident** x and y. Further, the vertices x and y are called **end vertices** and we say that x and y are **adjacent**. The **order** of G is $|V(G)|$. Two or more edges that join the same pair of vertices are called **parallel edges**. An edge that joins itself is a **loop**. A graph G is **simple** if G has no loops and parallel edges. If G is simple and $\omega_G(e) = \{x, y\}$, then we denote e by xy.

The **complement** $\overline{G}$ of a graph G is that graph with $V(\overline{G}) = V(G)$ and $xy \in E(\overline{G})$ if and only if $xy \notin E(G)$. A graph H is a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. H is an **induced subgraph** of G, denoted by $G[H]$, if, for every pair of $x, y \in V(H)$, $xy \in E(H)$ if and only if $xy \in E(G)$. Let x and y be a pair of non-adjacent vertices of G. Then $G + xy$ is the graph obtained from G by **adding the edge** xy.

The **open neighborhood** and the **closed neighborhood** of a vertex x of G are denoted by $N_G(x) = \{y \in V(G) | xy \in E(G)\}$ and $N_G[x] = N_G(x) \cup \{x\}$, respectively while $N_S(x)$ denotes either $N_G(x) \cap S$ if S is a subset of $V(G)$ or $N_G(x) \cap V(S)$ if S is a subgraph of G. The non-neighborhood of v in G denoted by $\overline{N_G}(x)$ is $V(G) - N_G[x]$. A degree of x in G, denoted by $d(x)$, is $|N_G(x)|$. A vertex of degree 1 is called an **end vertex**.

A **complete graph** is a simple graph in which every pair of vertices are adjacent. A complete graph of order n is denoted by K_n. A simple graph G is a **bipartite graph** if $V(G)$ can be partitioned into 2 non-empty subsets V_1 and V_2 such that no edge of G joins vertices in the same set. The sets V_1 and V_2 are called the **partite sets** of G. If G is a bipartite graph having partite sets V_1 and V_2 such that every vertex of V_1 is joined to every vertex of V_2, then G is called a **complete bipartite graph**. If $|V_i| = p_i$, for $1 \leq i \leq 2$, then we denote G by K_{p_1,p_2}. A complete bipartite graph which the cardinality of at least one of partitioned vertex set equals to one is called a **star**, denoted by $K_{1,n}$. The **center vertex** of star $K_{1,r}$ is either vertex of $K_{1,r}$ such that degree is equal r if $r \geq 2$ or one end vertex if $r = 1$. $K_{1,3}$ is called a **claw**. A graph is **claw-free** if it contains no $K_{1,3}$ as an induced subgraph .

The **double star** $S(m,n)$ is the graph obtained from the disjoint union of star $K_{1,m}$ and $K_{1,n}(m,n \geq 1)$ by joining the two central vertices.

A **walk** in a graph G is a finite, non-empty alternating sequence $W = v_0e_1v_1e_2...e_nv_n$ of vertices and edges such that for $1 \leq i \leq n$, the ends of edge e_i are v_{i-1} and v_i. W is said to be a walk from v_o to v_n. A **path** is a walk with distinct vertices. Two vertices x and y of G are **connected** if there is a path from x to y. A graph G is **connected** if every pair of vertices of G are connected otherwise G is disconnected. A maximal connected subgraph of G is called a **component** of G. The **distance** between two vertices x, y, denoted by $d(x,y)$, is the length of a shortest xy-path in G. The **diameter** of G, denoted by $diam(G)$, is the maximum distance between two vertices of G.

Two simple graphs G_1 and G_2 are **isomorphic** if there is a one-to-one function φ from $V(G_1)$ onto $V(G_2)$ such that $xy \in E(G_1)$ if and only if $\varphi(x)\varphi(y) \in E(G_2)$. If G_1 and G_2 are isomorphic, then we denote by $G_1 \cong G_2$.

For $M \subseteq E(G)$, M is a **matching** in G if no two edges of M have common end vertex. A **perfect matching** of a graph G is a matching covering all vertices in G.

Let G_1 and G_2 be vertex-disjoint graphs. Then the **union** of G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph having $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. We denote $G \cup G$ by $2G$. The **join** of G_1 and G_2, denoted by $G_1 + G_2$, is that graph consisting of the union $G_1 \cup G_2$, together with edges xy where $x \in V(G_1)$ and $y \in V(G_2)$.

A subset D of $V(G)$ is called a **dominating set** of G if every vertex of G either belongs to D or is adjacent to a vertex of D. We will write $D \succ G$ if D is a dominating set of G. Further, if $D = \{x\}$, then we say that x dominates G rather than $\{x\}$ dominates G and is denoted by $x \succ G$. Moreover, if D is a dominating set of $G[H]$, then we say that D dominates H and is denoted by $D \succ H$. The minimum cardinality of a dominating set of G is called the **domination number** of G, denoted by $\gamma(G)$.

A subset S of $V(G)$ is said to be an **independent set** if no two vertices in

S are adjacent. A dominating set D of G is called an **independent dominating set** if D is independent and the **independent domination number** denoted by $i(G)$ is the minimum cardinality of an independent dominating set for G.

A graph G is said to be k**-**γ**-critical** if $\gamma(G) = k$ but $\gamma(G + uv) < k$ for every pair of non-adjacent vertices u and v of G. A graph G is said to be k**-**(γ, t)**-critical** if $\gamma(G) = k$ but $\gamma(G + uv) < k$ for every pair of non-adjacent vertices u and v of G with $d(u, v) \leq t$.

All graph considered in this thesis are finite undirected and simple. Chapter 2 provides some basic background and preliminaries results related to our work. In Chapter 3, we present new classes of 3-$(\gamma, 2)$-critical graphs of diameter 3 which are not 3-γ-critical. Finally, Chapter 4 contains the characterization of 3-$(\gamma, 2)$-critical graphs which are not 3-γ-critical of order at most 8.

Chapter 2

Literature reviews

This chapter provides some basic background and results concerning k-γ-critical and k-(γ, t)-critical.

A concept of k-γ-critical was first introduced by Sumner and Blitch [?] in 1983. They showed that 1-γ-critical graph are complete graphs of order n and characterized 2-γ-critical as follows.

Theorem 2.1. *[?] A graph G is 2-γ-critical if and only if $\overline{G} \cong \bigcup_{i=1}^{n} K_{1,n_i} (n \geq 1)$.*

In the same paper, they gave a class of 3-γ-critical as follows. For positive integer $p \geq 6$, let $a + b + c = p - 3$ be any partition of $p - 3$. Let A, B, C be disjoint complete graphs of cardinality a, b and c, respectively, and such that $A \cup B \cup C$ is complete. Let G be a graph with $V(G) = A \cup B \cup C \cup \{v, u, w\}$ with $N_G(v) = A, N_G(u) = B$ and $N_G(w) = C$.

For any graph G with $\gamma(G) = k$, the diameter of G is at most $3k - 1$(see [?]). However, the upper bound of the diameter of k-γ-critical graphs was much lower than that as we can see in the next theorem.

Theorem 2.2. *[?] For $k \geq 2$, the diameter of a connected, k-γ-critical is at most $2k - 2$.*

The upper bound in Theorem 2.2 is not best possible. When $k \in \{3, 4\}$, such upper bound can be lower as in the next two theorems. These bounds are best possible.

Theorem 2.3. *[?] The diameter of a 3-γ-critical graph is at most 3.*

Theorem 2.4. *[?] The diameter of a 4-γ-critical graph is at most 5.*

Moreover, in [?], Favaron et al. proved that:

Theorem 2.5. *[?] For positive integer $k \geq 2$, there is a k-γ-critical having diameter $\left\lfloor \frac{3k}{2} - 1 \right\rfloor$.*

According to the definitions of dominating set and independent dominating set, it is easy to see that $\gamma(G) \leq i(G)$. However, the equality holds if G is claw-free, proved by Allan and Laskar [**?**].

Theorem 2.6. *[**?**] If G is claw-free, then $\gamma(G) = i(G)$.*

Sumner and Blitch [**?**] conjectured that if G is a k-γ-critical graph with $k \geq 3$, then $\gamma(G) = i(G)$. This conjecture was proved to be false for $k \geq 4$ and unsettled for $k = 3$. Ao [**?**] gave a counterexample to this conjecture for $k = 4$. Further, a class of k-γ-critical graphs with $\gamma(G) < i(G)$ where $k \geq 4$ was constructed in [**?**] by Ao et al.

In 1996, Henning et al. [**?**] extended the concept of k-γ-critical to k-(γ, t)-critical. They gave a complete characterization of 2-$(\gamma, 2)$-critical graphs as follows.

Theorem 2.7. *[**?**] A connected graph G is 2-$(\gamma, 2)$-critical if and only if either $\overline{G} \cong \bigcup_{i=1}^{r} K_{1,n_i}$ for $n_i \geq 1$ and $r \geq 1$ or $\overline{G} \cong S(m, n)$ for some positive integers m and n.*

They also provided an upper bound on the diameter of 3-$(\gamma, 2)$-critical and 4-$(\gamma, 2)$-critical as follows.

Theorem 2.8. *[**?**] The diameter of a 3-$(\gamma, 2)$-critical graph is at most 4.*

Theorem 2.9. *[**?**] The diameter of a 4-$(\gamma, 2)$-critical graph is at most 6.*

Further, they established a class $\mathcal{L}$ of 3-$(\gamma, 2)$-critical graphs with diameter 4 as follows. Let $A_1 \cong K_r (r \geq 2), A_2 \cong K_s (s \geq 1)$ and let A_3 be obtained from a complete graph $K_{2m} (m \geq 2)$ without the edges of a perfect matching. Let $u \in V(A_1)$ and $v \in V(A_3)$. Let G be obtained from the disjoint union of A_1, A_2 and A_3 by joining every vertex of A_2 to every vertex of $A_1 \cup A_3$ distinct from u and v. Then, they characterized 3-$(\gamma, 2)$-critical graphs with diameter 4 as follows.

Theorem 2.10. *[**?**] G is a 3-$(\gamma, 2)$-critical graph having diameter 4 if and only if $G \in \mathcal{L}$.*

In the same paper, Henning et al. gave a class of 3-$(\gamma, 2)$-critical graphs of diameter 3 which are not 3-γ-critical as follows. Let $A_1 \cong K_m (m \geq 2), A_2 \cong A_3 \cong K_n (n \geq 2)$ and $A_4 \cong K_1$. Let v be a vertex of A_1 and suppose $V(A_4) = \{u\}$. Let G be obtained from $A_1 \cup A_2 \cup A_3 \cup A_4$ by first joining every vertex of $A_1 - v$ to every vertex of A_2. Next join u to every vertex of A_2. Finally, if $V(A_2) = \{v_1, v_2, ..., v_n\}$

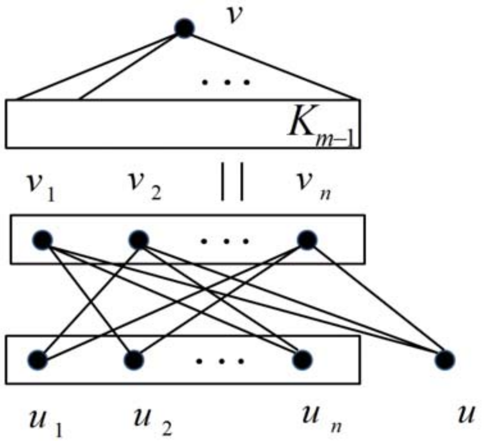


Figure 2.1: A 3-$(\gamma, 2)$-critical graph of diameter 3 which is not 3-γ-critical

and $V(A_3) = \{u_1, u_2, ..., u_n\}$, then join every v_i to every u_j for $i \neq j$. Figure 2.1 illustrates this construction.

Henning et al. concluded this paper by posing a problem to characterize 3-$(\gamma, 2)$-critical graphs of diameter 3 which are not 3-γ-critical and giving a conjecture that if G is a connected 3-$(\gamma, 2)$-critical graph, then $\gamma(G) = i(G)$. This conjecture is still unsettled.

In [?], Ananchuen gave an upper bound on the diameter of 2-(γ, t)-critical graphs and provided a characterization of such graphs for $t \geq 3$ as follows.

Theorem 2.11. *[?] Let G be a 2-(γ, t)-critical for $t \geq 2$. Then $diam(G) \leq 3$ if $t = 2$ and $diam(G) = 2$ if $t \geq 3$.*

Theorem 2.12. *[?] For an integer $t \geq 3$, G is 2-(γ, t)-critical if and only if $\overline{G} \cong \bigcup_{i=1}^{r} K_{1,n_i} (n \geq 1)$ for $n_i \geq 1$ and $r \geq 1$.*

In the same paper, she established an upper bound on the diameter of k-(γ, t)-critical graphs for $t \geq 3$ and then showed that 3-(γ, t)-critical graphs are 3-γ-critical for $t \geq 3$. This results are in the next three theorems.

Theorem 2.13. *[?] For integers $k \geq 4$ and $t \geq 3$, the diameter of k-(γ, t)-critical is at most $3k - 6$.*

Theorem 2.14. *[?] For an integer $t \geq 3$, the diameter of 3-(γ, t)-critical is at most 3.*

Theorem 2.15. *[?] For an integer $t \geq 3$, G is 3-(γ, t)-critical if and only if G is 3-γ-critical.*

In view of the results in this chapter, we have learnt that a characterization of 3-$(\gamma, 2)$-critical graphs of diameter 3 which are not 3-γ-critical is not known. In Chapter 3, we establish characterizations of some classes of 3-$(\gamma, 2)$-critical graphs of diameter 3 which are not 3-γ-critical. The characterization of 3-$(\gamma, 2)$-critical graphs of diameter 3 which are not 3-γ-critical of order at most 8 is provided in Chapter 4.

Chapter 3

Characterizations of some classes of 3-$(\gamma, 2)$-critical graphs which are not 3-γ-critical

This chapter contains the main results of this thesis. We present four new classes of 3-$(\gamma, 2)$-critical graphs of diameter 3 which are not 3-γ-critical. We also provide characterizations of some classes of such graphs.

3.1 Classes of 3-$(\gamma, 2)$-critical graphs which are not 3-γ-critical

In this section, we provide four classes of 3-$(\gamma, 2)$-critical graphs of diameter 3 which are not 3-γ-critical. Our first class is constructed as follows. For non-negative integers n_1, n_2, n_3 and n_4 with $n_1 \geq 1$, and $n_2 \geq n_3 \geq 2$, define a graph $J \in \mathcal{J}$ as follows. Set $V(J) = \{u\} \cup V_1 \cup V_2 \cup V_3$ where $|V_1| = n_1$, $|V_2| = n_2$ and $|V_3| = n_3 + 2n_4 + 1$. Partition V_3 into set $V^{'}, V^{''}$ and $\{v\}$ with $|V^{'}| = n_3$ and $|V^{''}| = 2n_4$. The edges of J are defined as follows. $G[V_1] = K_{n_1}$, $G[V_2] = K_{n_2}$, $G[V^{'}] = K_{n_3}$ and $G[V^{''}] = K_{2n_4}$- a perfect matching. Put $J[V_2 \cup V^{'}] = K_{n_2+n_3} - \bigcup_{i=1}^{n_3} K_{1,r_i}$ where $r_i \geq 1$ for $1 \leq i \leq n_3$, $\sum_{i=1}^{n_3} r_i = n_2$ and $V^{'} = \{c_1, c_2, ..., c_{n_3}\}$ is the set of centers of K_{1,r_i} in $\overline{J}[V_2 \cup V^{'}]$. Note that we have partitioned V_2 into $C_1, C_2, .., C_{n_3}$ where $|C_i| = r_i$ and $\overline{J}[\{c_i\} \cup C_i] = K_{1,r_i}$ for $1 \leq i \leq n_3$. Further, each vertex of V_1 is joined to every vertex of $\{u\} \cup V_2$. The vertex v is joined to every vertex of $V_2 \cup V^{''}$ and each vertex of $V^{''}$ is joined to every vertex of $V_2 \cup V^{'}$.

It is not difficult to show that J is a 3-$(\gamma, 2)$-critical graph of diameter 3 and u and v are the only pair of vertices of J with $d(u, v) = 3$ and $\gamma(J + uv) = 3$. Figure 3.1 illustrates our construction. Note that a " $=$ " in our diagram denotes the join between the corresponding graphs.

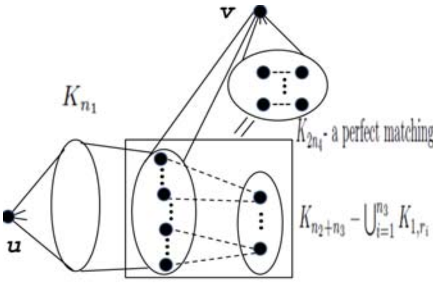


Figure 3.1: Graph $J \in \mathcal{J}$

We now establish the second class. For positive integers n_1, n_2, n_3 and n_4 where $n_1 = 1$ or $n_4 = 1$ and $n_2 = 1$ or $n_3 = 1$, define a graph $H \in \mathcal{H}_1$ as follows. Set $V(H) = \{u\} \cup \{a, b\} \cup V_2 \cup V_3$ where $|V_2| = n_1 + n_2$ and $|V_3| = n_3 + n_4$. Partition V_2 into sets $V_2^{'}$ and $V_2^{''}$ with $|V_2^{'}| = n_1$ and $|V_2^{''}| = n_2$. Partition V_3 into sets $V_3^{'}$ and $V_3^{''}$ with $|V_3^{'}| = n_3$ and $|V_3^{''}| = n_4$. The edges of H are defined as follows. $G[V_2^{'}] = K_{n_1}$, $G[V_2^{''}] = K_{n_2}$, $G[V_3^{'}] = K_{n_3}$ and $G[V_3^{''}] = K_{n_4}$. Further, the vertex a is joined to every vertex of $\{u\} \cup V_2^{'}$ and the vertex b is joined to every vertex of $\{u\} \cup V_2^{''}$. Finally, each vertex of $V_2^{'}$ is joined to every vertex of $V_3^{'}$, each vertex of $V_2^{''}$ is joined to every vertex of $V_3^{''}$ and each vertex of $V_3^{'}$ is joined to every vertex of $V_3^{''}$. It is easy to see that H is a 3-$(\gamma, 2)$-critical graph of diameter 3 and $\gamma(H + uz) = 3$ where $z \in V_3$. Figure 3.2 illustrates our construction.

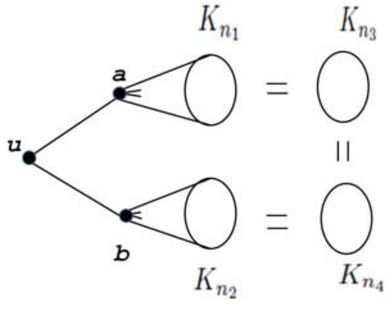


Figure 3.2: Graph $H \in \mathcal{H}_1$

A graph in the third class is constructed as follows. For positive integers n_1, n_2, n_3 and n_4 where $n_2 \geq n_1$ and $n_4 \geq n_3$, let $T = K_{n_1+n_2+n_3+n_4} - \bigcup_{i=1}^{n_1+n_3} K_{1,r_i}$ where $\sum_{i=1}^{n_1} r_i = n_2$ and $\sum_{i=n_1+1}^{n_3} r_i = n_4$. Put $X_1 = \{x \in V(T) | x$ is the center vertex of K_{1,r_i} in $\overline{T}$, $1 \leq i \leq n_1\}$, $X_2 = \{x \in V(T) | x$ is an end vertex of K_{1,r_i} in $\overline{T}$, $1 \leq i \leq n_1\}$, $Y_1 = \{x \in V(T) | x$ is the center vertex of K_{1,r_i} in $\overline{T}$, $n_1 + 1 \leq i \leq n_3\}$, and $Y_2 = \{x \in V(T) | x$ is an end vertex of K_{1,r_i} in $\overline{T}$, $n_1 + 1 \leq i \leq n_3\}$. Clearly,

$|X_1| = n_1$, $|X_2| = n_2$, $|Y_1| = n_3$ and $|Y_2| = n_4$.

Now let $H^{'}$ be a graph of order $n_1 + n_2 + n_3 + n_4 + 4$ obtained from T by adding the four new vertices, say, u, a, b and v. That is $V(H^{'}) = V(T) \cup \{u, v, a, b\}$. Choose $w \in X_1$. Then put $E(H^{'}) = E(T) \cup \{ax|x \in \{u\} \cup X_1 \cup X_2\} \cup \{bx|x \in \{u\} \cup Y_1 \cup Y_2\} \cup \{vx|x \in V(T) - \{w\}\}$. We now define a graph $H \in \mathcal{H}_2$ as follows.

Let

$$H = \begin{cases} H^{'} & \text{if } n_3 \geq 2 \\ H^{'} \text{or } H^{'} - \{wz\} & \text{if } n_1 = 1 \text{ and } n_3 = 1 \text{ where } Y_1 = \{z\}. \end{cases}$$

It is not difficult to show that $H \in \mathcal{H}_2$ is a 3-$(\gamma, 2)$-critical graph which is not 3-γ-critical. Note that $\gamma(H + uv) = 3$. Figures 3.3, 3.4, 3.5 and 3.6 illustrate our constructions.

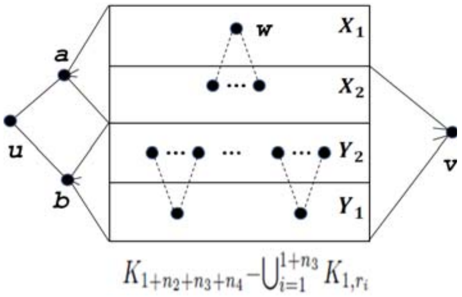


Figure 3.3: Graph $H = H^{'}$ when $|X_1| = n_1 = 1$ and $|Y_1| = n_3 \geq 2$

We conclude this section by establishing the fourth class. Let k be a positive integer where $k \geq 2$. For $1 \leq j \leq k$, let H_j be a graph of order at least 3 where $V(H_j)$ is partitioned to $2n_j + 1$ sets, say $V(H_0^j), V(H_1^j), ..., V(H_{2n_j}^j)$ for

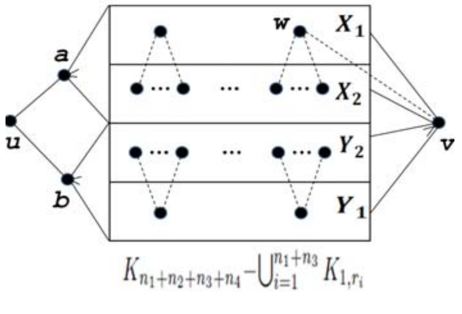


Figure 3.4: Graph $H = H^{'}$ when $|X_1| = n_1 \geq 2$ and $|Y_1| = n_3 \geq 2$

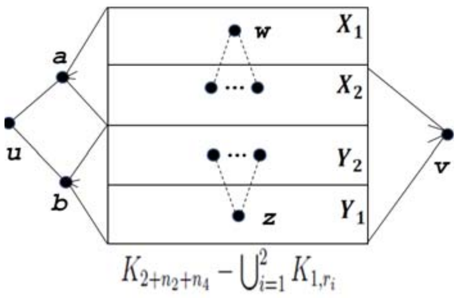


Figure 3.5: Graph $H = H^{'}$ when $|X_1| = n_1 = 1$ and $|Y_1| = n_3 = 1$

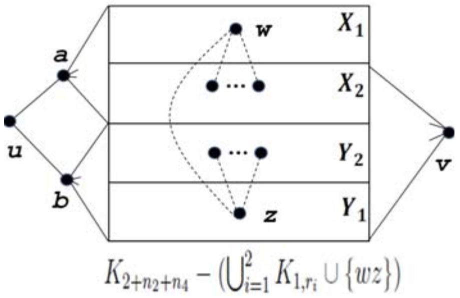


Figure 3.6: Graph $H = H^{'} - \{wz\}$ when $|X_1| = n_1 = 1$ and $|Y_1| = n_3 = 1$

some positive integer n_j where $|V(H_0^j)| = 1$ and for $1 \leq m \leq n_j$, $|V(H_{2m}^j)| \leq |V(H_{2m-1}^j)|$. The edges of H_j are defined as follows. For $1 \leq m \leq n_j$, each vertex of $V(H_{2m-1}^j)$ is adjacent to exactly one vertex of $V(H_{2m-2}^j)$ and exactly one vertex of $V(H_{2m}^j)$ in such a way that each vertex of $V(H_{2m}^j)$ is adjacent to at least one vertex of $V(H_{2m-1}^j)$. Figure 3.7 shows some examples of H_1, H_2, H_3 and H_4 where $|V(H_i)| = 9$ for $1 \leq i \leq 4$.

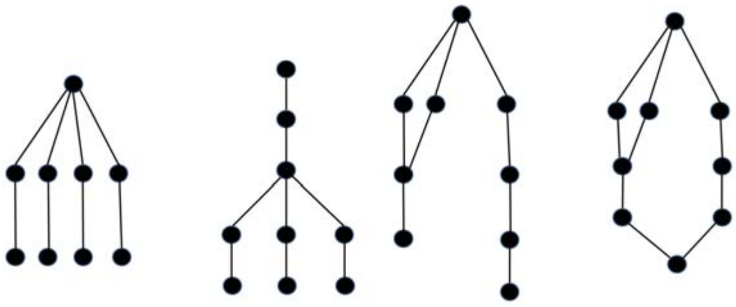


Figure 3.7: Graphs H_1, H_2, H_3 and H_4

Let k_1 and k_2 be positive integers where $k_1 + k_2 = k$. Put $T_1 = \bigcup_{j=1}^{k_1} H_j$ and $T_2 = \bigcup_{j=k_1+1}^{k} H_j$. Let either

$$Z = \bigcup_{j=1}^{k_1} \bigcup_{m=0}^{\lfloor \frac{n_j}{2} \rfloor} V(H^j_{4m}) \cup \bigcup_{j=k_1+1}^{k} \bigcup_{m=0}^{\lfloor \frac{n_j}{2} \rfloor} V(H^j_{4m+2})$$

and

$$W = \bigcup_{j=1}^{k_1} \bigcup_{m=0}^{\lfloor \frac{n_j}{2} \rfloor} V(H^j_{4m+2}) \cup \bigcup_{j=k_1+1}^{k} \bigcup_{m=0}^{\lfloor \frac{n_j}{2} \rfloor} V(H^j_{4m})$$

or

$$Z = \bigcup_{j=1}^{k_1+k_2} \bigcup_{m=0}^{\lfloor \frac{n_j}{2} \rfloor} V(H^j_{4m})$$

and

$$W = \bigcup_{j=1}^{k_1+k_2} \bigcup_{m=0}^{\lfloor \frac{n_j}{2} \rfloor} V(H^j_{4m+2})$$

and let $X = (V(T_1) \cup V(T_2)) - (Z \cup W)$. Put $n_0 = |V(T_1)| + |V(T_2)|$. Next, let $Q_0 = K_{n_0} - (E(T_1) \cup E(T_2))$ and for a positive integer $n_1 \geq 2$, let $Q_1 = K_{n_1} - \bigcup_{j=1}^{l_1} K_{1,r^1_j}$ where $l_1 + \sum_{j=1}^{l_1} r^1_j = n_1$ and $n_1 \geq 2l_1$. Define a graph $G \in \mathcal{H}^1_3$ as follows. Set $V(G) = V(Q_0) \cup V(Q_1) \cup \{u, a, b, s_1, s_2\}$ of order $n_0 + n_1 + 5$ and $E(G) = E(Q_0) \cup E(Q_1) \cup \{ax, s_1x | x \in Z \cap V(T_1)\} \cup \{ax, s_2x | x \in Z \cap V(T_2)\} \cup \{bx, s_2x | x \in W \cap V(T_1)\} \cup \{bx, s_1x | x \in W \cap V(T_2)\} \cup \{xy | x \in X, y \in \{a, b, s_1, s_2\} \cup V(Q_1)\} \cup \{xy | x \in \{a,\ s_1, s_2\} \cup Z \cup W, y \in V(Q_1)\} \cup \{ua, ub, s_1s_2\}$.

For a positive integer $n_2 \geq 2$, let $Q_2 = K_{n_2} - \bigcup_{j=1}^{l_2} K_{1,r^2_j}$ where $l_2 + \sum_{j=1}^{l_2} r^2_j = n_2$ and $n_2 \geq 2l_2$. Define a graph $G \in \mathcal{H}^2_3$ as follows. Set $V(G) = V(G') \cup V(Q_2)$ where $G' \in \mathcal{H}^1_3$ and $E(G) = E(G') \cup \big\{xy | x \in V(G') - \{u, a\}, y \in V(Q_2)\big\}$.

Put $\mathcal{H}_3 = \mathcal{H}^1_3 \cup \mathcal{H}^2_3$. It is not difficult to show that $G \in \mathcal{H}_3$ is a 3-$(\gamma, 2)$-critical graph which is not 3-γ-critical. Figures 3.8 and 3.9 illustrate our constructions. Note that in Figures 3.8 and 3.9, the vertex a joins to every vertex of $V(Q_1) \cup (Z \cap V(T_1)) \cup (Z \cap V(T_2)) \cup X$ and the vertex b joins to every vertex of $V(Q_2) \cup (W \cap V(T_1)) \cup (W \cap V(T_2)) \cup X$.

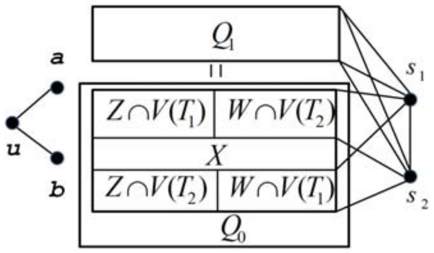


Figure 3.8: Graph $G \in \mathcal{H}^1_3$

3.2 Main results

We begin this section with some basic properties of k-$(\gamma, 2)$-critical graphs and 3-$(\gamma, 2)$-critical graphs of diameter 3 which are not 3-γ-critical.

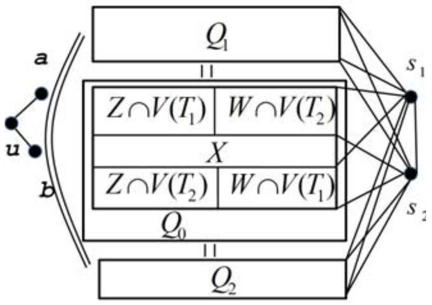


Figure 3.9: Graph $G \in \mathcal{H}_3^2$

For a pair of non-adjacent vertices x and y of G, D_{xy} denotes a minimum dominating set of $G + xy$. Our first result follows immediately by the definition of k-(γ, t)-critical graphs.

Lemma 3.2.1. *For integers $k \geq 2$ and $t \geq 2$, suppose G is a k-(γ, t)-critical graph and x and y are non-adjacent vertices of G with $d(x, y) \leq t$. Then*

(1) $|D_{xy}| = k - 1$ and $|D_{xy} \cap \{x, y\}| = 1$.

(2) If $x \in D_{xy}$ and $y \notin D_{xy}$, then no vertex of $D_{xy} - \{x\}$ is adjacent to y.

Let G be a 3-$(\gamma, 2)$-critical graph of diameter 3 which is not 3-γ-critical. Then there exist vertices u and v of G with $d(u, v) = 3$ and $\gamma(G + uv) = 3$. For $1 \leq i \leq 3$, let $V_i = \{x \in V(G) | d(u, x) = i\}$. Put $V_0 = \{u\}$. Note that $V_i \neq \emptyset$ and $V(G) = V_0 \cup V_1 \cup V_2 \cup V_3$. Further, for $1 \leq i \leq 3$, if $x \in V_i$, then there is a vertex $y \in V_{i-1}$ such that $xy \in E(G)$. Throughout the rest of our thesis, the symbols $G, u, v, V_i\ ; 0 \leq i \leq 3$, will refer specifically to this set up.

Our next result follows by the fact that $\gamma(G) = 3$ and each vertex of V_i is adjacent to some vertex of V_{i-1} for $1 \leq i \leq 3$.

Lemma 3.2.2. *Let G be a 3-$(\gamma, 2)$-critical graph of diameter 3 which is not 3-γ-critical. Then $|V_2| \geq 2$.*

Proof. Suppose that $V_2 = \{y\}$. Then each vertex of V_3 is adjacent to y . Thus $\{u, y\} \succ G$, a contradiction. Hence $|V_2| \geq 2$. This completes the proof of our lemma. $\square$

According to Lemma 3.2.1, if G is 3-$(\gamma, 2)$-critical and x and y are non-adjacent vertices of G with $d(x, y) = 2$, then $|D_{xy}| = 2$ and either $D_{xy} \cap \{x, y\} = \{x\}$ or $D_{xy} \cap \{x, y\} = \{y\}$. Then the next two easy results follow.

Lemma 3.2.3. *Let G be a 3-$(\gamma, 2)$-critical graph of diameter 3 which is not 3-γ-critical and $\{x, y\} \subseteq V(G)$ where $d(x, y) = 2$. Suppose $G[V_1]$ is complete and $x \in V_1$. Then $D_{xy} = \{x, z\}$ for some $z \in (V_2 \cup V_3) - \{y\}$.*

Proof. Consider $G + xy$. Then $D_{xy} = \{x, z\}$ or $D_{xy} = \{y, z\}$ for some $z \in V(G) - \{x, y\}$. If $D_{xy} = \{y, z\}$, then, in order to dominate u, $z \in \{u\} \cup V_1$. So $zx \in E(G)$, contradicting Lemma 3.2.1(1). Thus $D_{xy} = \{x, z\}$. Then, in order to dominate V_3, $z \in (V_2 \cup V_3) - \{y\}$. This completes the proof of our lemma. ☐

Lemma 3.2.4. *Let G be a 3-$(\gamma, 2)$-critical graph of diameter 3 which is not 3-γ-critical. Suppose $G[V_1]$ is complete and $x \in V_2$. Then $D_{ux} = \{x, z\}$ for some $z \in (V_2 \cup V_3) - \{x\}$.*

Proof. Clearly $d(u, x) = 2$. Consider $G + ux$. Then $D_{ux} = \{u, z\}$ or $D_{ux} = \{x, z\}$ for some $z \in V(G) - \{x, y\}$. If $D_{ux} = \{u, z\}$, then in order to dominate V_3, $z \in (V_2 \cup V_3) - \{x\}$ and thus $z \succ (V_2 \cup V_3) - \{x\}$. Hence, $\{z, y\} \succ G$ for some $y \in N_{V_1}(x)$, a contradiction. Therefore, $D_{ux} = \{x, z\}$. By Lemma 3.2.1(2), $z \in (V_2 \cup V_3) - \{x\}$. This completes the proof of our lemma. ☐

Lemma 3.2.5. *Let G be a 3-$(\gamma, 2)$-critical graph of diameter 3 which is not 3-γ-critical. Suppose $G[V_1]$ is complete. If x and y are vertices of $V_2 \cup V_3$ and $d(x, y) = 2$, then $|D_{xy} \cap (\{u\} \cup V_1)| = 1$. Further, $x \succ V_3 - \{y\}$ or $y \succ V_3 - \{x\}$.*

Proof. Consider $G + xy$. Then $D_{xy} = \{x, z\}$ or $D_{xy} = \{y, z\}$ for some $z \in V(G) - \{x, y\}$. Then, in order to dominate u, $z \in \{u\} \cup V_1$. Hence, $|D_{xy} \cap (\{u\} \cup V_1)| = 1$ as required. Clearly, z is not adjacent to any vertex of V_3. Thus $x \succ V_3 - \{y\}$ or $y \succ V_3 - \{x\}$. This completes the proof of our lemma. ☐

Our first theorem provides a characterization of G when $G[V_1]$ is complete.

Theorem 3.2.6. *Let G be a 3-$(\gamma, 2)$-critical graph of diameter 3 which is not 3-γ-critical. If $G[V_1]$ is complete and each vertex of V_1 is adjacent to every vertex of V_2. Then*

(1) $G[V_2]$ is complete.

(2) $|V_3| \geq 3$.

(3) v is the only vertex of G such that $\gamma(G+uv)=3$.

(4) $G \in \mathcal{J}$ defined in Section 3.1.

Proof. By our hypothesis we have the following observation.
Observation 1: Each vertex of V_1 dominates $\{u\} \cup V_1 \cup V_2$.

Claim 1: No vertex $x \in V_2 \cup V_3$ such that $x \succ V_3$.

It follows immediately by Observation 1 and the fact $\gamma(G)=3$.

(1) Suppose there exist $a, b \in V_2$ such that $ab \notin E(G)$. Note that $d(a,b) = 2$. Consider $G+ab$. Then $D_{ab} = \{a, x\}$ or $D_{ab} = \{b, x\}$ for some $x \in \{u\} \cup V_1$, by Lemma 3.2.5. Without loss of generality, we may assume that $D_{ab} = \{a, x\}$. Then $a \succ V_3 - \{b\} = V_3$, by Lemma 3.2.5. But this contradicts Claim 1. Hence, $G[V_2]$ is complete. This proves (1).

By (1), Observation 2 follows.

Observation 2: Each vertex of V_2 dominates $V_1 \cup V_2$.

(2) Suppose to the contrary that $|V_3| \leq 2$. If $V_3 = \{v\}$, then $v_0 \succ V_3$ where $v_0 \in N_{V_2}(v)$. But this contradicts Claim 1. Hence, $|V_3| = 2$. Let $\{v'\} = V_3 - \{v\}$. By Claim 1, $vv' \notin E(G)$. Further, there exists $x \in N_{V_2}(v') - N_{V_2}(v)$. Then $\{u, x\} \succ G + uv$ by Observation 2, a contradiction. Hence, $|V_3| \geq 3$. This proves (2).

Let $S = \{x \in V_3 | \gamma(G+ux) = 3\}$. Clearly $S \neq \phi$ since $v \in S$. In order to prove (3) and (4), we need to establish the following claims.

Claim 2: For each $s \in S$, $s \succ V_2$.

Let $s \in S$. Suppose to the contrary that there exists $y \in V_2$ such that $ys \notin E(G)$. Note that $d(s,y) = 2$ by (1) and the fact that $N_{V_2}(s) \neq \phi$. Consider $G + sy$. Then $D_{sy} = \{s, x\}$ or $D_{sy} = \{y, x\}$ for some $x \in \{u\} \cup V_1$ by Lemma 3.2.5. If $D_{sy} = \{s, x\}$, then $s \succ V_3$, contradicting Claim 1. Hence, $D_{sy} \neq \{s, x\}$. Therefore, $D_{sy} = \{y, x\}$. Then $y \succ V_3 - \{s\}$. It follows that $y \succ V(G) - \{u, s\}$ by Observation 2. Consequently, $\{u, y\} \succ G + us$. But this contradicts the fact that $s \in S$. Hence, $s \succ V_2$. This settles our claim.

Claim 3: For each $s \in S$ there exists $t \in V_3 - S$ such that $st \notin E(G)$ and $t \succ V_3 - \{s\}$.

Let $s \in S$. Choose $u_1 \in V_1$. Clearly, $d(u_1, s) = 2$. Consider $G + u_1s$. Then, by Lemma 3.2.3, $D_{u_1s} = \{u_1, t\}$ for some $t \in (V_2 \cup V_3) - \{s\}$. By Claim 2 and Lemma 3.2.1(2), $t \in V_3$ and thus $t \succ V_3 - \{s\}$. If $t \in S$, then $t \succ (V_2 \cup V_3) - \{s\}$ by Claim 2. But then $\{u, t\} \succ G + us$, a contradiction. Hence, $t \in V_3 - S$. This settles our claim.

Claim 4: If $x \in V_3 - S$ and $xs \notin E(G)$ for some $s \in S$, then $x \succ V_3 - \{s\}$. Further, there exists $y \in V_2$ such that $yx \notin E(G)$ and $y \succ V_3 - \{x\}$.

By Claim 2 and the fact that $N_{V_2}(x) \neq \phi$, $d(x, s) = 2$. By Lemma 3.2.5, $x \succ V_3 - \{s\}$ or $s \succ V_3 - \{x\}$. If $s \succ V_3 - \{x\}$, then $\{s, z\} \succ G + us$ for some $z \in N_{V_2}(x)$ by Observation 2, a contradiction. Hence, $x \succ V_3 - \{s\}$.

Suppose to the contrary that $x \succ V_2$. Then $x \succ (V_2 \cup V_3) - \{s\}$. So $\{u, x\} \succ G + us$, again a contradiction. Hence, there exists $y \in V_2$ such that $yx \notin E(G)$. Consider $G + yx$. Then $D_{yx} = \{y, z_1\}$ or $D_{yx} = \{x, z_1\}$ for some $z_1 \in \{u\} \cup V_1$ by Lemma 3.2.5. If $D_{yx} = \{x, z_1\}$, then no vertex of D_{yx} is adjacent to s, a contradiction. Hence, $D_{yx} \neq \{x, z_1\}$. Therefore, $D_{yx} = \{y, z_1\}$. Thus $y \succ V_3 - \{x\}$. This settles our claim.

Claim 5: For each $x \in V_2$, there exists a unique vertex $y \in V_3 - S$ such that $xy \notin E(G)$ and $x \succ V_3 - \{y\}$.

Let $x \in V_2$. Then, by Claims 1 and 2, there exists $y \in V_3 - S$ and $xy \notin E(G)$. Clearly, $d(x, y) = 2$. Consider $G + xy$. Then $D_{xy} = \{y, z\}$ or $D_{xy} = \{x, z\}$ for some $z \in \{u\} \cup V_1$ by Lemma 3.2.5. If $D_{yx} = \{y, z\}$, then $y \succ V_3$. But this contradicts Claim 1. Hence, $D_{xy} \neq \{y, z\}$. Therefore, $D_{xy} = \{x, z\}$. It follows that $x \succ V_3 - \{y\}$. This settles our claim.

Claim 6: $S = \{v\}$.

Suppose to the contrary that there exists $s \in S$ and $s \neq v$. By Claim 3, there are $x, y \in V_3 - S$ such that $xv \notin E(G)$, $ys \notin E(G)$, $x \succ V_3 - \{v\}$ and $y \succ V_3 - \{s\}$. Clearly $x \neq y$ and $xs \in E(G)$. By Claim 4, there exists $t \in V_2$ such that $tx \notin E(G)$ and $t \succ V_3 - \{x\}$. So $t \succ (V_1 \cup V_2 \cup V_3) - \{x\}$. Hence $\{s, t\} \succ G + us$, a contradiction. So $S = \{v\}$. This settles our claim.

Let $V^{'} = \{x \in V_3 | xy \notin E(G) \text{ for some } y \in V_2\}$ and $V^{''} = V_3 - (V^{'} \cup \{v\})$.

Note that for each $x \in V^{''}$, $x \succ V_2$ and $V^{'} \cap S = \phi$. Further, $V^{'} \neq \phi$ by Claim 5.

Claim 7: $|V^{'}| \geq 2$.

Suppose to the contrary that $V^{'} = \{x\}$. Then there exists $y \in V_2$ such that $yx \in E(G)$ since $x \in V_3$. So $y \succ V_3$. But this contradicts Claim 5. Hence, $|V^{'}| \geq 2$. This settles our claim.

Claim 8: For all $y \in V^{'}$, $yv \notin E(G)$.

Let $y \in V^{'}$. Then, by the definition of $V^{'}$, there exists $x \in V_2$ such that $xy \notin E(G)$. By Claim 5, $x \succ V_3 - \{y\}$. If $yv \in E(G)$, then $\{x, v\} \succ G + uv$, a contradiction. Hence, $yv \notin E(G)$. This settles our claim.

By Claims 4 and 8, each vertex of $V^{'}$ dominates $V_3 - \{v\}$. Thus following claims hold.

Claim 9: $G[V^{'}]$ is complete.

Claim 10: If $V^{''} \neq \phi$, then for each $x \in V^{''}$, $x \succ V_2 \cup V^{'}$.

Claim 11: $\overline{G}[V_2 \cup V^{'}] \cong \bigcup_{i=1}^{n} K_{1,r_i}$. Further, if x_0 is the center of K_{1,r_i} in $\overline{G}[V_2 \cup V^{'}]$, then $x_0 \in V^{'}$.

By Claim 5 and the fact that V_2 and $V^{'}$ are complete, $\gamma(G[V_2 \cup V^{'}]) = 2$. Let $x \in V_2$ and $y \in V^{'}$ such that $xy \notin E(G)$. Clearly, $d(x, y) = 2$. By Claim 5, $\gamma(G[V_2 \cup V^{'}] + xy) = 1$. Hence, $G[V_2 \cup V^{'}]$ is 2-γ-critical. By Theorem 2.1, $\overline{G}[V_2 \cup V^{'}] \cong \bigcup_{i=1}^{n} K_{1,r_i}$. Let x_0 be the center and let $x_1, x_2, ..., x_r$ be the end vertices of $K_{1,r}$ in $\overline{G}[V_2 \cup V^{'}]$. If $r = 1$, then we may assume with out loss of generality that $x_0 \in V^{'}$. In the case $r \geq 2$, $x_0 \in V^{'}$ by Claims 5 and 9 together with (1). This settles our claim.

It follows by Claims 4 and 11, that $|V_2| \geq |V^{'}|$.

Claim 12: If $V^{''} \neq \phi$, then for each $x \in V^{''}$, $xv \in E(G)$.

Let $x \in V^{''}$. Suppose to the contrary that $xv \notin E(G)$. By Claim 4, $x \succ V_3 - \{v\}$. So $\{u, x\} \succ G + uv$ by Claim 10, a contradiction. Hence, $xv \in E(G)$. This settles our claim.

Claim 13: If $V^{''} \neq \phi$, then $G[V^{''}] = K_{2k}$- a perfect matching for some $k \geq 1$.

Let $x \in V^{''}$. Then, by Claim 1, there exists $y \in V_2 \cup V_3$ such that $xy \notin E(G)$. By Claims 10 and 12, $y \in V^{''}$. Note that $d(x, y) = 2$. By Lemma 3.2.5, $x \succ V_3 - \{y\}$ or $y \succ V_3 - \{x\}$. Without loss of generality, we may assume that $x \succ V_3 - \{y\}$. Consider $G + u_1x$ where $u_1 \in V_1$. Then, by Lemma 3.2.3, $D_{u_1x} = \{u_1, c\}$ for some $c \in (V_2 \cup V_3) - \{x\}$. So $c \succ V_3 - \{x\}$. Thus $c = y$, by Claim 10 and Lemma 3.2.1(2). If $|V^{''}| = 2$, we are done. So suppose $|V^{''}| \geq 3$. Choose $x_1 \in V^{''} - \{x, y\}$. By applying similar arguments as above, it is not difficult to see that, $G[V^{''}] = K_{2n}$- a perfect matching. This settles our claim.

By Claim 6, (3) holds. Further, by our hypothesis, (1) and Claims 2,6,8,10,11, 12 and 13, (4) holds. □

The converse of Theorem 3.2.6(1) is not true. Figure 3.10 shows an example of 3-$(\gamma, 2)$-critical graph of diameter 3 which is not 3-γ-critical where $G[V_1]$ and $G[V_2]$ are complete but $G[V_1 \cup V_2]$ is not complete.

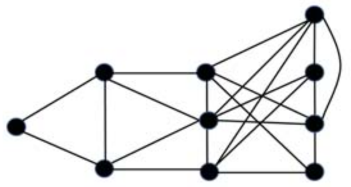


Figure 3.10: A 3-$(\gamma, 2)$-critical graph of order 10

In [2], Henning et al. gave an example of a class of 3-$(\gamma, 2)$-critical graphs of diameter 3 which is not 3-γ-critical as shown in Figure 2.1. It is easy to see that this class is a special case of graph in $\mathcal{J}$ when $n_2 = n_3$ and $n_4 = 0$.

Corollary 3.2.7. *Let G be a* 3-$(\gamma, 2)$-*critical graph of diameter 3 which is not* 3-γ-*critical. If* $G \in \mathcal{J}$ *where* $|V(G)| = 8$, *then* $|V_3| = 3$.

We now turn our attention to the case when $G[V_1] \cong 2K_1$.

Lemma 3.2.8. *Let G be a* 3-$(\gamma, 2)$-*critical graph of diameter 3 which is not* 3-γ-*critical. Suppose* $V_1 = \{a, b\}$ *and* $ab \notin E(G)$. *Let* $A = \{x \in V_2 | ax \in E(G)\}$ *and* $B = \{x \in V_2 | bx \in E(G)\}$. *Then*

(1) $A \neq \phi$ *and* $B \neq \phi$.

(2) $G[V_3]$ *is complete.*

(3) For each $x \in A - B$ and $y \in B - A$ either $d(x, y) = 1$ or $d(x, y) = 3$.

Proof. (1) Clearly, $A \neq \phi$ or $B \neq \phi$ since $V_2 \neq \phi$. Suppose to the contrary that $A \neq \phi$ but $B = \phi$. Then $xa \in E(G)$ and $xb \notin E(G)$ for all $x \in V_2$. But then $d(b, v) > 3$, a contradiction. This proves (1).

(2) Consider $G + ab$. Then $D_{ab} = \{a, x\}$ or $D_{ab} = \{b, x\}$ for some $x \in V(G) - \{a, b\}$. Without loss of generality, assume that $D_{ab} = \{a, x\}$. Then, in order to dominate V_3, $x \in V_2 \cup V_3$ and $x \succ V_3$. Suppose to the contrary that there exist $s, t \in V_3$ such that $st \notin E(G)$. Clearly, $x \notin \{s, t\}$ but $xs, xt \in E(G)$. Thus $d(s, t) = 2$. Consider $G + st$. Then $D_{st} = \{s, y\}$ or $D_{st} = \{t, y\}$ for some $y \in V(G) - \{s, t\}$. Without loss of generality, assume that $D_{st} = \{s, y\}$. Then, in order to dominate u, $y \in \{u, a, b\}$ and $y \succ \{u, a, b\}$. So $y = u$ since $ab \notin E(G)$ and thus $s \succ (V_2 \cup V_3) - \{t\}$. Clearly, $d(a, s) = 2$. Now consider $G + as$. Then $D_{as} = \{a, z\}$ or $D_{as} = \{s, z\}$ for some $z \in V(G) - \{a, s\}$. If $D_{as} = \{s, z\}$, then z must dominate u and t, which is not possible since $d(u, t) = 3$. Hence, $D_{as} \neq \{s, z\}$. Therefore, $D_{as} = \{a, z\}$. Since $a \in V_1$ and $ab \notin E(G)$, it follows that $z \succ \{b\} \cup (V_3 - \{s\})$. Thus $z \in V_2$. But this contradicts Lemma 3.2.1(2) since $s \succ V_2$. This proves (2).

(3) Let $x \in A - B$ and $y \in B - A$. Suppose that $d(x, y) = 2$. Consider $G + xy$. Then $D_{xy} = \{x, z\}$ or $D_{xy} = \{y, z\}$ for some $z \in V(G) - \{x, y\}$. Without loss of generality, assume that $D_{xy} = \{x, z\}$. Then $z \succ \{u, b\}$ and $zy \notin E(G)$ by Lemma 3.2.1(2). So $z = u$ and thus $x \succ (V_2 \cup V_3 \cup \{a\}) - \{y\}$. Hence, $\{b, x\} \succ G$, a contradiction. This proves (3) and completes the proof of our lemma. □

In order to establish our characterizations of 3-$(\gamma, 2)$-critical graph G with $G[V_1] = 2K_1$, we have to consider two cases. The case $A \cap B = \phi$ and the case $A \cap B \neq \phi$ where A and B are defined in Lemma 3.2.8. We begin with the case $A \cap B = \phi$.

Theorem 3.2.9. *Let G be a 3-$(\gamma, 2)$-critical graph of diameter 3 which is not 3-γ-critical. Further, let V_1, A and B be defined as in Lemma 3.2.8. If $A \cap B = \phi$, then $G \in \mathcal{H}_1$ or $G \in \mathcal{H}_2$ where $\mathcal{H}_1$ and $\mathcal{H}_2$ are defined in Section 3.1.*

Proof. Clearly, $V_2 = (A - B) \cup (B - A)$.

Claim 1: $A = A - B \neq \phi$ and $B = B - A \neq \phi$.

Our claim follows by Lemma 3.2.8(1) and the hypothesis that $A \cap B = \phi$.

Claim 2: No vertex of V_2 dominates V_3 if and only if $G[A-B]$ and $G[B-A]$ are complete.

Suppose $G[A-B]$ and $G[B-A]$ are complete. If there exists a vertex $x \in V_2$ such that $x \succ V_3$, then either $\{x, a\} \succ G$ or $\{x, b\} \succ G$, a contradiction. Hence, no vertex of V_2 dominates V_3. This proves the sufficiency.

We now assume that no vertex of V_2 dominates V_3. Suppose to the contrary that $G[A-B]$ or $G[B-A]$ is not complete. Without loss of generality, suppose $G[A-B]$ is not complete. Then there are $x, y \in A-B$ such that $xy \notin E(G)$. Clearly, $d(x, y) = 2$. Consider $G + xy$. Then $D_{xy} = \{x, z\}$ or $D_{xy} = \{y, z\}$ for some $z \in V(G) - \{x, y\}$. Thus $z \succ \{u, b\}$. So $z \in \{u\} \cup V_1$. Hence, $x \succ V_3$ or $y \succ V_3$, a contradiction. This proves the necessity and settles our claim.

We now distinguish two cases.

Case 1: No vertex of V_2 dominates V_3.

By Claim 2, $G[A-B]$ and $G[B-A]$ are complete.

Claim 3: For all $x \in A-B$ and for all $y \in B-A$, $xy \notin E(G)$.

Suppose to the contrary that there exist $x \in A-B$ and $y \in B-A$ such that $xy \in E(G)$. Then $d(a, y) = 2$. Consider $G + ay$. Then $D_{ay} = \{a, z\}$ or $D_{ay} = \{y, z\}$ for some $z \in V(G) - \{a, y\}$. If $D_{ay} = \{a, z\}$, then $z \succ \{b\} \cup V_3$. Thus $z \in B = B - A$. But this contradicts Lemma 3.2.1(2) since $G[B-A]$ is complete. Hence, $D_{ay} \neq \{a, z\}$. Therefore, $D_{ay} = \{y, z\}$. Clearly, $z \succ \{u\}$ and $za \notin E(G)$ by Lemma 3.2.1(2). So $z = b$. It follows that $y \succ V_3$, contradicting the hypothesis of Case 1. Hence, $xy \notin E(G)$ for all $x \in A-B$ and for all $y \in B-A$. This settles our claim.

Claim 4: For all $x \in V_3$, $N_{A-B}(x) = \phi$ or $N_{B-A}(x) = \phi$.

Suppose to the contrary that $N_{A-B}(x) \neq \phi$ and $N_{B-A}(x) \neq \phi$ for some $x \in V_3$. Let $z \in N_{A-B}(x)$ and $y \in N_{B-A}(x)$. Then $d(z, y) \leq 2$. By Claim 3, $d(z, y) = 2$, contradicting Lemma 3.2.8(3). This settles our claim.

Claim 5: For all $x \in V_3$, $x \succ A-B$ or $x \succ B-A$.

Let $x \in V_3$. Then there exists $z \in V_2$ such that $zx \in E(G)$. Without loss of generality, we may assume that $z \in A-B$. By Claim 4, $N_{B-A}(x) = \phi$. We need to show that $x \succ A-B$. Suppose to the contrary that there exists $w \in A-B$ such that $wx \notin E(G)$. Note that $d(u,w)=2$. Consider $G+uw$. Then $D_{uw}=\{w,t\}$ or $D_{uw}=\{u,t\}$ for some $t \in V(G)-\{u,w\}$. If $D_{uw}=\{w,t\}$, then t must dominate $\{x,b\}$. So $t \in B-A$ and $tx \in E(G)$, a contradiction. Hence, $D_{uw} \neq \{w,t\}$. Therefore, $D_{uw}=\{u,t\}$. Then $t \succ (V_2 \cup V_3)-\{w\}$. Thus $t \in V_2 \cup V_3$. Note that $B-A \neq \phi$ by Claim 1 and $(A-B)-\{w\} \neq \phi$ since $z \in (A-B)-\{w\}$. It then follows by Claim 3 that, $t \in V_3$ and $t \succ ((B-A) \cup (A-B))-\{w\}$. But this contradicts Claim 4. Hence, $x \succ A-B$. This settles our claim.

Let $C=\{x \in V_3 | x \succ A-B\}$ and $D=\{x \in V_3 | x \succ B-A\}$. Note that C and D are not empty otherwise $d(x,y)>3$ for $x \in A-B$ and $y \in B-A$. It also follows by Claims 4 and 5 that $C \cap D=\phi$. Further, $G[V_3]=G[C \cup D]$ is complete by Lemma 3.2.8(2).

Claim 6: $|A-B|=1$ or $|D|=1$ and $|B-A|=1$ or $|C|=1$.

Suppose that $|A-B| \geq 2$ and $|D| \geq 2$. Let $x \in A-B$ and $y \in D$. Clearly $d(x,y)=2$. Consider $G+xy$. Then $D_{xy}=\{x,z\}$ or $D_{xy}=\{y,z\}$ for some $z \in V(G)-\{x,y\}$. If $D_{xy}=\{x,z\}$, then z must dominate $\{u\} \cup (D-\{y\})$, which is not possible since $d(u,y^{'})=3$ for $y^{'} \in D-\{y\}$. Hence, $D_{xy} \neq \{x,z\}$. Therefore $D_{xy}=\{y,z\}$. Then z must dominate $(\{u\} \cup V_1 \cup (A-B))-\{x\}$. But this is not possible since $ab \notin E(G)$. Hence, $|A-B|=1$ or $|D|=1$. By a similar argument, $|B-A|=1$ or $|C|=1$. This settles our claim.

It follows by Claims 1-6 that G is isomorphic to a graph in $\mathcal{H}_1$.

Case 2: There exists a vertex of V_2, say c, such that $c \succ V_3$.

Without loss of generality, assume that $c \in A-B$. It follows by Claim 2 that either $G[A-B]$ or $G[B-A]$ is not complete.

Claim 7: For each $x \in B-A$, there exists $y \in (B-A)-\{x\}$ such that $xy \notin E(G)$ and either $x \succ V_3$ or $y \succ V_3$.

Let $x \in B-A$. We first suppose that $xx^{'} \in E(G)$ for some $x^{'} \in A-B$. Then $d(x,a)=2$. Consider $G+xa$. Then $D_{xa}=\{x,z\}$ or $D_{xa}=\{a,z\}$ for some $z \in V(G)-\{x,a\}$. Suppose first that $D_{xa}=\{x,z\}$. Then z must dominate $\{u\}$. By Lemma 3.2.1(2), $z=b$. So $x \succ (A-B) \cup V_3$. If $x \succ B-A$, then $\{x,a\} \succ G$, a contradiction. Hence, there exists $y \in (B-A)-\{x\}$ such that $xy \notin E(G)$ and

$x \succ V_3$ as required. We now suppose that $D_{xa} = \{a, z\}$. Then $z \succ \{b\} \cup V_3$. Thus $z \in (B - A) - \{x\}$, $zx \notin E(G)$ by Lemma 3.2.1(2) and $z \succ V_3$ as required.

We now suppose that x is not adjacent to any vertex of $A - B$. By Lemma 3.2.8(3), $d(x, c) = 3$. Then $N_G(x) \cap N_G(c) = \phi$. Since $c \succ V_3$, x is not adjacent to any vertex of V_3. We next show that c is adjacent to some vertex of $(B - A) - \{x\}$. Suppose this is not the case. Then for each $w \in (B - A) - \{x\}$, $N_{V_3}(w) = \phi$ by Lemma 3.2.8(3) and the fact that $c \succ V_3$. Consequently, x, b, u, a, c is a shortest $x - c$ path in G. But this contradicts the fact that $diam(G) = 3$. Hence, $cy \in E(G)$ for some $y \in (B - A) - \{x\}$. Thus $xy \notin E(G)$ since $d(x, c) = 3$. Now consider $G + xy$. Then $D_{xy} = \{x, s\}$ or $D_{xy} = \{y, s\}$ for some $s \in V(G) - \{x, y\}$. In either case, $s \succ \{u, a\}$. Then $s \in \{u, a\}$ by Lemma 3.2.1(2). Hence, $x \succ V_3$ or $y \succ V_3$ as required. This settles our claim.

Claim 8: For each $x \in A - B$, there exists $y \in (A - B) - \{x\}$ such that $xy \notin E(G)$ and $x \succ V_3$ or $y \succ V_3$.

By Claim 7, there is a vertex $c^{'} \in B - A$ such that $c^{'} \succ V_3$. By applying arguments as in the proof of Claim 7, our claim follows.

Claim 9: For each $x \in V_3$, $d(a, x) = 2$ and $d(b, x) = 2$.

Our claim follows by Claims 7 and 8.

Claim 10: $V_3 = \{v\}$.

Suppose that $|V_3| \geq 2$. Let $s, t \in V_3$ such that $s \neq t$. By Claim 9, $d(a, s) = 2$ and $d(b, s) = 2$. Consider $G + as$. Then $D_{as} = \{a, x\}$ or $D_{as} = \{s, x\}$ for some $x \in V(G) - \{a, s\}$. We first suppose that $D_{as} = \{a, x\}$. Then x dominates $\{b, t\}$. So $x \in B - A$ and $x \succ (B - A) \cup (V_3 - \{s\})$. But this contradicts Claim 7. Hence, $D_{as} = \{s, x\}$. Then x must dominate u and $xa \notin E(G)$ by Lemma 3.2.1(2). So $x = b$ and $s \succ (A - B) \cup V_3$. Now consider $G + bs$. Then $D_{bs} = \{b, y\}$ or $D_{bs} = \{s, y\}$ for some $y \in V(G) - \{b, s\}$. If $D_{bs} = \{b, y\}$, then $y \succ \{a\} \cup (A - B) \cup (V_3 - \{s\})$. Thus $y \in A - B$ and $ys \notin E(G)$ by Lemma 3.2.1(2). But this contradicts the fact that $s \succ (A - B) \cup V_3$. Hence, $D_{bs} \neq \{b, y\}$. Therefore, $D_{bs} = \{s, y\}$. Then $y \succ \{u, a\}$. It follows that $y = a$ by Lemma 3.2.1(2). Thus $s \succ (B - A) \cup V_3$. Hence, $s \succ V_2 \cup V_3$. Consequently, $\{s, u\} \succ G$, a contradiction. Hence, $|V_3| = 1$. This settles our claim.

Claim 11: $G[B - A]$ is 2-γ-critical.

By Claim 7, $\gamma(G[B-A]) \geq 2$. Let $y \in B-A$. Then there exists $z \in B-A$ such that $zy \notin E(G)$. Clearly, $d(y,z)=2$. Consider $G+yz$. Then the only vertex of $D_{yz}-\{y,z\}$ dominates $\{u,a\}$. Thus it must belong to $\{u,a\}$ by Lemma 3.2.1(2). Hence, $y \succ (B-A)-\{z\}$ or $z \succ (B-A)-\{y\}$. So $\gamma(G[B-A])=2$. It also follows that $\gamma(G[B-A]+yz)=1$. Hence, $G[B-A]$ is 2-γ-critical. This settles our claim.

Claim 12: $G[A-B]$ is 2-γ-critical.

Clearly, $\gamma(G[B-A]) \geq 2$ by Claim 8. By similar arguments as in the proof of Claim 11, our claim follows.

By Theorem 2.1, $\overline{G}[A-B] \cong \bigcup_{i=1}^{n} K_{1,r_i}$ and $\overline{G}[B-A] \cong \bigcup_{i=1}^{m} K_{1,s_i}$ where n, m, r_i, s_i are positive integers. Let X_1 and X_2 be the sets of all centers and the end vertices of K_{1,r_i} in $\overline{G}[A-B]$, respectively. Let Y_1 and Y_2 be the sets of all centers and the end vertices of K_{1,s_i} in $\overline{G}[B-A]$, respectively. Note that $G[X_i]$ and $G[Y_i]$ are complete where $1 \leq i \leq 2$.

Consider v. If $v \succ V_2$, then $\{u,v\} \succ G$, a contradiction. So there exists $w \in (A-B) \cup (B-A)$ such that $wv \notin E(G)$. Assume that $w \in A-B$.

Claim 13: w is the center of K_{1,r_i} in $\overline{G}(A-B)$ for some i.

It is easy to see that if $r_i=1$, then we may assume that w is the center of K_{1,r_i} in $\overline{G}(A-B)$ as required. We may now suppose that $r_i \geq 2$. Suppose to the contrary that w is an end vertex of K_{1,r_i} in $\overline{G}(A-B)$. Let x_0 and w_1 be the center and the other end vertex of K_{1,r_i} in $\overline{G}(A-B)$, respectively. Consider $G+x_0w$. Then $D_{x_0w}=\{x_0,z\}$ or $D_{x_0w}=\{w,z\}$ for some $z \in V(G)-\{x_0,w\}$. In either case, $z \succ \{u,b\}$. Thus $z \in \{u,b\}$ since $ab \notin E(G)$. If $D_{x_0w}=\{x_0,z\}$, then no vertex of D_{x_0w} is adjacent to w_1, a contradiction. Hence, $D_{x_0w}=\{w,z\}$. But then no vertex of D_{x_0w} is adjacent to v, again a contradiction. Hence, w is the center of K_{1,r_i} in $\overline{G}(A-B)$ for some i. This settles our claim.

Claim 14: $d(v,w)=2$. Further, $v \succ V_2-\{w\}$.

Suppose to the contrary that $d(v,w)=3$. Then $N_G(v) \cap N_G(w)=\phi$. Since $w \in X_1$ and $G[X_1]$ is complete, it follows that v is not adjacent to any vertex of $X_1-\{w\}$. By Claim 9, there exists $z \in X_2$ such that $zv \in E(G)$. Thus $z \in X_2$ and $zw \notin E(G)$. We first show that $v \succ X_2$. Suppose that there exists $t \in X_2$ such that $vt \notin E(G)$. Clearly $d(v, t) = 2$. Consider $G + vt$. Then $D_{vt} = \{v, x_1\}$ or $D_{vt} = \{t, x_1\}$ for some $x_1 \in V(G) - \{v, t\}$. If $D_{vt} = \{v, x_1\}$, then $x_1 \succ \{u, a, b\}$ and thus $x_1 = u$. Consequently, $v \succ V_2 - \{t\}$. Thus $vw \in E(G)$, a contradiction. Hence, $D_{vt} = \{t, x_1\}$. Then $x_1 \succ \{u, b\}$ and thus $x_1 \in \{u, b\}$. It follows that $t \succ A - B$. But this contradicts Claim 8. So $v \succ X_2$ as required. Since $N_G(v) \cap N_G(w) = \phi$ and $v \succ X_2$, it follows that $\overline{G}([A - B]) = K_{1,r}$ for some positive integer r. Hence, $X_1 = \{w\}$.

Consider $G + ab$. Then $D_{ab} = \{a, x_2\}$ or $D_{ab} = \{b, x_2\}$ for some $x_2 \in V(G) - \{a, b\}$. We first suppose that $D_{ab} = \{b, x_2\}$. Then $x_2 \in (B - A) \cup \{v\}$ and $x_2 \succ (A - B) \cup \{v\}$. If $x_2 = v$, then $vw \in E(G)$, a contradiction. So $x_2 \in B - A$. It follows that $N_G(v) \cap N_G(w) \neq \phi$, a contradiction. Hence, $D_{ab} = \{a, x_2\}$. Then $x_2 \in (A - B) \cup \{v\}$ and $x_2 \succ (B - A) \cup \{v\}$. Since $X_1 = \{w\}$ and $vw \notin E(G)$, $x_2 \in X_2 \cup \{v\}$. We next show that $v \succ B - A$. If $x_2 = v$, then $v \succ B - A$. We now assume that $x_2 \in X_2$, then $x_2 \succ B - A$. Suppose that there is $y \in Y_i$ such that $vy \notin E(G)$, for $1 \leq i \leq 2$. Consider $G + vy$. Then $D_{vy} = \{v, x_3\}$ or $D_{vy} = \{y, x_3\}$ for some $x_3 \in V(G) - \{v, y\}$. If $D_{vy} = \{v, x_3\}$, then $x_3 \succ \{u, a, b\}$ and thus $x_3 = u$. So $v \succ V_2 - \{y\}$. Hence, $vw \in E(G)$, a contradiction. Therefore, $D_{vy} = \{y, x_3\}$, then $x_3 \succ \{u, a\}$ and thus $x_3 \in \{u, a\}$. It follows that $y \succ B - A$. But this contradicts Claim 11. Hence, $v \succ B - A$ as required. In either case, $v \succ B - A$ and thus $v \succ X_2 \cup (B - A)$. It then follows by Lemma 3.2.8(3) that $z \succ B - A$ since $zv \in E(G)$. Consider $G + uz$. Then $D_{uz} = \{u, y_1\}$ or $D_{uz} = \{z, y_1\}$ for some $y_1 \in V(G) - \{u, z\}$. If $D_{uz} = \{u, y_1\}$, then $y_1 \succ (V_2 \cup \{v\}) - \{z\}$. By Lemma 3.2.1(2), $y_1 \notin (B - A) \cup \{v\} \cup X_2$. Thus $y_1 = w$. But then $wv \in E(G)$, a contradiction. Hence, $D_{uz} = \{z, y_1\}$, then $y_1 \succ \{b, w\}$. So $y_1 \in B - A$. It follows that $y_1 \in N_G(v) \cap N_G(w) = \phi$ since $v \succ B - A$, a contradiction. Hence, $d(v, w) = 2$ as required.

Now consider $G + vw$. Then $D_{vw} = \{v, x\}$ or $D_{vw} = \{w, x\}$ for some $x \in V(G) - \{v, w\}$. In either case, $x \succ \{u, b\}$. Thus $x \in \{u, b\}$ since $ab \notin E(G)$. If $D_{vw} = \{w, x\}$, then $w \succ A - B$ since no vertex of $\{u, b\}$ is adjacent to a vertex of $A - B$. But this contradicts Claim 12. Hence, $D_{vw} = \{v, x\}$. Therefore, $v \succ V_2 - \{w\}$. This settles our claim.

Claim 15: For $x \in (X_1 \cup X_2) - \{w\}$ and $y \in Y_1 \cup Y_2$, $xy \in E(G)$.

By Claim 14, $xv \in E(G)$ and $yv \in E(G)$. It then follows by Lemma 3.2.8(3) that $xy \in E(G)$.

Claim 16: If $|X_1| \geq 2$, then $wy \in E(G)$ for all $y \in Y_1 \cup Y_2$.

Let $w_1 \in X_1 - \{w\}$. Then there exists $x_1 \in X_2$ where x_1 is an end vertex of K_{1,r_i} centered at w_1 in $\overline{G}(A - B)$. Then $wx_1 \in E(G)$. By Claim 15, x_1 is adjacent to every vertex of $Y_1 \cup Y_2$. Thus $d(w, y) \leq 2$ for all $y \in Y_1 \cup Y_2$. It follows by Lemma 3.2.8(3), that $wy \in E(G)$ for all $y \in Y_1 \cup Y_2$. This settles our claim.

Claim 17: If $X_1 = \{w\}$, then $wz \in E(G)$ for some $z \in Y_1 \cup Y_2$.

Let $y \in Y_1$. Consider $G + uy$. Then $D_{uy} = \{u, x\}$ or $D_{uy} = \{y, x\}$ for some $x \in V(G) - \{u, y\}$. We first suppose that $D_{uy} = \{u, x\}$. Then $xy \notin E(G)$ and $x \succ (V_2 \cup \{v\}) - \{y\}$. Since $wv \notin E(G)$, $x \notin \{w, v\}$. By Claims 11 and 12, $x \in Y_2$. Clearly, $xw \in E(G)$ as required.

We now suppose that $D_{uy} = \{y, x\}$. Then $x \succ \{a\} \cup \overline{N}_{V_2}(y)$ but $x \neq a$ by Lemma 3.2.1(2). Thus $x \in A - B$. If $x = w$, then $w \succ \overline{N}_{V_2}(y)$. Thus $wy^{'} \in E(G)$ where $y^{'} \in Y_2$. If $x \in X_2$, then $yw \in E(G)$ as required since $xw \notin E(G)$. This settles our claim.

Claim 18: If $X_1 = \{w\}$, then $w \succ Y_1$ or $w \succ Y_2$.

It follows from Claim 17, Lemma 3.2.8(3) and the fact that $G[Y_i]$ is complete.

Claim 19: If $X_1 = \{w\}$ and $|Y_1| \geq 2$, then $w \succ B - A$.

By Claim 18, $w \succ Y_1$ or $w \succ Y_2$. We first assume that $w \succ Y_2$. Since $|Y_1| \geq 2$, there exist $x \in Y_1$ and $y \in Y_2$ such that $xy \in E(G)$. So $d(w, x) \leq 2$. By Lemma 3.2.8(3), $wx \in E(G)$. By Lemma 3.2.8(3) and the fact that $G[Y_1]$ is complete, $w \succ Y_1$. Hence, $w \succ B - A$ as required. By a similar argument, if $w \succ Y_1$, then $w \succ B - A$. This settles our claim.

Claim 20: Suppose $X_1 = \{w\}$ and $Y_1 = \{z\}$. If $wz \in E(G)$, then $w \succ Y_2$ or $|Y_2| = 1$.

Suppose to the contrary that w is not adjacent to some vertex of Y_2, say y_1, and $|Y_2| \geq 2$. Let $y_2 \in Y_2 - \{y_1\}$. Then $zy_i \notin E(G)$ for $1 \leq i \leq 2$. By Lemma 3.2.8(3) and the fact that $G[Y_2]$ is complete, w is not adjacent to any vertex of Y_2. Note that $d(a, y_1) = 2$ by Claim 15. Now consider $G + ay_1$. Then $D_{ay_1} = \{a, x\}$ or $D_{ay_1} = \{y_1, x\}$ for some $x \in V(G) - \{a, y_1\}$. We first suppose that $D_{ay_1} = \{a, x\}$. Then $x \succ \{b, v\}$. By Claims 11 and 15 together with Lemma 3.2.1(2), $x = z$. But then no vertex of D_{ay_1} is adjacent to y_2, a contradiction. Hence, $D_{ay_1} = \{y_1, x\}$.

Clearly, $x \succ \{u\}$. By Lemma 3.2.1(2), $x = b$. But then no vertex of D_{ay_1} is adjacent w, a contradiction. This settles our claim.

It follows by Claims 10-20 that if $|X_1| \geq 2$, then G is isomorphic to a graph in Figure 3.4. If $|X_1| = 1$ and $|Y_1| \geq 2$, then G is isomorphic to a graph in Figure 3.3. So we now suppose $X_1 = \{w\}$ and $Y_1 = \{z\}$. If $wz \in E(G)$ and $w \succ Y_2$, then G is isomorphic to a graph in Figure 3.5. If $wz \in E(G)$ and w is not adjacent to some vertex of Y_2, then $|Y_2| = 1$ by Claim 20 and thus G is isomorphic to a graph in Figure 3.6. Finally, if $wz \notin E(G)$, then G is isomorphic to a graph in Figure 3.6. Hence, G is in $\mathcal{H}_2$. This completes the proof of our theorem. $\square$

We now turn our attention to the case $A \cap B \neq \phi$.

Lemma 3.2.10. *Let G be a 3-$(\gamma, 2)$-critical graph of diameter 3 which is not 3-γ-critical. Further, let V_1, A and B be defined as in Lemma 3.2.8. Suppose $A \cap B \neq \phi$. Then*

(1) $G[A \cap B]$ is complete.

(2) $A \cap B \neq V_2$.

(3) For every vertex $c \in A \cap B$, there exist a unique vertex $x \in A - B$ and a unique vertex $y \in B - A$ such that $c \succ (V_1 \cup V_2 \cup V_3) - \{x, y\}$.

Proof. (1) Suppose to the contrary that $G[A \cap B]$ is not complete. Then there are $x, y \in A \cap B$ such that $xy \notin E(G)$. Clearly, $d(x, y) = 2$. Consider $G + xy$. Then $D_{xy} = \{x, s\}$ or $D_{xy} = \{y, s\}$ for some $s \in V(G) - \{x, y\}$. Without loss of generality, we may assume that $D_{xy} = \{x, s\}$. Then s must dominate u and $s \notin \{a, b\}$ by Lemma 3.2.1(2). So $s = u$ and $x \succ (V_2 \cup V_3) - \{y\}$. Hence, $\{a, x\} \succ G$, a contradiction. Therefore, $G[A \cap B]$ is complete. This proves (1).

(2) Suppose to the contrary that $A \cap B = V_2$. Then, by Lemma 3.2.8(2) and (1), $\{x, v\} \succ G + uv$ for each $x \in A \cap B$, a contradiction. Thus $A \cap B \neq V_2$. This proves (2).

(3) Let $c \in A \cap B$. If $c \succ (A - B) \cup (B - A)$, then $\{c, v\} \succ G + uv$ by (1) and Lemma 3.2.8(2), a contradiction. Hence, there exists a vertex $x \in (A - B) \cup (B - A)$ such that $cx \notin E(G)$. We may assume without loss of generality that $x \in A - B$. Consider $G + cx$. Then $D_{cx} = \{c, z\}$ or $D_{cx} = \{x, z\}$ for some $z \in V(G) - \{c, x\}$. In either case, $z \succ \{u\}$. We first suppose that $D_{cx} = \{x, z\}$. Then $z = u$ by

Lemma 3.2.1(2) and the fact that $c \in A \cap B$. Consequently, $x \succ (V_2 \cup V_3) - \{c\}$. Hence, $\{x, b\} \succ G$, a contradiction. Therefore, $D_{cx} = \{c, z\}$. Then $z \in \{u, b\}$ by Lemma 3.2.1(2). If $z = u$, then $c \succ (V_2 \cup V_3) - \{x\}$. But then $\{c, a\} \succ G$, again a contradiction. Hence, $z \neq u$. Therefore, $z = b$. Since b is not adjacent to any vertex of $(A - B) \cup V_3$, $c \succ ((A - B) - \{x\}) \cup V_3$. Hence, x is the unique vertex of $A - B$ such that $cx \notin E(G)$.

We next show that $c \succ (B - A) - \{y\}$ for some $y \in B - A$. It is easy to see that if $c \succ B - A$, then $\{c, a\} \succ G$, a contradiction. Hence, there exists $y \in B - A$ such that $cy \notin E(G)$. Consider $G + cy$. By similar arguments as above, $c \succ ((B - A) - \{y\}) \cup V_3$. Hence, y is the unique vertex of $B - A$ such $cy \notin E(G)$. It then follows that $c \succ (V_1 \cup V_2 \cup V_3) - \{x, y\}$ by (1) and the fact that $c \in A \cap B$. This proves (3) and completes the proof of our lemma. □

Lemma 3.2.11. *Let G be a 3-$(\gamma, 2)$-critical graph of diameter 3 which is not 3-γ-critical. Further, let V_1, A and B be defined as in Lemma 3.2.8. Suppose $A \cap B \neq \phi$. Then*

(1) $A - B \neq \phi$ and $B - A \neq \phi$.

(2) If $x \in A - B$ and $y \in B - A$ such that $\{x, y\} = \overline{N}_{V_2}(z)$ for some $z \in A \cap B$, then $xy \in E(G)$.

Proof. (1) It follows immediately by Lemma 3.2.10(3) since $A \cap B \neq \phi$.

(2) By Lemma 3.2.10(3), $z \succ (V_1 \cup V_2 \cup V_3) - \{x, y\}$. Consider $G + uz$. Then $D_{uz} = \{u, w\}$ or $D_{uz} = \{z, w\}$ for some $w \in V(G) - \{u, z\}$. We first suppose that $D_{uz} = \{u, w\}$. Then $w \in \{x, y\}$ by Lemma 3.2.1(2) and the fact that $z \succ (V_1 \cup V_2 \cup V_3) - \{x, y\}$. Further, $w \succ (V_2 \cup V_3) - \{z\}$. Hence, $xy \in E(G)$ as required. We now suppose that $D_{uz} = \{z, w\}$. Then $w \succ \{x, y\}$ since $\{x, y\} = \overline{N}_{V_2}(z)$. Thus $d(x, y) \leq 2$. By Lemma 3.2.8(3), $xy \in E(G)$. This proves (2) and completes the proof of our lemma. □

Lemma 3.2.12. *Let G be a 3-$(\gamma, 2)$-critical graph of diameter 3 which is not 3-γ-critical. Further, let V_1, A and B be defined as in Lemma 3.2.8 and $A \cap B \neq \phi$. If $|V_3| \geq 2$, then for each $s \in V_3$, there exist $z \in A - B$ and $x \in B - A$ such that $zs \notin E(G)$ and $xs \notin E(G)$. Further $z \succ (A - B) \cup (V_3 - \{s\})$ and $x \succ (B - A) \cup (V_3 - \{s\})$.*

Proof. Let $s_1 \in V_3$. Then there exists $s_2 \in V_3$ such that $s_1 \neq s_2$. By Lemma 3.2.10(3), $cs_1 \in E(G)$ for all $c \in A \cap B$. So $d(a, s_1) = 2$ and $d(b, s_1) = 2$. Consider

$G + as_1$. Then $D_{as_1} = \{a, x\}$ or $D_{as_1} = \{s_1, x\}$ for some $x \in V(G) - \{a, s_1\}$. We first suppose that $D_{as_1} = \{s_1, x\}$. Then $x \succ \{u\}$ and $xa \notin E(G)$. So $x = b$ and $s_1 \succ (A - B) \cup V_3$. Consider $G + bs_1$. Then $D_{bs_1} = \{b, y\}$ or $D_{bs_1} = \{s_1, y\}$ for some $y \in V(G) - \{b, s_1\}$. If $D_{bs_1} = \{b, y\}$, then $y \succ \{a, s_2\}$ and $ys_1 \notin E(G)$. Thus $y \notin A \cap B$ by Lemma 3.2.10(3). So $y \in A - B \subseteq N_G[s_1]$, contradicting Lemma 3.2.1(2). Hence, $D_{bs_1} \neq \{b, y\}$. Therefore, $D_{bs_1} = \{s_1, y\}$. Then $y \succ \{u, a\}$ and $yb \notin E(G)$. Thus $y = a$ and $s_1 \succ (B - A) \cup V_3$. By Lemma 3.2.10(3), $s_1 \succ A \cap B$. So $s_1 \succ V_2 \cup V_3$. Hence $\{u, s_1\} \succ G$, a contradiction. Thus $D_{as_1} \neq \{s_1, x\}$. Therefore, $D_{as_1} = \{a, x\}$. Then $x \succ \{b\} \cup (V_3 - \{s_1\})$ but $xs_1 \notin E(G)$ by Lemma 3.2.1(2). So $x \in B - A$ and $x \succ (B - A) \cup (V_3 - \{s_1\})$. Recall that $d(b, s_1) = 2$. By similar arguments as in the case $G + as_1$, when we consider $G + bs_1$, there is a vertex $z \in A - B$ such that $zs_1 \notin E(G)$ and $z \succ (A - B) \cup (V_3 - \{s_1\})$. This proves our lemma. $\square$

Lemma 3.2.13. *Let* G *be a* 3-$(\gamma, 2)$-*critical graph of diameter 3 which is not* 3-γ-*critical. Further, let* V_1, A *and* B *be defined as in Lemma 3.2.8. Suppose* $A \cap B \neq \phi$ *and* $|V_3| \geq 2$. *Let* $Z = \{x \in A - B | xy \notin E(G)$ *for some* $y \in V_3\}$, $W = \{x \in B - A | xy \notin E(G)$ *for some* $y \in V_3\}$, $Z' = (A - B) - Z$ *and* $W' = (B - A) - W$. *Then*

(1) $Z \neq \phi$ *and* $W \neq \phi$. *Further,* $G[Z]$ *and* $G[W]$ *are complete.*

(2) For each $z \in Z$, *there is only one vertex of* V_3, *say* x, *such that* $z \succ (A - B) \cup (V_3 - \{x\})$.

(3) For each $w \in W$, *there is only one vertex of* V_3, *say* y, *such that* $w \succ (B - A) \cup (V_3 - \{y\})$.

(4) For all $z \in Z$ *and for all* $w \in W$, $zw \in E(G)$.

(5) For each $z \in Z$, *there exists* $x \in A \cap B$ *such that* $zx \notin E(G)$. *Further,* $|A \cap B| \geq |Z|$.

(6) For each $w \in W$, *there exists* $y \in A \cap B$ *such that* $wy \notin E(G)$. *Further,* $|A \cap B| \geq |W|$.

(7) $|Z| \geq 2$ *and* $|W| \geq 2$.

Proof. (1) Clearly, $Z \neq \phi$ and $W \neq \phi$ by Lemma 3.2.12. We next show that $G[Z]$ is complete. Suppose to the contrary that there exist $x, y \in Z$ such that $xy \notin E(G)$. Clearly, $d(x, y) = 2$. Consider $G + xy$. Then $D_{xy} = \{x, s\}$ or $D_{xy} = \{y, s\}$ for some $s \in V(G) - \{a\}$ by Lemma 3.2.1(2). In order to dominate u, $s \in \{u, b\}$. So $x \succ V_3$ or $y \succ V_3$, contradicting the fact that $x, y \in Z$. Hence, $G[Z]$ is complete. By similar arguments, $G[W]$ is complete. This proves (1).

(2) Let $z \in Z$. Then there exists $x \in V_3$ such that $xz \notin E(G)$. Clearly, $d(z, x) = 2$ by Lemmas 3.2.8(2) and 3.2.12. Consider $G + zx$. Then $D_{zx} = \{z, t\}$ or $D_{zx} = \{x, t\}$ for some $t \in V(G) - \{z, x\}$. Suppose that $D_{zx} = \{x, t\}$. Then $tz \notin E(G)$. In order to dominate $\{u\} \cup V_1$, $t = u$ since $ab \notin E(G)$. Thus $x \succ B - A$. But this contradicts Lemma 3.2.12. Thus $D_{zx} \neq \{x, t\}$. Therefore, $D_{zx} = \{z, t\}$. Then, in order to dominate $\{u, b\}$, $t \in \{u, b\}$ and hence $z \succ (A - B) \cup (V_3 - \{x\})$. This proves (2).

(3) By similar arguments as in the proof of (2), (3) follows.

(4) Let $z \in Z$ and $w \in W$. Then, by (2) and (3), $z \succ V_3 - \{x\}$ and $w \succ V_3 - \{y\}$ for some $x, y \in V_3$. If $x = y$, then there exists $s \in V_3$ such that $sz \in E(G)$ and $sw \in E(G)$ since $|V_3| \geq 2$. By Lemma 3.2.8(3), $zw \in E(G)$. We now suppose that $x \neq y$. Then by Lemma 3.2.12, there exists $z_1 \in Z - \{z\}$ such that $z_1 y \notin E(G)$. By (2), $z_1 \succ V_3 - \{y\}$. Then $x \in N_{V_3}(z_1) \cap N_{V_3}(w)$. Thus $z_1 w \in E(G)$ by Lemma 3.2.8(3). It then follows by (1) that $z_1 \in N_G(z) \cap N_G(w)$. Hence, $zw \in E(G)$ by Lemma 3.2.8(3). This proves (4).

(5) Let $z \in Z$. Note that $d(b, z) = 2$ by (4). Consider $G + bz$. Then $D_{bz} = \{b, x\}$ or $D_{bz} = \{z, x\}$ for some $x \in V(G) - \{b, z\}$. If $D_{bz} = \{z, x\}$, then, in order to dominate u, $x \in \{u, a\}$ and thus $z \succ V_3$, a contradiction. Thus $D_{bz} \neq \{z, x\}$. Therefore, $D_{bz} = \{b, x\}$. Then, in order to dominate $\{a\} \cup V_3$, $x \in (A \cap B) \cup (A - B)$ and $xz \notin E(G)$ by Lemma 3.2.1(2). It follows by (2) that $x \in A \cap B$. It then follows by Lemma 3.2.10(3) that $|A \cap B| \geq |Z|$. This proves (5).

(6) By similar arguments as in the proof of (5), (6) follows.

(7) Suppose that $Z = \{z\}$. According to Lemma 3.2.12, $xz \notin E(G)$ for all $x \in V_3$. But this contradicts (2) since $z \succ V_3 - \{x\}$ and $|V_3| \geq 2$. Hence, $|Z| \geq 2$. By similar arguments, $|W| \geq 2$. This proves (7) and completes the proof of our lemma. $\square$

Corollary 3.2.14. *Let G be a 3-$(\gamma, 2)$-critical graph of diameter 3 which is not*

3-γ-critical. Further, let V_1, A and B be defined as in Lemma 3.2.8. Suppose $A \cap B \neq \phi$ and $|V_3| \geq 2$. Then for each $s \in V_3$, there exist $z \in Z$, $w \in W$ such that $zs \notin E(G)$ and $ws \notin E(G)$. Further, $z \succ (A - B) \cup (V_3 - \{s\})$ and $w \succ (B - A) \cup (V_3 - \{s\})$.

Proof. It follows from Lemmas 3.2.12, 3.2.13(2) and 3.2.13(3). □

Lemma 3.2.15. *Let G be a 3-$(\gamma, 2)$-critical graph of diameter 3 which is not 3-γ-critical. Further, let V_1, A and B be defined as in Lemma 3.2.8. Suppose $A \cap B \neq \phi$ and $|V_3| \geq 2$. Let $Z, Z^{'}, W$ and $W^{'}$ be defined as in Lemma 3.2.13. Then*

(1) $Z^{'} \neq \phi$ or $W^{'} \neq \phi$.

(2) Suppose $Z^{'} \neq \phi$ and $W^{'} \neq \phi$. Then for each $x \in Z^{'}$ and $y \in W^{'}$, $xy \in E(G)$.

(3) If $Z^{'} \neq \phi$, then for each $x \in Z^{'}$, $x \succ Z \cup (B - A) \cup V_3$.

(4) If $W^{'} \neq \phi$, then for each $w \in W^{'}$, $w \succ W \cup (A - B) \cup V_3$.

(5) If $Z^{'} \neq \phi$, then $G[Z^{'}]$ is 2-γ-critical.

(6) If $W^{'} \neq \phi$, then $G[W^{'}]$ is 2-γ-critical.

Proof. (1) Suppose to the contrary that $Z^{'} = \phi$ and $W^{'} = \phi$. Consider $G + ab$. Then we may assume without loss of generality that $D_{ab} = \{a, x\}$ for some $x \in V(G) - \{a, b\}$. So $x \succ (B - A) \cup V_3$. Thus $x \in Z \cup V_3$ by Lemma 3.2.1(2). But this contradicts Lemmas 3.2.12 and 3.2.13(2). Hence, $Z^{'} \neq \phi$ or $W^{'} \neq \phi$. This proves (1).

(2) Let $x \in Z^{'}$ and $y \in W^{'}$. By the definitions of $Z^{'}$ and $W^{'}$, $c \in N_G(x) \cap N_G(y)$ for each $c \in V_3$. It follows by Lemma 3.2.8(3) that $xy \in E(G)$.

(3) Let $x \in Z^{'}$. According to the definition of $Z^{'}$ together with (2) and Lemma 3.2.13(2), $x \succ Z \cup W^{'} \cup V_3$. Let $w \in W$. By Lemma 3.2.13(3) and the fact that $|V_3| \geq 2$, there exists $y \in V_3$ such that $y \in N_G(w) \cap N_G(x)$. Hence, $xw \in E(G)$ by Lemma 3.2.8(3). Therefore $x \succ W$ and hence $x \succ Z \cup (B - A) \cup V_3$. This proves (3).

(4) By similar arguments as in the proof of (3), (4) follows.

(5) Clearly, no vertex $x \in Z^{'}$ such that $x \succ Z^{'}$ otherwise $\{b, x\} \succ G$ by (3). So $\gamma(G[Z^{'}]) \geq 2$. Let $x \in Z^{'}$. Then there exists $y \in Z^{'}$ such that

$xy \notin E(G)$. Clearly, $d(x, y) = 2$. Consider $G + xy$. Then the only vertex of $D_{xy} - \{x, y\}$ dominates $\{u, b\}$. Thus it must belong to $\{u, b\}$ by Lemma 3.2.1(2). Hence, $x \succ Z' - \{y\}$ or $y \succ Z' - \{x\}$. So $\gamma(G[Z']) = 2$. It also follows that $\gamma(G[Z' + xy]) = 1$. Hence, Z' is 2-γ-critical. This proves (5).

(6) By similar arguments of (5), (6) follows. □

Lemma 3.2.16. *Let G be a 3-$(\gamma, 2)$-critical graph of diameter 3 which is not 3-γ-critical. Further, let V_1, A and B be defined as in Lemma 3.2.8. Suppose $A \cap B \neq \phi$ and $|V_3| \geq 2$. Let $S = \{x \in V_3 | \gamma(G + ux) = 3\}$. Then*

(1) $|S| \leq 2$.

(2) If $|S| = 2$, then $V_3 = S$.

(3) If $|S| = 2$ and $Z' \neq \phi$, then $x \succ A \cap B$ for all $x \in Z'$.

(4) If $|S| = 2$ and $W' \neq \phi$, then $y \succ A \cap B$ for all $y \in W'$.

Proof. (1) Let $c \in A \cap B$. Then, by Lemma 3.2.10(3), there exist $z \in A - B$ and $w \in B - A$ such that $c \succ (V_1 \cup V_2 \cup V_3) - \{z, w\}$. If there is a vertex $s \in S$ such that $sz \in E(G)$ and $sw \in E(G)$, then $\{s, c\} \succ G + us$, a contradiction. Hence, for each $s \in S$, $sz \notin E(G)$ or $sw \notin E(G)$. By Lemmas 3.2.13(2), 3.2.13(3), 3.2.15(3) and 3.2.15(4), it is easy to see that $|S| \leq 2$. This proves (1).

(2) Suppose that $V_3 \neq S$. Then there exists $t \in V_3 - S$ such that $\gamma(G + ut) = 2$. By Corollary 3.2.14, there exist $z \in Z$ and $w \in W$ such that $tz \notin E(G)$ and $tw \notin E(G)$. Let $S = \{s_1, s_2\}$. Then, by Lemmas 3.2.13(2) and 3.2.13(3), $z \succ S$ and $w \succ S$. Further, by Lemmas 3.2.13(5) and 3.2.13(6), there exist $c_1, c_2 \in A \cap B$ such that $c_1 z \notin E(G)$ and $c_2 w \notin E(G)$. If $c_1 = c_2$, then $\{c_1, s_1\} \succ G + us_1$, a contradiction. Thus $c_1 \neq c_2$. By Lemma 3.2.10(3), there exists $w_1 \in B - A$ such that $w_1 c_1 \notin E(G)$. According to Lemma 3.2.13(3), $w_1 s_1 \in E(G)$ or $w_1 s_2 \in E(G)$. Then either $\{c_1, s_1\} \succ G + us_1$ or $\{c_1, s_2\} \succ G + us_2$, a contradiction. This proves (2).

(3) Let $s \in S - \{v\}$ and let $z \in Z'$. Then $z \succ V_3$. Suppose to the contrary that there exists $c \in A \cap B$ such that $cz \notin E(G)$. Then, by Lemma 3.2.10(3), there exists $w \in B - A$ such that $wc \notin E(G)$. According to Lemma 3.2.13(3), $w \succ V_3 - \{y\}$ for some $y \in V_3$. So w is adjacent to some vertex of V_3, say s_1. Clearly, $s_1 = s$ or $s_1 = v$. It follows that $\{s_1, c\} \succ G + us_1$, a contradiction. Hence

$z \succ A \cap B$. This proves (3).

(4) By similar arguments as in the proof of (3), (4) follows. $\square$

Corollary 3.2.17. *Let G be a 3-$(\gamma, 2)$-critical graph of diameter 3 which is not 3-γ-critical. Further, let V_1, A and B be defined as in Lemma 3.2.8. Suppose $A \cap B \neq \phi$ and $|V_3| \geq 2$. If $|S| = 2$, then for each $c \in A \cap B$, there exist the unique vertex $x \in Z$ and the unique vertex $y \in W$ such that $c \succ (V_1 \cup V_2 \cup V_3) - \{x, y\}$. Further, $\overline{N}_{V_3}(x) \cap \overline{N}_{V_3}(y) = \phi$ but $\overline{N}_{V_3}(x) \cup \overline{N}_{V_3}(y) = V_3$.*

Proof. It follows from Lemmas 3.2.10(3), 3.2.13(2), 3.2.13(3), 3.2.16(2), 3.2.16(3) and 3.2.16(4) together with the fact that $\gamma(G + us) = 3$ for $s \in S$. $\square$

Recall that v is the vertex of $V_3 \subseteq V(G)$ which $d(u, v) = 3$ and $\gamma(G+uv) = 3$. By the definition of S in Lemma 3.2.16, $v \in S$. In what follows, for simplicity, if $|S| = 2$, then we shall denote $S = \{s_1, s_2\}$ where $v = s_1$.

Lemma 3.2.18. *Let G be a 3-$(\gamma, 2)$-critical graph of diameter 3 which is not 3-γ-critical. Let V_1, A and B be defined as in Lemma 3.2.8 where $A \cap B \neq \phi$. Let $S = \{s_1, s_2\}$ where S is defined as in Lemma 3.2.16 and let $Z, Z^{'}, W$ and $W^{'}$ be defined as in Lemma 3.2.13. Suppose H is a component of $\overline{G}[Z \cup (A \cap B) \cup W]$. Then*

(1) $Z \cap V(H) \neq \phi$, $(A \cap B) \cap V(H) \neq \phi$ and $W \cap V(H) \neq \phi$.

(2) For $z \in Z \cap V(H)$, $V(H)$ can be partitioned to $C_0, C_1, C_2, ..., C_{2n}$ for some positive integer n. Each of them is independent and $C_0 = \{z\}$ and for $1 \leq i \leq 2n$,

$$C_i \subseteq \begin{cases} A \cap B, & \text{if } i \equiv j(mod 4) \text{ for } j \in \{1, 3\} \\ W, & \text{if } i \equiv 2(mod 4) \\ Z, & \text{if } i \equiv 4(mod 4) \end{cases}$$

in such a way that for $x \in V(C_i)$ where $i \equiv j(mod4)$, $j \in \{1, 3\}$, there is exactly one vertex in $V(C_{i-1})$, say y, and exactly one vertex in $V(C_{i+1})$, say $y^{'}$, such that $y, x, y^{'}$ is a path in $\overline{G}$ and for $x \in V(C_i)$, $i \equiv j(mod4)$, $j \in \{2, 4\}$, there exists $t \in V(C_{i-1})$ such that $xt \in E(\overline{G})$. Further, for $1 \leq i \leq 2$, if $zs_i \notin E(G)$, then $z^{'}s_i \notin E(G)$ for all $z^{'} \in Z \cap V(H)$ and $ws_j \notin E(G)$ for all $w \in W \cap V(H)$ where $j \in \{1, 2\} - \{i\}$.

(3) For $w \in W \cap V(H)$, $V(H)$ can be partitioned to $C_0, C_1, C_2, ..., C_{2n}$ for some positive integer n. Each of them is independent and $C_0 = \{w\}$ and for $1 \leq i \leq 2n$,

$$C_i \subseteq \begin{cases} A \cap B, & \textit{if } i \equiv j(mod4) \textit{ for } j \in \{1,3\} \\ Z, & \textit{if } i \equiv 2(mod4) \\ W, & \textit{if } i \equiv 4(mod4) \end{cases}$$

in such a way that for $x \in V(C_i)$ where $i \equiv j(mod4)$, $j \in \{1,3\}$, there is exactly one vertex in $V(C_{i-1})$, say y, and exactly one vertex in $V(C_{i+1})$, say y', such that y, x, y' is a path in $\overline{G}$ and for $x \in V(C_i)$, $i \equiv j(mod4)$, $j \in \{2,4\}$, there exists $t \in V(C_{i-1})$ such that $xt \in E(\overline{G})$. Further, for $1 \leq i \leq 2$, if $ws_i \notin E(G)$, then $w's_i \notin E(G)$ for all $w' \in W \cap V(H)$ and $zs_j \notin E(G)$ for all $z \in Z \cap V(H)$ where $j \in \{1,2\} - \{i\}$.

(4) $\overline{G}[Z \cup (A \cap B) \cup W]$ contains $k \geq 2$ components. Further, there are $1 \leq k_1 \leq k-1$ components of $\overline{G}[Z \cup (A \cap B) \cup W]$ such that for each $z \in Z \cap \bigcup_{i=1}^{k_1} V(H_i)$, $zs_1 \in E(G)$ but $zs_2 \notin E(G)$ and for each $w \in W \cap \bigcup_{i=1}^{k_1} V(H_i)$, $ws_1 \notin E(G)$ but $ws_2 \in E(G)$ and for each $z \in Z - \bigcup_{i=1}^{k_1} V(H_i)$, $zs_1 \notin E(G)$ but $zs_2 \in E(G)$ and for each $w \in W - \bigcup_{i=1}^{k_1} V(H_i)$, $ws_1 \in E(G)$ but $ws_2 \notin E(G)$.

Proof. (1) It follows from Lemmas 3.2.13(5), 3.2.13(6) and Corollary 3.2.17.

(2) Let $z \in Z \cap V(H)$. Choose $y \in V(H)$ such that $d_{\overline{G}}(z, y) = max_{x \in V(H)} d(z, x) = k$. For a positive integer $1 \leq i \leq k$, let $C_i = \{x | d(z, x) = i\}$. Clearly, $y \in C_k$. Put $C_0 = \{z\}$. Then, by Lemmas 3.2.13(1) and 3.2.13(4), $G[Z \cup W]$ is complete. So $C_1 \subseteq A \cap B$. By Lemma 3.2.13(5), $C_1 \neq \phi$. By Lemma 3.2.10(1) and Corollary 3.2.17, $C_2 \neq \phi$ and $C_2 \subseteq W$. By Lemmas 3.2.13(1), 3.2.13(4) and 3.2.13(6), if $C_3 \neq \phi$, then $C_3 \subseteq A \cap B$. By Corollary 3.2.17, $C_4 \neq \phi$ and $C_4 \subseteq Z$. Continuing in this fashion, for $i \geq 1$, if $C_{4i+1} \neq \phi$, then $C_{4i+1} \subseteq A \cap B$ and $\phi \neq C_{4i+2} \subseteq W$ and if $C_{4i+3} \neq \phi$, then $C_{4i+3} \subseteq A \cap B$ and $\phi \neq C_{4i+4} \subseteq Z$. Clearly, $y \in Z \cup W$ by Corollary 3.2.17. Thus $k = 2n$ for some positive integer n. By Lemmas 3.2.10(1) and 3.2.13(1), $G[A \cap B], G[Z]$, and $G[W]$ are complete. Hence C_i is an independent set.

Note that, by our partition, each vertex of C_i for $1 \leq i \leq k$, is adjacent, in $\overline{G}$, to at least one vertex of C_{i-1}. However, if $i \equiv 1(mod4)$ or $i \equiv 3(mod4)$, then each vertex of C_i is adjacent, in $\overline{G}$, to exactly one vertex of C_{i-1} and to exactly one vertex of C_{i+1} by Corollary 3.2.17 and the fact $C_i \subseteq A \cap B$ for $i \equiv 1(mod4)$ or $i \equiv 3(mod4)$.

Now let $zs_i \notin E(G)$. Then, by Corollary 3.2.14, $zs_j \in E(G)$ for $j \in \{1,2\} - \{i\}$. We first show that each vertex of C_2 is not adjacent to s_j. Suppose this is not the case. Then there is $w \in C_2 \subseteq W$ such that $ws_j \in E(G)$. Since $w \in C_2$, there is $c \in C_1 \subseteq A \cap B$ such that $cw \in E(\overline{G})$ by our partition. Then, by Corollary 3.2.17, $\{z, w\} = \overline{N}_{V_2}(c)$ and thus $\{s_j, c\} \succ G + us_j$, a contradiction. Hence, $w^{'}s_j \notin E(G)$ for all $w^{'} \in C_2$. By Corollary 3.2.14, $w^{'}s_i \in E(G)$ for all $w^{'} \in C_2$.

If $C_3 = \phi$, then we are done. So suppose $C_3 \neq \phi$. By the above argument, $C_3 \subseteq A \cap B$ and $\phi \neq C_4 \subseteq Z$. Suppose to the contrary that there exists $z_1 \in C_4$ such that $z_1 s_i \in E(G)$. By our partition, there is $c_1 \in C_3$ such that $z_1 c_1 \in E(\overline{G})$. By Corollary 3.2.17 and our partition, there is $w_1 \in C_2$ such that $w_1 c_1 \in E(\overline{G})$ where $\{z_1, w_1\} = \overline{N}_{V_2}(c_1)$. Since $w_1 s_i \in E(G)$ and $z_1 s_i \in E(G)$, it follows that $\{c_1, s_i\} \succ G + us_i$ by Corollary 3.2.17, again a contradiction. Hence, each vertex $z^{'} \in C_4$, $z^{'}s_i \notin E(G)$ but $z^{'}s_j \in E(G)$ by Corollary 3.2.14. Continuing in this fashion, for all $z^{'} \in C_{4i+4} \subseteq Z$, $z^{'}s_i \notin E(G)$ but $z^{'}s_j \in E(G)$ and for all $w^{'} \in C_{4i+2} \subseteq W$, $w^{'}s_j \notin E(G)$ but $w^{'}s_i \in E(G)$. This proves (2).

(3) By similar arguments as in the proof of (2), (3) follows.

(4) Suppose that $\overline{G}[Z \cup (A \cap B) \cup W]$ contains exactly one component, say H. Let $z_1 \in Z \cap V(H)$. By Lemma 3.2.13(2), we may assume that $z_1 s_1 \in E(G)$ and $z_1 s_2 \notin E(G)$. Then, by (2), $z^{'}s_1 \in E(G)$ and $z^{'}s_2 \notin E(G)$ for all $z^{'} \in Z \cap V(H)$. Hence, s_1 is adjacent to every vertex of Z. But this contradicts Corollary 3.2.14. Hence, $\overline{G}[Z \cup (A \cap B) \cup W]$ contains $k \geq 2$ components.

Now let $H_1, H_2, ..., H_k$ be components of $\overline{G}[Z \cup (A \cap B) \cup W]$. Let $z \in Z \cap V(H_1)$. According to Lemma 3.2.13(2), we may assume that $zs_1 \in E(G)$ but $zs_2 \notin E(G)$. Then, by (2), each vertex of $Z \cap V(H_1)$, say $z^{'}$, $z^{'}s_1 \in E(G)$ but $z^{'}s_2 \notin E(G)$. Further, each vertex of $W \cap V(H_1)$, say w, $ws_1 \notin E(G)$ but $ws_2 \in E(G)$.

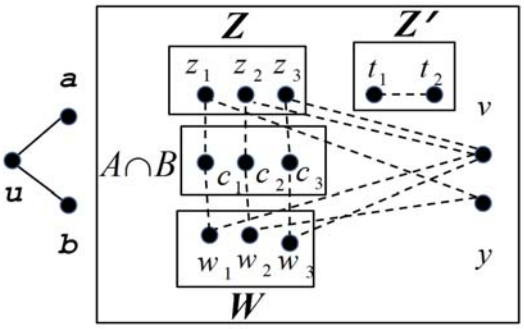
We may renumber the components of $\overline{G}[Z \cup (A \cap B) \cup W]$ in such a way that for $1 \leq i \leq k_1$, if $z \in Z \cap V(H_i)$, then $zs_1 \in (G)$ but $zs_2 \notin E(G)$ and if $w \in W \cap V(H_i)$, then $ws_1 \notin (G)$ but $ws_2 \in E(G)$. It is easy to see that $k_1 \geq 1$ and if $k_1 = k$, then $s_1 z \in E(G)$ for all $z \in Z$. But this contradicts Corollary 3.2.14. Hence, $k_1 \leq k - 1$. By (2) and Lemma 3.2.13(2), for $k_1 + 1 \leq i \leq k$, if $z \in Z \cap V(H_i)$, then $zs_1 \notin (G)$ but $zs_2 \in E(G)$ and if $w \in W \cap V(H_i)$, then $ws_1 \in (G)$ but $ws_2 \notin E(G)$. This proves (4) and completes the proof of our lemma. $\square$

Theorem 3.2.19. *Let G be a 3-$(\gamma, 2)$-critical graph of diameter 3 which is not 3-γ-critical. Further, let V_1, A and B be defined as in Lemma 3.2.8 and let S defined as in Lemma 3.2.16. If $A \cap B \neq \phi$ and $|S| = 2$, then $G \in \mathcal{H}_3$, defined in Section 3.1.*

Proof. Let $Z = \{x \in A - B | xy \notin E(G)$ for some $y \in V_3\}$, $W = \{x \in B - A | xy \notin E(G)$ for some $y \in V_3\}$, $Z' = (A - B) - Z$ and $W' = (B - A) - W$. According to Lemma 3.2.18(4), let $H_1, H_2, ..., H_{k_1}, H_{k_{1+1}}, ..., H_k$ be components of $\overline{G}[Z \cup (A \cap B) \cup W]$ where $1 \leq k_1 \leq k - 1$ and for each $x \in (Z \cap \bigcup_{i=1}^{k_1} V(H_i)) \cup (W - \bigcup_{i=1}^{k} V(H_i))$, $xs_1 \in E(G)$ but $xs_2 \notin E(G)$ and for each $x \in (Z - \bigcup_{i=1}^{k_1} V(H_i)) \cup (W \cap \bigcup_{i=1}^{k_1} V(H_i))$, $xs_1 \notin E(G)$ but $xs_2 \in E(G)$. Put $T_1 = \bigcup_{i=1}^{k_1} V(H_i)$ and $T_2 = \bigcup_{i=k_1+1}^{k} V(H_i)$. Then, by Lemmas 3.2.10(1), 3.2.13(1), 3.2.13(4), 3.2.13(5), 3.2.13(6) and 3.2.18 and Corollary 3.2.17, $G[Z \cup (A \cap B) \cup W] = K_{n_0} - (E(T_1) \cup E(T_2))$ where $n_0 = |V(T_1)| + |V(T_2)|$. By Lemma 3.2.15(1), we may assume without loss of generality that $Z' \neq \phi$. Then, by Lemma 3.2.15(5), and Theorem 2.1, $G[Z'] = K_{n_1} - \bigcup_{j=1}^{l_1} K_{1,r_j}$ where $n_1 = |Z'| = l_1 + \sum_{j=1}^{l_1} r_j$ and $n_1 \geq 2l_1$. If $W' = \phi$, then G is isomorphic to a graph in $\mathcal{H}_3^1$ by Lemmas 3.2.15(3), 3.2.16(3), 3.2.18(2) and 3.2.18(3).

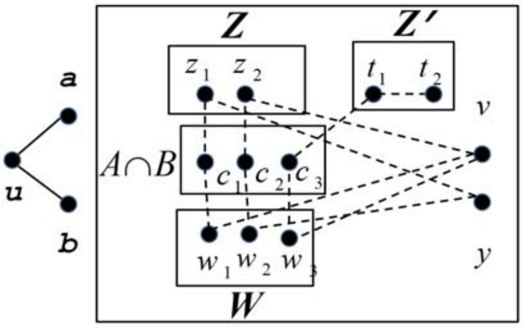
We now suppose that $W' \neq \phi$. Then, by Lemma 3.2.15(6), and Theorem 2.1, $G[W'] = K_{n_2} - \bigcup_{j=1}^{l_2} K_{1,t_j}$ where $n_2 = |W'| = l_2 + \sum_{j=1}^{l_2} t_j$ and $n_2 \geq 2l_2$. It then follows by Lemmas 3.2.15(2), 3.2.15(3), 3.2.15(4), 3.2.16(3), 3.2.16(4), 3.2.18(2) and 3.2.18(3) that G is isomorphic to a graph in $\mathcal{H}_3^2$. This completes the proof of our theorem. $\square$

In the case $S = \{v\}$ and $|V_3| \geq 2$. We cannot guarantee that for each $c \in A \cap B$, $\overline{N}_{V_2}(c) \subseteq Z \cup W$. Figure 3.11 shows an example of a 3-$(\gamma, 2)$-critical graph which is not 3-γ-critical when $\overline{N}_{V_2}(c) \subseteq Z \cup W$ and Figure 3.12 shows an example of a 3-$(\gamma, 2)$-critical graph which is not 3-γ-critical when $\overline{N}_{V_2}(c) \nsubseteq Z \cup W$. Further, for $c \in A \cap B$, the vertex v may not join to any vertex of $\overline{N}_{V_2}(c)$(see Figure 3.11 when $c = c_3$) or may join to exactly one vertex of $\overline{N}_{V_2}(c)$(see Figure 3.11 when $c \in \{c_1, c_2\}$ or see Figure 3.12 when $c \in A \cap B$). It seems not possible to characterize 3-$(\gamma, 2)$-critical graphs which are not 3-γ-critical in this case. Note that in Figures 3.11 and 3.12, the vertex a joins to every vertex of $Z \cup Z' \cup (A \cap B)$ and the vertex b joins to every vertex of $W \cup (A \cap B)$.



$K_{13} - \{c_1z_1, c_1w_1, c_2z_2, c_2w_2, c_3z_3, c_3w_3, t_1t_2, yz_1, yw_2, vz_2, vz_3, vw_1, vw_3\}$

Figure 3.11: A 3-$(\gamma, 2)$-critical graph with $\overline{N}_{V_2}(c) \subseteq Z \cup W$



$K_{12} - \{c_1z_1, c_1w_1, c_2z_2, c_2w_2, c_3w_3, c_3t_1, t_1t_2, yz_1, yw_2, vz_2, vw_1, vw_3\}$

Figure 3.12: A 3-$(\gamma, 2)$-critical graph with $\overline{N}_{V_2}(c) \nsubseteq Z \cup W$

Theorem 3.2.20. *Let G be a 3-$(\gamma, 2)$-critical graph of diameter 3 which is not 3-γ-critical. Further, let V_1, A and B be defined as in Lemma 3.2.8. Suppose $A \cap B \neq \phi$ and $V_3 = \{v\}$. Let $F_1 = \{x \in A - B | xy \notin E(G)$ for some $y \in A - B\}$, $F_2 = (A - B) - F_1$, $F_3 = \{x \in B - A | xy \notin E(G)$ for some $y \in B - A\}$, $F_3 = (B - A) - F_4$. Then*

(1) No vertex $x \in V_2 \cup \{v\}$ such that $x \succ V_2$.

(2) For each $x \in F_2 \cup F_4, xv \notin E(G)$.

(3) $F_1 \neq \phi$ or $F_3 \neq \phi$.

(4) For $i \in \{1, 3\}$, if $F_i \neq \phi$, then $G[F_i]$ is 2-γ-critical.

(5) Suppose $F_2 \neq \phi$ and $F_4 \neq \phi$. Then for each $x \in F_2$ and $y \in F_4$, $x \succ (A - B) \cup (B - A)$ and $y \succ (A - B) \cup (B - A)$.

(6) *Suppose* $F_2 \neq \phi$ *or* $F_4 \neq \phi$. *For* $i \in \{1,3\}$, *if* $F_i \neq \phi$, *then* $d(v,x) = 1$ *or* $d(v,x) = 3$ *for all* $x \in F_i$.

(7) *Suppose* $F_2 \neq \phi$ *and* $F_4 \neq \phi$. *For* $i \in \{1,3\}$, *if* $F_i \neq \phi$, *then* $v \succ F_i$ *and for* $x \in F_i$ *and* $y \in F_j$, $xy \in E(G)$ *where* $j \in \{1,3\} - \{i\}$.

(8) *Suppose* $F_2 = \phi$ *and* $F_4 = \phi$. *Then there is only vertex of* $F_1 \cup F_3$, *say* w, *such that* $v \succ V_2 - \{w\}$. *Further, if* $w \in F_1$, *then for each* $x \in F_1 - \{w\}$ *and* $y \in F_3$, $xy \in E(G)$ *and if* $w \in F_3$, *then for each* $y \in F_3 - \{w\}$ *and* $x \in F_1$, $xy \in E(G)$

Proof. (1) Suppose to the contrary that there exists $x \in V_2 \cup \{v\}$ such that $x \succ V_2$. If $x \in V_2$, then $\{u,x\} \succ G + uv$, a contradiction. If $x = v$, then $\{u,v\} \succ G$, a contradiction. This proves (1).

(2) Let $x \in F_2 \cup F_4$. Without loss of generality, assume that $x \in F_2$. Then, by definition of F_2 $x \succ A - B$. If $xv \in E(G)$, then $\{b,x\} \succ G$, a contradiction. So $xv \notin E(G)$. This proves (2).

(3) Suppose that $F_1 = \phi$ and $F_3 = \phi$. Then $F_2 \neq \phi$ and $F_4 \neq \phi$ by Lemma 3.2.11(1). Consider $G + ab$. Then $D_{ab} = \{a,x\}$ or $D_{ab} = \{b,x\}$ for some $x \in V(G) - \{a,b\}$. Without loss of generality, assume that $D_{ab} = \{a,x\}$. Then by Lemma 3.2.1(2), $x \in F_2 \cup \{v\}$ and $x \succ F_4 \cup \{v\}$. If $x \in F_2$, then $xv \in E(G)$. But this contradicts (2). Thus $x = v$ and $x \succ F_4$, which again contradicts (2). Hence, $F_1 \neq \phi$ or $F_3 \neq \phi$. This proves (3).

(4) Let $i \in \{1,3\}$. By the definition of F_i, $\gamma(G[F_i]) \geq 2$. We first suppose that $i = 1$. Let $y \in F_1$. Then there exists $t \in F_1$ such that $yt \notin E(G)$. Consider $G + yt$. Then the only vertex of $D_{yt} - \{y,t\}$ dominates $\{u,b\}$. Thus it must belong to $\{u,b\}$ since $ab \notin E(G)$. Hence, $y \succ F_1 - \{t\}$ or $t \succ F_1 - \{y\}$. So $\gamma(G[F_1]) = 2$. It also follows that $\gamma(G[F_1] + yt) = 1$. Hence, $G[F_1]$ is 2-γ-critical. By similar arguments, $G[F_3]$ is 2-γ-critical. This proves (4).

(5) By definitions of F_2 and F_4, for each $x \in F_2$, $x \succ A - B$ and for each $y \in F_4$, $y \succ B - A$. Let $c \in A \cap B$. Then, by Lemma 3.2.10(3), there exist $t \in A - B$ and $w \in B - A$ such that $ct \notin E(G)$ and $cw \notin E(G)$. By Lemma 3.2.11(2), $tw \in E(G)$. By Lemma 3.2.8(3), $xw \in E(G)$ for all $x \in F_2$ and $yt \in E(G)$ for all $y \in F_4$. Hence, $xy \in E(G)$ by Lemma 3.2.8(3). It follows that

$x \succ B-A$ and $y \succ A-B$. Hence, $x \succ (A-B)\cup(B-A)$ and $y \succ (A-B)\cup(B-A)$. This proves (5).

(6) Suppose that there exists $x \in F_i$, for $i \in \{1,3\}$ such that $d(x,v) = 2$. Consider $G + vx$. Then $D_{vx} = \{v,t\}$ or $D_{vx} = \{x,t\}$ for some $t \in V(G) - \{v,x\}$. If $D_{vx} = \{v,t\}$, then $t \succ \{u,a,b\}$. So $t = u$ and $v \succ V_2 - \{x\}$. Thus $v \succ F_2$ or $v \succ F_4$. But this contradicts (2). Therefore $D_{vx} = \{x,t\}$. Then $t \succ \{u,b\}$. Thus $t \in \{u,b\}$ and $x \succ F_1$, a contradiction. This proves (6).

(7) Suppose to the contrary that there exists $x \in F_i$, for $i \in \{1,3\}$ such that $xv \notin E(G)$. We may assume that $i = 1$. Then, by (6), $d(x,v) = 3$. So $xc \notin E(G)$ for all $c \in A\cap B$ since v is adjacent to every vertex of $A\cap B$ by Lemma 3.2.10(3). Further, $c \succ F_2$. It follows that for each $y \in F_2$, $y \succ A\cap B$. Hence, $y \succ V_2$ by (5). But this contradicts (1). Hence, $v \succ F_i$ for $i \in \{1,3\}$. By Lemma 3.2.8(3), $xy \in E(G)$ for all $x \in F_1$ and $y \in F_3$. This proves (7).

(8) By Lemma 3.2.11(1), $F_1 \neq \phi$ and $F_3 \neq \phi$. Then, by (4), $\overline{G}[F_1] \cong \bigcup_{i=1}^{n} K_{1,r_i}$ and $\overline{G}[F_3] \cong \bigcup_{i=1}^{m} K_{1,s_i}$ where n, m, r_i, s_i are positive integers. According to Lemma 3.2.10(3), $d(a,v) = 2$ and $d(b,v) = 2$. Further, there is $w \in F_1 \cup F_3$ such that $vw \notin E(G)$ otherwise $\{u,v\} \succ G$. We may suppose that $w \in F_1$. Then applying similar arguments as in the proof of Claims 13, 14 and 15 in Theorem 3.2.9, (8) follows. $\square$

Figures 3.13 - 3.17 show examples of 3-$(\gamma,2)$-critical graphs of diameter 3 which are not 3-γ-critical satisfying the hypothesis of Theorem 3.2.20. In order to characterize such graphs, we need to consider, up to isomorphism, at least 5 cases according to F_i, $i \in \{1,2,3,4\}$ by (3). So it seems not possible to characterize 3-$(\gamma,2)$-critical graphs of diameter 3 which are not 3-γ-critical satisfying the hypothesis of Theorem 3.2.20. Note that in Figures 3.13- 3.17, the vertex a joins to every vertex of $F_1 \cup F_2 \cup (A\cap B)$ and the vertex b joins to every vertex of $F_3 \cup F_4 \cup (A\cap B)$.

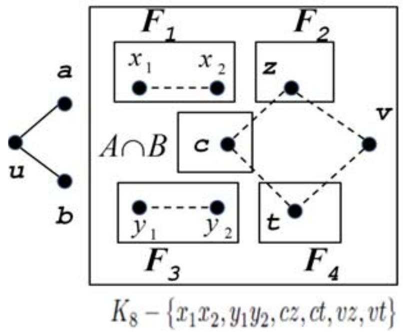


Figure 3.13: A 3-$(\gamma, 2)$-critical graph with $|F_1| = 2, |F_2| = 1, |F_3| = 2$ and $|F_4| = 1$

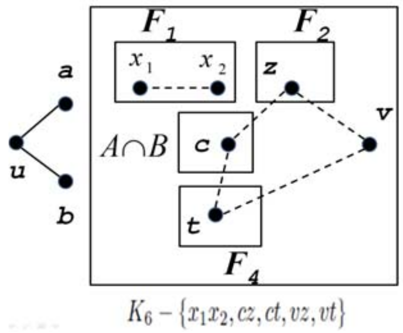


Figure 3.14: A 3-$(\gamma, 2)$-critical graph with $|F_1| = 2, |F_2| = 1, F_3 = \phi$ and $|F_4| = 1$

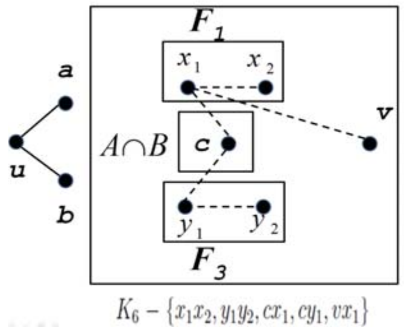


Figure 3.15: A 3-$(\gamma, 2)$-critical graph with $|F_1| = 2, F_2 = \phi, |F_3| = 2$ and $F_4 = \phi$

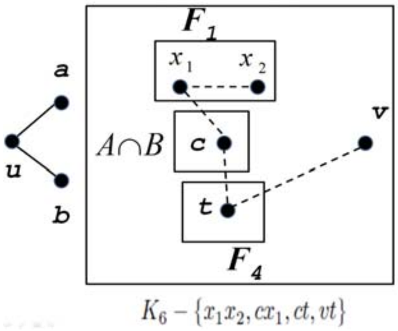


Figure 3.16: A 3-$(\gamma, 2)$-critical graph with $|F_1| = 2, F_2 = \phi, F_3 = \phi$ and $|F_4| = 1$

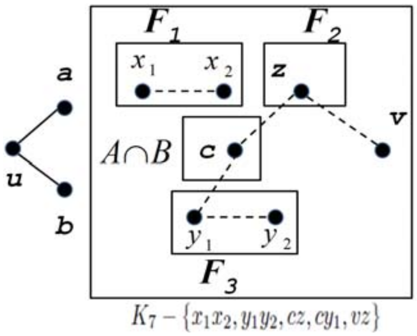


Figure 3.17: A 3-$(\gamma, 2)$-critical graph with $|F_1| = 2, |F_2| = 1, |F_3| = 2$ and $F_4 = \phi$

Chapter 4

Characterization of 3-$(\gamma,2)$-critical graphs which are not 3-γ-critical of small order

In this chapter, we apply lemmas and theorems in Chapter 3 to show that there are exactly 10 non-isomorphic 3-$(\gamma,2)$-critical graphs of diameter 3 which are not 3-γ-critical of order at most 8.

Recall that G is denoted a 3-$(\gamma,2)$-critical graph of diameter 3 which is not 3-γ-critical. Further, u and v are vertices of G where $d(u,v)=3$ and $\gamma(G+uv)=3$. For the sake of symmetry, we may suppose without loss of generality that $d(u)\le d(v)$. For $1\le i\le 3$, let $V_i=\{x\in V(G)|d(x,u)=i\}$. Then $|V_1|\le|V_2\cup V_3|-1$. If $V_2\cup V_3\subseteq N_G[v]$, then $\{u,v\}$ is a dominating set of G, a contradiction. Hence, $d(v)\le|V_2\cup V_3|-2$. It then follows that $|V_1|+2\le|V_2\cup V_3|$. Note also that $|V(G)|\ge 6$.

Lemma 4.1. *Let G be a 3-$(\gamma,2)$-critical graph of diameter 3 which is not 3-γ-critical. Then $|V(G)|\ge 7$.*

Proof. Since $|V_2\cup V_3|\ge|V_1|+2$, $|V(G)|=1+|V_1|+|V_2|+|V_3|\ge 3+2|V_1|$. If $|V_1|\ge 2$, then $|V(G)|\ge 7$ as required. So we now consider the case $|V_1|=1$. Then the only vertex of V_1 is adjacent to every vertex of V_2. By Theorem 3.2.6(2), $|V_3|\ge 3$. Because of Lemma 3.2.2, $|V(G)|=1+|V_1|+|V_2|+|V_3|\ge 1+1+2+3=7$. This completes the proof of our lemma. □

Lemma 4.2. *Let G be a 3-$(\gamma,2)$-critical graph of diameter 3 which is not 3-γ-critical. If $|V(G)|\le 8$, then $1\le|V_1|\le 2$. Further, if $|V_1|=1$, then $G\in\mathcal{J}$.*

Proof. Clearly, $|V_1|\ge 1$. If $|V_1|\ge 3$, then $|V_2\cup V_3|\ge|V_1|+2\ge 5$ and thus $|V(G)|\ge 9$, a contradiction. Hence, $1\le|V_1|\le 2$. We now suppose that $|V_1|=1$. Then the only vertex of V_1 is adjacent to every vertex of V_2. By Theorem 3.2.6(4), $G\in\mathcal{J}$. □

Theorem 4.3. *Let G be a 3-$(\gamma, 2)$-critical graph of diameter 3 which is not 3-γ-critical. If $|V(G)| = 7$, then G is isomorphic to the graph J_i, for some $1 \leq i \leq 2$, in Figure 4.1.*

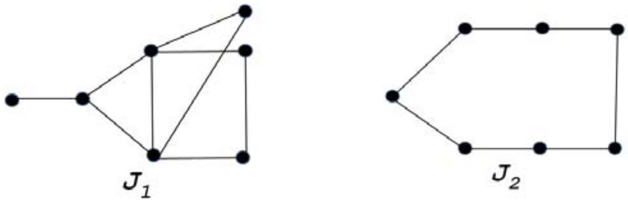


Figure 4.1: 3-$(\gamma, 2)$-critical graphs of order 7

Proof. By Lemma 4.2, $1 \leq |V_1| \leq 2$. Further, if $|V_1| = 1$, then G is isomorphic to the graph J_1. We now assume that $|V_1| = 2$. Then, by Lemma 3.2.2 and the fact that $|V_3| \geq 1$, $2 \leq |V_2| \leq 3$ and thus $1 \leq |V_3| \leq 2$. Put $|V_1| = \{a, b\}$. Let $A = \{x \in V_2 | ax \in E(G)\}$ and $B = \{x \in V_2 | bx \in E(G)\}$.

Claim 1: $ab \notin E(G)$.

Suppose to the contrary that $ab \in E(G)$. If each vertex of V_1 is adjacent to every vertex of V_2, then $|V_3| \geq 3$ by Theorem 3.2.6(2), a contradiction. Hence, there is a vertex of V_1 say a, such that a is not adjacent to some vertex of V_2, say y. Since $y \in V_2$ and $ya \notin E(G)$, it follows that $yb \in E(G)$. Clearly, $d(a, y) = 2$. Consider $G+ay$. By Lemma 3.2.3, $D_{ay} = \{a, z\}$ for some $z \in (V_2 \cup V_3) - \{y\}$. Then $zy \notin E(G)$ but $z \succ V_3$. If $z \succ V_2 - \{y\}$, then $\{b, z\} \succ G$, a contradiction. Hence, there exists a vertex $y_1 \in V_2 - \{y\}$ such that $zy_1 \notin E(G)$. Clearly, $y_1 a \in E(G)$ since $D_{ay} = \{a, z\}$.

We first show that $z \notin V_3$. Suppose this is not the case. Then $z \in V_3$. Thus there exists $w \in V_2 - \{y, y_1\}$ such that $wz \in E(G)$. Since $|V_2| \leq 3$ and $|V(G)| = 7$, It follows that $V_2 = \{y, y_1, w\}$ and $V_3 = \{z\} = \{v\}$. But then $d(v) = 1 < d(u)$, contradicting our hypothesis that $d(v) \geq d(u)$. Hence, $z \notin V_3$. Therefore, $z \in V_2$. Consequently, $V_2 = \{y, y_1, z\}$ and $V_3 = \{v\}$. Then $zv \in E(G)$ since $z \succ V_3$. If $vy_1 \in E(G)$, then $\{v, b\} \succ G$, a contradiction. Hence, $vy_1 \notin E(G)$. By our hypothesis that $d(v) \geq d(u)$, $vy \in E(G)$. But then $\{a, v\} \succ G$, again a contradiction. This settles our claim.

Claim 2: $A \cap B = \phi$.

Suppose to the contrary that $A \cap B \neq \phi$. Let $c \in A \cap B$. Then, by Lemma

3.2.10(3), there exist $x \in A-B$ and $y \in B-A$ such that $c \succ (V_1 \cup V_2 \cup V_3)-\{x, y\}$. Since $|V_2| \leq 3$, it follows that $A \cap B = \{c\}$, $A - B = \{x\}$ and $B - A = \{y\}$. By Lemma 3.2.11(2), $xy \in E(G)$. Because $\gamma(G) = 3$, $vx \notin E(G)$ or $vy \notin E(G)$. By our hypothesis that $d(v) \geq d(u)$, $vx \in E(G)$ or $vy \in E(G)$. Hence, $\{x, b\} \succ G$ or $\{y, a\} \succ G$, a contradiction. This settles our claim.

Since $A \cap B = \phi$, Theorem 3.2.9 together with the fact that $|V_2| \leq 3$ implies $G \in \mathcal{H}_1$. Hence, G is isomorphic to J_2. This completes the proof of our theorem. ☐

Theorem 4.4. *Let G be a 3-$(\gamma, 2)$-critical graph of diameter 3 which is not 3-γ-critical. If $|V(G)| = 8$, then G is isomorphic to the graph J_i, for some $1 \leq i \leq 8$, in Figure 4.2.*

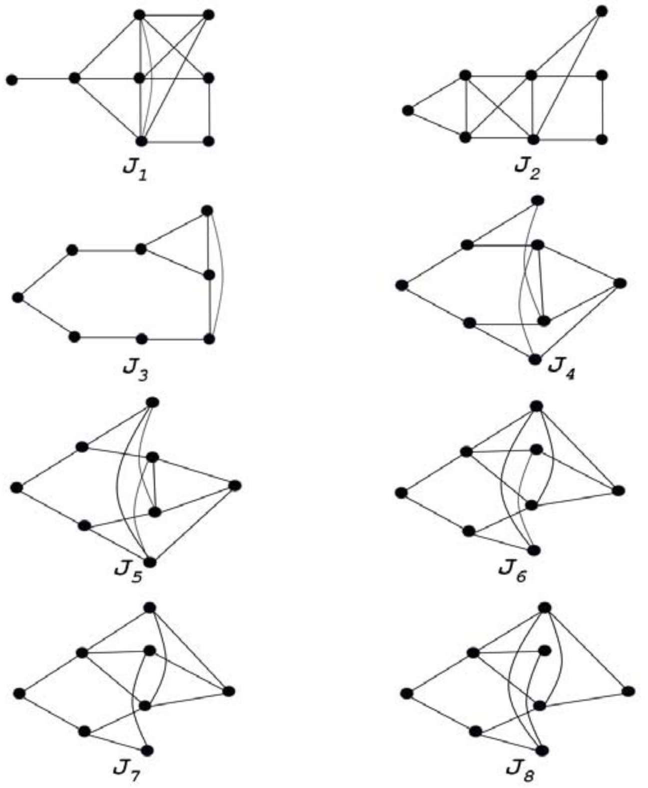


Figure 4.2: 3-$(\gamma, 2)$-critical graphs of order 8

Proof. By Lemmas 3.2.2 and 4.2, $1 \leq |V_1| \leq 2$ and $|V_2| \geq 2$. It then follows that $|V_3| \leq 4$ since $|V(G)| = 8$. If $G[V_1]$ is complete and each vertex of V_1 is adjacent to every vertex of V_2, then, by Theorem 3.2.6 and Corollary 3.2.7, G is isomorphic to J_1 when $|V_1| = 1$ and G is isomorphic to J_2 when $|V_1| = 2$. We next consider when $G[V_1]$ is not complete or there is a vertex of V_1 which is not adjacent to some vertex of V_2. In either case, it is easy to see that $|V_1| = 2$. Consequently, $2 \leq |V_2| \leq 4$ and $|V_3| \leq 3$. Now put $V_1 = \{a, b\}$. Let $A = \{x \in V_2 | ax \in E(G)\}$ and $B = \{x \in V_2 | bx \in E(G)\}$. We distinguish two cases.

Case 1: $G[V_1]$ is not complete.

Then $ab \notin E(G)$ and thus $G[V_3]$ is complete by Lemma 3.2.8(2).

Case 1.1: $A \cap B = \phi$.

Then, by Theorem 3.2.9, $G \in \mathcal{H}_1$ or $G \in \mathcal{H}_2$. We first suppose that $G \in \mathcal{H}_1$. Then $|V_3| \geq 2$ and thus $|V_2| \leq 3$. If $|V_2| = 2$, then G is isomorphic to J_3. If $|V_2| = 3$, then G is isomorphic to the graph in Figure 4.3.

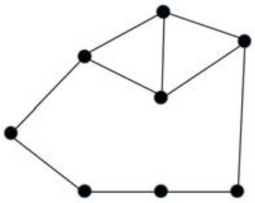


Figure 4.3: Graph $H \in \mathcal{H}_1$

It is easy to see that the graph in Figure 4.3 is isomorphic to the graph J_3.

We now assume that $G \in \mathcal{H}_2$. Then $|V_3| = 1$ and thus $|V_2| = 4$. Further, $|A - B| \geq 2$ and $|B - A| \geq 2$. Thus $|A - B| = 2$ and $|B - A| = 2$ since $|V_2| = 4$. Hence, G is isomorphic to either J_4 or J_5.

Case 1.2: $A \cap B \neq \phi$.

Then $|A \cap B| \geq 1$. It follows by Lemma 3.2.11(1) that $|A - B| \geq 1$ and $|B - A| \geq 1$. Since $|V_2| \leq 4$, Then $|A \cap B| \leq 2$. According to Lemma 3.2.13(7), $|V_3| = 1$. Hence, $|V_2| = 4$.

Claim 1: $|A \cap B| = 1$.

Suppose this is not the case. Then $|A \cap B| = 2$. Thus $|A - B| = |B - A| = 1$. Let $\{w\} = A - B$, $\{z\} = B - A$ and $\{x_1, x_2\} = A \cap B$. By Lemmas 3.2.10(1) and 3.2.10(3), $x_1x_2 \in E(G)$ and $x_iv \in E(G)$ for $1 \leq i \leq 2$ but $x_iw \notin E(G)$ and $x_iz \notin E(G)$ for $1 \leq i \leq 2$. Further, by Lemma 3.2.11(2), $wz \in E(G)$. Now consider $G + ab$. Without lost of generality, we may assume that $D_{ab} = \{a, t\}$

for some $t \in V(G) - \{a, b\}$. By Lemma 3.2.1(2), $t \notin N_G(b)$. Then $t \in \{w, v\}$. If $t = w$, then $wv \in E(G)$ and thus $\{w, b\} \succ G$, a contradiction. Hence, $t \neq w$. Therefore, $t = v$. Then $vz \in E(G)$ since $az \notin E(G)$. But then $\{z, a\} \succ G$, again a contradiction. This settles our claim.

Then $|A - B| = 2$ or $|B - A| = 2$ since $|V_2| = 4$. Without lost of generality, assume that $|A - B| = 2$. Then $|B - A| = 1$. Put $\{w_1, w_2\} = A - B$, $\{z\} = B - A$ and $\{x\} = A \cap B$. By Lemma 3.2.10(3), we may assume that $xw_1 \notin E(G)$ and $xz \notin E(G)$. Further, $xw_2 \in E(G)$ and $xv \in E(G)$. Then $zw_1 \in E(G)$ by Lemma 3.2.11(2). It follows that $zv \notin E(G)$ otherwise $\{z, a\} \succ G$.

Claim 2: $w_1w_2 \notin E(G)$.

Suppose not. Then $w_1w_2 \in E(G)$. Thus $d(w_2, z) \leq 2$ since $zw_1 \in E(G)$. By Lemma 3.2.8(3), $w_2z \in E(G)$. But then either $\{w_2, u\} \succ G$ if $w_2v \in E(G)$ or $\{w_2, u\} \succ G + uv$ if $w_2v \notin E(G)$, a contradiction. This proves our claim.

Claim 3: $w_2v \in E(G)$.

Suppose this is not the case. Then $w_2v \notin E(G)$. Then $d(w_2, v) = 2$ by the path w_2, x, v. Consider $G + w_2v$. If $D_{w_2v} \cap \{w_2, v\} = \{v\}$, then the only vertex of $D_{w_2v} - \{v\}$ dominates $\{u, a, b, z\}$. But this is not possible since $ab \notin E(G)$ and $z \in V_2$. Hence, $D_{w_2v} = \{w_2, t\}$ for some $t \in V(G) - \{w_2, v\}$. Then t dominates $\{w_1, u, b\}$. Thus $t \in V_1$ since $w_1 \in V_2$. But this is not possible since $bw_1 \notin E(G)$ and $ab \notin E(G)$. This contradicts the fact that $\gamma(G + w_2v) = 2$ and settles our claim.

We now distinguish two subcases.

Subcase 1.2.1: $vw_1 \in E(G)$.

Observe that $N_G(z) \cap N_G(w_2) = \phi$. By Lemma 3.2.8(3), $d(z, w_2) = 1$ or $d(z, w_2) = 3$. If $d(z, w_2) = 1$, then $zw_2 \in E(G)$ and thus G is isomorphic to J_6. Further, if $d(z, w_2) = 3$, then $zw_2 \notin E(G)$ and thus G is isomorphic to J_7.

Subcase 1.2.2: $vw_1 \notin E(G)$.

Consider $G + bw_2$. Let $\{s\} = D_{bw_2} - \{b, w_2\}$. We first suppose that $D_{bw_2} = \{b, s\}$. Then $s \notin N_G(w_2)$ by Lemma 3.2.1(2). Thus $s \in \{w_1, z\}$ since $s \succ \{a, v\}$. But then no vertex of D_{bw_2} is adjacent to v, a contradiction. Hence,

$D_{bw_2} = \{w_2, s\}$. Then $s \notin N_G(b)$ by Lemma 3.2.1(2). Thus $s = a$ since $s \succ \{u\}$. It follows that $w_2z \in E(G)$ since $sz \notin E(G)$. Hence, G is isomorphic to J_8.

Case 2: There exists a vertex of V_1, say a such that a is not adjacent to some vertex of V_2, say y.

By Case 1, we may assume that $ab \in E(G)$. Since $y \in V_2$ and $ay \notin E(G)$, it follows that $by \in E(G)$. Thus $y \in B - A$. Further, since $D_{uy} = \{y, y_2\}$, $z_1y \in E(G)$ or $z_1y_2 \in E(G)$. Thus $d(b, z_1) = 2$. Consider $G + ay$. By Lemma 3.2.3, $D_{ay} = \{a, z\}$ for some $z \in (V_2 \cup V_3) - \{y\}$. Then $zy \notin E(G)$ and $z \succ V_3$. If $z \succ V_2 - \{y\}$, then $\{z, b\} \succ G$, a contradiction. Hence, there is $y_1 \in V_2 - \{y\}$ such that $zy_1 \notin E(G)$. Then $ay_1 \in E(G)$ since $D_{ay} = \{a, z\}$. We now distinguish two cases.

Case 2.1: $|V_2| = 3$.

Then $|V_3| = 2$.

Claim 4: $z \in V_2$.

Suppose this is not the case. Then $z \in V_3$. Thus there exits $y_2 \in V_2 - \{y, y_1\}$ such that $zy_2 \in E(G)$. It follows that $by_1 \notin E(G)$ otherwise $\{b, z\} \succ G$. Consider $G + uy$. By Lemma 3.2.4, $D_{uy} = \{y, w\}$ for some $w \in (V_2 \cup V_3) - \{y\}$. Then $w \succ \{a, z\}$ since $ya \notin E(G)$ and $yz \notin E(G)$. Thus $w = y_2$ since $y_1z \notin E(G)$. Hence, $y_2a \in E(G)$. Now consider $G + uy_1$. By similar argument, $D_{uy_1} = \{y_1, y_2\}$ and thus $y_2b \in E(G)$.

Let $\{z_1\} = V_3 - \{z\}$. Since $D_{uy_1} = \{y_1, y_2\}$, $z_1y_1 \in E(G)$ or $z_1y_2 \in E(G)$. Thus $d(a, z_1) = 2$. Further, since $D_{uy} = \{y_1, y_2\}$, $z_1y \in E(G)$ or $z_1y_2 \in E(G)$. Thus $d(b, z_1) = 2$. Consider $G + az_1$. By Lemma 3.2.3, $D_{az_1} = \{a, w_1\}$ for some $w_1 \in (V_2 \cup V_3) - \{z_1\}$. Then $w_1 \succ \{y, z\}$ since $ay \notin E(G)$ and $az \notin E(G)$. Thus $w_1 = y_2$ since z is not adjacent to y and y_1. Hence, $y_2y \in E(G)$. We next consider $G + bz_1$. By similar arguments, $D_{bz_1} = \{b, y_2\}$ and thus $y_2y_1 \in E(G)$ since $by_1 \notin E(G)$. Now $y_2 \succ \{a, b, y, y_1, z\}$. Clearly, $\{y_2, z\} \succ G + uz$ and $\{y_2, z_1\} \succ G + uz_1$ since $z \succ V_3$ and $V_3 = \{z, z_1\}$. But this contradicts the fact that $\gamma(G + uv) = 3$ since $v \in \{z, z_1\}$ This settles our claim.

By our hypothesis that $|V_2| = 3$ and Claim 4, $V_2 = \{y, y_1, z\}$. Note that $by_1 \notin E(G)$ otherwise $\{b, z\} \succ G$. Let $\{v_1\} = V_3 - \{v\}$. Then $zv \in E(G)$ and $zv_1 \in E(G)$. Since $z \in V_2$, we may assume without loss of generality that

$za \in E(G)$.

Claim 5: $N_G(z) = \{v, v_1, a, b\}$.

By the above argument we only need to show that $zb \in E(G)$. Consider $G + uy_1$. By Lemma 3.2.4, $D_{uy_1} = \{y_1, w_2\}$ for some $w_2 \in (V_2 \cup V_3) - \{y_1\}$. Since $y_1 b \notin E(G)$ and $y_1 z \notin E(G)$, it follows that $w_2 \succ \{b, z\}$. Thus $w_2 \in V_2$ since $b \in V_1$ and $z \in V_2$. Consequently, $w_2 = z$ since $yz \notin E(G)$. Hence, $bz \in E(G)$ as required. This settles our claim.

Claim 6: $vv_1 \notin E(G)$.

Suppose to the contrary that $vv_1 \in E(G)$. Consider $G + av_1$. Then, by Lemma 3.2.3, $D_{av_1} = \{a, w_3\}$ for some $w_3 \in (V_2 \cup V_3) - \{v_1\}$. Clearly, $w_3 \succ \{v\}$ but $w_3 \notin N_G(v_1)$ by Lemma 3.2.1(2). Thus $w_3 \notin \{z, v\}$. Consequently, $w_3 = y$ or $w_3 = y_1$. If $w_3 = y$, then $\{v, a\} \succ G$, a contradiction. Hence, $w_3 = y_1$. But then $\{v, b\} \succ G$, again a contradiction. This settles our claim.

Claim 7: $d(v) = 2$.

Suppose this is not the case. Then $d(v) = 3$ since $|V_2 \cup V_3| = 5$ and by Claim 6. Thus $vy \in E(G)$ and $vy_1 \in E(G)$. By Claim 5, $\{v, z\} \succ G + uv$, a contradiction. Thus $d(v) \leq 2$. By our hypothesis that $d(v) \geq d(u) = 2$, $d(v) = 2$. This settles our claim.

Recall that $vz \in E(G)$. We may now assume without loss generality that $vy \in E(G)$ but $vy_1 \notin E(G)$. Note that $v_1 y \notin E(G)$ otherwise $\{y, a\} \succ G$. Now consider $G + av$. By Lemma 3.2.3, $D_{av} = \{a, w_4\}$ for some $w_4 \in (V_2 \cup V_3) - \{v\}$. Then $w_4 \succ \{y, v_1\}$ but $w_4 \notin N_G(v)$ by Lemma 3.2.1(2). Thus $w_4 \notin \{y, z\}$. Hence, $w_4 \in \{v_1, y_1\}$. Clearly, $w_4 \neq v_1$ since $v_1 y \notin E(G)$. Hence, $w_4 = y_1$. Then $y_1 y \in E(G)$ and $y_1 v_1 \in E(G)$. Hence, G is isomorphic to the graph in Figure 4.4.

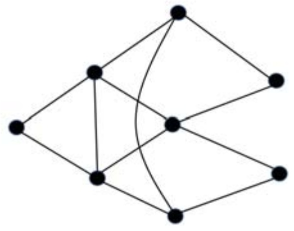


Figure 4.4: A 3-$(\gamma, 2)$-critical graph of order 8

It is easy to see that the graph in Figure 4.4 is isomorphic to the graph J_8.

Case 2.2: $|V_2| = 4$.

Then $|V_3| = 1$ and thus $V_3 = \{v\}$. Recall that $V_1 = \{a, b\}$ and $ab \in E(G)$.

Claim 8: $d(v) = 2$.

Clearly, $N_G(v) \subseteq V_2$ and $|N_G(v)| \leq |V_2| - 2 = 2$ otherwise $\{v, a\} \succ G$ or $\{v, b\} \succ G$ since each vertex of V_2 is adjacent to a or b. By our hypothesis that $d(v) \geq d(u) = 2$, $d(v) = 2$.

Claim 9: If $\{x_1, x_2\} = V_2 - N_G(v)$, then $N_{V_1}(x_1) \cap N_{V_1}(x_2) = \phi$.

If $x \in N_{V_1}(x_1) \cap N_{V_1}(x_2)$ where $x \in \{a, b\}$, then $\{x, v\} \succ G$, a contradiction. This settles our claim.

Put $N_G(v) = \{w_1, w_2\}$. We first suppose that $w_1w_2 \notin E(G)$. For $1 \leq i \leq 3$, set $V_i^{'} = \{x \in V(G)|\ d(v, x) = i\}$. Clearly, $u \in V_3^{'}$ and $\{v\} \cup V_1^{'} \cup V_2^{'} \cup V_3^{'}$ forms a partition set of $V(G)$. By applying Case 1, G is isomorphic to J_i for some $3 \leq i \leq 8$. We may now assume that $w_1w_2 \in E(G)$. Thus $G[\{v, w_1, w_2\}] = K_3$.

Claim 10: For $1 \leq i \leq 2$, $N_{V_1}(w_i) = \{a, b\}$.

Since $w_i \in V_2$, $w_ia \in E(G)$ or $w_ib \in E(G)$. Suppose without loss of generality that $w_ia \in E(G)$ and assume that $w_ib \notin E(G)$. Consider $G + bw_i$. By Lemma 3.2.3, $D_{bw_i} = \{b, x\}$ for some $x \in (V_2 \cup V_3) - \{w_i\}$. Then $x \succ \{v\}$ but $x \notin N_G(w_i)$ by Lemma 3.2.1(2). But this is not possible since $G[\{v\} \cup N_G(v)] = G[\{v, w_1, w_2\}] = K_3$. This settles our claim.

Recall that $y \in B - A$, $y_1 \in A$ and z is not adjacent to y and y_1. Further, $z \succ V_3$. Then $y \notin \{w_1, w_2\}$ by Claim 10. Since $z \succ \{v\}$ and $zy_1 \notin E(G)$, it follows that $y_1 \notin \{w_1, w_2\}$ since $G[\{v\} \cup N_G(v)] = K_3$. Hence, $V_2 - N_G(v) = \{y, y_1\}$. By Claim 9, $y_1b \notin E(G)$ since $yb \in E(G)$. Note that $z \in \{v, w_1, w_2\}$. It is easy to see that, for $1 \leq i \leq 2$, $w_iy \notin E(G)$ otherwise $\{w_i, a\} \succ G$ and $w_iy_1 \notin E(G)$ otherwise $\{w_i, b\} \succ G$. Consider $G + uy$. By Lemma 3.2.4, $D_{uy} = \{y, c\}$ where $c \in (V_2 \cup V_3) - \{y\}$. Clearly, $c \succ \{v\}$ since $yv \notin E(G)$. Then $c \in \{v, w_1, w_2\}$. Thus $yy_1 \in E(G)$ since y_1 is not adjacent to any vertex of $\{v, w_1, w_2\}$. Hence, G is isomorphic to the graph in Figure 4.5.

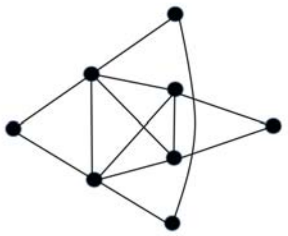


Figure 4.5: A 3-$(\gamma, 2)$-critical graph of order 8

It is easy to see that the graph in Figure 4.5 is isomorphic to the graph J_2. This completes the proof of Case 2.2 and hence our theorem. $\square$

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