DUALITY OF SEQUENCE SPACES OF INFINITE MATRICES

By

Mr. Suchat Samphavat

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree

Master of Science Program in Mathematics

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ภาวะคู่กันของปริภูมิลำดับของแมทริกซ์อนันต์

โดย

นายสุชาติ เสมประวัติ

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

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The Graduate School, Silpakorn University has approved and accredited the Thesis title of “Duality of sequence spaces of infinite matrices” submitted by MR.Suchat Samphavat as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics

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In this thesis, we define, for each $1 \leq r < \infty$, the set $\mathfrak{I}^r$ of infinite complex matrices as follows:

$$\mathfrak{I}^r := \left\{ \left\{ \left[a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty} : \left[\sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \right|^r \right] \in B\left(l^2\right) \right\}.$$

We first show as a preliminary that equipped with the norm

$$\left\| \left\{ \left[a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty} \right\|_r := \left\| \left[\sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \right|^r \right] \right\|^{1/r},$$

the set $\mathfrak{I}^r$ is a Banach space. The main goal of this research is to decompose the dual $\left(\mathfrak{I}^r\right)^*$ of $\mathfrak{I}^r$ as an l^1 direct- sum of its two closed subspaces by a way analogous to the classical theorem of Dixmier on decomposing the dual $B\left(l^2\right)^*$ of $B\left(l^2\right)$.

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Student's signature	Academic Year 2011
Thesis Advisor's signature	

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ในวิทยานิพนธ์นี้ เรานิยามเซต $\mathfrak{I}^r$ ของเมทริกซ์อนันต์ของจำนวนเชิงซ้อน สำหรับ $1 \le r < \infty$ ดังนี้

$$\mathfrak{I}^r := \left\{ \left\{ \left[a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty} : \left[\sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \right|^r \right] \in B\left(l^2\right) \right\}$$

ในขั้นแรก เราได้แสดงว่าเซต $\mathfrak{I}^r$ พร้อมด้วยนอร์มที่นิยามโดย

$$\left\| \left\{ \left[a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty} \right\|_r := \left\| \left[\sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \right|^r \right] \right\|^{1/r}$$

เป็นปริภูมิบานาค จุดประสงค์หลักของวิทยานิพนธ์นี้คือการแยกปริภูมิคู่กัน $\left(\mathfrak{I}^r\right)^*$ ของ $\mathfrak{I}^r$ เป็นส่วน ในรูปของ l^1 ผลบวกตรงของ 2 ปริภูมิย่อยปิดของ $\left(\mathfrak{I}^r\right)^*$ ในทำนองเดียวกันกับทฤษฎีบทของดิกซ์เมียร์ที่ว่าด้วยการแยกปริภูมิคู่กัน $B\left(l^2\right)^*$ ของ $B\left(l^2\right)$

ภาควิชาคณิตศาสตร์ บัณฑิตวิทยาลัย มหาวิทยาลัยศิลปากร

ลายมือชื่อนักศึกษา.. ปีการศึกษา 2554

ลายมือชื่ออาจารย์ที่ปรึกษาวิทยานิพนธ์ ....

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Chapter 1

Introduction and Preliminaries

A beautiful decomposition of the dual $\mathcal{B}(l^2)^*$ of the Banach algebra $\mathcal{B}(l^2)$ of bounded linear operators on l^2, under the usual multiplication, was established by Dixmier (see [5] and [6]). He proved that every bounded linear functional f in $\mathcal{B}(l^2)^*$ can uniquely be decomposed as the sum $f = g + h$ of two bounded linear functionals h and g on $\mathcal{B}(l^2)$ such that g is a *trace* functional defined associated with a trace class operator T by $g(S) = \text{trace}(ST)$ for all $S \in \mathcal{B}(l^2)$, and h vanishes on the ideal $\mathcal{K}(l^2)$ of compact operators on l^2. The most interesting part of the theorem of Dixmier mentioned above is that the norm of the decomposition $f = g + h$ of each f in $\mathcal{B}(l^2)^*$ is additive, i.e., $\|f\| = \|g\| + \|h\|$. A part of Schatten's theorem (see [12]) states that $\mathcal{K}(l^2)^* \cong \mathcal{C}^1$, where $\mathcal{C}^1$ denotes the class of all trace class operators on l^2. From the theorem of Schatten, it can easily be deduced that the space of all trace functionals on $\mathcal{B}(l^2)$ is isometrically isomorphic to the dual $\mathcal{K}(l^2)^*$ of $\mathcal{K}(l^2)$. Thus the theorem of Dixmier mentioned above can be symbolized as $\mathcal{B}(l^2)^* = \mathcal{K}(l^2)^* \oplus_1 \mathcal{K}(l^2)_s$, where $\mathcal{K}(l^2)_s$ denotes the space of all linear functionals on $\mathcal{B}(l^2)$ vanishing on $\mathcal{K}(l^2)$, which are called *singular functionals* on $\mathcal{K}(l^2)$, and the notation "$\oplus_1$" is referred to as the l^1 direct-sum.

Let $1 \le p, q, r < \infty$. An infinite scalar matrix $A = [a_{jk}]$ is said to *define a linear operator from l^p into l^q* if for every $x = \{x_k\}_{k=1}^\infty$ in l^p the series $\sum_{k=1}^{\infty} a_{jk}x_k$ converges for all j, and the sequence $Ax := \left\{\sum_{k=1}^{\infty} a_{jk}x_k\right\}_{j=1}^{\infty}$ is a member of l^q. If a matrix A defines a linear operator from l^p into l^q, we then call the operator $x \mapsto Ax$ the *linear operator defined by A*. In this case, it can be shown by the uniform boundedness principle that the linear operator defined by A is bounded. Let $\mathcal{M}(l^p, l^q)$ be the set of all infinite matrices which define linear operators from l^p into l^q. For each matrix A, we call A a *bounded matrix* and define $\|A\|$ to be the norm of the linear operator defined by A if $A \in \mathcal{M}(l^p, l^q)$ and call A an *unbounded matrix* and define $\|A\|$ to be ∞ otherwise. It is well-known that $\mathcal{M}(l^p, l^q)$ is a Banach space under the norm $\|\cdot\|$. Indeed, it coincides with the set of matrix representations of all bounded linear operators from l^p into l^q with respect to the standard Schauder bases of l^p and l^q, which is isometrically isomorphic to the Banach space $\mathcal{B}(l^p, l^q)$ of all bounded linear operators from l^p into l^q. A matrix A is called a *compact matrix* if the linear operator defined by A is a compact operator.

For each matrix $A = [a_{ji}]$ and positive integer n, let $A_{n_\lrcorner} = [b_{ji}]$ be the matrix with $b_{ji} = a_{ji}$ for all $1 \leq j, i \leq n$ and $b_{ji} = 0$ otherwise, and let $A_{n^\ulcorner} = [c_{ji}]$ be the matrix with $c_{ji} = a_{ji}$ for all $j, i \geq n$ and $c_{ji} = 0$ otherwise. The following are well-known facts about infinite matrices which are useful for the research.

Theorem 1.1.

(1) *If* $[a_{ji}]$ *and* $[b_{ji}]$ *are scalar matrices such that* $|a_{ji}| \leq b_{ji}$ *for all* j, i, *then* $\|[a_{ji}]\| \leq \|[|a_{ji}|]\| \leq \|[b_{ji}]\|$.

(2) *A matrix* A *belongs to* $\mathcal{B}(l^p, l^q)$ *if and only if* $\sup\limits_n \|A_{n_\lrcorner}\| < \infty$.

(3) *For every matrix* A, $\|A_{n_\lrcorner}\| \nearrow \|A\|$.

(4) *For each* $A \in \mathcal{B}(l^2)$ *and positive integer* n, $\|A_{n_\lrcorner} + A_{n^\ulcorner}\| = \max\{\|A_{n_\lrcorner}\|, \|A_{n^\ulcorner}\|\}$.

(5) *A matrix* A *is compact as an operator on* l^2 *if and only if* $\|A_{n_\lrcorner} - A\| \to 0$.

The *Schur product* or *Hardamard product* or *entry-wise product* of two scalar matrices $A = [a_{jk}]$ and $B = [b_{jk}]$ having the same size is defined by the matrix $A \bullet B := [a_{jk} b_{jk}]$. In [13], Schur proved that Banach space $\mathcal{B}(l^2)$ is a commutative Banach algebra (without identity) under the operator norm and the Schur product multiplication. After that, Bennett extended in [1] the result of Schur referred to above. He showed for each $1 \leq p, q < \infty$ that the Banach space $\mathcal{B}(l^p, l^q)$ under the Schur product operation is also a Banach algebra. These beautiful results of Bennett motivated Chaisuriya and Ong [2] to study some classes of infinite matrices over Banach algebras with identity. In [2], for a fixed Banach algebra $\mathscr{B}$ with identity and $1 \leq p, q, r < \infty$, the authors defined the class $\mathcal{S}^r_{p,q}(\mathscr{B})$ of matrices $A = [a_{jk}]$ over $\mathscr{B}$ such that *the absolute Schur* $r^{th}-power$ $A^{[r]} := [\|a_{jk}\|^r]$ defines a linear operator from l^p into l^q. And then they proved that it is a Banach algebra under the *the absolute Schur r-norm* defined by

$$|||A|||_{p,q,r} = \left\|A^{[r]}\right\|^{1/r}$$

and the Schur product, which is straightforwardly generalized to the setting of matrices over the Banach algebra $\mathscr{B}$ by using the multiplication in $\mathscr{B}$. The authors also provided a beautiful relationship, which follows from the results of Schur and Bennett mentioned above, between the algebra $\mathcal{B}(l^p, l^q)$ of all bounded operators form l^p into l^q and the algebra $\mathcal{S}^r_{p,q}(\mathbb{C})$. They found that $\mathcal{B}(l^p, l^q)$ is contained in $\mathcal{S}^r_{p,q}(\mathbb{C})$ as a non-closed ideal for all $r \geq 2$.

In [8], Livshits, Ong and Wang studied the duality of the absolute Schur algebras $\mathcal{S}^r_{2,2}(\mathbb{C})$ by a way analogous to Dixmier's theorem and Schatten's theorem mentioned in the first paragraph. The authors defined the class $\mathcal{K}^r$ of infinite matrices A such that $A^{[r]}$ is compact as an operator on l^2 for playing the role as the class $\mathcal{K}(l^2)$ of all compact operators on l^2. They also constructed a class $\mathcal{M}^r$ of infinite matrices for playing the role as the class $\mathcal{C}^1$ of all trace class operators, which is known as the dual of $\mathcal{K}(l^2)$. They obtained that $(\mathcal{K}^r)^* \cong \mathcal{M}^r$ and that each bounded linear functional φ on $\mathcal{S}^r_{2,2}(\mathbb{C})$ can uniquely be decomposed as the sum $\varphi = \rho + \psi$, where ρ is determined by a unique matrix in $\mathcal{M}^r$ under a certain way and ψ is a singular functional on $\mathcal{K}^r$. Furthermore, the decomposition $\varphi = \rho + \psi$ satisfies $\|\varphi\| = \|\rho\| + \|\psi\|$. Schatten's theorem also states that the trace class operators form a predual of $\mathcal{B}(l^2)$. An analogue of this result on the setting of Livshits, Ong and Wang: $(\mathcal{M}^r)^* \cong \mathcal{S}^r_{2,2}(\mathbb{C})$, was also obtained.

From the beautiful result of Chaisuriya and Ong that the absolute Schur algebra $\mathcal{S}^2_{2,2}(\mathbb{C})$ contains $\mathcal{B}(l^2)$ as a non-closed ideal, Rakbud and Ong defined three sequence spaces of matrices from $\mathcal{S}^2_{2,2}(\mathbb{C})$ in [11] as follows:

$$\mathscr{O}_b = \left\{ \{A_k\}_{k=1}^{\infty} \subseteq \mathcal{S}^2_{2,2}(\mathbb{C}) : \text{ the sequence } \left\{ \sum_{k=1}^{n} A_k^{[2]} \right\}_{n=1}^{\infty} \text{ is bounded in } \mathcal{B}(l^2) \right\},$$

$$\mathscr{O}_c = \left\{ \{A_k\}_{k=1}^{\infty} \subseteq \mathcal{S}^2_{2,2}(\mathbb{C}) : \text{ the sequence } \left\{ \sum_{k=1}^{n} A_k^{[2]} \right\}_{n=1}^{\infty} \text{ converges in } \mathcal{B}(l^2) \right\},$$

and

$$\mathscr{O}_\kappa = \left\{ \left\{ \left[a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty} \subseteq \mathcal{S}^2_{2,2}(\mathbb{C}) : \text{ the matrix } \left[\sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \right|^2 \right] \in \mathcal{K}(l^2) \right\}.$$

The authors obtained the inclusion relation among these three spaces as follows: $\mathscr{O}_\kappa \subsetneq \mathscr{O}_c \subsetneq \mathscr{O}_b$. They defined naturally a norm on these three spaces by

$$|||\{A_k\}_{k=1}^{\infty}||| = \left(\sup_n \left\| \sum_{k=1}^{n} A_k^{[2]} \right\| \right)^{1/2}$$

and showed that all three sequence spaces equipped with this norm are Banach spaces. It was observed that because of the non-closedness of $\mathcal{B}(l^2)$ in $\mathcal{S}^2_{2,2}(\mathbb{C})$, the restrictions of these sequence spaces to $\mathcal{B}(l^2)$ are all not complete. The study on this paper was mainly focused on the sequence spaces $\mathscr{O}_c$ and $\mathscr{O}_\kappa$. The authors studied sequential convergence in these two sequence spaces and duality and preduality of $\mathscr{O}_\kappa$.

From the idea of Rakbud and Ong referred to above, we obtain a way analogous to the classical sequence spaces l^p to define sequence spaces of infinite matrices as follows. Let $\mathscr{M}_\infty$ be the vector space of all infinite complex matrices. For each $1 \leq r < \infty$, let

$$\mathscr{L}^r = \left\{ \left\{ \left[a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty} \subseteq \mathscr{M}_\infty : \left[\sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \right|^r \right] \in \mathcal{B}(l^2) \right\}.$$

In this research, we study some elementary properties and provide some results on duality of the sequence spaces $\mathscr{L}^r$. The main goal is to establish a decomposition theorem for the dual space $(\mathscr{L}^r)^*$ of $\mathscr{L}^r$ by a way analogous to the theorem of Dixmier mentioned in the first paragraph.

Chapter 2

Theoretical Background

In this chapter, we provide some theoretical background which is necessary for the research.

Throughout this thesis, we let $\mathbb{C}$ and $\mathbb{R}$ denote the sets of all complex numbers and real numbers respectively.

2.1 Banach Spaces

Definition 2.1.1. [9] Let X be a vector space over a scalar field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). A *norm* on X is a real-valued function $\|\cdot\|$ on X satisfying the following properties:

(*i*) $\|x\| \geq 0$;

(*ii*) $\|x\| = 0$ if and only if $x = 0$;

(*iii*) $\|\alpha x\| = |\alpha| \, \|x\|$;

(*iv*) $\|x + y\| \leq \|x\| + \|y\|$ (*Triangle inequality*),

where x and y are arbitrary vectors in X and α is any scalar in $\mathbb{K}$. A *normed space* is a pair $(X, \|\cdot\|)$ of a non-empty set X and a norm $\|\cdot\|$ on X. It may be sometimes written just X as a normed space by omitting the norm on X.

Definition 2.1.2. [9] A sequence $\{x_n\}_{n=1}^{\infty}$ in a normed space X is said to *converge* or to be *convergent* if there is a point x in X satisfying the following property: for any $\epsilon > 0$, there is a positive integer N such that

$$\|x - x_n\| < \epsilon \quad \text{for all} \;\; n \geq N.$$

In this situation, we write $\lim_{n\to\infty} x_n = x$, or simply $x_n \to x$ and call x the *limit* of $\{x_n\}_{n=1}^{\infty}$.

Definition 2.1.3. [9] A sequence $\{x_n\}_{n=1}^{\infty}$ in a normed space X is said to be *bounded* if there is a positive real number c such that $\|x_n\| \leq c$ for all positive integer n.

Definition 2.1.4. [9] A sequence $\{x_n\}_{n=1}^{\infty}$ in a normed space X is said to be a *Cauchy sequence* in X if for any $\epsilon > 0$, there is a positive integer N such that

$$\|x_m - x_n\| < \epsilon$$

for all $m, n \geq N$. A normed space X is said to be a *Banach space* if it is *complete* under the metric d defined by $d(x, y) = \|x - y\|$, that is, every Cauchy sequence converges to an element in X.

Definition 2.1.5. [9] Let X and Y be vector spaces over the same scalar field. A function $T : X \to Y$ is said to be a *linear operator* or *linear function* or *linear transformation* if

$$T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2$$

for every $x_1, x_2 \in X$ and any scalars α and β.

Definition 2.1.6. [9] Let X and Y be normed spaces over the same scalar field. A linear operator $T : X \to Y$ is said to be *bounded* if $T(B)$ is bounded for all bounded subsets B of X.

Definition 2.1.7. Let T be a linear operator from a normed space X into a normed space Y. Then the range of T is denoted by ran T. We call the set $\{x \in X : Tx = 0\}$ the *kernel* of T and denote by ker T.

Theorem 2.1.8. [9] *Let $T : X \to Y$ be a linear operator from a normed space X into a normed space Y. Then the following are equivalent.*

(1) *T is bounded.*

(2) *T is continuous.*

(3) *There is a constant $M > 0$ such that $\|Tx\| \leq M \|x\|$ for all $x \in X$.*

Let $\mathcal{B}(X, Y)$ be the set of all bounded linear operators from a normed space X into a normed space Y. We denote $\mathcal{B}(X, X)$ by just $\mathcal{B}(X)$.

Definition 2.1.9. [9] Let X and Y be normed spaces. For each T in $\mathcal{B}(X, Y)$, the *norm* or *operator norm* $\|T\|$ of T is the nonnegative real number $\sup\{\|Tx\| : x \in X, \|x\| \leq 1\}$. The operator norm on $\mathcal{B}(X, Y)$ is the map $T \mapsto \|T\|$.

From Theorem 2.1.8, the following corollary is immediately obtained.

Corollary 2.1.10. [9] *If T is a bounded linear operator from a normed space X into a normed space Y, then $\|Tx\| \leq \|T\| \|x\|$ for all x in X. Furthermore, the number $\|T\|$ is the smallest nonnegative real number M such that $\|Tx\| \leq M \|x\|$ for all $x \in X$.*

Definition 2.1.11. [9] Let T be a linear operator from a normed space X onto a normed space Y. The operator T is an *isometric isomorphism* if $\|T(x)\| = \|x\|$ whenever $x \in X$.

Notice that the condition $\|T(x)\| = \|x\|$ for all $x \in X$ implies T is an one-to-one function.

Theorem 2.1.12. [9] *If X is a normed space and Y is a Banach space, then the set $\mathcal{B}(X,Y)$ equipped with the operator norm is a Banach space.*

Theorem 2.1.13. [9] (The Uniform Boundedness Principle) *Let $\mathcal{F}$ be a nonempty family of bounded linear operators from a Banach space X into a normed space Y. If* $\sup\{\|Tx\| : T \in \mathcal{F}\}$ *is finite for each x in X, then* $\sup\{\|T\| : T \in \mathcal{F}\}$ *is finite.*

Definition 2.1.14. [9] A normed space X is said to be the *direct sum* of its two subspaces Y and Z, written by $X = Y \oplus Z$, if each $x \in X$ has a unique representation of the form $x = y + z$, where $y \in Y$ and $z \in Z$. If, in addition, the condition $\|x\| = \|y\| + \|z\|$ is satisfied for all $x \in X$, we say specifically that X is the l^1 *direct-sum* of Y and Z and write $X = Y \oplus_1 Z$ in this situation.

Theorem 2.1.15. [9] *Let X be a normed space and Y and Z be subspaces of X. Then $X = Y \oplus Z$ if and only if for $X \cap Y = \{0\}$ and for every x in X, there are $y \in Y$ and $z \in Z$ such that $x = y + z$.*

Definition 2.1.16. [9] A *linear functional* f is a linear operator from a normed space X into the scalar filed $\mathbb{K}$ of X, where $\mathbb{K}$ is regarded as a normed space under the usual norm on $\mathbb{K}$.

If X is a normed space, then the set of all bounded linear functionals on X is denoted by X^*. By Theorem 2.1.11, the normed space X^* is immediately a Banach space.

Theorem 2.1.17. [9] (Hahn-Banach extension theorem) *Let X be a Banach space and Y a closed subspace of X. If f_0 is a bounded linear functional on Y, then there is a unique bounded linear functional f on X such that $f(x) = f_0(x)$ for all $x \in Y$ and $\|f\| = \|f_0\|$.*

Definition 2.1.18. [9] Let X be a normed space and Y a subspace of X. The *annihilator* of Y, denoted by $Y^\perp$, is the set $\{f \in X^* : f(x) = 0 \text{ for all } x \in Y\}$.

Theorem 2.1.19. [9] *If X is a normed space and Y is a subspace of X, then $Y^\perp$ is a closed subspace of X^*.*

Definition 2.1.20. [9] Let X and Y be Banach spaces. A linear operator $T : X \to Y$ is *compact* if $\overline{T(B)}$ is compact for all bounded subset B of X. The set of all compact operators from X into Y will be denoted by $\mathcal{K}(X, Y)$. For the case where $X = Y$, we write $\mathcal{K}(X)$ instead of $\mathcal{K}(X, Y)$.

Proposition 2.1.21. [9] *Let X and Y be Banach spaces. Then the following hold.*

(1) $\mathcal{K}(X, Y) \subseteq \mathcal{B}(X, Y)$.

(2) $\mathcal{K}(X, Y)$ *is a closed subspace of* $\mathcal{B}(X, Y)$.

(3) *If $X = Y$, then $\mathcal{K}(X)$ is an ideal of $\mathcal{B}(X)$.*

Definition 2.1.22. [9] A linear operator T from a Banach space X into a Banach space Y is said to be of *finite rank* if $T(X)$ is finite dimensional.

Theorem 2.1.23. [9] *A finite rank operator from a Banach space X into a Banach space Y is bounded if and only if it is compact.*

2.2 l^p Spaces

Definition 2.2.1. [9] For $1 \leq p \leq \infty$ and a sequence $\{\lambda_k\}_{k=1}^{\infty}$ of complex numbers, the *p-norm* of $\{\lambda_k\}_{k=1}^{\infty}$ is defined by

$$\|\{\lambda_k\}_{k=1}^{\infty}\|_p = \begin{cases} \left(\displaystyle\sum_{k=1}^{\infty} |\lambda_k|^p\right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \sup\{|\lambda_k| : k = 1, 2, 3, ...\} & \text{if } p = \infty. \end{cases}$$

For each $1 \leq p < \infty$, let

$$l^p = \left\{ \{\lambda_k\}_{k=1}^{\infty} \subseteq \mathbb{C} : \sum_{k=1}^{\infty} |\lambda_k|^p < \infty \right\}$$

and

$$l^{\infty} = \{\{\lambda_k\}_{k=1}^{\infty} \subseteq \mathbb{C} : \sup\{|\lambda_k| : k = 1, 2, 3, ...\} < \infty\}.$$

Theorem 2.2.2. [9] (Hölder's inequality) *For any $1 \leq p \leq \infty$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and sequences $\mathbf{x}$ and $\mathbf{y}$ of complex numbers,* $\|\mathbf{xy}\|_1 \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q$.

In particular, Hölder's inequality is also called Cauchy-Schwartz's inequality when $p = q = 2$. From Hölder's inequality, the following Minkowski's inequality is obtained.

Theorem 2.2.3. [9] (Minkowski's inequality) *For any* $1 \leq p \leq \infty$ *and sequences* $\mathbf{x}$ *and* $\mathbf{y}$ *of complex numbers,* $\|\mathbf{x}+\mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p$.

Theorem 2.2.4. [9] *For any* $1 \leq p \leq \infty$*, the set* l^p *endowed with the p-norm* $\|\cdot\|_p$ *is a Banach space.*

For $1 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, we define, for each $\mathbf{x} = \{x_k\}_{k=1}^{\infty} \in l^q$, the function $f_{\mathbf{x}} : l^p \to \mathbb{C}$ by

$$f_{\mathbf{x}}(\{y_k\}_{k=1}^{\infty}) = \sum_{k=1}^{\infty} x_k y_k \text{ for all } \{y_k\}_{k=1}^{\infty} \in l^p.$$

By Hölder's inequality, we have that the function $f_{\mathbf{x}}$ is well-defined.

Theorem 2.2.5. [9] *Let* $1 \leq p < \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*. Then* l^q *is isometrically isomorphic to* $(l^p)^*$ *by the isomorphism defined by* $\mathbf{x} \mapsto f_{\mathbf{x}}$.

The following result is closely related to the duality theorem stated above.

Theorem 2.2.6. [9] *Let* $1 \leq p < \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*. Then a sequence* $\{x_k\}_{k=1}^{\infty}$ *of complex numbers belongs to* l^q *if and only if* $\{x_k y_k\}_{k=1}^{\infty}$ *belongs to* l^1 *for all* $\{y_k\}_{k=1}^{\infty}$ *in* l^p.

2.3 Hilbert Spaces

Definition 2.3.1. [4] Let $\mathcal{H}$ be a vector space over a scalar $\mathbb{K}$ ($\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$), a *semi-inner product* on $\mathcal{H}$ is a function $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{K}$ having the following properties:

(i) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$;

(ii) $\langle x, x \rangle \geqslant 0$;

(iii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

If $\langle \cdot, \cdot \rangle$ has the following additional property:

(iv) if $\langle x, x \rangle = 0$, then $x = 0$,

we call $\langle \cdot, \cdot \rangle$ an *inner product* on $\mathcal{H}$.

From *(i)*, we have $\langle 0, y \rangle = \langle 0x, y \rangle = 0\langle x, y \rangle = 0$, and similarly, $\langle x, 0 \rangle = 0$. In particular, $\langle 0, 0 \rangle = 0$. Hence if $\langle \cdot, \cdot \rangle$ is an inner-product, then $\langle x, x \rangle = 0$ if and only if $x = 0$. If $\langle \cdot, \cdot \rangle$ is an inner-product on $\mathcal{H}$, then

$$\|x\| = \langle x, x \rangle^{1/2}$$

defines a norm on $\mathcal{H}$. We call a vector space $\mathcal{H}$ equipped with an inner product on $\mathcal{H}$ an *inner product space*. Every inner product space is a normed space under the norm defined by $\|x\| = \langle x, x\rangle^{1/2}$. If $\mathcal{H}$ equipped with the norm $\|\cdot\|$ is a Banach space, we call $\mathcal{H}$ a *Hilbert space*.

Let, in the sequel, $\mathcal{H}$ and $\mathcal{L}$ be Hilbert spaces.

Definition 2.3.2. [4] If $f, g \in \mathcal{H}$, then f and g are *orthogonal* if $\langle f, g\rangle = 0$, in symbols, $f \perp g$. If $A, B \subseteq \mathcal{H}$, we say that A and B are *orthogonal* and write $A \perp B$ provided $f \perp g$ for every f in A and g in B. If $A \subseteq \mathcal{H}$ and $f \in \mathcal{H}$ satisfying $\{f\} \perp A$, then we write $f \perp A$. If $A \subseteq \mathcal{H}$, then the set $A^\perp$ is defined by $A^\perp = \{h \in \mathcal{H} : h \perp g \text{ for all } g \in A\}$.

Definition 2.3.3. [4] An *orthonormal set* in $\mathcal{H}$ is a subset $\mathcal{E}$ of $\mathcal{H}$ having the following properties:

(i) for $e \in \mathcal{E}$, $\|e\| = 1$;

(ii) if $e_1, e_2 \in \mathcal{E}$ and $e_1 \neq e_2$, then $e_1 \perp e_2$.

An *orthonormal basis* for $\mathcal{H}$ is a maximal orthonormal set.

Proposition 2.3.4. [4] *If $\mathcal{E}$ is an orthonormal set in $\mathcal{H}$, then there is an orthonormal basis for $\mathcal{H}$ that contains $\mathcal{E}$.*

Theorem 2.3.5. [4] *If $\mathcal{E}$ is an orthonormal set in $\mathcal{H}$ and $h \in \mathcal{H}$, then $\{e \in \mathcal{E} : \langle h, e\rangle \neq 0\}$ is countable.*

Theorem 2.3.6. [4] *Let $\mathcal{E}$ be an orthonormal set in $\mathcal{H}$. Then the following statements are equivalent.*

(1) *$\mathcal{E}$ is an orthonormal basis.*

(2) *If $h \in \mathcal{H}$ and $h \perp \mathcal{E}$, then $h = 0$.*

(3) *$\bigvee \mathcal{E} = \mathcal{H}$, where $\bigvee \mathcal{E}$ is the smallest closed subspace of $\mathcal{H}$ containing $\mathcal{E}$.*

(4) *$h = \sum\{\langle h, e\rangle e : e \in \mathcal{E}\}$ for all $h \in \mathcal{H}$, where $\sum\{\langle h, e\rangle e : e \in \mathcal{E}\}$ denotes the limit of the net $\left\{\sum_{e\in F}\langle h, e\rangle e : F \text{ is a finite subset of } \mathcal{E}\right\}$.*

Theorem 2.3.7. *Any two orthonormal bases of $\mathcal{H}$ have the same cardinality.*

Definition 2.3.8. [4] The *dimension* of $\mathcal{H}$ is the cardinality of an orthornomal basis and is denoted by $\dim \mathcal{H}$.

Definition 2.3.9. [4] A subset D of $\mathcal{H}$ is said to be *dense* in $\mathcal{H}$ if $\overline{D} = \mathcal{H}$. $\mathcal{H}$ is said to be *separable* if it has a countable subset which is dense in $\mathcal{H}$.

Theorem 2.3.10. [4] *Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Then $\mathcal{H}$ is separable if and only if* $\dim\mathcal{H}=\aleph_0$, *where $\aleph_0$ is the cardinality of the set of all positive integers.*

Definition 2.3.11. [4] A function $u : \mathcal{H} \times \mathcal{L} \to \mathbb{K}$ is *a sesquilinear form* if for h, g in $\mathcal{H}$, k, f in $\mathcal{L}$, and α, β in $\mathbb{K}$,

(*i*) $u(\alpha h + \beta g, k) = \alpha u(h, k) + \beta u(g, k)$;

(*ii*) $u(h, \alpha k + \beta f) = \overline{\alpha} u(h, k) + \overline{\beta} u(h, f)$.

Definition 2.3.12. [4] A sesquilinear form u is *bounded* if there is a constant M such that $|u(h, k)| \leq M \|h\| \|k\|$ for all h in $\mathcal{H}$ and k in $\mathcal{L}$. The constant M is called a *bound* for u.

Theorem 2.3.13. [4] *If $u : \mathcal{H} \times \mathcal{L} \to \mathbb{K}$ is a bounded sesquilinear form with a bound M, then there are unique operators A in $\mathcal{B}(\mathcal{H}, \mathcal{L})$ and B in $\mathcal{B}(\mathcal{L}, \mathcal{H})$ such that*

$$u(h, k) = \langle Ah, k\rangle = \langle h, Bk\rangle$$

for all h in $\mathcal{H}$ and k in $\mathcal{L}$ and both $\|A\|$ and $\|B\|$ are not greater than M.

Definition 2.3.14. [4] If $A \in \mathcal{B}(\mathcal{H}, \mathcal{L})$, then the unique operator B in $\mathcal{B}(\mathcal{L}, \mathcal{H})$ satisfying $\langle Ah, k\rangle = \langle h, Bk\rangle$ is called the *adjoint* of A and is denoted by A^*.

Proposition 2.3.15. [4] *If $A, B \in \mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})$ and $\alpha \in \mathbb{K}$, then the following hold.*

(1) $(\alpha A + B)^* = \bar{\alpha} A^* + B^*$.

(2) $(AB)^* = B^* A^*$.

(3) $A^{**} = (A^*)^* = A$.

(4) *If A is invertible in $\mathcal{B}(\mathcal{H})$, then* $(A^*)^{-1} = (A^{-1})^*$.

Theorem 2.3.16. [4] *Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space with an orthonormal basis $\{e_n\}$. If $T \in \mathcal{B}(\mathcal{H})$ with the matrix representation $A = [a_{ji}]$ with respect to the basis $\{e_n\}$, then the matrix representation of T^* with respect to $\{e_n\}$ is the matrix $[\overline{a_{ji}}]^t$.*

Definition 2.3.17. [4] A bounded linear operator A on a Hilbert space $\mathcal{H}$ is said to be *self-adjoint* if $A = A^*$.

Theorem 2.3.18. [4] *If $T \in \mathcal{B}(\mathcal{H}, \mathcal{L})$, the following statements are equivalent.*

(1) *T is compact.*

(2) *T^* is compact.*

(3) *There is a sequence $\{T_n\}$of operators of finite rank such that $\|T - T_n\| \to 0$.*

Definition 2.3.19. [4] If $A \in \mathcal{B}(\mathcal{H})$, a scalar α is an *eigenvalue* of A if $\ker(A - \alpha I) \neq \{0\}$.

Definition 2.3.20. [4] If $T \in \mathcal{B}(\mathcal{H})$, then T is *positive* if $\langle Th, h\rangle \geq 0$ for all $h \in \mathcal{H}$.

In symbols, this is denoted by $T \geq 0$. Note that every positive operator on a complex Hilbert space is self-adjoint.

Theorem 2.3.21. [4] *If T is a positive compact operator on a Hilbert space $\mathcal{H}$, then there is a unique positive compact operator A such that $A^2 = T$.*

Definition 2.3.22. [4] If T is a positive compact operator on a Hilbert space $\mathcal{H}$, then the unique positive compact operator A such that $A^2 = T$ according to Theorem 2.3.21 is called the *positive square root of T* and denoted by $|T|$.

Definition 2.3.23. [4] A *partial isometry* is a linear operator W such that $\|Wh\| = \|h\|$ for all $h \in (\ker W)^\perp$. The space $(\ker W)^\perp$ is called the *initial space* of W and the space ran W is called the *final space* of W.

Theorem 2.3.24. [4] (Polar Decomposition) *If $T \in \mathcal{B}(\mathcal{H})$, then there is a partial isometry W with $(\ker T)^\perp$ as its initial space and $\overline{\operatorname{ran} T}$ as its final space such that $T = W|T|$. Moreover, if $T = UP$ where $P \geq 0$ and U is a partial isometry with $\ker U = \ker P$, then $P = |T|$ and $U = W$.*

Theorem 2.3.25. [4] (Spectral Theorem) *If T is a compact self-adjoint operator on $\mathcal{H}$, then T has only a countable number of distinct eigenvalues. If $\{\lambda_1, \lambda_2, ...\}$ are the distinct nonzero eigenvalue of T with $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq ...$, and P_n is the projection of $\mathcal{H}$ onto $\ker(T - \lambda_n)$, then $P_nP_m = P_mP_n = 0$ if $n \neq m$, each λ_n is an real, and*

$$T = \sum_{n=1}^{\infty} \lambda_n P_n,$$

where the series converges to T in the metric defined by the norm of $\mathcal{B}(\mathcal{H})$.

Corollary 2.3.26. [4] *With the notation of Spectral Theorem. One has the following.*

(1) $\ker T = \overline{\operatorname{span}(\bigcup_{n=1}^{\infty} P_n\mathcal{H})} = (\operatorname{ran} T\ )^{\perp}$;

(2) *each* P_n *has finite rank;*

(3) $\|T\| = \sup\{|\lambda_n| : n \geq 1\}$ *and* $\lambda_n \to 0$ *as* $n \to \infty$.

Let T be a compact self-adjoint operator on $\mathcal{H}$. By Spectral Theorem, T has precisely finite or countable number of distinct eigenvalues. Let $\{\lambda_n\}_{n=1}^{\infty}$ be the sequence of eigenvalues of T with $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq ....$ For each n, let N_n be the dimension of $\ker(T - \lambda_n)$, and let $\{\mu_n\}_{n=1}^{\infty} = \{\underbrace{\lambda_1, ..., \lambda_1}_{N_1}, \underbrace{\lambda_2, ..., \lambda_2}_{N_2}, ...\}$. If T has only k eigenvalues, then we let $\mu_n = 0$ for all for $n > N_1 + N_2 + ... + N_k$.

Corollary 2.3.27. [4] *If* T *is a compact self-adjoint operator on* $\mathcal{H}$*, then there is an orthonormal basis* $\{e_n\}$ *for* $(\ker T)^{\perp}$ *such that*

$$Th = \sum_{n=1}^{\infty} \mu_n \langle h, e_n \rangle e_n$$

for all $h \in \mathcal{H}$.

2.4 Schatten p-Classes

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and K a compact operator on $\mathcal{H}$. Since $0 \leq \|Kh\|^2 = \langle Kh, Kh \rangle = \langle K^*Kh, h \rangle$ for all $h \in \mathcal{H}$, it follows that K^*K is a positive compact operator. Whence, by Theorem 2.3.21, there is a unique positive compact operator $|K|$ such that $|K|^2 = K^*K$. Since $|K|$ is positive, $|K|$ is self-adjoint. Thus, by Corollary 2.3.27, we have $|K|h = \sum_{n=1}^{\infty} \mu_n \langle h, e_n \rangle e_n$ for all $h \in \mathcal{H}$, where $\{\mu_n\}_{n=1}^{\infty}$ is the sequence of eigenvalues of $|K|$ and $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis for $(\ker |K|)^{\perp}$. Notice that $\ker K = \ker |K|$ due to the fact that $\|Kh\|^2 = \langle Kh, Kh \rangle = \langle h, K^*Kh \rangle = \langle h, |K|^2 h \rangle = \langle |K|h, |K|h \rangle = \||K|h\|^2$ for all $h \in \mathcal{H}$. We call the sequence $\{\mu_n\}_{n=1}^{\infty}$ *the sequence of singular values* of the compact operator K and denote μ_n by $s_n(K)$ for all n. By Corollary 2.3.26, we have $\|K\| = s_1(K) \geq s_2(K) \geq ... \geq 0$ and $\lim_{n\to\infty} s_n(K) = 0$.

For $1 \leq p \leq \infty$, let

$$\mathcal{C}^p = \{K \in \mathcal{K}(\mathcal{H}) : \{s_k(K)\}_{k=1}^{\infty} \in l^p\}.$$

The set $\mathcal{C}^p$ is called the *Schatten p-class*. We define, for $1 \leq p < \infty$, the norm $\|\cdot\|_p$ on $\mathcal{C}^p$ by

$$\|K\|_p = \left(\sum_{n=1}^{\infty} s_n(K)^p \right)^{1/p}.$$

For $p = \infty$, we define $\|K\|_{\infty} = \sup_n s_n(K)$. It is obvious that for any compact operator K on $\mathcal{H}$, $\|K\|_{\infty} = s_1(K) = \|K\|$. Thus $\mathcal{C}^{\infty} = \mathcal{K}(\mathcal{H})$.

Theorem 2.4.1. [5] *If $K \in \mathcal{C}^1$, then for each othonormal basis $\{e_n\}$ of $\mathcal{H}$ the sum $\sum_{n=1}^{\infty} \langle Ke_n, e_n\rangle$ is absolutely convergent and*

$$\sum_{n=1}^{\infty} \langle Ke_n, e_n\rangle = \sum_{n=1}^{\infty} s_n(K)\langle Ue_n, e_n\rangle,$$

where U is the unique partial isometry such that $K = U|K|$.

Definition 2.4.2. [5] For each $K \in \mathcal{C}^1$, the number

$$\sum_{n=1}^{\infty} \langle Ke_n, e_n\rangle,$$

where $\{e_n\}$ is an othonormal basis of $\mathcal{H}$, is called the *trace* of K.

Remark 2.4.3. [5] *If $\{e_n\}$ is an ordered othonormal basis of $\mathcal{H}$ and $K \in \mathcal{C}^1$ with the matrix representation A with respect to $\{e_n\}$, then the trace of K is exactly the sum of all entries in the main diagonal of A.*

Theorem 2.4.4. [5] *For each $1 < p \leq \infty$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $(\mathcal{C}^p)^* \cong \mathcal{C}^q$.*

Theorem 2.4.5. [5] $(\mathcal{C}^1)^* \cong \mathcal{B}(\mathcal{H})$.

Chapter 3

Duality of Sequence Spaces of Infinite Matrices

3.1 Basic Results

Recall that, for any $1 \le r < \infty$, the set $\mathscr{L}^r$ of sequences of infinite matrices is defined by

$$\mathscr{L}^r = \left\{ \left\{ \left[a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty} \subseteq \mathscr{M}_\infty : \left[\sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \right|^r \right] \in \mathcal{B}(l^2) \right\}.$$

It is clear that if $\{A_k\}_{k=1}^\infty \in \mathscr{L}^r$, then A_k is necessarily a member of the absolute Schur algebra $\mathcal{S}_{2,2}^r(\mathbb{C})$ for all k.

The following theorem was first stated in [3] by A. Charearnpol. It is a generalization of the characterization of the sequence spaces $\mathcal{O}_b$ provided by J. Rakbud and S.-C. Ong in [11].

Theorem 3.1.1. *Let* $\left\{ \left[a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty}$ *be a sequence in* $\mathcal{B}(l^2)$ *with* $a_{ji}^{(k)} \ge 0$ *for all* i, j, k.

(1) *The sequence* $\left\{ \sum_{k=1}^{n} \left[a_{ji}^{(k)} \right] \right\}_{n=1}^{\infty}$ *is bounded in* $\mathcal{B}(l^2)$ *if and only if* $\left[\sum_{k=1}^{\infty} a_{ji}^{(k)} \right] \in \mathcal{B}(l^2)$.

(2) *If* $\left[\sum_{k=1}^{\infty} a_{ji}^{(k)} \right] \in \mathcal{B}(l^2)$, *then* $\left\| \left[\sum_{k=1}^{\infty} a_{ji}^{(k)} \right] \right\| = \sup_n \left\| \sum_{k=1}^{n} \left[a_{ji}^{(k)} \right] \right\|$.

From the above theorem, the following characterization of the set $\mathscr{L}^r$ is immediately obtained.

Corollary 3.1.2. *Let* $\{A_k\}_{k=1}^\infty$ *be a sequence in* $\mathscr{M}_\infty$ *and* $1 \le r < \infty$. *Then the following are equivalent:*

(1) $\{A_k\}_{k=1}^\infty$ *belongs to* $\mathscr{L}^r$;

(2) $A_k \in \mathcal{S}^r_{2,2}(\mathbb{C})$ *for all* k *and the sequence* $\left\{\sum_{k=1}^{n} A_k^{[r]}\right\}_{n=1}^{\infty}$ *is bounded in* $\mathcal{B}(l^2)$*;*

(3) *the sequence* $\left\{\left\|\sum_{k=1}^{n} A_k^{[r]}\right\|\right\}_{n=1}^{\infty}$ *is bounded.*

For any sequence $\left\{\left[a_{ji}^{(k)}\right]\right\}_{k=1}^{\infty}$ in $\mathscr{M}_\infty$ and $1 \le r < \infty$, we define

$$\left\|\left|\left\{\left[a_{ji}^{(k)}\right]\right\}_{k=1}^{\infty}\right\|\right|_r = \begin{cases} \left\|\left[\sum_{k=1}^{\infty}\left|a_{ji}^{(k)}\right|^r\right]\right\|^{1/r} & \text{if } \left\{\left[a_{ji}^{(k)}\right]\right\}_{k=1}^{\infty} \in \mathscr{L}^r, \\ \infty & \text{otherwise.} \end{cases}$$

The following Hölder-type inequality was first established in [2] by Chaisuriya and Ong. It is useful for the research.

Theorem 3.1.3. (Hölder-type inequality) *For any* $A, B \in \mathscr{M}_\infty$ *and* $1 < r < \infty$ *with* $\frac{1}{r} + \frac{1}{r^*} = 1$*,*

$$\left\|(A \bullet B)^{[1]}\right\| \le \left\|A^{[r]}\right\|^{1/r} \left\|B^{[r^*]}\right\|^{1/r^*}$$

under the conventions that $\infty \cdot 0 = 0 \cdot \infty = 0$*,* $\infty \cdot \alpha = \alpha \cdot \infty = \infty$ *for all positive real number* α *and* $\infty \cdot \infty = \infty$*.*

The Hölder and Minkowski-type inequalities below are extensions of the ones in [11].

Theorem 3.1.4. (Hölder-type inequality for sequences of matrices) *For any sequences* $\{A_k\}_{k=1}^{\infty}$ *and* $\{B_k\}_{k=1}^{\infty}$ *in* $\mathscr{M}_\infty$*,*

$$\left|\left|\left|\{A_k \bullet B_k\}_{k=1}^{\infty}\right|\right|\right|_1 \le \left|\left|\left|\{A_k\}_{k=1}^{\infty}\right|\right|\right|_r \left|\left|\left|\{B_k\}_{k=1}^{\infty}\right|\right|\right|_{r^*},$$

where $1 < r < \infty$ *with* $\frac{1}{r} + \frac{1}{r^*} = 1$*, under the same convention as in Theorem* 3.1.3.

Proof. Let $\left\{A_k = \left[a_{ji}^{(k)}\right]\right\}_{k=1}^{\infty}$ and $\left\{B_k = \left[b_{ji}^{(k)}\right]\right\}_{k=1}^{\infty}$ be sequences in $\mathscr{M}_\infty$. If either $\left|\left|\left|\{A_k\}_{k=1}^{\infty}\right|\right|\right|_r$ or $\left|\left|\left|\{B_k\}_{k=1}^{\infty}\right|\right|\right|_{r^*}$ is ∞, then we are done. Suppose that both $\left|\left|\left|\{A_k\}_{k=1}^{\infty}\right|\right|\right|_r$ and $\left|\left|\left|\{B_k\}_{k=1}^{\infty}\right|\right|\right|_{r^*}$ are finite. Then $\left[\sum_{k=1}^{\infty}\left|a_{ji}^{(k)}\right|^r\right]$ and $\left[\sum_{k=1}^{\infty}\left|b_{ji}^{(k)}\right|^{r^*}\right]$ belong to $\in \mathcal{B}(l^2)$. Thus, by Hölder's inequality, we have for each i, j that

$$\sum_{k=1}^{\infty}\left|a_{ji}^{(k)} b_{ji}^{(k)}\right| \le \left(\sum_{k=1}^{\infty}\left|a_{ji}^{(k)}\right|^r\right)^{1/r}\left(\sum_{k=1}^{\infty}\left|b_{ji}^{(k)}\right|^{r^*}\right)^{1/r^*} < \infty.$$

Hence the matrix $\left[\sum_{k=1}^{\infty}\left|a_{ji}^{(k)}b_{ji}^{(k)}\right|\right] \in \mathscr{M}_{\infty}$. We want to show that $\left[\sum_{k=1}^{\infty}\left|a_{ji}^{(k)}b_{ji}^{(k)}\right|\right] \in \mathcal{B}(l^2)$ and $\|\{A_k \bullet B_k\}_{k=1}^{\infty}\|_1 \leq \|\{A_k\}_{k=1}^{\infty}\|_r \|\{B_k\}_{k=1}^{\infty}\|_{r^*}$. By the Hölder-type inequality, we have

$$\begin{aligned}
\left\|\left[\sum_{k=1}^{\infty}\left|a_{ji}^{(k)}b_{ji}^{(k)}\right|\right]\right\| &\leq \left\|\left[\left(\sum_{k=1}^{\infty}\left|a_{ji}^{(k)}\right|^r\right)^{1/r}\left(\sum_{k=1}^{\infty}\left|b_{ji}^{(k)}\right|^{r^*}\right)^{1/r^*}\right]\right\| \\
&= \left\|\left[\left(\sum_{k=1}^{\infty}\left|a_{ji}^{(k)}\right|^r\right)^{1/r}\right]\bullet\left[\left(\sum_{k=1}^{\infty}\left|b_{ji}^{(k)}\right|^{r^*}\right)^{1/r^*}\right]\right\| \\
&\leq \left\|\left[\sum_{k=1}^{\infty}\left|a_{ji}^{(k)}\right|^r\right]\right\|^{1/r}\left\|\left[\sum_{k=1}^{\infty}\left|b_{ji}^{(k)}\right|^{r^*}\right]\right\|^{1/r^*} \\
&= \|\{A_k\}_{k=1}^{\infty}\|_r \|\{B_k\}_{k=1}^{\infty}\|_{r^*} < \infty.
\end{aligned}$$

This implies that $\left[\sum_{k=1}^{\infty}\left|a_{ji}^{(k)}b_{ji}^{(k)}\right|\right] \in \mathcal{B}(l^2)$, which is equivalent to that $\{A_k \bullet B_k\}_{k=1}^{\infty} \in \mathscr{L}^1$, and $\|\{A_k \bullet B_k\}_{k=1}^{\infty}\|_1 \leq \|\{A_k\}_{k=1}^{\infty}\|_r \|\{B_k\}_{k=1}^{\infty}\|_{r^*}$. □

Theorem 3.1.5. (Minkowski-type inequality for sequences of matrices) *For any sequences $\{A_k\}_{k=1}^{\infty}$ and $\{B_k\}_{k=1}^{\infty}$ in $\mathscr{M}_{\infty}$ and $1 \leq r < \infty$,*

$$\|\{A_k + B_k\}_{k=1}^{\infty}\|_r \leq \|\{A_k\}_{k=1}^{\infty}\|_r + \|\{B_k\}_{k=1}^{\infty}\|_r$$

under the conventions that $\infty + \alpha = \alpha + \infty = \infty$ for all non-negative real number α and $\infty + \infty = \infty$.

Proof. For the case where either $\|\{A_k\}_{k=1}^{\infty}\|_r = \infty$ or $\|\{B_k\}_{k=1}^{\infty}\|_r = \infty$, there is nothing to prove. Suppose that both $\|\{A_k\}_{k=1}^{\infty}\|_r$ and $\|\{B_k\}_{k=1}^{\infty}\|_r$ are finite. We assume first that $1 < r < \infty$. Then by the Hölder-type inequality for sequences of matrices, we have for each positive integer n that

$$\begin{aligned}
\left\|\sum_{k=1}^{n}(A_k + B_k)^{[r]}\right\| &= \left\|\sum_{k=1}^{n}(A_k + B_k)^{[1]} \bullet (A_k + B_k)^{[r-1]}\right\| \\
&\leq \left\|\sum_{k=1}^{n}\left(A_k^{[1]} + B_k^{[1]}\right) \bullet (A_k + B_k)^{[r-1]}\right\| \\
&= \left\|\sum_{k=1}^{n}A_k^{[1]} \bullet (A_k + B_k)^{[r-1]} + \sum_{k=1}^{n}B_k^{[1]} \bullet (A_k + B_k)^{[r-1]}\right\| \\
&\leq \left\|\sum_{k=1}^{n}A_k^{[1]} \bullet (A_k + B_k)^{[r-1]}\right\| + \left\|\sum_{k=1}^{n}B_k^{[1]} \bullet (A_k + B_k)^{[r-1]}\right\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left\|\sum_{k=1}^{n} A_k^{[r]}\right\|^{1/r} \left\|\sum_{k=1}^{n} \left((A_k+B_k)^{[r-1]}\right)^{[r^*]}\right\|^{1/r^*} \\
&\quad + \left\|\sum_{k=1}^{n} B_k^{[r]}\right\|^{1/r} \left\|\sum_{k=1}^{n} \left((A_k+B_k)^{[r-1]}\right)^{[r^*]}\right\|^{1/r^*} \\
&= \left(\left\|\sum_{k=1}^{n} A_k^{[r]}\right\|^{1/r} + \left\|\sum_{k=1}^{n} B_k^{[r]}\right\|^{1/r}\right) \left\|\sum_{k=1}^{n} (A_k+B_k)^{[r]}\right\|^{1/r^*},
\end{aligned}
$$

where $\dfrac{1}{r}+\dfrac{1}{r^*}=1$, which implies that

$$
\begin{aligned}
\left\|\sum_{k=1}^{n} (A_k+B_k)^{[r]}\right\|^{1/r} &\leq \left\|\sum_{k=1}^{n} A_k^{[r]}\right\|^{1/r} + \left\|\sum_{k=1}^{n} B_k^{[r]}\right\|^{1/r} \\
&\leq |||\{A_k\}_{k=1}^{\infty}|||_r + |||\{B_k\}_{k=1}^{\infty}|||_r \,.
\end{aligned}
$$

For the case where $r=1$, we easily have for each positive integer n that

$$
\begin{aligned}
\left\|\sum_{k=1}^{n} (A_k+B_k)^{[1]}\right\| &\leq \left\|\sum_{k=1}^{n} A_k^{[1]} + \sum_{k=1}^{n} B_k^{[1]}\right\| \\
&\leq \left\|\sum_{k=1}^{n} A_k^{[1]}\right\| + \left\|\sum_{k=1}^{n} B_k^{[1]}\right\| \\
&\leq |||\{A_k\}_{k=1}^{\infty}|||_1 + |||\{B_k\}_{k=1}^{\infty}|||_1
\end{aligned}
$$

as well. Thus, by Corollary 3.1.2, the sequence $\{A_k+B_k\}_{k=1}^{\infty}$ belongs to $\mathscr{L}^r$ and

$$
|||\{A_k+B_k\}_{k=1}^{\infty}|||_r \leq |||\{A_k\}_{k=1}^{\infty}|||_r + |||\{B_k\}_{k=1}^{\infty}|||_r
$$

for all $1 \leq r < \infty$. The proof is complete. □

The following lemma was first stated and proved in [10]. It is a beautiful consequence of the Hölder-type inequality.

Lemma 3.1.6. *For any $1 \leq r < \infty$ and matrices A and B in $\mathcal{S}_{2,2}^r(\mathbb{C})$,*

$$
\left\|A^{[r]} - B^{[r]}\right\| \leq \left(|||A|||_{2,2,r} + |||B|||_{2,2,r}\right) |||A-B|||_{2,2,r} \,.
$$

The proposition below was first stated and proved in [10] as well. We can see that it follows easily from the lemma above.

Proposition 3.1.7. *For any $1 \leq r < \infty$, the map $A \mapsto A^{[r]}$ from $\mathcal{S}_{2,2}^r(\mathbb{C})$ into $\mathcal{B}(l^2)$ is continuous.*

Theorem 3.1.8. *For each $1 \leq r < \infty$, the set $\mathscr{L}^r$ equipped with the norm $|||\cdot|||_r$ is a Banach space.*

Proof. From Minkowski's inequality for sequences of matrices, we have that the set $\mathscr{L}^r$ endowed with the norm $|||\cdot|||_r$ is a normed space. To see that it is a Banach space, let $\left\{\mathbf{A}_n = \left\{A_k^{(n)}\right\}_{k=1}^{\infty}\right\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathscr{L}^r$. As we have for each k that

$$\left|\left|\left|A_k^{(n)} - A_k^{(m)}\right|\right|\right|_{2,2,r} \leq |||\mathbf{A}_n - \mathbf{A}_m|||_r \text{ for all } n, m,$$

it follows that the sequence $\left\{A_k^{(n)}\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{S}_{2,2}^r(\mathbb{C})$ for all k. So, for each k, we obtain by the completeness of $\mathcal{S}_{2,2}^r(\mathbb{C})$ that there exists an A_k in $\mathcal{S}_{2,2}^r(\mathbb{C})$ such that $A_k^{(n)} \to A_k$. Let $\mathbf{A} = \{A_k\}_{k=1}^{\infty}$. We claim that $\mathbf{A} \in \mathscr{L}^r$ and $\mathbf{A}_n \to \mathbf{A}$. To prove these, let $\epsilon > 0$ be given. Then there is a positive integer N such that for each positive integer K,

$$\left\|\sum_{k=1}^{K}\left(A_k^{(n)} - A_k^{(m)}\right)^{[r]}\right\|^{1/r} \leq |||\mathbf{A}_n - \mathbf{A}_m|||_r < \frac{\epsilon}{2} \text{ for all } n, m \geq N. \qquad (*)$$

Since $A_k^{(m)} \to A_k$ in $\mathcal{S}_{2,2}^r(\mathbb{C})$ for all k, it follows for each fixed n that $A_k^{(n)} - A_k^{(m)} \to A_k^{(n)} - A_k$ in $\mathcal{S}_{2,2}^r(\mathbb{C})$ for all k. Thus, by Proposition 3.1.7, we obtain for each fixed n that $\left(A_k^{(n)} - A_k^{(m)}\right)^{[r]} \to \left(A_k^{(n)} - A_k\right)^{[r]}$ in $\mathcal{B}(l^2)$ for all k. From this we have for each fixed n and K that $\sum_{k=1}^{K}\left(A_k^{(n)} - A_k^{(m)}\right)^{[r]} \to \sum_{k=1}^{K}\left(A_k^{(n)} - A_k\right)^{[r]}$ in $\mathcal{B}(l^2)$. Whence, by taking the limits as $m \to \infty$ on both sides of $(*)$, we obtain by the continuity of the operator norm on $\mathcal{B}(l^2)$ that for each $n \geq N$,

$$\left\|\sum_{k=1}^{K}\left(A_k^{(n)} - A_k\right)^{[r]}\right\|^{1/r} \leq \frac{\epsilon}{2} \text{ for all } K \geq 1.$$

Therefore, by Theorem 3.1.1,

$$|||\mathbf{A}_n - \mathbf{A}|||_r = \sup_K \left\|\sum_{k=1}^{K}\left(A_k^{(n)} - A_k\right)^{[r]}\right\|^{1/r} < \epsilon \text{ for all } n \geq N. \qquad (**)$$

The inequality $(**)$ yields that $\mathbf{A}_N - \mathbf{A}$ belongs to $\mathscr{L}^r$, which implies that $\mathbf{A} = \mathbf{A}_N - (\mathbf{A}_N - \mathbf{A})$ is an element of $\mathscr{L}^r$. Consequently, by $(**)$ again, we get $\mathbf{A}_n \to \mathbf{A}$. □

3.2 Duality

In this section, we study the duality of the sequence spaces $\mathscr{L}^r$. The aim is to decompose the dual space $(\mathscr{L}^r)^*$ of $\mathscr{L}^r$ as an l^1 direct-sum of its two closed subspaces. Before getting the results, we need some notational conventions.

For any $z \in \mathbb{C}$, we define the function $\operatorname{sgn}(\cdot)$ on $\mathbb{C}$ by

$$\operatorname{sgn}(z) = \begin{cases} \dfrac{\overline{z}}{|z|} & \text{if } z \neq 0, \\ 1 & \text{if } z = 0. \end{cases}$$

For any sequences $\mathbf{A} = \{A_k\}_{k=1}^{\infty}$ and $\mathbf{B} = \{B_k\}_{k=1}^{\infty}$ in $\mathscr{M}_\infty$ and any positive integer n, we let $\mathbf{A} \bullet \mathbf{B} = \{A_k \bullet B_k\}_{k=1}^{\infty}$, $\mathbf{A}_{n\lrcorner} = \{(A_k)_{n\lrcorner}\}_{k=1}^{\infty}$, $\mathbf{A}_{n\ulcorner} = \{(A_k)_{n\ulcorner}\}_{k=1}^{\infty}$, and $\mathbf{A}_{n]} = \{A_1, A_2, ..., A_n, 0, 0, ...\}$. It is clear that $\left(\mathbf{A}_{K]}\right)_{n\lrcorner} = (\mathbf{A}_{n\lrcorner})_{K]}$ for all positive integers n and K. Notice that for each $1 \le r < \infty$, if $\mathbf{A} = \left\{A_k = \left[a_{ji}^{(k)}\right]\right\}_{k=1}^{\infty} \in \mathscr{L}^r$, then each of the following holds true:

$$(i)\ \ \|\|\mathbf{A}_{n\lrcorner}\|\|_r = \left\| \left[\sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \right|^r \right]_{n\lrcorner} \right\|^{1/r},$$

$$(ii)\ \ \left\|\left\|\mathbf{A}_{K]}\right\|\right\|_r = \left\| \sum_{k=1}^{K} A_k^{[r]} \right\|^{1/r} = \left\| \left[\sum_{k=1}^{K} \left| a_{ji}^{(k)} \right|^r \right] \right\|^{1/r} \text{ and}$$

$$(iii)\ \ \|\|\mathbf{A}_{n\lrcorner} - \mathbf{A}\|\|_r = \|\|\{(A_k)_{n\lrcorner} - A_k\}_{k=1}^{\infty}\|\|_r = \left\| \left[\sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \right|^r \right]_{n\lrcorner} - \left[\sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \right|^r \right] \right\|^{1/r},$$

for all n and K. The first and second equations imply that $\|\|\mathbf{A}\|\|_r = \sup_n \|\|\mathbf{A}_{n\lrcorner}\|\|_r$ and $\|\|\mathbf{A}\|\|_r = \sup_K \left\|\left\|\mathbf{A}_{K]}\right\|\right\|_r$ respectively. And the last one implies that the matrix $\left[\sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \right|^r \right]$ is compact if and only if $\|\|\mathbf{A}_{n\lrcorner} - \mathbf{A}\|\|_r \to 0$. For each $A = [a_{ji}]$ in $\mathscr{M}_\infty$ and positive integer k, let $\sum A = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ji}$ if the series converges, let $\mathrm{sgn}A$=[$\mathrm{sgn}(a_{ji})$], and let $s(A;k)$ be the sequence whose k-th term is the matrix A and all other terms are 0. Finally, for any $\lambda \in \mathbb{C}$ and pair (j,i) of positive integers, let $E(\lambda;(j,i))$ be the matrix whose (j,i)-th entry is the number λ and all other entries are 0.

On the classical sequence spaces l^p, there is a result closely related to their duality as follows: for $1 \le p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, a sequence $\mathbf{x}$ belongs to l^q if and only if $\mathbf{x}$ "Schur multiplies"every $\mathbf{y}$ in l^p into l^1. An analogue of this result is also obtained for our sequence spaces $\mathscr{L}^r$ of infinite matrices.

Theorem 3.2.1. *Let* $1 < r < \infty$ *with* $\frac{1}{r} + \frac{1}{r^*} = 1$.

(1) $\{A_k\}_{k=1}^{\infty} \in \mathscr{L}^{r^*}$ *if and only if* $\{A_k \bullet B_k\}_{k=1}^{\infty} \in \mathscr{L}^1$ *for all* $\{B_k\}_{k=1}^{\infty} \in \mathscr{L}^r$.

(2) *If* $\{A_k\}_{k=1}^{\infty} \in \mathscr{L}^{r^*}$, *then*

$$\|\|\{A_k\}_{k=1}^{\infty}\|\|_{r^*} = \sup\{\|\|\{A_k \bullet B_k\}_{k=1}^{\infty}\|\|_1 : \{B_k\}_{k=1}^{\infty} \in \mathscr{L}^r, \|\|\{B_k\}_{k=1}^{\infty}\|\|_r \le 1\}.$$

Proof. (1). Let $\mathbf{A} = \left\{A_k = \left[a_{ji}^{(k)}\right]\right\}_{k=1}^{\infty}$ be a sequence in $\mathscr{M}_\infty$. Suppose that $\{A_k \bullet B_k\}_{k=1}^{\infty} \in \mathscr{L}^1$ for all $\{B_k\}_{k=1}^{\infty} \in \mathscr{L}^r$. We want to show that $\{A_k\}_{k=1}^{\infty} \in \mathscr{L}^{r^*}$. By the assumption, a map $\Psi : \mathscr{L}^r \to \mathscr{L}^1$ can be defined as follows: $\Psi(\{B_k\}_{k=1}^{\infty}) = \{A_k \bullet B_k\}_{k=1}^{\infty}$ for all $\{B_k\}_{k=1}^{\infty} \in \mathscr{L}^r$. For any positive integer n, let $\Psi_n : \mathscr{L}^r \to \mathscr{L}^1$ be defined by $\Psi_n(\{B_k\}_{k=1}^{\infty}) = \mathbf{A}_{n]} \bullet \mathbf{B}$ for all $\mathbf{B} = \{B_k\}_{k=1}^{\infty} \in \mathscr{L}^r$. Then by the Hölder-type inequality for sequences of matrices, we have for every $\mathbf{B} = \{B_k\}_{k=1}^{\infty} \in \mathscr{L}^r$ that

$$|||\Psi_n(\{B_k\}_{k=1}^{\infty})|||_1 = \left|\left|\left|\mathbf{A}_{n]} \bullet \mathbf{B}\right|\right|\right|_1 \leq \left|\left|\left|\mathbf{A}_{n]}\right|\right|\right|_{r^*} |||\mathbf{B}|||_r .$$

So the operator Ψ_n is bounded for all n. For each $\left\{B_k = \left[b_{ji}^{(k)}\right]\right\}_{k=1}^{\infty} \in \mathscr{L}^r$, we have

$$\begin{aligned} |||\Psi_n(\{B_k\}_{k=1}^{\infty})|||_1 &= \left\|\left[\sum_{k=1}^{n}\left|a_{ji}^{(k)} b_{ji}^{(k)}\right|\right]\right\| \leq \left\|\left[\sum_{k=1}^{\infty}\left|a_{ji}^{(k)} b_{ji}^{(k)}\right|\right]\right\| \\ &= |||\{A_k \bullet B_k\}_{k=1}^{\infty}|||_1 \text{ for all } n. \end{aligned}$$

Hence, by the uniform boundedness principle, the set $\{\|\Psi_n\| : n = 1, 2, 3, ...\}$ is bounded. For every $\mathbf{B} = \{B_k\}_{k=1}^{\infty} \in \mathscr{L}^r$ with $|||\mathbf{B}|||_r \leq 1$, we have by Theorem 3.1.1 that

$$\begin{aligned} |||\Psi(\mathbf{B})|||_1 &= \sup_n \left\|\sum_{k=1}^{n}(A_k \bullet B_k)^{[1]}\right\| = \sup_n \left|\left|\left|\mathbf{A}_{n]} \bullet \mathbf{B}\right|\right|\right|_1 \\ &= \sup_n |||\Psi_n(\mathbf{B})|||_1 \leq \sup_n \|\Psi_n\| . \end{aligned}$$

Thus, by the boundedness of the set $\{\|\Psi_n\| : n = 1, 2, 3, ...\}$, the operator Ψ is bounded. Next, let $\mathbf{D} = \left\{A_k^{[r^*-1]}\right\}_{k=1}^{\infty}$. Then $(\mathbf{D}_{n\lrcorner})_{K]} \in \mathscr{L}^r$ for all n, K. Thus

$$\begin{aligned} \left\|\left(\sum_{k=1}^{K} A_k^{[r^*]}\right)_{n\lrcorner}\right\| &= \left\|\sum_{k=1}^{K}\left(A_k^{[r^*]}\right)_{n\lrcorner}\right\| = \left\|\sum_{k=1}^{K} A_k^{[1]} \bullet \left(A_k^{[r^*-1]}\right)_{n\lrcorner}\right\| \\ &= \left|\left|\left|\mathbf{A} \bullet (\mathbf{D}_{n\lrcorner})_{K]}\right|\right|\right|_1 = \left|\left|\left|\Psi\left((\mathbf{D}_{n\lrcorner})_{K]}\right)\right|\right|\right|_1 \\ &\leq \|\Psi\| \left|\left|\left|(\mathbf{D}_{n\lrcorner})_{K]}\right|\right|\right|_r = \|\Psi\| \left\|\left(\sum_{k=1}^{K}\left(A_k^{[r^*-1]}\right)^{[r]}\right)_{n\lrcorner}\right\|^{1/r} \\ &= \|\Psi\| \left\|\left(\sum_{k=1}^{K} A_k^{[r^*]}\right)_{n\lrcorner}\right\|^{1/r} \text{ for all } n, K. \qquad (\star) \end{aligned}$$

It follows that

$$\left\|\left(\sum_{k=1}^{K} A_k^{[r^*]}\right)_{n\lrcorner}\right\| \leq \|\Psi\|^{r^*} \text{ for all } n, K.$$

Whence, by Theorem 1.1(2), we obtain for each K that $\sum_{k=1}^{K} A_k^{[r^*]} \in \mathcal{B}(l^2)$ and by Theorem 1.1(3),

$$\left\|\sum_{k=1}^{K} A_k^{[r^*]}\right\| = \sup_n \left\|\left(\sum_{k=1}^{K} A_k^{[r^*]}\right)_{n\lrcorner}\right\| \leq \|\Psi\|^{r^*} .$$

Therefore, by Corollary 3.1.2, the sequence $\mathbf{A}$ belongs to $\mathscr{L}^{r^*}$. Conversely, suppose that $\mathbf{A} \in \mathscr{L}^{r^*}$. Then for any $\{B_k\}_{k=1}^{\infty} \in \mathscr{L}^r$, we have by the Hölder-type inequality for sequences of matrices that $\{A_k \bullet B_k\}_{k=1}^{\infty} \in \mathscr{L}^1$.

(2). Suppose that $\{A_k\}_{k=1}^{\infty} \in \mathscr{L}^{r^*}$. Then by (1), the linear operator $\Psi : \mathscr{L}^r \to \mathscr{L}^1$ defined by $\{B_k\}_{k=1}^{\infty} \mapsto \{A_k \bullet B_k\}_{k=1}^{\infty}$ is well-defined, and by the Hölder-type inequality for sequences of matrices, it is obvious that $\|\Psi\| \leq |||\{A_k\}_{k=1}^{\infty}|||_{r^*}$. By the same argument as given in the proof of (1) (see the argument to obtain the inequality ($\star$)), we have

$$\left\|\sum_{k=1}^{n} A_k^{[r^*]}\right\|^{1/r^*} \leq \|\Psi\| \text{ for all } n.$$

It follows from Theorem 3.1.1 that $|||\{A_k\}_{k=1}^{\infty}|||_{r^*} \leq \|\Psi\|$. Consequently, we obtain

$$|||\{A_k\}_{k=1}^{\infty}|||_{r^*} = \|\Psi\| = \sup\{|||\{A_k \bullet B_k\}_{k=1}^{\infty}|||_1 : \{B_k\}_{k=1}^{\infty} \in \mathscr{L}^r, |||\{B_k\}_{k=1}^{\infty}|||_r \leq 1\}$$

as required. The proof is complete. □

For each $1 \leq r < \infty$, let

$$\mathscr{L}_{\kappa}^r = \left\{ \left\{ \left[a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty} \subseteq \mathscr{M}_{\infty} : \left[\sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \right|^r \right] \in \mathcal{K}(l^2) \right\}.$$

The following results on the sets $\mathscr{L}_{\kappa}^r$ are evident.

(*i*) $\mathscr{L}_{\kappa}^r \subsetneq \mathscr{L}^r$.

(*ii*) A sequence $\mathbf{A}$ in $\mathscr{M}_{\infty}$ belongs to $\mathscr{L}_{\kappa}^r$ if and only if $|||\mathbf{A} - \mathbf{A}_{n\lrcorner}|||_r \to 0$.

(*iii*) If a sequence $\mathbf{A}$ belongs to $\mathscr{L}_{\kappa}^r$, then $\mathbf{A} - \mathbf{A}_{n\ulcorner}$ belongs to $\mathscr{L}_{\kappa}^r$ for all n.

The following theorem is a more general version of the characterization of the sequence space $\mathcal{O}_{\kappa}$ provided by Rakbud et al. in [11].

Theorem 3.2.2. *Let* $\left\{ A_k = \left[a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty}$ *be a sequence in* $\mathscr{M}_{\infty}$ *with* $a_{jk}^{(k)} \geq 0$ *for all* i, j, k. *Then* $\left[\sum_{k=1}^{\infty} a_{ji}^{(k)} \right] \in \mathcal{K}(l^2)$ *if and only if* $A_k \in \mathcal{K}(l^2)$ *for all* k *and the sequence* $\left\{ \sum_{k=1}^{n} A_k \right\}_{k=1}^{\infty}$ *converges in* $\mathcal{B}(l^2)$.

Proof. Suppose that the matrix $A = \left[\sum_{k=1}^{\infty} a_{ji}^{(k)} \right]$ is compact. Then for each k, we have by Theorem 1.1(1) that $A_k \in \mathcal{B}(l^2)$ and

$$\|A_k - (A_k)_{n\lrcorner}\| \leq \|A - A_{n\lrcorner}\| \to 0.$$

Thus A_k is compact for all k. To see that the sequence $\left\{ \sum_{k=1}^{n} A_k \right\}_{k=1}^{\infty}$ converges in $\mathcal{B}(l^2)$, let $\epsilon > 0$ be given. Then by the compactness of the matrix A, there exists a positive integer N such that $\|A_{N_\lrcorner} - A\| < \dfrac{\epsilon}{3}$. As the series $\sum_{k=1}^{\infty} a_{ji}^{(k)}$ converges for all $1 \le j, i \le N$, there is a positive integer K_0 such that for each $1 \le j, i \le N$, $\sum_{k=K}^{\infty} a_{ji}^{(k)} < \dfrac{\epsilon}{3N^{3/2}}$ for all $K \ge K_0$. Hence for each $K \ge K_0$,

$$\begin{aligned}\left\|\sum_{k=1}^{K} A_k - A\right\| &\le \left\|A_{N_\lrcorner} - \left(\sum_{k=1}^{K} A_k\right)_{N_\lrcorner}\right\| + \left\|\left(\sum_{k=1}^{K} A_k\right)_{N_\lrcorner} - \sum_{k=1}^{K} A_k\right\| \\ &\quad + \|A_{N_\lrcorner} - A\| \\ &\le \left\{\sum_{j=1}^{N}\left(\sum_{i=1}^{N}\sum_{k=K}^{\infty} a_{ji}^{(k)}\right)^2\right\}^{1/2} + 2\,\|A_{N_\lrcorner} - A\| \\ &< \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon.\end{aligned}$$

This yields $\sum_{k=1}^{\infty} A_k = A$ in $\mathcal{B}(l^2)$. Conversely, suppose that A_k is compact for all k and that the sequence $\left\{\sum_{k=1}^{n} A_k\right\}_{n=1}^{\infty}$ converges in $\mathcal{B}(l^2)$. It is clear that $\sum_{k=1}^{\infty} A_k = \left[\sum_{k=1}^{\infty} a_{ji}^{(k)}\right]$. Since $\mathcal{K}(l^2)$ is closed in $\mathcal{B}(l^2)$, it follows that $\sum_{k=1}^{\infty} A_k$ is compact. Thus we obtain that $\left[\sum_{k=1}^{\infty} a_{ji}^{(k)}\right]$ is compact as required. □

The following characterization of the set $\mathscr{L}_\kappa^r$ is an immediate consequence of Theorem 3.2.2 above.

Corollary 3.2.3. *Let $\{A_k\}_{k=1}^{\infty}$ be a sequence in $\mathscr{M}_\infty$ and $1 \le r < \infty$. Then $\{A_k\}_{k=1}^{\infty} \in \mathscr{L}_\kappa^r$ if and only if $A_k^{[r]}$ is compact for all k and the sequence $\left\{\sum_{k=1}^{n} A_k^{[r]}\right\}_{k=1}^{\infty}$ converges in $\mathcal{B}(l^2)$.*

Theorem 3.2.4. *For each $1 \le r < \infty$, the set $\mathscr{L}_\kappa^r$ is a Banach subspace of $\mathscr{L}^r$.*

Proof. For any matrix $A \in \mathscr{M}_\infty$ and positive integer n, we let here for convenience $A_{\lrcorner n} = A - A_{n-1_\lrcorner}$. We will show first that $\mathscr{L}_\kappa^r$ is a normed subspace of $\mathscr{L}^r$. Let $\{A_k\}_{k=1}^{\infty}, \{B_k\}_{k=1}^{\infty} \in \mathscr{L}_\kappa^r$. Then

$$\begin{aligned}\|\!|\{(A_k + B_k)_{\lrcorner n}\}_{k=1}^{\infty}|\!\|_r &= \|\!|\{(A_k)_{\lrcorner n}\}_{k=1}^{\infty} + \{(B_k)_{\lrcorner n}\}_{k=1}^{\infty}|\!\|_r \\ &\le \|\!|\{(A_k)_{\lrcorner n}\}_{k=1}^{\infty}|\!\|_r + \|\!|\{(B_k)_{\lrcorner n}\}_{k=1}^{\infty}|\!\|_r \to 0.\end{aligned}$$

Thus $\mathscr{L}_\kappa^r$ is closed under addition. It clear that $\lambda\{A_k\}_{k=1}^{\infty} \in \mathscr{L}_\kappa^r$ for any complex number λ. Hence $\mathscr{L}_\kappa^r$ is a normed subspace of $\mathscr{L}^r$. To show that $\mathscr{L}_\kappa^r$ is a Banach space, it suffices to show that $\mathscr{L}_\kappa^r$ is a closed subspace of $\mathscr{L}^r$. Suppose that

$\left\{\mathbf{A}_n = \left\{A_k^{(n)}\right\}_{k=1}^{\infty}\right\}_{n=1}^{\infty}$ is a sequence in $\mathscr{L}_\kappa^r$ converging to an element $\mathbf{A} = \{A_k\}_{k=1}^{\infty}$ in $\mathscr{L}^r$, and let $\epsilon > 0$ be given. Then there is a positive integer N such that

$$|||\mathbf{A}_N - \mathbf{A}|||_r < \frac{\epsilon}{2}.$$

Due to the membership of $\mathbf{A}_N$ in $\mathscr{L}_\kappa^r$, we have that there exists a positive integer J_0 such that

$$\left|\left|\left|\left\{\left(A_k^{(N)}\right)_{\lrcorner J}\right\}_{k=1}^{\infty}\right|\right|\right|_r < \frac{\epsilon}{2} \text{ for all } J \geq J_0.$$

It follows that

$$\begin{aligned}
|||\{(A_k)_{\lrcorner J}\}_{k=1}^{\infty}|||_r &\leq \left|\left|\left|\left\{\left(A_k^{(N)}\right)_{\lrcorner J}\right\}_{k=1}^{\infty} - \{(A_k)_{\lrcorner J}\}_{k=1}^{\infty}\right|\right|\right|_r + \left|\left|\left|\left\{\left(A_k^{(N)}\right)_{\lrcorner J}\right\}_{k=1}^{\infty}\right|\right|\right|_r \\
&= \left|\left|\left|\left\{\left(A_k^{(N)} - A_k\right)_{\lrcorner J}\right\}_{k=1}^{\infty}\right|\right|\right|_r + \left|\left|\left|\left\{\left(A_k^{(N)}\right)_{\lrcorner J}\right\}_{k=1}^{\infty}\right|\right|\right|_r \\
&\leq |||\mathbf{A}_N - \mathbf{A}|||_r + \left|\left|\left|\left\{\left(A_k^{(N)}\right)_{\lrcorner J}\right\}_{k=1}^{\infty}\right|\right|\right|_r \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for all } J \geq J_0.
\end{aligned}$$

Consequently, $\{A_k\}_{k=1}^{\infty}$ belongs to $\mathscr{L}_\kappa^r$. □

The following is the main theorem in this thesis. It tells us that the annihilator $(\mathscr{L}_\kappa^r)^{\perp}$ of $\mathscr{L}_\kappa^r$ is complemented in $(\mathscr{L}^r)^*$. Furthermore, the norm of the decomposition of any bounded linear functional on $\mathscr{L}^r$ is additive.

Theorem 3.2.5. *Let $1 \leq r < \infty$. Then the following hold.*

(1) *The annihilator $(\mathscr{L}_\kappa^r)^{\perp}$ of $\mathscr{L}_\kappa^r$ is a non-trivial closed subspace of the dual $(\mathscr{L}^r)^*$ of $\mathscr{L}^r$.*

(2) *There is a subspace $\mathscr{P}$ of $(\mathscr{L}^r)^*$ such that $\mathscr{P}$ is isometrically isomorphic to $(\mathscr{L}_\kappa^r)^*$ and $(\mathscr{L}^r)^* = \mathscr{P} \oplus (\mathscr{L}_\kappa^r)^{\perp}$.*

(3) *For any $f \in (\mathscr{L}^r)^*$, the decomposition $f = g + h$, where $g \in \mathscr{P}$ and $h \in (\mathscr{L}_\kappa^r)^{\perp}$, satisfies $\|f\| = \|g\| + \|h\|$.*

Proof. (1). By the Hahn-Banach extension theorem, we have in general that if A is a non-trivial closed subspace of a Banach space X, then the annihilator $A^{\perp}$ of A is a non-trivial closed subspace of the dual X^* of X. Thus, by this fact, the assertion (1) holds.

(2). Let $\varphi \in (\mathscr{L}^r)^*$. For each k, let $\varphi_k : \mathcal{S}_{2,2}^r(\mathbb{C}) \to \mathbb{C}$ be defined by $\varphi_k(A) = \varphi(s(A; k))$ for all $A \in \mathcal{S}_{2,2}^r(\mathbb{C})$. It is easy to see that φ_k is linear and $\|\varphi_k\| \leq \|\varphi\|$ for all k. Hence, for each k, the map φ_k belongs to $(\mathcal{S}_{2,2}^r(\mathbb{C}))^*$. Next, let $B_k^{(\varphi)} = [\varphi_k(E(1; (j, i)))]$ for all k, and let $\mathbf{B}^{(\varphi)} = \left\{B_k^{(\varphi)}\right\}_{k=1}^{\infty}$. We want to show first that $\sum_{k=1}^{\infty} \sum \left(A_k \bullet B_k^{(\varphi)}\right)^{[1]} < \infty$ for all $\{A_k\}_{k=1}^{\infty} \in \mathscr{L}^r$. To see this, let $\mathbf{A} = \left\{A_k = \left[a_{ji}^{(k)}\right]\right\}_{k=1}^{\infty} \in \mathscr{L}^r$. Notice that $\mathbf{A} \bullet \mathbf{B}^{(\varphi)}$ belongs to $\mathscr{L}^r$ due to the fact that $\sum_{k=1}^{\infty} \left|a_{ji}^{(k)} \varphi_k(E(1;(j,i)))\right|^r \leq \|\varphi\|^r \sum_{k=1}^{\infty} \left|a_{ji}^{(k)}\right|^r$ for all j,i. For each k, let $\widetilde{A_k} = \left[\left(\operatorname{sgn}\left(\varphi_k\left(E\left(a_{ji}^{(k)};(j,i)\right)\right)\right)\right) a_{ji}^{(k)}\right]$, and let $\widetilde{\mathbf{A}} = \left\{\widetilde{A_k}\right\}_{k=1}^{\infty}$. Then $\widetilde{\mathbf{A}} \in \mathscr{L}^r$ with the same norm as $\mathbf{A}$. Let ν, μ and K be positive integers, and let $n = \max\{\nu, \mu\}$. Then

$$\begin{aligned}
\sum_{k=1}^{K}\sum_{j=1}^{\nu}\sum_{i=1}^{\mu}\left|a_{ji}^{(k)}\varphi_k(E(1;(j,i)))\right| &< \sum_{k=1}^{K}\sum_{j=1}^{n}\sum_{i=1}^{n}\left|\varphi_k\left(E\left(a_{ji}^{(k)};(j,i)\right)\right)\right| \\
&= \sum_{k=1}^{K}\sum_{j=1}^{n}\sum_{i=1}^{n}\left(\operatorname{sgn}\left(\varphi_k\left(E\left(a_{ji}^{(k)};(j,i)\right)\right)\right)\right)\varphi_k\left(E\left(a_{ji}^{(k)};(j,i)\right)\right) \\
&= \sum_{k=1}^{K}\sum_{j=1}^{n}\sum_{i=1}^{n}\varphi_k\left(E\left(\left(\operatorname{sgn}\left(\varphi_k\left(E\left(a_{ji}^{(k)};(j,i)\right)\right)\right)\right)a_{ji}^{(k)};(j,i)\right)\right) \\
&= \sum_{k=1}^{K}\varphi_k\left(\sum_{j=1}^{n}\sum_{i=1}^{n}E\left(\left(\operatorname{sgn}\left(\varphi_k\left(E\left(a_{ji}^{(k)};(j,i)\right)\right)\right)\right)a_{ji}^{(k)};(j,i)\right)\right) \\
&= \sum_{k=1}^{K}\varphi_k\left(\left(\widetilde{A_k}\right)_{n\lrcorner}\right) = \sum_{k=1}^{K}\varphi\left(s\left(\left(\widetilde{A_k}\right)_{n\lrcorner};k\right)\right) \\
&= \varphi\left(\sum_{k=1}^{K}s\left(\left(\widetilde{A_k}\right)_{n\lrcorner};k\right)\right) = \varphi\left(\left(\widetilde{\mathbf{A}}_{n\lrcorner}\right)_{K]}\right) \\
&\leq \|\varphi\|\left\|\left(\widetilde{\mathbf{A}}_{n\lrcorner}\right)_{K]}\right\|_r \leq \|\varphi\|\left\|\widetilde{\mathbf{A}}\right\|_r = \|\varphi\|\,\|\mathbf{A}\|_r .
\end{aligned}$$

It follows that

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left|a_{ji}^{(k)}\varphi_k(E(1;(j,i)))\right| \leq \|\varphi\|\,\|\mathbf{A}\|_r .$$

From this result, we can define a bounded linear functional ψ_φ on $\mathscr{L}^r$ by $\{A_k\}_{k=1}^{\infty} \mapsto \sum_{k=1}^{\infty}\sum A_k \bullet B_k^{(\varphi)}$ with $\|\psi_\varphi\| \leq \|\varphi\|$. Notice that for any $\mathbf{A} = \{A_k\}_{k=1}^{\infty} \in \mathscr{L}^r$ and positive integer K, we have by the absolute convergence of the series $\sum A_k \bullet B_k^{(\varphi)}$ $(k = 1, 2, ..., K)$ that

$$\begin{aligned}
\psi_\varphi\left(\mathbf{A}_{K]}\right) &= \sum_{k=1}^{K}\sum A_k \bullet B_k^{(\varphi)} = \sum_{k=1}^{K}\lim_{n\to\infty}\sum\left(A_k \bullet B_k^{(\varphi)}\right)_{n\lrcorner} \\
&= \lim_{n\to\infty}\sum_{k=1}^{K}\sum\left(A_k \bullet B_k^{(\varphi)}\right)_{n\lrcorner} = \lim_{n\to\infty}\sum_{k=1}^{K}\varphi_k((A_k)_{n\lrcorner}) \\
&= \lim_{n\to\infty}\sum_{k=1}^{K}\varphi(s((A_k)_{n\lrcorner};k)) = \lim_{n\to\infty}\varphi\left(\sum_{k=1}^{K}s((A_k)_{n\lrcorner};k)\right) \\
&= \lim_{n\to\infty}\varphi\left((\mathbf{A}_{n\lrcorner})_{K]}\right) = \lim_{n\to\infty}\varphi\left((\mathbf{A}_{K]})_{n\lrcorner}\right) . \qquad (\S)
\end{aligned}$$

Next, let $\rho_\varphi = \varphi - \psi_\varphi$. We will show that $\rho_\varphi \in (\mathscr{L}^r_\kappa)^\perp$. To see this, let $\mathbf{A} \in \mathscr{L}^r_\kappa$. Then $\mathbf{A}_{K]} \in \mathscr{L}^r_\kappa$, and thus $\left\|\left|\left(\mathbf{A}_{K]}\right)_{n_\lrcorner} - \mathbf{A}_{K]}\right|\right\|_r \to 0$ for all K. Whence, by (§) and the continuity of φ, we get $\psi_\varphi\left(\mathbf{A}_{K]}\right) = \lim_{n\to\infty} \varphi\left(\left(\mathbf{A}_{K]}\right)_{n_\lrcorner}\right) = \varphi\left(\mathbf{A}_{K]}\right)$ for all K. Since by Corollary 3.2.3, we have $\left\|\left|\mathbf{A}_{K]} - \mathbf{A}\right|\right\|_r \to 0$, it follows from the continuity of ψ_φ and φ that $\psi_\varphi(\mathbf{A}) = \psi_\varphi\left(\lim_{K\to\infty} \mathbf{A}_{K]}\right) = \lim_{K\to\infty} \psi_\varphi\left(\mathbf{A}_{K]}\right) = \lim_{K\to\infty} \varphi\left(\mathbf{A}_{K]}\right) = \varphi\left(\lim_{K\to\infty} \mathbf{A}_{K]}\right) = \varphi(\mathbf{A})$. Hence $\psi_\varphi = \varphi$ on $\mathscr{L}^r_\kappa$, which implies that $\rho_\varphi \in (\mathscr{L}^r_\kappa)^\perp$. Put $\mathscr{P} = \{\psi_\varphi : \varphi \in (\mathscr{L}^r)^*\}$. We claim that $(\mathscr{L}^r)^* = \mathscr{P} \oplus (\mathscr{L}^r_\kappa)^\perp$ and $\mathscr{P}$ is isometrically isomorphic to $(\mathscr{L}^r_\kappa)^*$. From the definition of $\mathscr{P}$, we have already had that $(\mathscr{L}^r)^* = \mathscr{P} + (\mathscr{L}^r_\kappa)^\perp$. The decomposition $(\mathscr{L}^r)^* = \mathscr{P} \oplus (\mathscr{L}^r_\kappa)^\perp$ will be obtained once it can be shown that $\mathscr{P} \cap (\mathscr{L}^r_\kappa)^\perp = \{0\}$. To see this, let $\psi_\varphi \in \mathscr{P} \cap (\mathscr{L}^r_\kappa)^\perp$ for some $\varphi \in (\mathscr{L}^r)^*$. Then for every $\mathbf{A} = \{A_k\}_{k=1}^\infty \in \mathscr{L}^r$, we have by the absolute convergence of the series $\sum_{k=1}^\infty \sum A_k \bullet B_k^{(\varphi)}$ and the fact that the sequence $\mathbf{A}_{n_\lrcorner} \in \mathscr{L}^r_\kappa$ for all n that $\psi_\varphi(\mathbf{A}) = \lim_{n\to\infty} \psi_\varphi\left(\mathbf{A}_{n_\lrcorner}\right) = 0$. Therefore, $\psi_\varphi = 0$, which yields $\mathscr{P} \cap (\mathscr{L}^r_\kappa)^\perp = \{0\}$. Accordingly, we have $(\mathscr{L}^r)^* = \mathscr{P} \oplus (\mathscr{L}^r_\kappa)^\perp$ as asserted. The rest is to prove hat $\mathscr{P}$ is isometrically isomorphic to $(\mathscr{L}^r_\kappa)^*$. To get this, we need to show first that $\left\|\psi_\varphi|_{\mathscr{L}^r_\kappa}\right\| = \|\psi_\varphi\|$. It is obvious that $\left\|\psi_\varphi|_{\mathscr{L}^r_\kappa}\right\| \le \|\psi_\varphi\|$. To have that $\left\|\psi_\varphi|_{\mathscr{L}^r_\kappa}\right\| \ge \|\psi_\varphi\|$, let $\epsilon > 0$ be given. Then there is a sequence $\mathbf{A} = \{A_k\}_{k=1}^\infty \in \mathscr{L}^r$ such that $\|\|\mathbf{A}\|\|_r \le 1$ and $\|\psi_\varphi\| < |\psi_\varphi(\mathbf{A})| + \epsilon$. Thus, by the absolute convergence of the series $\sum_{k=1}^\infty \sum A_k \bullet B_k^{(\varphi)}$, there is a positive integer n such that $\|\psi_\varphi\| < \left|\psi_\varphi\left(\mathbf{A}_{n_\lrcorner}\right)\right| + \epsilon < \left\|\psi_\psi|_{\mathscr{L}^r_\kappa}\right\| + \epsilon$ for all $\epsilon > 0$. This implies that $\|\psi_\varphi\| \le \left\|\psi_\psi|_{\mathscr{L}^r_\kappa}\right\|$, and hence we obtain $\left\|\psi_\varphi|_{\mathscr{L}^r_\kappa}\right\| = \|\psi_\varphi\|$ as desired. From this result, the map $\psi_\varphi \mapsto \psi_\varphi|_{\mathscr{L}^r_\kappa}$ is now an isometric isomorphism from $\mathscr{P}$ into $(\mathscr{L}^r_\kappa)^*$. To see that it is onto, let $\varphi_0 \in (\mathscr{L}^r_\kappa)^*$. We then have by the Hahn Banach extension theorem that φ_0 can extend uniquely to a bounded linear functional φ on $\mathscr{L}^r$ with $\|\varphi\| = \|\varphi_0\|$. Since ψ_φ agrees with φ on $\mathscr{L}^r_\kappa$, it follows $\psi_\varphi|_{\mathscr{L}^r_\kappa} = \varphi_0$. Consequently, the map $\psi_\varphi \mapsto \psi_\varphi|_{\mathscr{L}^r_\kappa}$ is an isometric isomorphism from $\mathscr{P}$ onto $(\mathscr{L}^r_\kappa)^*$.

(3). Let $\varphi = \psi_\varphi + \rho_\varphi \in (\mathscr{L}^r)^*$. It is apparent that $\|\varphi\| \le \|\psi_\varphi\| + \|\rho_\varphi\|$. We want to show that the reverse inequality holds. To prove this, let $\epsilon > 0$ be given. Then there is a sequence $\mathbf{A} = \{A_k\}_{k=1}^\infty \in \mathscr{L}^r$ with $\|\|\mathbf{A}\|\|_r \le 1$ such that $|\psi_\varphi(\mathbf{A})| > \|\psi_\varphi\| - \frac{\epsilon}{3}$. From this, we have by the absolute convergence of the series $\sum_{k=1}^\infty \sum A_k \bullet B_k^{(\varphi)}$ that there exists a positive integer N such that $\left|\psi_\varphi\left(\mathbf{A}_{N_\lrcorner}\right)\right| > \|\psi_\varphi\| - \frac{\epsilon}{3}$. Let $\mathbf{C} = \left(\operatorname{sgn}\psi_\varphi\left(\mathbf{A}_{N_\lrcorner}\right)\right)\mathbf{A}_{N_\lrcorner}$. Then $\|\|\mathbf{C}\|\|_r = \|\|\mathbf{A}_{N_\lrcorner}\|\|_r \le \|\|\mathbf{A}\|\|_r \le 1$ and $\psi_\varphi(\mathbf{C}) = \left|\psi_\varphi\left(\mathbf{A}_{N_\lrcorner}\right)\right| > \|\psi_\varphi\| - \frac{\epsilon}{3}$. Next, let $\mathbf{D} = \{D_k\}_{k=1}^\infty \in \mathscr{L}^r$ be such that $\|\|\mathbf{D}\|\|_r \le 1$, $\rho_\varphi(\mathbf{D}) > 0$ and $\rho_\varphi(\mathbf{D}) > \|\rho_\varphi\| - \frac{\epsilon}{3}$. Then by the absolute convergence of the series $\sum_{k=1}^\infty \sum D_k \bullet B_k^{(\varphi)}$, we have $\psi_\varphi(\mathbf{D}_{n_\ulcorner}) \to 0$. Whence there is a positive integer $J > N$ such that $|\psi_\varphi(\mathbf{D}_{J_\ulcorner})| < \frac{\epsilon}{3}$. Since $\mathbf{D} - \mathbf{D}_{J_\ulcorner} \in \mathscr{L}^r_\kappa$, it follows that $\rho_\varphi(\mathbf{D}_{J_\ulcorner}) = \rho_\varphi(\mathbf{D})$. Let $\mathbf{E} = \mathbf{C} + \mathbf{D}_{J_\ulcorner}$. Then

$\mathbf{E} \in \mathscr{L}^r$ and by Theorem 1.1(4), we have $\|\mathbf{E}\|_r = \max\{\|\mathbf{C}\|_r, \|\mathbf{D}_{J_\Gamma}\|_r\} \leq 1$. Thus

$$\begin{aligned}
\|\varphi\| &\geq |\varphi(\mathbf{E})| = |\psi_\varphi(\mathbf{E}) + \rho_\varphi(\mathbf{E})| \\
&= |\psi_\varphi(\mathbf{C}) + \psi_\varphi(\mathbf{D}_{J_\Gamma}) + \rho_\varphi(\mathbf{C}) + \rho_\varphi(\mathbf{D}_{J_\Gamma})| \\
&= |\psi_\varphi(\mathbf{C}) + \psi_\varphi(\mathbf{D}_{J_\Gamma}) + \rho_\varphi(\mathbf{D})| \\
&\geq \psi_\varphi(\mathbf{C}) + \rho_\varphi(\mathbf{D}) - |\psi_\varphi(\mathbf{D}_{J_\Gamma})| \\
&> \|\psi_\varphi\| - \frac{\epsilon}{3} + \|\rho_\varphi\| - \frac{\epsilon}{3} - \frac{\epsilon}{3} \\
&= \|\psi_\varphi\| + \|\rho_\varphi\| - \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ was given arbitrarily, it follows that $\|\psi_\varphi\| + \|\rho_\varphi\| \leq \|\varphi\|$. Hence the equation $\|\varphi\| = \|\psi_\varphi\| + \|\rho_\varphi\|$ is obtained. □

Remark 3.2.6. *Since $(\mathscr{L}^r_\kappa)^*$ is isometrically isomorphic to $\mathscr{P}$, we may treat $(\mathscr{L}^r_\kappa)^*$ as a subspace of $(\mathscr{L}^r)^*$. Thus Theorem 3.2.5 can be symbolized analogously to Dixmier's theorem as follows:*

$$(\mathscr{L}^r)^* = (\mathscr{L}^r_\kappa)^* \oplus_1 (\mathscr{L}^r_\kappa)_s .$$

Chapter 4

Conclusion

Let $\mathscr{M}_\infty$ be the set of all infinite complex matrices. For each $1 \leq r < \infty$, we define a class of sequences of infinite complex matrices $\mathscr{L}^r$ as follows:

$$\mathscr{L}^r = \left\{ \left\{ \left[a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty} \subseteq \mathscr{M}_\infty : \left[\sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \right|^r \right] \in \mathcal{B}(l^2) \right\},$$

and for any sequence $\left\{ \left[a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty}$ in $\mathscr{M}_\infty$, we define

$$\left\| \left\{ \left[a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty} \right\|_r = \begin{cases} \left\| \left[\sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \right|^r \right] \right\|^{1/r} & \text{if } \left\{ \left[a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty} \in \mathscr{L}^r, \\ \infty & \text{otherwise.} \end{cases}$$

In this thesis, we study some elementary properties and provide some results on the duality of $\mathscr{L}^r$. The main goal is to decompose the dual space $(\mathscr{L}^r)^*$ of $\mathscr{L}^r$ as an l^1 direct-sum of its two closed subspaces by a way analogous to a beautiful theorem of Dixmier on decomposing the dual $\mathcal{B}(l^2)^*$ of $\mathcal{B}(l^2)$. The following are the results.

We first obtain some characterizations of the sets $\mathscr{L}^r$.

Theorem 4.1. *Let $\{A_k\}_{k=1}^{\infty}$ be a sequence in $\mathscr{M}_\infty$ and $1 \leq r < \infty$. Then the following are equivalent:*

(1) *$\{A_k\}_{k=1}^{\infty}$ belongs to $\mathscr{L}^r$;*

(2) *$A_k \in \mathcal{S}_{2,2}^r(\mathbb{C})$ for all k and the sequence $\left\{ \sum_{k=1}^{n} A_k^{[r]} \right\}_{n=1}^{\infty}$ is bounded in $\mathcal{B}(l^2)$;*

(3) *the sequence $\left\{ \left\| \sum_{k=1}^{n} A_k^{[r]} \right\| \right\}_{n=1}^{\infty}$ is bounded.*

To obtain that $\|\cdot\|_r$ is precisely a norm on $\mathscr{L}^r$, the following Hölder-type inequality is constructed.

Theorem 4.2. (Hölder-type inequality for sequences of matrices) *For any sequences* $\{A_k\}_{k=1}^{\infty}$ *and* $\{B_k\}_{k=1}^{\infty}$ *in* $\mathscr{M}_\infty$,

$$|||\{A_k \bullet B_k\}_{k=1}^{\infty}|||_1 \le |||\{A_k\}_{k=1}^{\infty}|||_r \, |||\{B_k\}_{k=1}^{\infty}|||_{r^*} ,$$

where $1 < r < \infty$ *with* $\dfrac{1}{r} + \dfrac{1}{r^*} = 1$, *under the conventions that* $\infty \cdot 0 = 0 \cdot \infty = 0$, $\infty \cdot \alpha = \alpha \cdot \infty = \infty$ *for all positive real number* α *and* $\infty \cdot \infty = \infty$.

From the Hölder-type inequality, the corresponding Minkowski's inequality is obtained.

Theorem 4.3. (Minkowski-type inequality for sequences of matrices) *For any sequences* $\{A_k\}_{k=1}^{\infty}$ *and* $\{B_k\}_{k=1}^{\infty}$ *in* $\mathscr{M}_\infty$ *and* $1 \le r < \infty$,

$$|||\{A_k + B_k\}_{k=1}^{\infty}|||_r \le |||\{A_k\}_{k=1}^{\infty}|||_r + |||\{B_k\}_{k=1}^{\infty}|||_r ,$$

under the conventions that $\infty + \alpha = \alpha + \infty = \infty$ *for all non-negative real number* α *and* $\infty + \infty = \infty$.

From the Minkowski's inequality, we have that the set $\mathscr{L}^r$ equipped with the norm $|||\cdot|||_r$ is a normed space. A Rieze-fischer-type theorem for completeness of the sequence spaces $\mathscr{L}^r$ is obtained below.

Theorem 4.4. *For each* $1 \le r < \infty$, *the set* $\mathscr{L}^r$ *equipped with the norm* $|||\cdot|||_r$ *is a Banach space.*

For the classical sequence spaces l^p $(1 \le p < \infty)$, there is a result closely related to their duality stating that for every $1 \le p < \infty$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$, a sequence $\{x_k\}_{k=1}^{\infty}$ belongs to l^q if and only if $\{x_k y_k\}_{k=1}^{\infty} \in l^1$ for all $\{y_k\} \in l^p$. We obtain a similar duality-type result for the sequence spaces $\mathscr{L}^r$ as follows.

Theorem 4.5. *Let* $1 < r < \infty$ *with* $\dfrac{1}{r} + \dfrac{1}{r^*} = 1$.

(1) *A sequence* $\{A_k\}_{k=1}^{\infty} \in \mathscr{L}^{r^*}$ *if and only if* $\{A_k \bullet B_k\}_{k=1}^{\infty} \in \mathscr{L}^1$ *for all* $\{B_k\}_{k=1}^{\infty} \in \mathscr{L}^r$.

(2) *If* $\{A_k\}_{k=1}^{\infty} \in \mathscr{L}^{r^*}$, *then*

$$|||\{A_k\}_{k=1}^{\infty}|||_{r^*} = \sup\{|||\{A_k \bullet B_k\}_{k=1}^{\infty}|||_1 : \{B_k\}_{k=1}^{\infty} \in \mathscr{L}^r, |||\{B_k\}_{k=1}^{\infty}|||_r \le 1\}.$$

Next, we define the class

$$\mathscr{L}_\kappa^r = \left\{ \left\{ \left[a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty} \subseteq \mathscr{M}_\infty : \left[\sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \right|^r \right] \in \mathcal{K}(l^2) \right\},$$

We obtain a characterization of $\mathscr{L}_\kappa^r$ as follows.

Theorem 4.6. *Let $\{A_k\}_{k=1}^{\infty}$ be a sequence in $\mathscr{M}_\infty$ and $1 \le r < \infty$. Then $\{A_k\}_{k=1}^{\infty} \in \mathscr{L}_\kappa^r$ if and only if $A_k^{[r]}$ is compact for all k and the sequence $\left\{\sum_{k=1}^{n} A_k^{[r]}\right\}_{k=1}^{\infty}$ converges in $\mathcal{B}(l^2)$.*

Theorem 4.7. *For each $1 \le r < \infty$, the set $\mathscr{L}_\kappa^r$ is a Banach subspace of $\mathscr{L}^r$.*

We finally obtain a decomposition theorem for the dual $(\mathscr{L}^r)^*$ of $\mathscr{L}^r$ as follows.

Theorem 4.8. Let $1 \le r < \infty$. Then the following hold.

(1) *The annihilator $(\mathscr{L}_\kappa^r)^\perp$ of $\mathscr{L}_\kappa^r$ is a non-trivial closed subspace of the dual $(\mathscr{L}^r)^*$ of $\mathscr{L}^r$.*

(2) *There is a subspace $\mathscr{P}$ of $(\mathscr{L}^r)^*$ such that $\mathscr{P}$ is isometrically isomorphic to $(\mathscr{L}_\kappa^r)^*$ and $(\mathscr{L}^r)^* = \mathscr{P} \oplus (\mathscr{L}_\kappa^r)^\perp$.*

(3) *For any $f \in (\mathscr{L}^r)^*$, the decomposition $f = g + h$, where $g \in \mathscr{P}$ and $h \in (\mathscr{L}_\kappa^r)^\perp$, satisfies $\|f\| = \|g\| + \|h\|$.*

Notice that Theorem 4.8 can be symbolized analogously to Dixmier's theorem as follows:

$$(\mathscr{L}^r)^* = (\mathscr{L}_\kappa^r)^* \oplus_1 (\mathscr{L}_\kappa^r)_s .$$

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