

ภาคผนวก

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1. Criteria for system of three second-order ordinary differential equations to be reduced to a linear system via restricted class of point transformation.
2. Application of group analysis to classification of systems of three second-order ordinary differential equations (submitted).

Criteria for System of Three Second-Order Ordinary Differential Equations to Be Reduced to a Linear System via Restricted Class of Point Transformation

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ABSTRACT

This paper is devoted to the study of the linearization problem of system of three second-order ordinary differential equations $y_1'' = F_1(x, y_1, y_2, y_3, y_1', y_2', y_3')$, $y_2'' = F_2(x, y_1, y_2, y_3, y_1', y_2', y_3')$ and $y_3'' = F_3(x, y_1, y_2, y_3, y_1', y_2', y_3')$. The necessary conditions for linearization by general point transformation $t = \varphi(x, y_1, y_2, y_3)$, $u_1(t) = \psi_1(x, y_1, y_2, y_3)$, $u_2(t) = \psi_2(x, y_1, y_2, y_3)$ and $u_3(t) = \psi_3(x, y_1, y_2, y_3)$ are found. The sufficient conditions for linearization by restricted class of point transformation $t = \varphi(x)$, $u_1(t) = \psi_1(x, y_1)$, $u_2(t) = \psi_2(x, y_1, y_2)$ and $u_3(t) = \psi_3(x, y_1, y_2, y_3)$ are obtained. Moreover, the procedure for obtaining the linearizing transformation is provided in explicit forms. Examples demonstrating the procedure of using the linearization theorems are presented.

KEYWORDS

Linearization Problem; Point Transformation; System of Three Second-Order Ordinary Differential Equation

1. Introduction

In general, physical applications of differential equations are in the form of nonlinear equations, which are very difficult to solve explicitly. Most of the solutions are approximate solutions. In solving nonlinear ordinary differential equations, one of the solving methods is reducing nonlinear ordinary differential equations to be linear ordinary differential equations, which makes the method easier and then we have exact solution of original equation.

The first linearization problem for single second-order ordinary differential equations was solved by S. Lie [1]. He found the general form of all ordinary differential equations of second-order that can be reduced to a linear equation by changing the independent and dependent variables. He showed that any linearizable second-order equation should be at most cubic in the first-order derivative and provided a linearization test in terms of its coefficients. The linearization criterion is written through relative invariants of the equivalence group. A. M. Tresse [2] and R. Liouville [3] treated the equivalence problem for second-order ordinary differential equations in terms of relative invariants of the equivalence group of point transformations. In [4] an infinitesimal technique for obtaining relative invariants was applied to the linearization problem. A different approach to tackling the equivalence problem of second-order ordinary differential equations was developed by E. Cartan [5]. The idea of his approach was to associate with every differential equation a uniquely defined geometric structure of a certain form.

The linearization problem for a system of second-order ordinary differential equations was studied in [6,7]. In

[6], Wafo and Mahomed found the criteria for linearization of a system of two second-order ordinary differential equations which are related with the existence of an admitted four-dimensional Lie algebra. In [8], Aminova and Aminov gave the necessary and sufficient conditions for a system of second-order ordinary differential equations to be equivalent to the free particle equations. Particular class of systems of two ($n = 2$) second-order ordinary differential equations was considered by Mahomed and Qadir [9] and they also provided the construction of the linearizing point transformation by using complex variables. Some first-order and second-order relative invariants with respect to point transformations for a system of two ordinary differential equations were obtained in [10]. In [11], Sookmee and Meleshko proposed a new method of linearizing a system of equations, where a given system of equations is reduced to a single equation to which the Lie theorem on linearization is applied. In [12], necessary and sufficient conditions for a system of two second-order ordinary differential equations to be equivalent to the simplest equations were obtained by using the implementation of Cartan's method. Linearization criteria for a system of two second-order ordinary under general point transformation were obtained in [13]. In [7], linearization criteria for a system of two second-order ordinary differential equations to be equivalent to the linear system with constant coefficients matrix via fiber preserving point transformations were achieved.

Nowadays, the linearization problem of a system of three second-order ordinary differential equations to be equivalent to linear system via point transformations is open. Hence, it's worth solving this problem as essential part of a study of differential equations.

The manuscript is organized as follows. In Section 2, the necessary conditions of linearization of a system of three second-order ordinary differential equations in the general case of point transformations are presented. In particular, in Section 3, sufficient conditions for linearizing restricted class of point transformations are obtained. Examples which illustrate the procedure of using the linearization theorems are presented in Section 4.

2. Necessary Conditions for Linearization

We begin with investigating the necessary conditions for linearization. We consider a system of three second-order ordinary differential equations

$$\begin{aligned}y_1'' &= F_1(x, y_1, y_2, y_3, y_1', y_2', y_3'), \\y_2'' &= F_2(x, y_1, y_2, y_3, y_1', y_2', y_3'), \\y_3'' &= F_3(x, y_1, y_2, y_3, y_1', y_2', y_3'),\end{aligned}\tag{1}$$

which can be transformed to a linear system

$$u_1''(t) = 0, \quad u_2''(t) = 0, \quad u_3''(t) = 0,\tag{2}$$

under the point transformation

$$\begin{aligned}t &= \varphi(x, y_1, y_2, y_3), \\u_1(t) &= \psi_1(x, y_1, y_2, y_3), \\u_2(t) &= \psi_2(x, y_1, y_2, y_3), \\u_3(t) &= \psi_3(x, y_1, y_2, y_3)\end{aligned}\tag{3}$$

So that we arrive at the following theorem.

Theorem 1. Any system of three second-order ordinary differential Equation (1) obtained from a linear system (2) by a point transformation (3) has to be the form

$$\begin{aligned}y_1'' &= a_{11}y_1^3 + (a_{12}y_2' + a_{13}y_3' + a_{14})y_1^2 + (a_{15}y_2'' + (a_{16}y_3' + a_{17})y_2' + a_{18}y_3'' + a_{19}y_3' + a_{110})y_1' \\&\quad + a_{111}y_2^2 + (a_{112}y_3' + a_{113})y_2' + a_{114}y_3^2 + a_{115}y_3' + a_{116}, \\y_2'' &= a_{15}y_2^3 + (a_{12}y_1' + a_{16}y_3' + a_{24})y_2^2 + (a_{11}y_1^2 + (a_{13}y_3' + a_{27})y_1' + a_{18}y_3'' + a_{29}y_3' + a_{210})y_2' \\&\quad + a_{211}y_1^2 + (a_{212}y_3' + a_{213})y_1' + a_{214}y_3^2 + a_{215}y_3' + a_{216}, \\y_3'' &= a_{18}y_3^3 + (a_{13}y_1' + a_{16}y_2' + a_{34})y_3^2 + (a_{11}y_1^2 + (a_{12}y_2' + a_{37})y_1' + a_{15}y_2'' + a_{39}y_2' + a_{310})y_3' \\&\quad + a_{311}y_1^2 + (a_{312}y_2' + a_{313})y_1' + a_{314}y_2^2 + a_{315}y_2' + a_{316}.\end{aligned}\tag{4}$$

where

$$a_{11} = (h_4\varphi_{y_1y_1} + h_2\psi_{1y_1y_1} + h_3\psi_{2y_1y_1} + h_4\psi_{3y_1y_1})/\Delta, \quad (5)$$

$$a_{12} = (2h_1\varphi_{y_1y_2} + 2h_2\psi_{1y_1y_2} + 2h_3\psi_{2y_1y_2} + 2h_4\psi_{3y_1y_2})/\Delta, \quad (6)$$

$$a_{13} = (2h_4\varphi_{y_1y_3} + 2h_2\psi_{1y_1y_3} + 2h_3\psi_{2y_1y_3} + 2h_4\psi_{3y_1y_3})/\Delta, \quad (7)$$

$$a_{14} = (2h_4\varphi_{xy_1} + h_5\varphi_{y_1y_1} + 2h_2\psi_{1xy_1} + h_6\psi_{1y_1y_1} + 2h_3\psi_{2xy_1} + h_7\psi_{2y_1y_1} + 2h_4\psi_{3xy_1} + h_8\psi_{3y_1y_1})/\Delta, \quad (8)$$

$$a_{15} = (h_4\varphi_{y_2y_2} + h_2\psi_{1y_2y_2} + h_3\psi_{2y_2y_2} + h_4\psi_{3y_2y_2})/\Delta, \quad (9)$$

$$a_{16} = (2h_4\varphi_{y_2y_3} + 2h_2\psi_{1y_2y_3} + 2h_3\psi_{2y_2y_3} + 2h_4\psi_{3y_2y_3})/\Delta, \quad (10)$$

$$a_{17} = (2h_4\varphi_{xy_2} + 2h_5\varphi_{y_1y_2} + 2h_2\psi_{1xy_2} + 2h_6\psi_{1y_1y_2} + 2h_3\psi_{2xy_2} + 2h_7\psi_{2y_1y_2} + 2h_4\psi_{3xy_2} + 2h_8\psi_{3y_1y_2})/\Delta, \quad (11)$$

$$a_{18} = (h_4\varphi_{y_3y_3} + h_2\psi_{1y_3y_3} + h_3\psi_{2y_3y_3} + h_4\psi_{3y_3y_3})/\Delta, \quad (12)$$

$$a_{19} = (2h_4\varphi_{xy_3} + 2h_5\varphi_{y_1y_3} + 2h_2\psi_{1xy_3} + 2h_6\psi_{1y_1y_3} + 2h_3\psi_{2xy_3} + 2h_7\psi_{2y_1y_3} + 2h_4\psi_{3xy_3} + 2h_8\psi_{3y_1y_3})/\Delta, \quad (13)$$

$$a_{110} = (2h_5\varphi_{xy_1} + h_4\varphi_{xx} + 2h_6\psi_{1xy_1} + h_2\psi_{1xx} + 2h_7\psi_{2xy_1} + h_3\psi_{2xx} + 2h_8\psi_{3xy_1} + h_4\psi_{3xx})/\Delta, \quad (14)$$

$$a_{111} = (h_5\varphi_{y_2y_2} + h_6\psi_{1y_2y_2} + h_7\psi_{2y_2y_2} + h_8\psi_{3y_2y_2})/\Delta, \quad (15)$$

$$a_{112} = (2h_5\varphi_{y_2y_3} + 2h_6\psi_{1y_2y_3} + 2h_7\psi_{2y_2y_3} + 2h_8\psi_{3y_2y_3})/\Delta, \quad (16)$$

$$a_{113} = (2h_5\varphi_{xy_2} + 2h_6\psi_{1xy_2} + 2h_7\psi_{2xy_2} + 2h_8\psi_{3xy_2})/\Delta, \quad (17)$$

$$a_{114} = (h_5\varphi_{y_3y_3} + h_6\psi_{1y_3y_3} + h_7\psi_{2y_3y_3} + h_8\psi_{3y_3y_3})/\Delta, \quad (18)$$

$$a_{115} = (2h_5\varphi_{xy_3} + 2h_6\psi_{1xy_3} + 2h_7\psi_{2xy_3} + 2h_8\psi_{3xy_3})/\Delta, \quad (19)$$

$$a_{116} = (h_5\varphi_{xx} + h_6\psi_{1xx} + h_7\psi_{2xx} + h_8\psi_{3xx})/\Delta, \quad (20)$$

$$a_{24} = (2h_4\varphi_{xy_2} + h_9\varphi_{y_2y_2} + 2h_2\psi_{1xy_2} + h_{10}\psi_{1y_2y_2} + 2h_3\psi_{2xy_2} + h_{11}\psi_{2y_2y_2} + 2h_4\psi_{3xy_2} + h_{12}\psi_{3y_2y_2})/\Delta, \quad (21)$$

$$a_{27} = (2h_4\varphi_{xy_1} + 2h_9\varphi_{y_1y_2} + 2h_2\psi_{1xy_1} + 2h_{10}\psi_{1y_1y_2} + 2h_3\psi_{2xy_1} + 2h_{11}\psi_{2y_1y_2} + 2h_4\psi_{3xy_1} + 2h_{12}\psi_{3y_1y_2})/\Delta, \quad (22)$$

$$a_{29} = (2h_4\varphi_{xy_3} + 2h_9\varphi_{y_2y_3} + 2h_2\psi_{1xy_3} + 2h_{10}\psi_{1y_2y_3} + 2h_3\psi_{2xy_3} + 2h_{11}\psi_{2y_2y_3} + 2h_4\psi_{3xy_3} + 2h_{12}\psi_{3y_2y_3})/\Delta, \quad (23)$$

$$a_{210} = (2h_5\varphi_{xy_2} + h_4\varphi_{xx} + 2h_{10}\psi_{1xy_2} + h_2\psi_{1xx} + 2h_{11}\psi_{2xy_2} + h_3\psi_{2xx} + 2h_{12}\psi_{3xy_2} + h_4\psi_{3xx})/\Delta, \quad (24)$$

$$a_{211} = (h_9\varphi_{y_1y_1} + h_{10}\psi_{1y_1y_1} + h_{11}\psi_{2y_1y_1} + h_{12}\psi_{3y_1y_1})/\Delta, \quad (25)$$

$$a_{212} = (2h_9\varphi_{y_1y_3} + 2h_{10}\psi_{1y_1y_3} + 2h_{11}\psi_{2y_1y_3} + 2h_{12}\psi_{3y_1y_3})/\Delta, \quad (26)$$

$$a_{213} = (2h_9\varphi_{xy_1} + 2h_{10}\psi_{1xy_1} + 2h_{11}\psi_{2xy_1} + 2h_{12}\psi_{3xy_1})/\Delta, \quad (27)$$

$$a_{214} = (h_9\varphi_{y_3y_3} + h_{10}\psi_{1y_3y_3} + h_{11}\psi_{2y_3y_3} + h_{12}\psi_{3y_3y_3})/\Delta, \quad (28)$$

$$a_{215} = (2h_9\varphi_{xy_3} + 2h_{10}\psi_{1xy_3} + 2h_{11}\psi_{2xy_3} + 2h_{12}\psi_{3xy_3})/\Delta, \quad (29)$$

$$a_{216} = (h_9\varphi_{xx} + h_{10}\psi_{1xx} + h_{11}\psi_{2xx} + h_{12}\psi_{3xx})/\Delta, \quad (30)$$

$$a_{34} = (2h_4\varphi_{xy_3} + h_{13}\varphi_{y_3y_3} + 2h_2\psi_{1xy_3} + h_{14}\psi_{1y_3y_3} + 2h_3\psi_{2xy_3} + h_{15}\psi_{2y_3y_3} + 2h_4\psi_{3xy_3} + h_{16}\psi_{3y_3y_3})/\Delta, \quad (31)$$

$$a_{37} = (2h_4\varphi_{xy_1} + 2h_{13}\varphi_{y_1y_3} + 2h_2\psi_{1xy_1} + 2h_{14}\psi_{1y_1y_3} + 2h_3\psi_{2xy_1} + 2h_{15}\psi_{2y_1y_3} + 2h_4\psi_{3xy_1} + 2h_{16}\psi_{3y_1y_3})/\Delta, \quad (32)$$

$$a_{39} = (2h_4\varphi_{xy_2} + 2h_{13}\varphi_{y_2y_3} + 2h_2\psi_{1xy_2} + 2h_{14}\psi_{1y_2y_3} + 2h_3\psi_{2xy_2} + 2h_{15}\psi_{2y_2y_3} + 2h_4\psi_{3xy_2} + 2h_{16}\psi_{3y_2y_3})/\Delta, \quad (33)$$

$$a_{310} = (2h_{13}\varphi_{xy_3} + h_4\varphi_{xx} + 2h_{14}\psi_{1xy_3} + h_2\psi_{1xx} + 2h_{15}\psi_{2xy_3} + h_3\psi_{2xx} + 2h_{16}\psi_{3xy_3} + h_4\psi_{3xx})/\Delta, \quad (34)$$

$$a_{311} = (h_{13}\varphi_{y_1y_1} + h_{14}\psi_{1y_1y_1} + h_{15}\psi_{2y_1y_1} + h_{16}\psi_{3y_1y_1})/\Delta, \quad (35)$$

$$a_{312} = (2h_{13}\varphi_{y_1y_2} + 2h_{14}\psi_{1y_1y_2} + 2h_{15}\psi_{2y_1y_2} + 2h_{16}\psi_{3y_1y_2})/\Delta, \quad (36)$$

$$a_{313} = (2h_{13}\varphi_{xy_1} + 2h_{14}\psi_{1xy_1} + 2h_{15}\psi_{2xy_1} + 2h_{16}\psi_{3xy_1})/\Delta, \quad (37)$$

$$a_{314} = (h_{13}\varphi_{y_2y_2} + h_{14}\psi_{1y_2y_2} + h_{15}\psi_{2y_2y_2} + h_{16}\psi_{3y_2y_2})/\Delta, \quad (38)$$

$$a_{315} = (2h_{13}\varphi_{xy_2} + 2h_{14}\psi_{1xy_2} + 2h_{15}\psi_{2xy_2} + 2h_{16}\psi_{3xy_2})/\Delta, \quad (39)$$

$$a_{316} = (h_{13}\varphi_{xx} + h_{14}\psi_{1xx} + h_{15}\psi_{2xx} + h_{16}\psi_{3xx})/\Delta, \quad (40)$$

$$h_1 = (\psi_{1y_1}\psi_{2y_2}\psi_{3y_3} - \psi_{1y_1}\psi_{2y_3}\psi_{3y_2} - \psi_{1y_2}\psi_{2y_1}\psi_{3y_3} + \psi_{1y_2}\psi_{2y_3}\psi_{3y_1} + \psi_{1y_3}\psi_{2y_1}\psi_{3y_2} - \psi_{1y_3}\psi_{2y_2}\psi_{3y_1}),$$

$$h_2 = (-\varphi_{y_1}\psi_{2y_2}\psi_{3y_3} + \varphi_{y_1}\psi_{2y_3}\psi_{3y_2} + \varphi_{y_2}\psi_{2y_1}\psi_{3y_3} - \varphi_{y_2}\psi_{2y_3}\psi_{3y_1} - \varphi_{y_3}\psi_{2y_1}\psi_{3y_2} + \varphi_{y_3}\psi_{2y_2}\psi_{3y_1}),$$

$$h_3 = (\varphi_{y_1}\psi_{1y_2}\psi_{3y_3} - \varphi_{y_1}\psi_{1y_3}\psi_{3y_2} - \varphi_{y_2}\psi_{1y_1}\psi_{3y_3} + \varphi_{y_2}\psi_{1y_3}\psi_{3y_1} + \varphi_{y_3}\psi_{1y_1}\psi_{3y_2} - \varphi_{y_3}\psi_{1y_2}\psi_{3y_1}),$$

$$h_4 = (-\varphi_{y_1}\psi_{1y_2}\psi_{2y_3} + \varphi_{y_1}\psi_{1y_3}\psi_{2y_2} + \varphi_{y_2}\psi_{1y_1}\psi_{2y_3} - \varphi_{y_2}\psi_{1y_3}\psi_{2y_1} - \varphi_{y_3}\psi_{1y_1}\psi_{2y_2} + \varphi_{y_3}\psi_{1y_2}\psi_{2y_1}),$$

$$h_5 = (\psi_{1x}\psi_{2y_2}\psi_{3y_3} - \psi_{1x}\psi_{2y_3}\psi_{3y_2} - \psi_{1y_2}\psi_{2x}\psi_{3y_3} + \psi_{1y_2}\psi_{2y_3}\psi_{3x} + \psi_{1y_3}\psi_{2x}\psi_{3y_2} - \psi_{1y_3}\psi_{2y_2}\psi_{3x}),$$

$$h_6 = (-\varphi_x\psi_{2y_2}\psi_{3y_3} + \varphi_x\psi_{2y_3}\psi_{3y_2} + \varphi_{y_2}\psi_{2x}\psi_{3y_3} - \varphi_{y_2}\psi_{2y_3}\psi_{3x} - \varphi_{y_3}\psi_{2x}\psi_{3y_2} + \varphi_{y_3}\psi_{2y_2}\psi_{3x}),$$

$$h_7 = (\varphi_x\psi_{1y_2}\psi_{3y_3} - \varphi_x\psi_{1y_3}\psi_{3y_2} - \varphi_{y_2}\psi_{1x}\psi_{3y_3} + \varphi_{y_2}\psi_{1y_3}\psi_{3x} + \varphi_{y_3}\psi_{1x}\psi_{3y_2} - \varphi_{y_3}\psi_{1y_2}\psi_{3x}),$$

$$h_8 = (-\varphi_x\psi_{1y_2}\psi_{2y_3} + \varphi_x\psi_{1y_3}\psi_{2y_2} + \varphi_{y_2}\psi_{1x}\psi_{2y_3} - \varphi_{y_2}\psi_{1y_3}\psi_{2x} - \varphi_{y_3}\psi_{1x}\psi_{2y_2} + \varphi_{y_3}\psi_{1y_2}\psi_{2x}),$$

$$h_9 = (-\psi_{1x}\psi_{2y_1}\psi_{3y_3} + \psi_{1x}\psi_{2y_3}\psi_{3y_1} + \psi_{1y_1}\psi_{2x}\psi_{3y_3} - \psi_{1y_1}\psi_{2y_3}\psi_{3x} - \psi_{1y_3}\psi_{2x}\psi_{3y_1} + \psi_{1y_3}\psi_{2y_1}\psi_{3x}),$$

$$h_{10} = (\varphi_x\psi_{2y_1}\psi_{3y_3} - \varphi_x\psi_{2y_3}\psi_{3y_1} - \varphi_{y_1}\psi_{2x}\psi_{3y_3} + \varphi_{y_1}\psi_{2y_3}\psi_{3x} + \varphi_{y_3}\psi_{2x}\psi_{3y_1} - \varphi_{y_3}\psi_{2y_1}\psi_{3x}),$$

$$h_{11} = (-\varphi_x\psi_{1y_1}\psi_{3y_3} + \varphi_x\psi_{1y_3}\psi_{3y_1} + \varphi_{y_1}\psi_{1x}\psi_{3y_3} - \varphi_{y_1}\psi_{1y_3}\psi_{3x} - \varphi_{y_3}\psi_{1x}\psi_{3y_1} + \varphi_{y_3}\psi_{1y_1}\psi_{3x}),$$

$$h_{12} = (\varphi_x\psi_{1y_1}\psi_{2y_3} - \varphi_x\psi_{1y_3}\psi_{2y_1} - \varphi_{y_1}\psi_{1x}\psi_{2y_3} + \varphi_{y_1}\psi_{1y_3}\psi_{2x} + \varphi_{y_3}\psi_{1x}\psi_{2y_1} - \varphi_{y_3}\psi_{1y_1}\psi_{2x}),$$

$$h_{13} = (\psi_{1x}\psi_{2y_1}\psi_{3y_2} - \psi_{1x}\psi_{2y_2}\psi_{3y_1} - \psi_{1y_1}\psi_{2x}\psi_{3y_2} + \psi_{1y_1}\psi_{2y_2}\psi_{3x} + \psi_{1y_2}\psi_{2x}\psi_{3y_1} - \psi_{1y_2}\psi_{2y_1}\psi_{3x}),$$

$$h_{14} = (-\varphi_x\psi_{2y_1}\psi_{3y_2} + \varphi_x\psi_{2y_2}\psi_{3y_1} + \varphi_{y_1}\psi_{2x}\psi_{3y_2} - \varphi_{y_1}\psi_{2y_2}\psi_{3x} - \varphi_{y_2}\psi_{2x}\psi_{3y_1} + \varphi_{y_2}\psi_{2y_1}\psi_{3x}),$$

$$h_{15} = (\varphi_x\psi_{1y_1}\psi_{3y_2} - \varphi_x\psi_{1y_2}\psi_{3y_1} - \varphi_{y_1}\psi_{1x}\psi_{3y_2} + \varphi_{y_1}\psi_{1y_2}\psi_{3x} + \varphi_{y_2}\psi_{1x}\psi_{3y_1} - \varphi_{y_2}\psi_{1y_1}\psi_{3x}),$$

$$h_{16} = (-\varphi_x\psi_{1y_1}\psi_{2y_2} + \varphi_x\psi_{1y_2}\psi_{2y_1} + \varphi_{y_1}\psi_{1x}\psi_{2y_2} - \varphi_{y_1}\psi_{1y_2}\psi_{2x} - \varphi_{y_2}\psi_{1x}\psi_{2y_1} + \varphi_{y_2}\psi_{1y_1}\psi_{2x})$$

and

$$\begin{aligned} \Delta = & \left(\varphi_x \psi_{1y_1} \psi_{2y_2} \psi_{3y_3} - \varphi_x \psi_{1y_1} \psi_{2y_3} \psi_{3y_2} - \varphi_x \psi_{1y_2} \psi_{2y_1} \psi_{3y_3} + \varphi_x \psi_{1y_2} \psi_{2y_3} \psi_{3y_1} + \varphi_x \psi_{1y_3} \psi_{2y_1} \psi_{3y_2} \right. \\ & - \varphi_x \psi_{1y_3} \psi_{2y_2} \psi_{3y_1} - \varphi_{y_1} \psi_{1x} \psi_{2y_2} \psi_{3y_3} + \varphi_{y_1} \psi_{1x} \psi_{2y_3} \psi_{3y_2} + \varphi_{y_1} \psi_{1y_2} \psi_{2x} \psi_{3y_3} - \varphi_{y_1} \psi_{1y_2} \psi_{2y_3} \psi_{3x} \\ & - \varphi_{y_1} \psi_{1y_3} \psi_{2x} \psi_{3y_2} + \varphi_{y_1} \psi_{1y_3} \psi_{2y_2} \psi_{3x} + \varphi_{y_2} \psi_{1x} \psi_{2y_1} \psi_{3y_3} - \varphi_{y_2} \psi_{1x} \psi_{2y_3} \psi_{3y_1} - \varphi_{y_2} \psi_{1y_1} \psi_{2x} \psi_{3y_3} \\ & + \varphi_{y_2} \psi_{1y_1} \psi_{2y_3} \psi_{3x} + \varphi_{y_2} \psi_{1y_3} \psi_{2x} \psi_{3y_1} - \varphi_{y_2} \psi_{1y_3} \psi_{2y_1} \psi_{3x} - \varphi_{y_3} \psi_{1x} \psi_{2y_1} \psi_{3y_2} + \varphi_{y_3} \psi_{1x} \psi_{2y_2} \psi_{3y_1} \\ & \left. + \varphi_{y_3} \psi_{1y_1} \psi_{2x} \psi_{3y_2} - \varphi_{y_3} \psi_{1y_1} \psi_{2y_2} \psi_{3x} - \varphi_{y_3} \psi_{1y_2} \psi_{2x} \psi_{3y_1} + \varphi_{y_3} \psi_{1y_2} \psi_{2y_1} \psi_{3x} \right) \neq 0. \end{aligned}$$

Proof. Applying a point transformation (3), one obtains the following transformation of the first-order derivatives

$$u'_1(t) = \frac{D_x \psi_1}{D_x \varphi} = \frac{\psi_{1x} + y'_1 \psi_{1y_1} + y'_2 \psi_{1y_2} + y'_3 \psi_{1y_3}}{\varphi_x + y'_1 \varphi_{y_1} + y'_2 \varphi_{y_2} + y'_3 \varphi_{y_3}} = g_1(x, y_1, y_2, y_3, y'_1, y'_2, y'_3),$$

$$u'_2(t) = \frac{D_x \psi_2}{D_x \varphi} = \frac{\psi_{2x} + y'_1 \psi_{2y_1} + y'_2 \psi_{2y_2} + y'_3 \psi_{2y_3}}{\varphi_x + y'_1 \varphi_{y_1} + y'_2 \varphi_{y_2} + y'_3 \varphi_{y_3}} = g_2(x, y_1, y_2, y_3, y'_1, y'_2, y'_3),$$

$$u'_3(t) = \frac{D_x \psi_3}{D_x \varphi} = \frac{\psi_{3x} + y'_1 \psi_{3y_1} + y'_2 \psi_{3y_2} + y'_3 \psi_{3y_3}}{\varphi_x + y'_1 \varphi_{y_1} + y'_2 \varphi_{y_2} + y'_3 \varphi_{y_3}} = g_3(x, y_1, y_2, y_3, y'_1, y'_2, y'_3).$$

The transformed second-order derivatives are

$$u''_1(t) = \frac{d^2 u_1}{dt^2} = \frac{D_x g_1}{D_x \varphi} \quad (41)$$

$$= \left(g_{1x} + y'_1 g_{1y_1} + y'_2 g_{1y_2} + y'_3 g_{1y_3} + y''_1 g_{1y'_1} + y''_2 g_{1y'_2} + y''_3 g_{1y'_3} \right) / \left(\varphi_x + y'_1 \varphi_{y_1} + y'_2 \varphi_{y_2} + y'_3 \varphi_{y_3} \right)$$

$$u''_2(t) = \frac{d^2 u_2}{dt^2} = \frac{D_x g_2}{D_x \varphi} \quad (42)$$

$$= \left(g_{2x} + y'_1 g_{2y_1} + y'_2 g_{2y_2} + y'_3 g_{2y_3} + y''_1 g_{2y'_1} + y''_2 g_{2y'_2} + y''_3 g_{2y'_3} \right) / \left(\varphi_x + y'_1 \varphi_{y_1} + y'_2 \varphi_{y_2} + y'_3 \varphi_{y_3} \right)$$

$$u''_3(t) = \frac{d^2 u_3}{dt^2} = \frac{D_x g_3}{D_x \varphi} \quad (43)$$

$$= \left(g_{3x} + y'_1 g_{3y_1} + y'_2 g_{3y_2} + y'_3 g_{3y_3} + y''_1 g_{3y'_1} + y''_2 g_{3y'_2} + y''_3 g_{3y'_3} \right) / \left(\varphi_x + y'_1 \varphi_{y_1} + y'_2 \varphi_{y_2} + y'_3 \varphi_{y_3} \right)$$

where

$$\begin{aligned} g_{1x} = & \left((\varphi_x + y'_1 \varphi_{y_1} + y'_2 \varphi_{y_2} + y'_3 \varphi_{y_3}) (\psi_{1xx} + y'_1 \psi_{1xy_1} + y'_2 \psi_{1xy_2} + y'_3 \psi_{1xy_3}) \right. \\ & \left. - (\psi_{1x} + y'_1 \psi_{1y_1} + y'_2 \psi_{1y_2} + y'_3 \psi_{1y_3}) (\varphi_{xx} + y'_1 \varphi_{xy_1} + y'_2 \varphi_{xy_2} + y'_3 \varphi_{xy_3}) \right) / \left(\varphi_x + y'_1 \varphi_{y_1} + y'_2 \varphi_{y_2} + y'_3 \varphi_{y_3} \right)^2, \\ g_{1y_1} = & \left((\varphi_x + y'_1 \varphi_{y_1} + y'_2 \varphi_{y_2} + y'_3 \varphi_{y_3}) (\psi_{1xy_1} + y'_1 \psi_{1y_1y_1} + y'_2 \psi_{1y_1y_2} + y'_3 \psi_{1y_1y_3}) \right. \\ & \left. - (\psi_{1x} + y'_1 \psi_{1y_1} + y'_2 \psi_{1y_2} + y'_3 \psi_{1y_3}) (\varphi_{xy_1} + y'_1 \varphi_{y_1y_1} + y'_2 \varphi_{y_1y_2} + y'_3 \varphi_{y_1y_3}) \right) / \left(\varphi_x + y'_1 \varphi_{y_1} + y'_2 \varphi_{y_2} + y'_3 \varphi_{y_3} \right)^2, \\ g_{1y_2} = & \left((\varphi_x + y'_1 \varphi_{y_1} + y'_2 \varphi_{y_2} + y'_3 \varphi_{y_3}) (\psi_{1xy_2} + y'_1 \psi_{1y_1y_2} + y'_2 \psi_{1y_2y_2} + y'_3 \psi_{1y_2y_3}) \right. \\ & \left. - (\psi_{1x} + y'_1 \psi_{1y_1} + y'_2 \psi_{1y_2} + y'_3 \psi_{1y_3}) (\varphi_{xy_2} + y'_1 \varphi_{y_1y_2} + y'_2 \varphi_{y_2y_2} + y'_3 \varphi_{y_2y_3}) \right) / \left(\varphi_x + y'_1 \varphi_{y_1} + y'_2 \varphi_{y_2} + y'_3 \varphi_{y_3} \right)^2, \\ g_{1y_3} = & \left((\varphi_x + y'_1 \varphi_{y_1} + y'_2 \varphi_{y_2} + y'_3 \varphi_{y_3}) (\psi_{1xy_3} + y'_1 \psi_{1y_1y_3} + y'_2 \psi_{1y_2y_3} + y'_3 \psi_{1y_3y_3}) \right. \\ & \left. - (\psi_{1x} + y'_1 \psi_{1y_1} + y'_2 \psi_{1y_2} + y'_3 \psi_{1y_3}) (\varphi_{xy_3} + y'_1 \varphi_{y_1y_3} + y'_2 \varphi_{y_2y_3} + y'_3 \varphi_{y_3y_3}) \right) / \left(\varphi_x + y'_1 \varphi_{y_1} + y'_2 \varphi_{y_2} + y'_3 \varphi_{y_3} \right)^2, \\ g_{1y'_1} = & \left((\varphi_x + y'_1 \varphi_{y_1} + y'_2 \varphi_{y_2} + y'_3 \varphi_{y_3}) \psi_{1y_1} \right. \\ & \left. - (\psi_{1x} + y'_1 \psi_{1y_1} + y'_2 \psi_{1y_2} + y'_3 \psi_{1y_3}) \varphi_{y_1} \right) / \left(\varphi_x + y'_1 \varphi_{y_1} + y'_2 \varphi_{y_2} + y'_3 \varphi_{y_3} \right)^2, \end{aligned}$$

$$\begin{aligned} g_{3y_2} &= \left((\varphi_x + y_1' \varphi_{y_1} + y_2' \varphi_{y_2} + y_3' \varphi_{y_3}) \psi_{3y_2} \right. \\ &\quad \left. - (\psi_{3x} + y_1' \psi_{3y_1} + y_2' \psi_{3y_2} + y_3' \psi_{3y_3}) \varphi_{y_2} \right) / \left(\varphi_x + y_1' \varphi_{y_1} + y_2' \varphi_{y_2} + y_3' \varphi_{y_3} \right)^2, \\ g_{3y_3} &= \left((\varphi_x + y_1' \varphi_{y_1} + y_2' \varphi_{y_2} + y_3' \varphi_{y_3}) \psi_{3y_3} \right. \\ &\quad \left. - (\psi_{3x} + y_1' \psi_{3y_1} + y_2' \psi_{3y_2} + y_3' \psi_{3y_3}) \varphi_{y_3} \right) / \left(\varphi_x + y_1' \varphi_{y_1} + y_2' \varphi_{y_2} + y_3' \varphi_{y_3} \right)^2, \end{aligned}$$

and

$$D_x = \frac{\partial}{\partial x} + y_1' \frac{\partial}{\partial y_1} + y_2' \frac{\partial}{\partial y_2} + y_3' \frac{\partial}{\partial y_3} + y_1'' \frac{\partial}{\partial y_1'} + y_2'' \frac{\partial}{\partial y_2'} + y_3'' \frac{\partial}{\partial y_3'} + \dots$$

is a total derivatives. Replacing $g_{1x}, g_{1y_1}, g_{1y_2}, g_{1y_3}, g_{1y_1'}, g_{1y_2'}, g_{1y_3'}, g_{2x}, g_{2y_1}, g_{2y_2}, g_{2y_3}, g_{2y_1'}, g_{2y_2'}, g_{2y_3'}, g_{3x}, g_{3y_1}, g_{3y_2}, g_{3y_3}, g_{3y_1'}, g_{3y_2'}, g_{3y_3}'$ into Equations (41)-(43), one gets the system (4). \square

3. Sufficient Conditions for Linearization

We have shown in the previous section that every linearizable system of three second-order ordinary differential equations belongs to the class of systems (4). In this section, we formulate the theorem containing sufficient conditions for linearization under the restricted class of point transformation

$$\begin{aligned} t &= \varphi(x), \\ u_1(t) &= \psi_1(x, y_1), \\ u_2(t) &= \psi_2(x, y_1, y_2), \\ u_3(t) &= \psi_3(x, y_1, y_2, y_3) \end{aligned} \quad (44)$$

We arrive at the following theorem.

Theorem 2 System (4) is linearizable by restricted class of point transformation (44) if and only if its coefficients satisfied the following equations

$$\begin{aligned} a_{11} = 0, a_{12} = 0, a_{13} = 0, a_{15} = 0, a_{16} = 0, a_{17} = 0, a_{18} = 0, a_{19} = 0, \\ a_{111} = 0, a_{112} = 0, a_{113} = 0, a_{114} = 0, a_{115} = 0, a_{29} = 0, a_{212} = 0, a_{214} = 0, a_{215} = 0, \end{aligned} \quad (45)$$

$$\begin{aligned} a_{14y_2} = 0, a_{14y_3} = 0, a_{110y_2} = 0, a_{110y_3} = 0, \\ a_{116y_2} = 0, a_{116y_3} = 0, a_{24y_3} = 0, a_{27y_3} = 0, \end{aligned} \quad (46)$$

$$a_{210y_3} = 0, a_{211y_3} = 0, a_{213y_3} = 0, a_{216y_3} = 0,$$

$$a_{110x} = \left(4a_{116y_1} + 2a_{310x} - 4a_{316y_3} + a_{110}^2 - 4a_{116}a_{14} + 2a_{116}a_{37} + 2a_{216}a_{39} - a_{310}^2 + 4a_{316}a_{34} \right) / 2, \quad (47)$$

$$a_{110y_1} = 2a_{14x}, \quad (48)$$

$$a_{116y_1y_1} = a_{116y_1}a_{14} + a_{14xx} - a_{14x}a_{110} + a_{14y_1}a_{116}, \quad (49)$$

$$a_{24y_1} = a_{27y_2} / 2, \quad (50)$$

$$a_{210x} = \left(4a_{216y_2} + 2a_{310x} - 4a_{316y_3} - 2a_{116}a_{27} + 2a_{116}a_{37} + a_{210}^2 - 4a_{216}a_{24} + 2a_{216}a_{39} - a_{310}^2 + 4a_{316}a_{34} \right) / 2, \quad (51)$$

$$a_{210y_1} = a_{27x}, \quad (52)$$

$$a_{210y_2} = 2a_{24x}, \quad (53)$$

$$a_{211x} = \left(2a_{213y_1} - 2a_{110}a_{211} + 2a_{14}a_{213} + 2a_{210}a_{211} - a_{213}a_{27} \right) / 4, \quad (54)$$

$$a_{211y_2} = \left(2a_{27y_1} + 2a_{14}a_{27} + 4a_{211}a_{24} - a_{27}^2 \right) / 4, \quad (55)$$

$$a_{213x} = (4a_{216y_1} + a_{110}a_{213} - 4a_{116}a_{211} + a_{210}a_{213} - 2a_{216}a_{27})/2, \quad (56)$$

$$a_{213y_2} = (2a_{27x} + a_{110}a_{27} - a_{210}a_{27} + 2a_{213}a_{24})/2, \quad (57)$$

$$a_{216y_1y_2} = (a_{116y_1}a_{27} + 2a_{216y_1}a_{24} + a_{27xx} - a_{27x}a_{210} + a_{27y_1}a_{116} + a_{27y_2}a_{216})/2, \quad (58)$$

$$a_{216y_2y_2} = (2a_{216y_2}a_{24} + 2a_{24xx} - 2a_{24x}a_{210} + 2a_{24y_2}a_{216} + a_{27y_2}a_{116})/2, \quad (59)$$

$$a_{34y_1} = a_{37y_3}/2, \quad (60)$$

$$a_{34y_2} = a_{39y_3}/2, \quad (61)$$

$$a_{37y_2} = a_{39y_1}, \quad (62)$$

$$a_{310y_1} = a_{37x}, \quad (63)$$

$$a_{310y_2} = a_{39x}, \quad (64)$$

$$a_{310y_3} = 2a_{34x}, \quad (65)$$

$$a_{311x} = (2a_{313y_1} - 2a_{110}a_{311} + 2a_{14}a_{313} + 2a_{211}a_{315} - a_{213}a_{312} + 2a_{310}a_{311} - a_{313}a_{37})/4, \quad (66)$$

$$a_{311y_2} = (2a_{312y_1} + 2a_{14}a_{312} + 4a_{211}a_{314} - a_{27}a_{312} + 2a_{311}a_{39} - a_{312}a_{37})/4, \quad (67)$$

$$a_{311y_3} = (2a_{37y_1} + 2a_{14}a_{37} + 2a_{211}a_{39} + 4a_{311}a_{34} - a_{37}^2)/4, \quad (68)$$

$$a_{312x} = (2a_{315y_1} - a_{210}a_{312} + a_{27}a_{315} + a_{310}a_{312} - a_{315}a_{37})/2, \quad (69)$$

$$a_{312y_2} = (4a_{314y_1} - 2a_{24}a_{312} + 2a_{27}a_{314} + a_{312}a_{39} - 2a_{314}a_{37})/2, \quad (70)$$

$$a_{312y_3} = (2a_{39y_1} + a_{27}a_{39} + 2a_{312}a_{34} - a_{37}a_{39})/2, \quad (71)$$

$$a_{313x} = (4a_{316y_1} + a_{110}a_{313} - 4a_{116}a_{311} + a_{213}a_{315} - 2a_{216}a_{312} + a_{310}a_{313} - 2a_{316}a_{37})/2, \quad (72)$$

$$a_{313y_2} = (2a_{315y_1} + a_{110}a_{312} - a_{210}a_{312} + 2a_{213}a_{314} + a_{313}a_{39} - a_{315}a_{37})/2, \quad (73)$$

$$a_{313y_3} = (2a_{37x} + a_{110}a_{37} + a_{213}a_{39} - a_{310}a_{37} + 2a_{313}a_{34})/2, \quad (74)$$

$$a_{314x} = (2a_{315y_2} - 2a_{210}a_{314} + 2a_{24}a_{315} + 2a_{310}a_{314} - a_{315}a_{39})/4, \quad (75)$$

$$a_{314y_3} = (2a_{39y_2} + 2a_{24}a_{39} + 4a_{314}a_{34} - a_{39}^2)/4, \quad (76)$$

$$a_{315x} = (4a_{316y_2} - 2a_{116}a_{312} + a_{210}a_{315} - 4a_{216}a_{314} + a_{310}a_{315} - 2a_{316}a_{39})/2, \quad (77)$$

$$a_{315y_3} = (2a_{39x} + a_{210}a_{39} - a_{310}a_{39} + 2a_{315}a_{34})/2, \quad (78)$$

$$a_{316y_1y_3} = (a_{116y_1}a_{37} + a_{216y_1}a_{39} + 2a_{316y_1}a_{34} + a_{37xx} - a_{37x}a_{310} + a_{37y_1}a_{116} + a_{37y_3}a_{316} + a_{39y_1}a_{216})/2, \quad (79)$$

$$a_{316y_2y_3} = (a_{216y_2}a_{39} + 2a_{316y_2}a_{34} + a_{39xx} - a_{39x}a_{310} + a_{39y_1}a_{116} + a_{39y_2}a_{216} + a_{39y_3}a_{316})/2, \quad (80)$$

$$a_{316y_3y_3} = (2a_{316y_3}a_{34} + 2a_{34x} - 2a_{34x}a_{310} + 2a_{34y_3}a_{316} + a_{37y_3}a_{116} + a_{39y_3}a_{216})/2. \quad (81)$$

Proof. Since $\varphi_{y_1}, \varphi_{y_2}, \varphi_{y_3}, \psi_{1y_2}, \psi_{1y_3}$ and ψ_{2y_3} are 0, then $\Delta = \varphi_x \psi_{3y_3} v \neq 0$ where $v = \psi_{1y_1} \psi_{2y_2}$. Considering equation $(\psi_{2y_2})_{y_3} = 0$ one obtains $v_{y_3} = 0$. From Equations (5)-(7), (9)-(13), (15)-(19), (23), (26) and (28)-(29) one gets the conditions

$$a_{11} = a_{12} = a_{13} = a_{15} = a_{16} = a_{17} = a_{18} = a_{19} = a_{111} = 0,$$

$$a_{112} = a_{113} = a_{114} = a_{115} = a_{29} = a_{212} = a_{214} = a_{215} = 0.$$

The results from Equations (31)-(34), (8), (21), (22), (25), (27), (38), (36),(37), (14), (20), (24), (37), (39), (30) and (40) can be rewritten as

$$\begin{aligned} \varphi_{xx} &= \varphi_x (2v_x + a_{110}v + a_{210}v)/(2v), \\ \psi_{1xx} &= (2\psi_{1x}v_x + \psi_{1x}a_{110}v + \psi_{1x}a_{210}v - 2\psi_{1y_1}a_{116}v)/(2v), \\ \psi_{1xy_1} &= \psi_{1y_1} (2v_x - a_{110}v + a_{210}v)/(4v), \\ \psi_{1y_1y_1} &= -\psi_{1y_1}a_{14}, \\ \psi_{2xx} &= (2\psi_{1y_1}\psi_{2x}v_x + \psi_{1y_1}\psi_{2x}a_{110}v + \psi_{1y_1}\psi_{2x}a_{210}v - 2\psi_{1y_1}\psi_{2y_1}a_{116}v - 2a_{216}v^2)/(2\psi_{1y_1}v), \\ \psi_{2xy_1} &= (2\psi_{1y_1}\psi_{2y_1}v_x - \psi_{1y_1}\psi_{2y_1}a_{110}v + \psi_{1y_1}\psi_{2y_1}a_{210}v - 2a_{213}v^2)/(4\psi_{1y_1}v), \\ \psi_{2y_1y_1} &= -(\psi_{1y_1}\psi_{2y_1}a_{14} + a_{211}v)/\psi_{1y_1}, \\ \psi_{3xx} &= (2\psi_{3x}v_x + \psi_{3x}a_{110}v + \psi_{3x}a_{210}v - 2\psi_{3y_1}a_{116}v - 2\psi_{3y_2}a_{216}v - 2\psi_{3y_3}a_{316}v)/(2v), \\ \psi_{3xy_1} &= (2\psi_{3y_1}v_x - \psi_{3y_1}a_{110}v + \psi_{3y_1}a_{210}v - 2\psi_{3y_2}a_{213}v - 2\psi_{3y_3}a_{313}v)/(4v), \\ \psi_{3xy_2} &= (2\psi_{3y_2}v_x + \psi_{3y_2}a_{110}v - \psi_{3y_2}a_{210}v - 2\psi_{3y_3}a_{315}v)/(4v), \\ \psi_{3xy_3} &= (\psi_{3y_3} (2v_x + a_{110}v + a_{210}v - 2a_{310}v))/(4v), \\ \psi_{3y_1y_1} &= -(\psi_{3y_1}a_{14} + \psi_{3y_2}a_{211} + \psi_{3y_3}a_{311}), \\ \psi_{3y_1y_2} &= -(\psi_{3y_2}a_{27} + \psi_{3y_3}a_{312})/2, \\ \psi_{3y_1y_3} &= -\psi_{3y_3}a_{37}/2, \\ \psi_{3y_2y_2} &= -(\psi_{3y_2}a_{24} + \psi_{3y_3}a_{314}), \\ \psi_{3y_2y_3} &= -\psi_{3y_3}a_{39}/2, \\ \psi_{3y_3y_3} &= -\psi_{3y_3}a_{34}, \\ v_{y_1} &= -v(2a_{14} + a_{27})/2, \\ v_{y_2} &= -a_{24}v. \end{aligned}$$

The relation $(v_{y_2})_{y_3} = 0, (\psi_{1y_1y_1})_{y_2} = 0, (\psi_{1y_1y_1})_{y_3} = 0, (\psi_{2y_1y_1})_{y_3} = 0, (v_{y_1})_{y_3} = 0, (\varphi_{xx})_{y_3} = 0, (\psi_{1xy_1})_{y_3} = 0,$
 $(\psi_{1xx})_{y_3} = 0, (\psi_{2xy_1})_{y_3} = 0, (\psi_{2xx})_{y_3} = 0, (\varphi_{xx})_{y_2} = 0, (\varphi_{xx})_{y_1} = 0, (\psi_{1xy_1})_{y_2} = 0, (\psi_{1xx})_{y_2} = 0$ yield the conditions

$$\begin{aligned} a_{24y_3} &= 0, a_{14y_2} = 0, a_{14y_3} = 0, a_{211y_3} = 0, a_{27y_3} = 0, \\ a_{110y_3} &= -a_{210y_3}, a_{210y_3} = 0, a_{116y_3} = 0, a_{213y_3} = 0, a_{216y_3} = 0, \\ a_{110y_2} &= -a_{210y_2} + 2a_{24x}, a_{110y_1} = 2a_{14x} - a_{210y_1} + a_{27x}, \\ a_{110y_1} &= 2a_{14x} - a_{210y_1} + a_{27x}, a_{210y_2} = 2a_{24x}, a_{116y_2} = 0, \end{aligned}$$

respectively. Comparing the mixed derivatives

$$\begin{aligned} (\psi_{3xy_3})_{y_3} &= (\psi_{3y_3y_3})_x, (v_{y_1})_{y_2} = (v_{y_2})_{y_1}, (\psi_{3y_2y_3})_{y_3} = (\psi_{3y_3y_3})_{y_2}, \\ (\psi_{3y_1y_3})_{y_2} &= (\psi_{3y_2y_3})_{y_1}, (\psi_{3y_1y_3})_{y_3} = (\psi_{3y_3y_3})_{y_1}, (\psi_{1xy_1})_{y_1} = (\psi_{1y_1y_1})_x, \\ (\psi_{3xy_3})_{y_1} &= (\psi_{3y_1y_3})_x, (\psi_{3xy_3})_{y_2} = (\psi_{3y_2y_3})_x \end{aligned}$$

one arrives at the following conditions

$$a_{310y_3} = 2a_{34x}, a_{24y_1} = \frac{a_{27y_2}}{2}, a_{34y_2} = \frac{a_{39y_3}}{2}, a_{37y_2} = a_{39y_1},$$

$$a_{34y_1} = \frac{a_{37y_3}}{2}, a_{210y_1} = a_{27x}, a_{310y_1} = a_{37x}, a_{310y_2} = a_{39x},$$

respectively. The equations $(\psi_{2y_2})_{x_1} = (\psi_{2x_1})_{y_2}$,

$$(\psi_{2y_2})_{y_1y_1} = (\psi_{2y_1y_1})_{y_2}, (\psi_{2x_1})_{y_1} = (\psi_{2y_1y_1})_x, (\psi_{3x_1})_{y_3} = (\psi_{3x_1y_3})_{y_1},$$

$$(\psi_{3x_2})_{y_3} = (\psi_{3x_2y_3})_{y_2}, (\psi_{3x_2})_{y_1} = (\psi_{3y_1y_2})_x, (\psi_{3x_2})_{y_2} = (\psi_{3y_2y_2})_x,$$

$$(\psi_{3y_1y_1})_{y_3} = (\psi_{3y_1y_3})_{y_1}, (\psi_{3y_1y_2})_{y_3} = (\psi_{3y_1y_3})_{y_2}, (\psi_{3y_1y_2})_{y_2} = (\psi_{3y_2y_2})_{y_1},$$

$$(\psi_{3y_2y_2})_{y_3} = (\psi_{3y_2y_3})_{y_2}, (\psi_{3x_1})_{y_2} = (\psi_{3x_2})_{y_1}, (\psi_{3x_1})_{y_1} = (\psi_{3y_1y_1})_x,$$

$$(\psi_{3y_1y_1})_{y_2} = (\psi_{3y_1y_2})_{y_1}, (\psi_{1xx})_{y_1} = (\psi_{2x_1})_x, (\psi_{1xx})_{y_1y_1} = (\psi_{2y_1y_1})_{xx},$$

$$(\psi_{3xx})_{y_2} = (\psi_{3x_2})_x, (\psi_{3xx})_{y_3} = (\psi_{3x_3})_x, (\psi_{2xx})_{y_1} = (\psi_{2x_1})_x,$$

$$(\psi_{3xx})_{y_1} = (\psi_{3x_1})_x, (\psi_{3xx})_{y_1y_3} = (\psi_{3y_1y_3})_{xx}, (\psi_{3xx})_{y_2y_3} = (\psi_{3y_2y_3})_{xx},$$

$(\psi_{3xx})_{y_3y_3} = (\psi_{3y_3y_3})_{xx}$, provide the following conditions:

$$2a_{213y_2} = 2a_{27x} + a_{110}a_{27} - a_{210}a_{27} + 2a_{213}a_{24},$$

$$4a_{211y_2} = 2a_{27y_1} + 2a_{14}a_{27} + 4a_{211}a_{24}v - a_{27}^2,$$

$$4a_{211x} = 2a_{213y_1} - 2a_{110}a_{211} + 2a_{14}a_{213} + 2a_{210}a_{211} - a_{213}a_{27},$$

$$2a_{313y_3} = 2a_{37x} + a_{110}a_{37} + a_{213}a_{39} - a_{310}a_{37} + 2a_{313}a_{34},$$

$$2a_{315y_3} = 2a_{39x} + a_{210}a_{39} - a_{310}a_{39} + 2a_{315}a_{34},$$

$$2a_{312x} = 2a_{315y_1} - a_{210}a_{312} + a_{27}a_{315} + a_{310}a_{312} - a_{315}a_{37},$$

$$4a_{314x} = 2a_{315y_2} - 2a_{210}a_{314} + 2a_{24}a_{315} + 2a_{310}a_{314} - a_{315}a_{39},$$

$$4a_{311y_3} = 2a_{37y_1} + 2a_{14}a_{37} + 2a_{211}a_{39} + 4a_{311}a_{34} - a_{37}^2,$$

$$2a_{312y_3} = 2a_{39y_1} + a_{27}a_{39} + 2a_{312}a_{34} - a_{37}a_{39},$$

$$2a_{312y_2} = 4a_{314y_1} - 2a_{24}a_{312} + 2a_{27}a_{314} + a_{312}a_{39} - 2a_{314}a_{37},$$

$$4a_{314y_3} = 2a_{39y_2} + 2a_{24}a_{39} + 4a_{314}a_{34} - a_{39}^2,$$

$$2a_{313y_2} = 2a_{315y_1} + a_{110}a_{312} - a_{210}a_{312} + 2a_{213}a_{314} + a_{313}a_{39} - a_{315}a_{37},$$

$$4a_{311x} = 2a_{313y_1} - 2a_{110}a_{311} + 2a_{14}a_{313} + 2a_{211}a_{315} - a_{213}a_{312} + 2a_{310}a_{311} - a_{313}a_{37},$$

$$4a_{311y_2} = 2a_{312y_1} + 2a_{14}a_{312} + 4a_{211}a_{314} - a_{27}a_{312} + 2a_{311}a_{39} - a_{312}a_{37},$$

$$2a_{110x} = 4a_{116y_1} + 2a_{210x} - 4a_{216y_2} + a_{110}^2 - 4a_{116}a_{14} + 2a_{116}a_{27} - a_{210}^2 + 4a_{216}a_{24},$$

$$a_{116y_1y_1} = a_{116y_1}a_{14} + a_{14xx} - a_{14x}a_{110} + a_{14y_1}a_{116},$$

$$2a_{315x} = 4a_{316y_2} - 2a_{116}a_{312} + a_{210}a_{315} - 4a_{216}a_{314} + a_{310}a_{315} - 2a_{316}a_{39},$$

$$2a_{210x} = 4a_{216y_2} + 2a_{310x} - 4a_{316y_3} - 2a_{116}a_{27} + 2a_{116}a_{37} + a_{210}^2 - 4a_{216}a_{24} + 2a_{216}a_{39} - a_{310}^2 + 4a_{316}a_{34},$$

$$\begin{aligned}
 2a_{213x} &= 4a_{216y_1} + a_{110}a_{213} - 4a_{116}a_{211} + a_{210}a_{213} - 2a_{216}a_{27}, \\
 2a_{313x} &= 4a_{316y_1} + a_{110}a_{313} - 4a_{116}a_{311} + a_{213}a_{315} - 2a_{216}a_{312} + a_{310}a_{313} - 2a_{316}a_{37}, \\
 a_{316y_1y_3} &= a_{116y_1}a_{37} + a_{216y_1}a_{39} + 2a_{316y_1}a_{34} + a_{37xx} - a_{37x}a_{310} + a_{37y_1}a_{116} + a_{37y_3}a_{316} + a_{39y_1}a_{216}, \\
 2a_{316y_2y_3} &= a_{216y_2}a_{39} + 2a_{316y_2}a_{34} + a_{39x} - a_{39x}a_{310} + a_{39y_1}a_{116} + a_{39y_2}a_{216} + a_{39y_3}a_{316}, \\
 2a_{316y_3y_3} &= 2a_{316y_3}a_{34} + 2a_{34xx} - 2a_{34x}a_{310} + 2a_{34y_3}a_{316} + a_{37y_3}a_{116} + a_{39y_3}a_{216}.
 \end{aligned}$$

Mixing the derivative $(\psi_{2y_2})_{xx} = (\psi_{2xx})_{y_2}$, one obtains the derivative

$$\begin{aligned}
 v_{xx} &= (12v_x^2 + 4v_x v(a_{110} + a_{210}) \\
 &\quad - v^2(4a_{110x} - 4a_{210x} + 16a_{216y_2} - a_{110}^2 - 2a_{110}a_{210} - 8a_{116}a_{27} + 3a_{210}^2 - 16a_{216}a_{24})) / (8v).
 \end{aligned}$$

Comparing the mixed derivatives $(v_{y_1})_{xx} = (v_{xx})_{y_1}$, $(v_{y_2})_{xx} = (v_{xx})_{y_2}$ one gets the conditions

$$\begin{aligned}
 2a_{216y_1y_2} &= a_{116y_1}a_{27} + 2a_{216y_1}a_{24} + a_{27xx} - a_{27x}a_{210} + a_{27y_1}a_{116} + a_{27y_2}a_{216}, \\
 2a_{216y_2y_2} &= 2a_{216y_2}a_{24} + 2a_{24xx} - 2a_{24x}a_{210} + 2a_{24y_2}a_{216} + a_{27y_2}a_{116}.
 \end{aligned}$$

All obtained results satisfied the equations $(\psi_{2y_2})_{y_3} = 0$,

$$\begin{aligned}
 (\psi_{3xx})_{y_1y_1} &= (\psi_{3y_1y_1})_{xx}, (\psi_{3xx})_{y_1y_2} = (\psi_{3y_1y_2})_{xx}, (\psi_{3xx})_{y_2y_2} = (\psi_{3y_2y_2})_{xx}, (\psi_{3xy_1})_{y_2} = (\psi_{3y_1y_2})_x, \\
 (\psi_{3xy_1})_{y_3} &= (\psi_{3y_1y_3})_x, (\psi_{3xy_1})_{y_2y_2} = (\psi_{3y_2y_2})_{xy_1}, (\psi_{3xy_1})_{y_2y_3} = (\psi_{3y_2y_3})_{xy_1}, (\psi_{3xy_1})_{y_3y_3} = (\psi_{3y_3y_3})_{xy_1}, \\
 (\psi_{3xy_2})_{y_1y_3} &= (\psi_{3y_1y_3})_{xy_2}, (\psi_{3xy_2})_{y_3} = (\psi_{3y_2y_3})_x, (\psi_{3xy_2})_{y_3y_3} = (\psi_{3y_3y_3})_{xy_2}, (\psi_{3xy_3})_{y_1y_1} = (\psi_{3y_1y_1})_{xy_3}, \\
 (\psi_{3xy_3})_{y_1y_2} &= (\psi_{3y_1y_2})_{xy_3}, (\psi_{3xy_3})_{y_2y_2} = (\psi_{3y_2y_2})_{xy_3}, (\psi_{3y_1y_1})_{y_2y_2} = (\psi_{3y_2y_2})_{y_1y_1}, (\psi_{3y_1y_1})_{y_2y_3} = (\psi_{3y_2y_3})_{y_1y_1}, \\
 (\psi_{3y_1y_1})_{y_3y_3} &= (\psi_{3y_3y_3})_{y_1y_1}, (\psi_{3y_1y_2})_{y_3} = (\psi_{3y_2y_3})_{y_1}, (\psi_{3y_1y_2})_{y_3y_3} = (\psi_{3y_3y_3})_{y_1y_2}, (\psi_{3y_1y_3})_{y_2y_2} = (\psi_{3y_2y_2})_{y_1y_3}, \\
 (\psi_{3y_2y_2})_{y_3y_3} &= (\psi_{3y_3y_3})_{y_2y_2}, (\psi_{3y_2y_3})_{y_3} = (\psi_{3y_3y_3})_{y_2}, (v_{xx})_{y_3} = 0.
 \end{aligned}$$

□

4. Linearizing Transformation

By the prove of Theorem 2, we arrive at the following corollary.

Corollary 3. Provided that the sufficient conditions in Theorem 2 are satisfied, the transformation (44) mapping system (4) to a linear system (2) is obtained by solving the following compatible system of equations for the functions $\varphi(x), \psi_1(x, y_1), \psi_2(x, y_1, y_2)$ and $\psi_3(x, y_1, y_2, y_3)$:

$$v_{y_1} = -v(2a_{14} + a_{27})/2, \tag{82}$$

$$v_{y_2} = -va_{24}, \tag{83}$$

$$\begin{aligned}
 v_{xx} &= (12v_x^2 + 4v_x v(a_{110} + a_{210}) \\
 &\quad + v^2(-8a_{116y_1} - 8a_{216y_2}a_{110}^2 + 2a_{110}a_{210} + 8a_{116}a_{14} + 4a_{116}a_{27} - a_{210}^2 + 8a_{216}a_{24})) / (8v),
 \end{aligned}
 \tag{84}$$

$$\varphi_{xx} = (2\varphi_x v_x + \varphi_x v(a_{110} + a_{210})) / (2v), \tag{85}$$

$$\psi_{1xx} = (2\psi_{1x} v_x + \psi_{1x} v(a_{110} + a_{210}) - 2v\psi_{1y_1} a_{116}) / (2v), \tag{86}$$

$$\psi_{1xy_1} = (2v_x \psi_{1y_1} + v\psi_{1y_1}(-a_{110} + a_{210})) / (4v), \tag{87}$$

$$\psi_{1y_1y_1} = -\psi_{1y_1} a_{14}, \tag{88}$$

$$\psi_{2y_2} = v/\psi_{1y_1}, \quad (89)$$

$$\psi_{2xx} = (2\psi_{2x} v_x \psi_{1y_1} + \psi_{2x} v \psi_{1y_1} (a_{110} + a_{210}) - 2\psi_{2y_1} v \psi_{1y_1} a_{116} - 2v^2 a_{216}) / (2v \psi_{1y_1}), \quad (90)$$

$$\psi_{2xy_1} = (2\psi_{2y_1} v_x \psi_{1y_1} + \psi_{2y_1} v \psi_{1y_1} (-a_{110} + a_{210}) - 2v^2 a_{213}) / (4v \psi_{1y_1}), \quad (91)$$

$$\psi_{2y_1y_1} = (-\psi_{2y_1} \psi_{1y_1} a_{14} - v a_{211}) / \psi_{1y_1}, \quad (92)$$

$$\psi_{3xx} = (2\psi_{3x} v_x + \psi_{3x} v (a_{110} + a_{210}) - 2\psi_{3y_1} v a_{116} - 2\psi_{3y_2} v a_{216} - 2\psi_{3y_3} v a_{316}) / (2v), \quad (93)$$

$$\psi_{3xy_1} = (2\psi_{3y_1} v_x + \psi_{3y_1} v (-a_{110} + a_{210}) - 2\psi_{3y_2} v a_{213} - 2\psi_{3y_3} v a_{313}) / (4v), \quad (94)$$

$$\psi_{3xy_2} = (2\psi_{3y_2} v_x + \psi_{3y_2} v (a_{110} - a_{210}) - 2\psi_{3y_3} v a_{315}) / (4v), \quad (95)$$

$$\psi_{3xy_3} = (2\psi_{3y_3} v_x + \psi_{3y_3} v (a_{110} + a_{210} - 2a_{310})) / (4v), \quad (96)$$

$$\psi_{3y_1y_1} = -\psi_{3y_1} a_{14} - \psi_{3y_2} a_{211} - \psi_{3y_3} a_{311}, \quad (97)$$

$$\psi_{3y_1y_2} = (-\psi_{3y_2} a_{27} - \psi_{3y_3} a_{312}) / 2, \quad (98)$$

$$\psi_{3y_1y_3} = (-\psi_{3y_3} a_{37}) / 2, \quad (99)$$

$$\psi_{3y_2y_2} = -\psi_{3y_2} a_{24} - \psi_{3y_3} a_{314}, \quad (100)$$

$$\psi_{3y_2y_3} = (-\psi_{3y_3} a_{39}) / 2, \quad (101)$$

$$\psi_{3y_3y_3} = -\psi_{3y_3} a_{34}. \quad (102)$$

5. Examples

Example 1. Consider the system of nonlinear ordinary differential equation.

$$\begin{aligned} y_1'' + \frac{1}{2} y_1'^2 + y_1' &= 0, \\ y_2'' + \frac{1}{2} y_2'^2 + y_1' y_2' + y_2' &= 0, \\ y_3'' + y_3'^2 + y_1' y_3' + y_2' y_3' + y_3' &= 0. \end{aligned} \quad (103)$$

It is a system of the form (4) with the coefficients

$$\begin{aligned} a_{11} &= 0, a_{12} = 0, a_{13} = 0, a_{14} = -(1/2), a_{15} = 0, a_{16} = 0, a_{17} = 0, a_{18} = 0, a_{19} = 0, \\ a_{110} &= -1, a_{111} = 0, a_{112} = 0, a_{113} = 0, a_{114} = 0, a_{115} = 0, a_{116} = 0, \\ a_{24} &= -(1/2), a_{27} = -1, a_{29} = 0, \\ a_{210} &= -1, a_{211} = 0, a_{212} = 0, a_{213} = 0, a_{214} = 0, a_{215} = 0, a_{216} = 0, \\ a_{34} &= -1, a_{37} = -1, a_{39} = -1, \\ a_{310} &= -1, a_{311} = 0, a_{312} = 0, a_{313} = 0, a_{314} = 0, a_{315} = 0, a_{316} = 0. \end{aligned} \quad (104)$$

One can check that the coefficients (104) obey the conditions (45)-(81). Thus, the Equation (103) is linearizable. We have

$$v_{y_1} = v, \quad (105)$$

$$v_{y_2} = v/2, \quad (106)$$

$$v_{xx} = v_x (3v_x - 2v) / (2v), \quad (107)$$

$$\varphi_{xx} = \varphi_x (v_x - v) / v, \quad (108)$$

$$\psi_{1xx} = \psi_{1x} (v_x - v) / v, \quad (109)$$

$$\psi_{1x_1} = (\psi_{1y_1} v_x) / (2v), \quad (110)$$

$$\psi_{1y_1y_1} = \psi_{1y_1} / 2, \quad (111)$$

$$\psi_{2y_2} = v / \psi_{1y_1}, \quad (112)$$

$$\psi_{2xx} = \psi_{2x} (v_x - v) / v, \quad (113)$$

$$\psi_{2x_1} = (\psi_{2y_1} v_x) / (2v), \quad (114)$$

$$\psi_{2y_1y_1} = \psi_{2y_1} / 2, \quad (115)$$

$$\psi_{3xx} = \psi_{3x} (v_x - v) / (v), \quad (116)$$

$$\psi_{3x_1} = (\psi_{3y_1} v_x) / (2v), \quad (117)$$

$$\psi_{3x_2} = (\psi_{3y_2} v_x) / (2v), \quad (118)$$

$$\psi_{3x_3} = (\psi_{3y_3} v_x) / (2v), \quad (119)$$

$$\psi_{3y_1y_1} = \psi_{3y_1} / 2, \quad (120)$$

$$\psi_{3y_1y_2} = \psi_{3y_2} / 2, \quad (121)$$

$$\psi_{3y_1y_3} = \psi_{3y_3} / 2, \quad (122)$$

$$\psi_{3y_2y_2} = \psi_{3y_2} / 2, \quad (123)$$

$$\psi_{3y_2y_3} = \psi_{3y_3} / 2, \quad (124)$$

$$\psi_{3y_3y_3} = \psi_{3y_3}. \quad (125)$$

From (105) and (106) we get

$$v = e^{y_1 + c_1(x, y_2)}$$

and

$$v = e^{\frac{y_2}{2} + c_2(x, y_1)},$$

respectively. Since one can use any particular solution, we can take

$$v = e^{2x + y_1 + \frac{y_2}{2}},$$

and this solution satisfies (107). Now the Equation (108) becomes

$$\varphi_{xx} = \varphi_x$$

and yields

$$\varphi_x = e^{x+c_3}, c_3 = \text{const.}$$

Therefore

$$\varphi = e^{x+c_3} + c_4.$$

Since one can use any particular solution, we set $c_3 = 0, c_4 = 0$ and take

$$\varphi = e^x.$$

Now the Equations (109)-(111) are written

$$\psi_{1xx} = \psi_{1x}, \quad (126)$$

$$\psi_{1x_1} = \psi_{1y_1}, \quad (127)$$

$$\psi_{1y_1y_1} = \frac{\psi_{1y_1}}{2}. \quad (128)$$

To consider (127) and (128), one takes

$$\psi_{1y_1} = e^{\frac{x+y_1}{2}},$$

then

$$\psi_1 = 2e^{\frac{x+y_1}{2}} + c_5(x).$$

Since one can use any particular solution, we set $c_5(x) = 0$ and take

$$\psi_1 = 2e^{\frac{x+y_1}{2}},$$

this solution satisfies (126). Now the Equations (112)-(115) are written as

$$\psi_{2y_2} = \frac{e^{\frac{x+y_1+y_2}{2}}}{e^{\frac{y_1}{2}}}, \quad (129)$$

$$\psi_{2xx} = \psi_{2x}, \quad (130)$$

$$\psi_{2xy_1} = \psi_{2y_1}, \quad (131)$$

$$\psi_{2y_1y_1} = \frac{\psi_{2y_1}}{2}. \quad (132)$$

To consider (129), after integration, one finds

$$\psi_2 = 2e^{\frac{x+y_1+y_2}{2}} + c_6(x, y_1).$$

Since one can use any particular solution, we set $c_6(x, y_1) = 0$ and take

$$\psi_2 = 2e^{\frac{x+y_1+y_2}{2}},$$

this solution satisfies (130)-(132). Now the Equations (116)-(125) are written

$$\psi_{3xx} = \psi_{3x}, \quad (133)$$

$$\psi_{3xy_1} = \psi_{3y_1}, \quad (134)$$

$$\psi_{3xy_2} = \psi_{3y_2}, \quad (135)$$

$$\psi_{3xy_3} = \psi_{3y_3}, \quad (136)$$

$$\psi_{3y_1y_1} = \frac{\psi_{3y_1}}{2}, \quad (137)$$

$$\psi_{3y_1y_2} = \frac{\psi_{3y_2}}{2}, \quad (138)$$

$$\psi_{3y_1y_3} = \frac{\psi_{3y_3}}{2}, \quad (139)$$

$$\psi_{3y_2y_2} = \frac{\psi_{3y_2}}{2}, \quad (140)$$

$$\psi_{3y_2y_3} = \frac{\psi_{3y_3}}{2}, \quad (141)$$

$$\psi_{3y_3y_3} = \psi_{3y_3}. \quad (142)$$

To consider (136), (139), (141) and (142), one obtains

$$\psi_{3y_3} = e^{\frac{x+y_1+y_2}{2}+y_3}$$

so that

$$\psi_3 = e^{\frac{x+y_1+y_2}{2}+y_3} + c_7(x, y_1, y_2).$$

Since one can use any particular solution, we set $c_7(x, y_1, y_2) = 0$ and take

$$\psi_3 = e^{\frac{x+y_1+y_2}{2}+y_3},$$

this solution satisfies (133)-(135), (137)-(138) and (140). Then, one obtains the following transformations

$$t = e^x, u_1(t) = 2e^{\frac{x+y_1}{2}}, u_2(t) = 2e^{\frac{x+y_1+y_2}{2}}, u_3(t) = e^{\frac{x+y_1+y_2}{2}+y_3}. \quad (143)$$

Hence, the system (103) is mapped by the transformations (143) to the linear system

$$u_1''(t) = 0, u_2''(t) = 0, u_3''(t) = 0.$$

The solution of this linear system is

$$u_1(t) = c_1t + c_2,$$

$$u_2(t) = c_3t + c_4,$$

$$u_3(t) = c_5t + c_6$$

where $c_i, (i = 1, 2, \dots, 6)$ are arbitrary constants. By using the transformation (143), one finds

$$2e^{\frac{x+y_1}{2}} = c_1e^x + c_2, 2e^{\frac{x+y_1+y_2}{2}} = c_3e^x + c_4, e^{\frac{x+y_1+y_2}{2}+y_3} = c_5e^x + c_6.$$

Hence, the solution of the system (103) is

$$y_1 = 2 \ln \left(\frac{c_1 + c_2 e^{-x}}{2} \right),$$

$$y_2 = 2 \ln \left(\frac{c_3 + c_4 e^{-x}}{c_1 + c_2 e^{-x}} \right),$$

$$y_3 = \ln \left(\frac{2(c_5 + c_6 e^{-x})}{c_3 + c_4 e^{-x}} \right).$$

Example 2. Consider the system of nonlinear ordinary differential equation

$$y_1'' + \frac{1}{2}y_1'^2 + y_1' = 0,$$

$$y_2'' + y_2'^2 + y_1'y_2' + y_2' = 0,$$

$$y_3'' + y_3'^2 + y_1'y_3' + y_3' = 0.$$

(144)

It is a system of the form (4) with the coefficients

$$\begin{aligned} a_{11} = 0, a_{12} = 0, a_{13} = 0, a_{14} = (-1)/2, a_{15} = 0, a_{16} = 0, \\ a_{17} = 0, a_{18} = 0, a_{19} = 0, a_{110} = -1, a_{111} = 0, a_{112} = 0, \\ a_{113} = 0, a_{114} = 0, a_{115} = 0, a_{116} = 0, a_{24} = -1, a_{27} = -1, \\ a_{29} = 0, a_{210} = -1, a_{211} = 0, a_{212} = 0, a_{213} = 0, a_{214} = 0, \\ a_{215} = 0, a_{216} = 0, a_{34} = -1, a_{37} = -1, a_{39} = 0, a_{310} = -1, \\ a_{311} = 0, a_{312} = 0, a_{313} = 0, a_{314} = 0, a_{315} = 0, a_{316} = 0. \end{aligned} \quad (145)$$

One can check that the coefficients (145) obey the conditions (45)-(81). Thus, the Equation (144) is linearizable. We have

$$v_{y_1} = v, \quad (146)$$

$$v_{y_2} = v, \quad (147)$$

$$v_{xx} = v_x(3v_x - 2v)/(2v), \quad (148)$$

$$\varphi_{xx} = \varphi_x(v_x - v)/v, \quad (149)$$

$$\psi_{1xx} = \psi_{1x}(v_x - v)/v, \quad (150)$$

$$\psi_{1xy_1} = (\psi_{1y_1} v_x)/(2v), \quad (151)$$

$$\psi_{1y_1y_1} = \psi_{1y_1}/2, \quad (152)$$

$$\psi_{2y_2} = v/\psi_{1y_1}, \quad (153)$$

$$\psi_{2xx} = \psi_{2x}(v_x - v)/v, \quad (154)$$

$$\psi_{2xy_1} = (\psi_{2y_1} v_x)/(2v), \quad (155)$$

$$\psi_{2y_1y_1} = \psi_{2y_1}/2, \quad (156)$$

$$\psi_{3xx} = \psi_{3x}(v_x - v)/(v), \quad (157)$$

$$\psi_{3xy_1} = (\psi_{3y_1} v_x)/(2v), \quad (158)$$

$$\psi_{3xy_2} = (\psi_{3y_2} v_x)/(2v), \quad (159)$$

$$\psi_{3xy_3} = (\psi_{3y_3} v_x)/(2v), \quad (160)$$

$$\psi_{3y_1y_1} = \psi_{3y_1}/2, \quad (161)$$

$$\psi_{3y_1y_2} = \psi_{3y_2}/2, \quad (162)$$

$$\psi_{3y_1y_3} = \psi_{3y_3}/2, \quad (163)$$

$$\psi_{3y_2y_2} = \psi_{3y_2}, \quad (164)$$

$$\psi_{3y_2y_3} = 0, \quad (165)$$

$$\psi_{3y_3y_3} = \psi_{3y_3}. \quad (166)$$

From (146) and (147) we get

$$v = e^{y_1 + c_1(x, y_2)}$$

and

$$v = e^{y_2 + c_2(x, y_1)},$$

respectively. Since one can use any particular solution, we can take

$$v = e^{2x + y_1 + y_2}$$

and this solution satisfies (148). Now the Equation (149) becomes

$$\varphi_{xx} = \varphi_x$$

and yields

$$\varphi_x = e^{x + c_3}, \quad c_3 = \text{const.}$$

Thus

$$\varphi = e^{x+c_3} + c_4.$$

Since one can use any particular solution, we set $c_3 = 0, c_4 = 0$ and take

$$\varphi = e^x.$$

Now the Equations (150)-(152) are written

$$\psi_{1xx} = \psi_{1x}, \quad (167)$$

$$\psi_{1xy_1} = \psi_{1y_1}, \quad (168)$$

$$\psi_{1y_1y_1} = \frac{\psi_{1y_1}}{2}. \quad (169)$$

To consider (168) and (169), one gets

$$\psi_{1y_1} = e^{\frac{x+y_1}{2}},$$

Then

$$\psi_1 = 2e^{\frac{x+y_1}{2}} + c_5(x).$$

Since one can use any particular solution, we set $c_5(x) = 0$ and take

$$\psi_1 = 2e^{\frac{x+y_1}{2}},$$

this solution satisfies (167). Now the Equations (153)-(156) are written as

$$\psi_{2y_2} = \frac{e^{x+y_1+y_2}}{e^2}, \quad (170)$$

$$\psi_{2xx} = \psi_{2x}, \quad (171)$$

$$\psi_{2xy_1} = \psi_{2y_1}, \quad (172)$$

$$\psi_{2y_1y_1} = \frac{\psi_{2y_1}}{2}. \quad (173)$$

To consider (170), after integration, one finds

$$\psi_2 = e^{\frac{x+y_1+y_2}{2}} + c_6(x, y_1).$$

Since one can use any particular solution, we set $c_6(x, y_1) = 0$ and take

$$\psi_2 = e^{\frac{x+y_1+y_2}{2}},$$

this solution satisfies (171)-(173). Now the Equations (157)-(166) are written

$$\psi_{3xx} = \psi_{3x}, \quad (174)$$

$$\psi_{3xy_1} = \psi_{3y_1}, \quad (175)$$

$$\psi_{3xy_2} = \psi_{3y_2}, \quad (176)$$

$$\psi_{3xy_3} = \psi_{3y_3}, \quad (177)$$

$$\psi_{3y_1y_1} = \frac{\psi_{3y_1}}{2}, \quad (178)$$

$$\psi_{3y_1y_2} = \frac{\psi_{3y_2}}{2}, \quad (179)$$

$$\psi_{3y_1y_3} = \frac{\psi_{3y_3}}{2}, \quad (180)$$

$$\psi_{3y_2y_2} = \psi_{3y_2}, \quad (181)$$

$$\psi_{3y_2y_3} = 0, \quad (182)$$

$$\psi_{3y_3y_3} = \psi_{3y_3}. \quad (183)$$

To consider (177), (180) and (183), one takes

$$\psi_{3y_3} = e^{\frac{x+y_1}{2}+y_3}$$

so that

$$\psi_3 = e^{\frac{x+y_1}{2}+y_3} + c_7(x, y_1, y_2).$$

Since one can use any particular solution, we set $c_7(x, y_1, y_2) = 0$ and take

$$\psi_3 = e^{\frac{x+y_1}{2}+y_3},$$

this solution satisfies (174)-(176), (178)-(179), (181)-(182). Then, one obtains the following transformations

$$t = e^x, u_1(t) = 2e^{\frac{y_1}{2}}, u_2(t) = e^{\frac{y_1}{2}+y_2}, u_3(t) = e^{\frac{y_1}{2}+y_3}. \quad (184)$$

Hence, the system (144) is mapped by the transformations (184) to the linear system

$$u_1''(t) = 0, u_2''(t) = 0, u_3''(t) = 0.$$

The solution of this linear system is

$$u_1(t) = c_1t + c_2,$$

$$u_2(t) = c_3t + c_4,$$

$$u_3(t) = c_5t + c_6$$

where $c_i, (i=1, 2, \dots, 6)$ re arbitrary constants. By using the transformation (184), one finds

$$2e^{\frac{y_1}{2}} = c_1e^x + c_2,$$

$$e^{\frac{y_1}{2}+y_2} = c_3e^x + c_4,$$

$$e^{\frac{y_1}{2}+y_3} = c_5e^x + c_6.$$

Hence, the solution of the system (144) is

$$y_1 = 2 \ln \left(\frac{c_1 + c_2 e^{-x}}{2} \right),$$

$$y_2 = \ln \left(\frac{2(c_3 + c_4 e^{-x})}{c_1 + c_2 e^{-x}} \right),$$

$$y_3 = \ln \left(\frac{2(c_5 + c_6 e^{-x})}{c_1 + c_2 e^{-x}} \right).$$

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Application of Group Analysis to Classification of Systems of Three Second-Order Ordinary Differential Equations

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Abstract

Here we give a complete group classification of the general case of linear systems of three second-order ordinary differential equations excluding the case of systems which are studied in the literature. This is given as the initial step in the study of nonlinear systems of three second-order ordinary differential equations. In addition the complete group classification of a system of three linear second-order ordinary differential equations is done. Four cases of linear systems of equations are obtained.

Keywords: Group classification, linear equations, admitted Lie group, equivalence transformation

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1. Introduction

Systems of ordinary differential equations appear in the modeling of natural phenomena and have generated sufficient interest in theoretical studies of these equations and their symmetry properties. For instance, one of the popular problems in the analysis of differential equations in the 19th century was that of group classification of ordinary differential equations (see the works by S. Lie [1, 2]). Group classification of differential equations here means to classify given differential equations with respect to arbitrary elements.

In this paper we consider the complete group classification of systems of three linear second-order ordinary differential equations. Systems of second-order ordinary differential equations appear in the study of many applications. The study of their symmetry structure constitutes an important field of application of the group symmetry analysis method. Earlier studies in this area initiated by Lie [3] gave a complete group classification of a scalar ordinary differential equation of the form $y'' = f(x, y)$. Later on L.V.Ovsiannikov [4] did the group classification using a different approach. In this approach the classification was obtained by directly solving the determining equations and exploiting the equivalence transformations. In more recent works the same approach was used in [5] for the group classification of more general types of equations. We note here that in the general case of a scalar ordinary differential equation, $y'' = f(x, y, y')$, the application of the method that involves directly solving the determining equations gives rise to some difficulties. The group classification of such equations [6] is based on the enumeration of all possible Lie algebras of operators acting on the plane (x, y) . In the works by Lie [3] is the

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classification of all non similar Lie algebras (under complex change of variables) in two complex variables. In 1992, Gonzalez-Lopez et al. ordered the Lie classification of realizations of complex Lie algebras and extended it to the real case [7]. In the literature and references therein is a large amount of results on the dimension and structure of symmetry algebras of linearizable ordinary differential equations (see [6, 8, 9, 10, 11, 12]).

It is apparent from these sources that there is a significant number of studies on symmetry properties of scalar ordinary differential equations but equally not as much on the group classification of systems of three linear second-order equations. In recent works [13, 14, 15, 16, 17] the authors focused on the study of systems of second-order ordinary differential equations with constant coefficients of the form

$$y'' = My, \quad (1)$$

where M is a matrix with constant entries.

In the general case of systems of two linear second-order ordinary differential equations the more advanced results are obtained in [12], where the canonical form

$$y'' = \begin{pmatrix} a(x) & b(x) \\ c(x) & -a(x) \end{pmatrix} y, \quad (2)$$

was used by the authors to obtain several admitted Lie groups. We note that the list of all distinguished representatives of systems of two linear second-order ordinary differential equations found in [12] was not exhaustive and hence this formed the basis of the paper [18] where the complete group classification of two linear second-order ordinary differential equations using Ovsianikov's approach was performed. It is also worth to note here that the form (2) allows one to apply an algebraic approach for group classification. The algebraic approach takes into account algebraic properties of an admitted Lie group and the knowledge of the algebraic structure of admitted Lie algebras that can significantly simplify the group classification. In particular, the group classification of a single second-order ordinary differential equation, done by the founder of the group analysis method, S.Lie [1, 3], cannot be performed without using the algebraic structure of admitted Lie groups. Recently the algebraic properties for group classification was applied in [19, 20, 21, 22, 23, 24, 25, 26]. We also note that the use of the algebraic structure of admitted Lie groups completely simplified the group classification of equations describing behavior of fluids with internal inertia in [27, 28]. The study of [29] showed that the problem of classification of systems of two linear second-order ordinary differential equations using the algebraic approach leads to the study of more cases than those found in [18], where Ovsianikov's approach was applied.

The system considered in the current paper is a generalization; it is a system of three linear second-order ordinary differential equations. We exclude from our consideration the study of systems of second-order ordinary differential equations with constant coefficients and the degenerate case given as follows:

$$y'' = F(x, y, z, u), \quad z'' = G(x, y, z, u), \quad u'' = 0,$$

or

$$y'' = F(x, y, z, u), \quad z'' = G(x, y, z), \quad u'' = H(x, y, z) = 0.$$

The results found here are new and have not been reported in the literature as far as we are aware.

The paper is organized as follows:

The first part of the paper deals with the preliminary study of systems of three second-order nonlinear equations of the form $y'' = F(x, y, z, u)$, $z'' = G(x, y, z, u)$ and $u'' = H(x, y, z, u)$ using Ovsianikov's approach [4]. This approach involves simplifying one generator and finding the

associated functions. These functions are then used to solve the determining equations. Using this approach one can classify second-order nonlinear differential equations but this case has been left for a current project. The second part of the paper gives a complete treatment of systems of second-order linear equations. This is then followed by the results and conclusion.

2. Preliminary study of systems of nonlinear equations

A system of three second-order nonlinear differential equations of the form

$$y'' = F(x, y, z, u), \quad z'' = G(x, y, z, u), \quad u'' = H(x, y, z, u) \quad (3)$$

is considered in this section. In matrix form equations (3) are given by

$$\mathbf{y}'' = \mathbf{F}(x, \mathbf{y}), \quad (4)$$

where

$$\mathbf{y} = \begin{pmatrix} y \\ z \\ u \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} F(x, y, z, u) \\ G(x, y, z, u) \\ H(x, y, z, u) \end{pmatrix}.$$

We consider a system of nonlinear equations here as it will later allow us to separate equations given into their respective classes. We exclude from the study the degenerate systems which are equivalent with respect to change of the dependent and independent variables to one of the classes: (a) the class where $F_z = 0$, $F_u = 0$; (b) the class where $F_u = 0$, $G_u = 0$. The class (a) is characterized by the property that one of the equations has the form $y'' = F(x, y)$. For the class (b) two equations are of the form

$$y'' = F(x, y, z), \quad z'' = G(x, y, z).$$

Hence for these classes the group classification is reduced to the study of a single equation or a system of two equations. It is noted that group classifications of linear equations with the number of equations $n \leq 2$ is complete.

2.1. Equivalence transformations

Calculations show that the equivalence Lie group is defined by the generators:

$$\begin{aligned} X_1^e &= y\partial_y + F\partial_F, & X_2^e &= z\partial_z + G\partial_G, & X_3^e &= u\partial_u + H\partial_H, \\ X_4^e &= y\partial_z + F\partial_G, & X_5^e &= z\partial_z + G\partial_G, & X_6^e &= u\partial_z + H\partial_G, \\ X_7^e &= y\partial_u + F\partial_H, & X_8^e &= z\partial_u + G\partial_H, & X_9^e &= u\partial_u + H\partial_H, \\ X_{10}^e &= \phi_1(x)\partial_y + \phi_1''(x)\partial_F, & X_{11}^e &= \phi_2(x)\partial_z + \phi_2''(x)\partial_G, \\ & X_{12}^e &= \phi_3(x)\partial_u + \phi_3''(x)\partial_H, \\ X_{13}^e &= 2\xi(x)\partial_x + \xi'(x)(y\partial_y + z\partial_z + u\partial_u \\ & - 3F\partial_F - 3G\partial_G - 3H\partial_H) + \xi'''(x)(y\partial_F + z\partial_G + u\partial_H), \end{aligned}$$

where $\xi(x)$ and $\phi_i(x)$, ($i = 1, 2, 3$) are arbitrary functions.

The transformations related with the generators X_i^e , ($i = 1, 2, \dots, 9$) correspond to the linear change of the dependent variables $\tilde{\mathbf{y}} = P\mathbf{y}$ with a constant nonsingular matrix P . The transformations corresponding to the generators X_i^e , ($i = 10, 11, 12$) define the change

$$\tilde{y} = y + \varphi_1(x), \quad \tilde{z} = z + \varphi_2(x), \quad \tilde{u} = u + \varphi_3(x).$$

The equivalence transformation related with the generator X_{13}^e is

$$\tilde{x} = \varphi(x), \quad \tilde{y} = y\psi(x), \quad \tilde{z} = z\psi(x), \quad \tilde{u} = u\psi(x),$$

where the functions $\varphi(x)$ and $\psi(x)$ satisfy the condition

$$\frac{\varphi''}{\varphi'} = 2\frac{\psi'}{\psi}. \quad (5)$$

2.2. Determining equations

Consider the generator

$$X = \xi(x, y, z, u)\frac{\partial}{\partial x} + \eta_1(x, y, z, u)\frac{\partial}{\partial y} + \eta_2(x, y, z, u)\frac{\partial}{\partial z} + \eta_3(x, y, z, u)\frac{\partial}{\partial u}.$$

According to the Lie algorithm [11], X is admitted by system (3) if it satisfies the associated determining equations. The first part of the determining equation is given by

$$\begin{aligned} 3F\xi_1 + G\xi_2 + H\xi_3 &= 3\xi_1''y + \xi_2''z + \xi_3''u - \xi_0'' - 2h_1', \\ F\xi_1 + 3G\xi_2 + H\xi_3 &= \xi y + 3\xi_2''z + \xi_3''u - \xi_0'' - 2h_2', \\ F\xi_1 + G\xi_2 + 3H\xi_3 &= \xi_1''y + \xi_2''z + 3\xi_3''u - \xi_0'' - 2h_3', \\ F\xi_2 &= \xi_2''y + h_{12}', \quad F\xi_3 = \xi_3''y + h_{13}', \\ G\xi_1 &= \xi_1''z + h_{21}', \quad G\xi_3 = \xi_3''z + h_{23}', \\ H\xi_1 &= \xi_1''u + h_{31}', \quad H\xi_2 = \xi_2''u + h_{32}' \end{aligned}$$

where

$$\xi(x, y, z, u) = \xi_1(x)y + \xi_2(x)z + \xi_3(x)u + \xi_0(x)$$

and the functions $h_i = h_i(x)$, and $h_{ij} = h_{ij}(x)$.

From these equations one can conclude that $\xi_1^2 + \xi_2^2 + \xi_3^2 \neq 0$ only for the case where two of the equations are equivalent to the free particle equation, for instance,

$$G = 0, \quad H = 0.$$

Hence we consider the case where

$$\xi_1 = 0, \quad \xi_2 = 0, \quad \xi_3 = 0.$$

The determining equations in this case are given by

$$\begin{aligned} F_y(y(\xi' + k_1) + zk_2 + uk_3 + \zeta_1) + F_z(yk_4 + (\xi' + k_5)z + uk_6 + \zeta_2) \\ + F_u(yk_7 + zk_8 + (\xi' + k_9)u + \zeta_3) \\ + 2F_x\xi - \xi'''y + (3\xi' - k_1)F - k_2G - k_3H - \zeta_1'' = 0, \\ G_y(y(\xi' + k_1) + zk_2 + uk_3 + \zeta_1) + G_z(yk_4 + (\xi' + k_5)z + uk_6 + \zeta_2) \\ + G_u(yk_7 + zk_8 + (\xi' + k_9)u + \zeta_3) \\ + 2G_x\xi - \xi'''z - k_4F + (3\xi' - k_5)G - k_6H - \zeta_2'' = 0, \\ H_y(y(\xi' + k_1) + zk_2 + uk_3 + \zeta_1) + H_z(yk_4 + (\xi' + k_5)z + uk_6 + \zeta_2) \\ + H_u(yk_7 + zk_8 + (\xi' + k_9)u + \zeta_3) \\ + 2H_x\xi - \xi'''u - k_7F - k_8G + (3\xi' - k_9)H - \zeta_3'' = 0, \end{aligned}$$

where an admitted generator has the form

$$X = 2\xi \frac{\partial}{\partial x} + ((\xi' + k_1)y + k_2z + k_3u + \zeta_1) \frac{\partial}{\partial y} \\ + (k_4y + (\xi' + k_5)z + k_6u + \zeta_2) \frac{\partial}{\partial z} \\ + (k_7y + k_8z + (\xi' + k_9)u + \zeta_3) \frac{\partial}{\partial u}$$

with $\xi = \xi(x)$, $\zeta_i = \zeta_i(x)$ and k_i , $i = 1, 2, 3$ constant.

For further analysis the study of the determining equations is separated into the cases:

- (a) the case in which there is at least one admitted generator with $\xi \neq 0$;
- (b) the case in which for all admitted generators $\xi = 0$.

2.3. Case $\xi \neq 0$

We consider the generator X_o for which $\xi \neq 0$. Using the equivalence transformation:

$$y_1 = y + \varphi_1(x), \quad z_1 = z + \varphi_2, \quad u_1 = u + \varphi_3(x),$$

the generator X_o becomes

$$X_o = 2\xi \frac{\partial}{\partial x} + ((\xi' + k_1)y_1 + k_2z_1 + k_3u_1 + \tilde{\zeta}_1) \frac{\partial}{\partial y_1} \\ + (k_4y_1 + (\xi' + k_5)z_1 + k_6u_1 + \tilde{\zeta}_2) \frac{\partial}{\partial z_1} \\ + (k_7y_1 + k_8z_1 + (\xi' + k_9)u_1 + \tilde{\zeta}_3) \frac{\partial}{\partial u_1},$$

where

$$\tilde{\zeta}_1 = 2\xi\varphi_1' - (\xi' + k_1)\varphi_1 - k_2\varphi_2 - k_3\varphi_3 + \zeta_1, \\ \tilde{\zeta}_2 = 2\xi\varphi_2' - k_4\varphi_1 - (\xi' + k_5)\varphi_2 - k_6\varphi_3 + \zeta_2, \\ \tilde{\zeta}_3 = 2\xi\varphi_3' - k_7\varphi_1 - k_8\varphi_2 - (\xi' + k_9)\varphi_3 + \zeta_3.$$

The functions $\varphi_i(x)$ ($i = 1, 2, 3$) can be chosen such that

$$\tilde{\zeta}_1 = 0, \quad \tilde{\zeta}_2 = 0, \quad \tilde{\zeta}_3 = 0.$$

Then without loss of generality one can assume that the generator X_o has the form

$$X_o = 2\xi \frac{\partial}{\partial x} + ((\xi' + k_1)y + k_2z + k_3u) \frac{\partial}{\partial y} + (k_4y + (\xi' + k_5)z + k_6u) \frac{\partial}{\partial z} \\ + (k_7y + k_8z + (\xi' + k_9)u) \frac{\partial}{\partial u}.$$

The equivalence transformation

$$x_1 = \alpha(x), \quad y_1 = y\beta(x), \quad z_1 = z\beta(x), \quad u_1 = u\beta(x),$$

where

$$\alpha''\beta = 2\alpha'\beta', \quad (\alpha'\beta \neq 0),$$

reduces the generator X_o to

$$X_o = 2\alpha'\xi \frac{\partial}{\partial x_1} + ((2\xi\beta'/\beta + \xi' + k_1)y_1 + k_2z_1 + k_3u_1) \frac{\partial}{\partial y_1} \\ + (k_4y_1 + (2\xi\beta'/\beta + \xi' + k_5)z_1 + k_6u_1) \frac{\partial}{\partial z_1} \\ + (k_7y_1 + k_8z_1 + (2\xi\beta'/\beta + \xi' + k_9)u_1) \frac{\partial}{\partial u_1}.$$

Choosing $\beta(x)$ such that $2\xi\beta'/\beta + \xi' = 0$ makes the generator X_o become

$$X_o = 2\alpha'\xi \frac{\partial}{\partial x_1} + (k_1y_1 + k_2z_1 + k_3u_1) \frac{\partial}{\partial y_1} + (k_4y_1 + k_5z_1 + k_6u_1) \frac{\partial}{\partial z_1} \\ + (k_7y_1 + k_8z_1 + k_9u_1) \frac{\partial}{\partial u_1}.$$

Note that in this case

$$\frac{d(\alpha'\xi)}{dx_1} = 0.$$

This means

$$\frac{d(\alpha'\xi)}{dx_1} = \frac{(\alpha'\xi)'}{\alpha'} = \xi' + \frac{\alpha''}{\alpha'}\xi = -2\xi\frac{\beta'}{\beta} + 2\frac{\beta'}{\beta}\xi = 0.$$

Hence without loss of generality we can assume that the generator X_o has the form

$$X_o = k\frac{\partial}{\partial x} + (k_1y + k_2z + k_3u)\frac{\partial}{\partial y} + (k_4y + k_5z + k_6u)\frac{\partial}{\partial z} + (k_7y + k_8z + k_9u)\frac{\partial}{\partial u},$$

where $k = 2\alpha'\xi \neq 0$ is constant. We rewrite the generator X_o in the form

$$X_o = \partial_x + (a_{11}y + a_{12}z + a_{13}u)\frac{\partial}{\partial y} + (a_{21}y + a_{22}z + a_{23}u)\frac{\partial}{\partial z} + (a_{31}y + a_{32}z + a_{33}u)\frac{\partial}{\partial u}.$$

The determining equations become

$$\begin{aligned} & (a_{11}y + a_{12}z + a_{13}u)F_y + (a_{21}y + a_{22}z + a_{23}u)F_z \\ & + (a_{31}y + a_{32}z + a_{33}u)F_u + F_x = a_{11}F + a_{12}G + a_{13}H, \\ & (a_{11}y + a_{12}z + a_{13}u)G_y + (a_{21}y + a_{22}z + a_{23}u)G_z \\ & + (a_{31}y + a_{32}z + a_{33}u)G_u + G_x = a_{21}F + a_{22}G + a_{23}H, \\ & (a_{11}y + a_{12}z + a_{13}u)H_y + (a_{21}y + a_{22}z + a_{23}u)H_z \\ & + (a_{31}y + a_{32}z + a_{33}u)H_u + H_x = a_{31}F + a_{32}G + a_{33}H. \end{aligned} \quad (6)$$

Here a_{ij} , ($i, j = 1, 2, 3$) are constant. In matrix form these equations are rewritten as

$$\mathbf{F}_x + ((\mathbf{A}\mathbf{y}) \cdot \nabla) \mathbf{F} - \mathbf{A}\mathbf{F} = 0, \quad (7)$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \nabla = \begin{pmatrix} \partial_y \\ \partial_z \\ \partial_u \end{pmatrix}.$$

Here “ \cdot ” means the scalar product

$$\mathbf{b} \cdot \nabla = b_i \partial_{y_i},$$

where a vector $\mathbf{b} = (b_1, b_2, b_3)$, $\mathbf{y} = (y_1, y_2, y_3)$, and it is also used standard agreement: summation with respect to a repeat index.

Further simplifications are related with simplifications of the matrix A .

We apply the change $\tilde{\mathbf{y}} = P\mathbf{y}$ where P is a nonsingular matrix with constant entries

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}.$$

Equations (3) become

$$\tilde{\mathbf{y}}'' = \tilde{\mathbf{F}}(x, \tilde{\mathbf{y}})$$

where

$$\tilde{\mathbf{F}}(x, \tilde{\mathbf{y}}) = P\mathbf{F}(x, P^{-1}\tilde{\mathbf{y}}).$$

The partial derivatives ∂_y , ∂_z and ∂_u are changed as follows:

$$\nabla = P^t \tilde{\nabla}.$$

Hence equations (7) become

$$\begin{aligned} & \left((AP^{-1}\tilde{y}) \cdot P^t \tilde{\nabla} \right) (P^{-1}\tilde{F}) + P^{-1}\tilde{F}_x - AP^{-1}\tilde{F} \\ &= P^{-1} \left(\left((PAP^{-1}\tilde{y}) \cdot \tilde{\nabla} \right) \tilde{F} + \tilde{F}_x - PAP^{-1}\tilde{F} \right) \\ &= P^{-1} \left(\left((\tilde{A}\tilde{y})^t \tilde{\nabla} \right) \cdot \tilde{F} + \tilde{F}_x - \tilde{A}\tilde{F} \right) = 0 \end{aligned}$$

where

$$\tilde{A} = PAP^{-1}.$$

This means that the change $\tilde{y} = Py$ reduces equation (7) to the same form with the matrix A changed. The infinitesimal generator is also changed as

$$X_o = \partial_x + (\tilde{A}\tilde{y}) \cdot \tilde{\nabla}.$$

Using this change matrix A can be presented in the Jordan form. For a real-valued 3×3 matrix A , if the matrix P also has real-valued entries, then the Jordan matrix is one of the following four types:

$$J_1 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & d \end{pmatrix}, \quad J_2 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & -c & b \end{pmatrix}, \quad J_3 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 1 \\ 0 & 0 & b \end{pmatrix}, \quad J_4 = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}, \quad (8)$$

where a , b , c and $d > 0$ are real numbers.

2.3.1. Case $A = J_1$

We assume that

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & d \end{pmatrix}.$$

In this case the equations for the functions F , G and H are

$$\begin{aligned} ayF_y + bzF_z + duF_u + F_x &= aF, \\ ayG_y + bzG_z + duG_u + G_x &= bG, \\ ayH_y + bzH_z + duH_u + H_x &= dH. \end{aligned}$$

The general solution of these equations is

$$\begin{aligned} F(x, y, z, u) &= e^{ax} f(s, v, w), \quad G(x, y, z, u) = e^{bx} g(s, v, w), \\ H(x, y, z, u) &= e^{dx} h(s, v, w) \end{aligned} \quad (9)$$

where

$$s = ye^{-ax}, \quad v = ze^{-bx}, \quad w = ue^{-dx}.$$

The admitted generator is

$$X_0 = \frac{\partial}{\partial x} + ay \frac{\partial}{\partial y} + bz \frac{\partial}{\partial z} + du \frac{\partial}{\partial u}.$$

2.3.2. Case $A = J_2$

We assume that

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & -c & b \end{pmatrix}.$$

In this case equations (6) become

$$\begin{aligned} ayF_y + (bz + cu)F_z + (-cz + bu)F_u + F_x &= aF, \\ ayG_y + (bz + cu)G_z + (-cz + bu)G_u + G_x &= bG + cH, \\ ayH_y + (bz + cu)H_z + (-cz + bu)H_u + H_x &= cG + bH. \end{aligned} \quad (10)$$

Here again we introduce the variables

$$\begin{aligned} s &= ye^{-ax}, \quad v = e^{-bx}(z \cos(cx) - u \sin(cx)), \\ w &= e^{-bx}(z \sin(cx) + u \cos(cx)), \end{aligned}$$

equations (10) become

$$F_x - aF = 0, \quad G_x - bG - cH = 0, \quad H_x + cG - bH = 0.$$

The general solution of these equations is

$$\begin{aligned} F(x, y, z, u) &= e^{ax}f(s, v, w), \\ G(x, y, z, u) &= e^{bx}(\cos(cx)g(s, v, w) + \sin(cx)h(s, v, w)), \\ H(x, y, z, u) &= e^{bx}(-\sin(cx)g(s, v, w) + \cos(cx)h(s, v, w)) \end{aligned} \quad (11)$$

where $f(s, v, w)$, $g(s, v, w)$ and $h(s, v, w)$ are arbitrary functions. The admitted generator is

$$X_0 = \frac{\partial}{\partial x} + ay\frac{\partial}{\partial y} + (bz + cu)\frac{\partial}{\partial z} + (-cz + bu)\frac{\partial}{\partial u}.$$

2.3.3. Case $A = J_3$

We assume that

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 1 \\ 0 & 0 & b \end{pmatrix}.$$

In this case equations (6) become

$$\begin{aligned} ayF_y + (bz + cu)F_z + buF_u + F_x &= aF, \\ ayG_y + (bz + cu)G_z + buG_u + G_x &= bG + H, \\ ayH_y + (bz + cu)H_z + buH_u + H_x &= bH. \end{aligned} \quad (12)$$

As in the previous case we introduce the variables

$$s = ye^{-ax}, \quad v = e^{-bx}(z - ux), \quad w = e^{-bx}u$$

so that equations (12) become

$$F_x - aF = 0, \quad G_x - bG - H = 0, \quad H_x - bH = 0.$$

The general solution of these equations is

$$\begin{aligned} F(x, y, z, u) &= e^{ax}f(s, v, w), \\ G(x, y, z, u) &= e^{bx}(h(s, v, w)x + g(s, v, w)), \\ H(x, y, z, u) &= e^{bx}h(s, v, w), \end{aligned} \quad (13)$$

where $f(s, v, w)$, $g(s, v, w)$ and $h(s, v, w)$ are arbitrary functions. The admitted generator is

$$X_0 = \frac{\partial}{\partial x} + ay\frac{\partial}{\partial y} + (bz + cu)\frac{\partial}{\partial z} + bu\frac{\partial}{\partial u}.$$

2.3.4. Case $A = J_4$

We assume that

$$A = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}.$$

In this case equations (6) become

$$\begin{aligned} (ay + z)F_y + (az + u)F_z + auF_u + F_x &= aF, \\ (ay + z)G_y + (az + u)G_z + auG_u + G_x &= aG + H, \\ (ay + z)H_y + (az + u)H_z + auH_u + H_x &= aH. \end{aligned} \quad (14)$$

Introducing the variables

$$s = e^{-ax} \left(y - xz + \frac{1}{2}x^2u \right), \quad v = e^{-ax} (z - xu), \quad w = e^{-ax}u$$

so that equations (14) become

$$F_x - aF - G = 0, \quad G_x - aG - H = 0, \quad H_x - aH = 0.$$

The general solution of these equations is

$$\begin{aligned} F(x, y, z, u) &= e^{ax} \left(h(s, v, w) \frac{x^2}{2} + g(s, v, w) + f(s, v, w) \right), \\ G(x, y, z, u) &= e^{ax} (h(s, v, w)x + g(s, v, w)), \\ H(x, y, z, u) &= e^{bx}h(s, v, w) \end{aligned} \quad (15)$$

where $f(s, v, w)$, $g(s, v, w)$ and $h(s, v, w)$ are arbitrary functions. The admitted generator is

$$X_0 = \frac{\partial}{\partial x} + (ay + z)\frac{\partial}{\partial y} + (az + u)\frac{\partial}{\partial z} + au\frac{\partial}{\partial u}.$$

2.4. Case $\xi = 0$

Substituting $\xi = 0$ into the determining equations we find that

$$\begin{aligned} (a_{11}y + a_{12}z + a_{13}u + \zeta_1)F_y + (a_{21}y + a_{22}z + a_{23}u + \zeta_2)F_z \\ + (a_{31}y + a_{32}z + a_{33}u + \zeta_3)F_u &= a_{11}F + a_{12}G + a_{13}H + \zeta_1'', \\ (a_{11}y + a_{12}z + a_{13}u + \zeta_1)G_y + (a_{21}y + a_{22}z + a_{23}u + \zeta_2)G_z \\ + (a_{31}y + a_{32}z + a_{33}u + \zeta_3)G_u &= a_{21}F + a_{22}G + a_{23}H + \zeta_2'', \\ (a_{11}y + a_{12}z + a_{13}u + \zeta_1)H_y + (a_{21}y + a_{22}z + a_{23}u + \zeta_2)H_z \\ + (a_{31}y + a_{32}z + a_{33}u + \zeta_3)H_u &= a_{31}F + a_{32}G + a_{33}H + \zeta_3'', \end{aligned} \quad (16)$$

or in matrix form

$$((Ay + \mathbf{h}) \cdot \nabla) \mathbf{F} = A\mathbf{F} + \mathbf{h}'' \quad (17)$$

where

$$A = \begin{pmatrix} k_1 & k_2 & k_3 \\ k_4 & k_5 & k_6 \\ k_7 & k_8 & k_9 \end{pmatrix}, \quad \mathbf{h}(x) = \begin{pmatrix} \zeta_1(x) \\ \zeta_2(x) \\ \zeta_3(x) \end{pmatrix}.$$

Similarly to the case where $\xi \neq 0$ we use the Jordan forms (8) of the matrix A . The admitted generator has the form

$$\begin{aligned} X_o &= (k_1y + k_2z + k_3u + \zeta_1(x))\partial_y + (k_4y + k_5z + k_6u + \zeta_2(x))\partial_z \\ &\quad + (k_7y + k_8z + k_9u + \zeta_3(x))\partial_u. \end{aligned}$$

2.4.1. Case $A = J_1$

Assuming that $A = J_1$, equations (16) for the functions F and G are

$$\begin{aligned}(ay + h_1)F_y + (bz + h_2)F_z + (du + h_3)F_u &= aF + h_1'', \\ (ay + h_1)G_y + (bz + h_2)G_z + (du + h_3)G_u &= bG + h_2'', \\ (ay + h_1)H_y + (bz + h_2)H_z + (du + h_3)H_u &= dH + h_3''.\end{aligned}$$

The general solution of these equations depends on a value of a, b and d :

- Case: $a \neq 0, b \neq 0, d \neq 0$

$$\begin{aligned}aF + h_1'' &= (ay + h_1)f(x, s, v), & bG + h_2'' &= (bz + h_2)g(x, s, v), \\ dH + h_3'' &= (du + h_3)h(x, s, v), & s &= (bz + h_2)^a(ay + h_1)^{-b}, \\ & & v &= (du + h_3)^b(bz + h_2)^{-d}.\end{aligned}$$

- Case: $a \neq 0, b \neq 0, d = 0$

$$\begin{aligned}aF + h_1'' &= (ay + h_1)f(x, s, v), & bG + h_2'' &= (bz + h_2)g(x, s, v), \\ H &= \frac{h_3''}{a} \ln(ay + h_1) + h(x, s, v), & s &= (bz + h_2)^a(ay + h_1)^{-b}, \\ & & v &= u - \frac{h_3}{b} \ln(bz + h_2).\end{aligned}$$

- Case: $a \neq 0, b = 0, d = 0$

$$\begin{aligned}aF + h_1'' &= (ay + h_1)f(x, s, v), & G &= \frac{h_2''}{a} \ln(ay + h_1) + g(x, s, v), \\ H &= \frac{h_3''}{a} \ln(ay + h_1) + h(x, s, v), & s &= z - \frac{h_2}{a} \ln(ay + h_1), \\ & & v &= u - \frac{h_3}{a} \ln(ay + h_1).\end{aligned}$$

- Case: $a = 0, b = 0, d = 0, h_1 \neq 0$

$$\begin{aligned}F &= \frac{h_1''}{h_1} y + f(x, s, v), & G &= \frac{h_2''}{h_1} y + g(x, s, v), \\ H &= \frac{h_3''}{h_1} y + h(x, s, v), & s &= z - \frac{h_2}{h_1} y, & v &= u - \frac{h_3}{h_1} y.\end{aligned}$$

- Case: $a = 0, b = 0, d = 0, h_1 = 0, h_2 \neq 0$

$$F = f(x, s), \quad G = \frac{h_2''}{h_2} z + g(x, s), \quad H = \frac{h_3''}{h_2} z + h(x, s), \quad s = u - \frac{h_3}{h_2} z.$$

Here $f(x, s, v), g(x, s, v)$ and $h(x, s, v)$ are arbitrary functions.

2.5. Case $A = J_2$

In this case equations (16) become

$$\begin{aligned}(ay + h_1)F_y + (bz + cu + h_2)F_z + (-cz + bu + h_3)F_u &= aF + h_1'', \\ (ay + h_1)G_y + (bz + cu + h_2)G_z + (-cz + bu + h_3)G_u &= bG + cH + h_2'', \\ (ay + h_1)H_y + (bz + cu + h_2)H_z + (-cz + bu + h_3)H_u &= -cG + bH + h_3''.\end{aligned} \tag{18}$$

- Case: $a \neq 0$

Introducing the variables

$$\begin{aligned}y &= \bar{y} - \frac{h_1}{a}, & z &= \bar{z} - (b^2 + c^2)^{-1}(bh_2 - ch_3), \\ u &= \bar{u} - (b^2 + c^2)^{-1}(ch_2 + bh_3),\end{aligned}$$

$$\begin{aligned} F &= \bar{F} - \frac{h_1''}{a}, \quad G = \bar{G} - (b^2 + c^2)^{-1}(bh_2'' - ch_3''), \\ H &= \bar{H} - (b^2 + c^2)^{-1}(ch_2'' + bh_3'') \end{aligned}$$

equations (4) become

$$\begin{aligned} a\bar{y}\bar{F}_{\bar{y}} + (b\bar{z} + c\bar{u})\bar{F}_{\bar{z}} + (-c\bar{z} + b\bar{u})\bar{F}_{\bar{u}} &= a\bar{F}, \\ a\bar{y}\bar{G}_{\bar{y}} + (b\bar{z} + c\bar{u})\bar{G}_{\bar{z}} + (-c\bar{z} + b\bar{u})\bar{G}_{\bar{u}} &= b\bar{G} + c\bar{H}, \\ a\bar{y}\bar{H}_{\bar{y}} + (b\bar{z} + c\bar{u})\bar{H}_{\bar{z}} + (-c\bar{z} + b\bar{u})\bar{H}_{\bar{u}} &= -c\bar{G} + b\bar{H}. \end{aligned}$$

In the variables

$$\bar{y} = ve^{as}, \quad \bar{z} = we^{bs} \sin(cs), \quad \bar{u} = we^{bs} \cos(cs)$$

these equations are

$$\bar{F}_s = a\bar{F}, \quad \bar{G}_s = b\bar{G} + c\bar{H}, \quad \bar{H}_s = -c\bar{G} + b\bar{H}.$$

The general solution of the last set of equations is

$$\begin{aligned} \bar{F}(x, y, z, u) &= e^{as} f(x, v, w), \\ \bar{G}(x, y, z, u) &= e^{as} (\cos(cs)g(x, v, w) + \sin(cs)h(x, v, w)), \\ \bar{H}(x, y, z, u) &= e^{bs} (-\sin(cs)g(x, v, w) + \cos(cs)h(x, v, w)) \end{aligned}$$

where $f(x, v, w)$, $g(x, v, w)$ and $h(x, v, w)$ are arbitrary functions.

- Case: $a = 0$

In this case equations (4) become

$$\begin{aligned} h_1 F_y + (bz + cu + h_2)F_z + (-cz + bu + h_3)F_u &= h_1'', \\ h_1 G_y + (bz + cu + h_2)G_z + (-cz + bu + h_3)G_u &= bG + cH + h_2'', \\ h_1 H_y + (bz + cu + h_2)H_z + (-cz + bu + h_3)H_u &= -cG + bH + h_3''. \end{aligned} \tag{19}$$

– If $h_1 \neq 0$, introducing the variables

$$\begin{aligned} y &= \bar{y}h_1, \quad z = \bar{z} - (b^2 + c^2)^{-1}(bh_2 - ch_3), \\ u &= \bar{u} - (b^2 + c^2)^{-1}(ch_2 + bh_3), \\ F &= \bar{F}, \quad G = \bar{G} - (b^2 + c^2)^{-1}(bh_2'' - ch_3''), \\ H &= \bar{H} - (b^2 + c^2)^{-1}(ch_2'' + bh_3'') \end{aligned}$$

equations (19) become

$$\begin{aligned} \bar{F}_{\bar{y}} + (b\bar{z} + c\bar{u})\bar{F}_{\bar{z}} + (-c\bar{z} + b\bar{u})\bar{F}_{\bar{u}} &= h_1'', \\ \bar{G}_{\bar{y}} + (b\bar{z} + c\bar{u})\bar{G}_{\bar{z}} + (-c\bar{z} + b\bar{u})\bar{G}_{\bar{u}} &= b\bar{G} + c\bar{H}, \\ \bar{H}_{\bar{y}} + (b\bar{z} + c\bar{u})\bar{H}_{\bar{z}} + (-c\bar{z} + b\bar{u})\bar{H}_{\bar{u}} &= -c\bar{G} + b\bar{H}. \end{aligned}$$

In the variables

$$\begin{aligned} s &= \bar{y}, \quad v = e^{-b\bar{y}} (\bar{z} \cos(c\bar{y}) - \bar{u} \sin(c\bar{y})), \\ w &= e^{-b\bar{y}} (\bar{z} \sin(c\bar{y}) + \bar{u} \cos(c\bar{y})) \end{aligned}$$

these equations become

$$\bar{F}_s = h_1'', \quad \bar{G}_s = b\bar{G} + c\bar{H}, \quad \bar{H}_s = -c\bar{G} + b\bar{H}.$$

The general solution of the last set of equations is

$$\begin{aligned} \bar{F}(x, y, z, u) &= h_1''s + f(x, v, w), \\ \bar{G}(x, y, z, u) &= e^{bs} (\cos(cs)g(x, v, w) + \sin(cs)h(x, v, w)), \\ \bar{H}(x, y, z, u) &= e^{bs} (-\sin(cs)g(x, v, w) + \cos(cs)h(x, v, w)). \end{aligned}$$

– If $h_1 = 0$, in this case equations (19) become

$$\begin{aligned}(bz + cu + h_2)F_z + (-cz + bu + h_3)F_u &= 0, \\ (bz + cu + h_2)G_z + (-cz + bu + h_3)G_u &= bG + cH + h_2'', \\ (bz + cu + h_2)H_z + (-cz + bu + h_3)H_u &= -cG + bH + h_3''.\end{aligned}\tag{20}$$

Introducing the variables

$$\begin{aligned}z &= \bar{z} - (b^2 + c^2)^{-1}(bh_2 - ch_3), \quad u = \bar{u} - (b^2 + c^2)^{-1}(ch_2 + bh_3), \\ F &= \bar{F}, \quad G = \bar{G} - (b^2 + c^2)^{-1}(bh_2'' - ch_3''), \\ H &= \bar{H} - (b^2 + c^2)^{-1}(ch_2'' + bh_3'')\end{aligned}$$

equations (20) become

$$\begin{aligned}(b\bar{z} + c\bar{u})\bar{F}_{\bar{z}} + (-c\bar{z} + b\bar{u})\bar{F}_{\bar{u}} &= 0, \\ (b\bar{z} + c\bar{u})\bar{G}_{\bar{z}} + (-c\bar{z} + b\bar{u})\bar{G}_{\bar{u}} &= b\bar{G} + c\bar{H}, \\ (b\bar{z} + c\bar{u})\bar{H}_{\bar{z}} + (-c\bar{z} + b\bar{u})\bar{H}_{\bar{u}} &= -c\bar{G} + b\bar{H}.\end{aligned}$$

In the variables

$$\bar{y} = w, \quad \bar{z} = ve^{bs} \sin(cs), \quad \bar{u} = ve^{bs} \cos(cs)$$

these equations become

$$\bar{F}_s = 0, \quad \bar{G}_s = b\bar{G} + c\bar{H}, \quad \bar{H}_s = -c\bar{G} + b\bar{H}$$

and their general solution is

$$\begin{aligned}\bar{F}(x, y, z, u) &= f(x, v, w), \\ \bar{G}(x, y, z, u) &= e^{bs} (\cos(cs)g(x, v, w) + \sin(cs)h(x, v, w)), \\ \bar{H}(x, y, z, u) &= e^{bs} (-\sin(cs)g(x, v, w) + \cos(cs)h(x, v, w)).\end{aligned}$$

2.5.1. Case $A = J_3$

In this case equations (16) become

$$\begin{aligned}(ay + h_1)F_y + (bz + u + h_2)F_z + (bu + h_3)F_u &= aF + h_1'', \\ (ay + h_1)G_y + (bz + u + h_2)G_z + (bu + h_3)G_u &= bG + H + h_2'', \\ (ay + h_1)H_y + (bz + u + h_2)H_z + (bu + h_3)H_u &= bH + h_3''.\end{aligned}\tag{21}$$

- Case: $a \neq 0, b \neq 0$

Introducing the variables

$$\begin{aligned}y &= \bar{y} - \frac{h_1}{a}, \quad z = \bar{z} - \frac{h_2}{b} + \frac{h_3}{b^2}, \quad u = \bar{u} - \frac{h_3}{b}, \\ F &= \bar{F} - \frac{h_1''}{a}, \quad G = \bar{G} + \frac{h_3''}{b^2} - \frac{h_2''}{b}, \quad H = \bar{H} - \frac{h_3''}{b}\end{aligned}$$

equations (21) become

$$\begin{aligned}a\bar{y}\bar{F}_{\bar{y}} + (b\bar{z} + \bar{u})\bar{F}_{\bar{z}} + b\bar{u}\bar{F}_{\bar{u}} &= a\bar{F}, \\ a\bar{y}\bar{G}_{\bar{y}} + (b\bar{z} + \bar{u})\bar{G}_{\bar{z}} + b\bar{u}\bar{G}_{\bar{u}} &= b\bar{G} + \bar{H}, \\ a\bar{y}\bar{H}_{\bar{y}} + (b\bar{z} + \bar{u})\bar{H}_{\bar{z}} + b\bar{u}\bar{H}_{\bar{u}} &= b\bar{H}.\end{aligned}$$

In the variables

$$\bar{y} = e^{as}, \quad \bar{z} = e^{bs}(sw + v), \quad \bar{u} = e^{bs}w$$

these equations are

$$\bar{F}_s = a\bar{F}, \quad \bar{G}_s = b\bar{G} + c\bar{H}, \quad \bar{H}_s = b\bar{H}$$

with general solution

$$\begin{aligned} \bar{F}(x, y, z, u) &= e^{as}f(x, v, w), \\ \bar{G}(x, y, z, u) &= e^{bs}(h(x, v, w)s + g(x, v, w)), \\ \bar{H}(x, y, z, u) &= e^{bs}h(x, v, w) \end{aligned}$$

where $f(x, v, w), g(x, v, w)$ and $h(x, v, w)$ are arbitrary functions.

- Case: $a \neq 0, b = 0$

In this case equations (21) become

$$\begin{aligned} (ay + h_1)F_y + (u + h_2)F_z + h_3F_u &= aF + h_1'', \\ (ay + h_1)G_y + (u + h_2)G_z + h_3G_u &= H + h_2'', \\ (ay + h_1)H_y + (u + h_2)H_z + h_3H_u &= h_3''. \end{aligned} \tag{22}$$

Introducing the variables

$$\begin{aligned} y &= \bar{y} - \frac{h_1}{a}, \quad z = \bar{z}, \quad u = \bar{u} - h_2, \\ F &= \bar{F} - \frac{h_1''}{a}, \quad G = \bar{G}, \quad H = \bar{H} - h_2'' \end{aligned}$$

equations (22) become

$$\begin{aligned} a\bar{y}\bar{F}_{\bar{y}} + \bar{u}\bar{F}_{\bar{z}} + h_3\bar{F}_{\bar{u}} &= a\bar{F}, \\ a\bar{y}\bar{G}_{\bar{y}} + \bar{u}\bar{G}_{\bar{z}} + h_3\bar{G}_{\bar{u}} &= \bar{H}, \\ a\bar{y}\bar{H}_{\bar{y}} + \bar{u}\bar{H}_{\bar{z}} + h_3\bar{H}_{\bar{u}} &= h_3''. \end{aligned} \tag{23}$$

– If $h_3 \neq 0$, then in the variables

$$\bar{y} = e^{as}, \quad \bar{z} = \frac{h_3s^2}{2} + ws + v, \quad \bar{u} = h_3s + w$$

equations (23) become

$$\bar{F}_s = a\bar{F}, \quad \bar{G}_s = \bar{H}, \quad \bar{H}_s = h_3''$$

with general solution

$$\begin{aligned} \bar{F}(x, y, z, u) &= e^{as}f(x, v, w), \\ \bar{G}(x, y, z, u) &= \frac{h_3s^2}{2} + h(x, v, w)s + g(x, v, w), \\ \bar{H}(x, y, z, u) &= h_3''s + h(x, v, w). \end{aligned}$$

– If $h_3 = 0$ then in the variables

$$\bar{y} = e^{as}, \quad \bar{z} = ws + v, \quad \bar{u} = w$$

equations (23) become

$$\bar{F}_s = a\bar{F}, \quad \bar{G}_s = \bar{H}, \quad \bar{H}_s = 0$$

and their general solution is

$$\begin{aligned} \bar{F}(x, y, z, u) &= e^{as}f(x, v, w), \\ \bar{G}(x, y, z, u) &= h(x, v, w)s + g(x, v, w), \\ \bar{H}(x, y, z, u) &= h(x, v, w). \end{aligned}$$

• Case: $a = 0, b \neq 0$

In this case equations (21) become

$$\begin{aligned} h_1 F_y + (bz + u + h_2)F_z + (bu + h_3)F_u &= h_1'', \\ h_1 G_y + (bz + u + h_2)G_z + (bu + h_3)G_u &= bG + H + h_2'', \\ h_1 H_y + (bz + u + h_2)H_z + (bu + h_3)H_u &= bH + h_3''. \end{aligned} \quad (24)$$

Introducing the variables

$$\begin{aligned} y &= \bar{y}, \quad z = \bar{z} - \frac{h_2}{b} + \frac{h_3}{b^2}, \quad u = \bar{u} - \frac{h_3}{b}, \\ F &= \bar{F}, \quad G = \bar{G} - \frac{h_2''}{b} + \frac{h_3''}{b^2}, \quad H = \bar{H} - \frac{h_3''}{b} \end{aligned}$$

equations (24) become

$$\begin{aligned} h_1 \bar{F}_{\bar{y}} + (b\bar{z} + \bar{u})\bar{F}_{\bar{z}} + b\bar{u}\bar{F}_{\bar{u}} &= h_1'', \\ h_1 \bar{G}_{\bar{y}} + (b\bar{z} + \bar{u})\bar{G}_{\bar{z}} + b\bar{u}\bar{G}_{\bar{u}} &= b\bar{G} + \bar{H}, \\ h_1 \bar{H}_{\bar{y}} + (b\bar{z} + \bar{u})\bar{H}_{\bar{z}} + b\bar{u}\bar{H}_{\bar{u}} &= b\bar{H}. \end{aligned} \quad (25)$$

– If $h_1 \neq 0$, then in the variables

$$\bar{y} = h_1 s, \quad \bar{z} = e^{bs}(ws + v), \quad \bar{u} = e^{bs}w$$

equations (25) become

$$\bar{F}_s = h_1'', \quad \bar{G}_s = b\bar{G} + \bar{H}, \quad \bar{H}_s = b\bar{H}.$$

The general solution of the last set of equations is

$$\begin{aligned} \bar{F}(x, y, z, u) &= h_1''s + f(x, v, w), \\ \bar{G}(x, y, z, u) &= e^{bs}(h(x, v, w)s + g(x, v, w)), \\ \bar{H}(x, y, z, u) &= e^{bs}h(x, v, w). \end{aligned}$$

– If $h_1 = 0$, then in the variables

$$\bar{y} = w, \quad \bar{z} = e^{bs}(w + v), \quad \bar{u} = e^{bs}w$$

equations (25) become

$$\bar{F}_s = 0, \quad \bar{G}_s = b\bar{G} + \bar{H}, \quad \bar{H}_s = b\bar{H}$$

and their general solution is

$$\begin{aligned} \bar{F}(x, y, z, u) &= f(x, v, w), \\ \bar{G}(x, y, z, u) &= e^{bs}(h(x, v, w)s + g(x, v, w)), \\ \bar{H}(x, y, z, u) &= e^{bs}h(x, v, w). \end{aligned}$$

- Case: $a = 0, b = 0$

In this case equations (21) become

$$\begin{aligned} h_1 F_y + (u + h_2) F_z + h_3 F_u &= h_1'', \\ h_1 G_y + (u + h_2) G_z + h_3 G_u &= H + h_2'', \\ h_1 H_y + (u + h_2) H_z + h_3 H_u &= h_3''. \end{aligned} \tag{26}$$

Introducing the variables

$$\begin{aligned} y &= \bar{y}, \quad z = \bar{z}, \quad u = \bar{u} - h_2, \\ F &= \bar{F}, \quad G = \bar{G}, \quad H = \bar{H} - h_2'' \end{aligned}$$

equations (26) become

$$\begin{aligned} h_1 \bar{F}_{\bar{y}} + \bar{u} \bar{F}_{\bar{z}} + h_3 \bar{F}_{\bar{u}} &= h_1'', \\ h_1 \bar{G}_{\bar{y}} + \bar{u} \bar{G}_{\bar{z}} + h_3 \bar{G}_{\bar{u}} &= \bar{H}, \\ h_1 \bar{H}_{\bar{y}} + \bar{u} \bar{H}_{\bar{z}} + h_3 \bar{H}_{\bar{u}} &= h_3''. \end{aligned} \tag{27}$$

– If $h_1 \neq 0, h_3 \neq 0$, then in the variables

$$\bar{y} = h_1 s, \quad \bar{z} = \frac{h_3 s^2}{2} + ws + v, \quad \bar{u} = h_3 s + w$$

equations (27) become

$$\bar{F}_s = h_1'', \quad \bar{G}_s = \bar{H}, \quad \bar{H}_s = h_3''.$$

The general solution of the last set of equations is

$$\begin{aligned} \bar{F}(x, y, z, u) &= h_1'' s + f(x, v, w), \\ \bar{G}(x, y, z, u) &= \frac{h_3'' s^2}{2} + h(x, v, w)s + g(x, v, w), \\ \bar{H}(x, y, z, u) &= h_3'' s + h(x, v, w). \end{aligned}$$

– If $h_1 \neq 0, h_3 = 0$, then in the variables

$$\bar{y} = h_1 s, \quad \bar{z} = ws + v, \quad \bar{u} = w$$

equations (27) become

$$\bar{F}_s = h_1'', \quad \bar{G}_s = \bar{H}, \quad \bar{H}_s = 0.$$

The general solution of the last set of equations is

$$\begin{aligned} \bar{F}(x, y, z, u) &= h_1'' s + f(x, v, w), \\ \bar{G}(x, y, z, u) &= h(x, v, w)s + g(x, v, w), \\ \bar{H}(x, y, z, u) &= h(x, v, w). \end{aligned}$$

– If $h_1 = 0, h_3 \neq 0$, then in the variables

$$\bar{y} = w, \quad \bar{z} = \frac{h_3 s^2}{2} + ws + v, \quad \bar{u} = h_3 s + w$$

equations (27) become

$$\bar{F}_s = 0, \quad \bar{G}_s = \bar{H}, \quad \bar{H}_s = h_3''$$

with general solution

$$\begin{aligned} \bar{F}(x, y, z, u) &= f(x, v, w), \\ \bar{G}(x, y, z, u) &= \frac{h_3'' s^2}{2} + h(x, v, w)s + g(x, v, w), \\ \bar{H}(x, y, z, u) &= h_3'' s + h(x, v, w). \end{aligned}$$

– If $h_1 = 0, h_3 = 0$, then in the variables

$$\bar{y} = v, \quad \bar{z} = ws + v, \quad \bar{u} = w$$

equations (27) become

$$\bar{F}_s = 0, \quad \bar{G}_s = \bar{H}, \quad \bar{H}_s = 0$$

and their general solution is

$$\begin{aligned} \bar{F}(x, y, z, u) &= f(x, v, w), \\ \bar{G}(x, y, z, u) &= h(x, v, w)s + g(x, v, w), \\ \bar{H}(x, y, z, u) &= h(x, v, w). \end{aligned}$$

2.5.2. Case $A = J_4$

In this case equations (16) become

$$\begin{aligned} (ay + z + h_1)F_y + (az + u + h_2)F_z + (au + h_3)F_u &= aF + G + h_1'', \\ (ay + z + h_1)G_y + (az + u + h_2)G_z + (au + h_3)G_u &= aG + H + h_2'', \\ (ay + z + h_1)H_y + (az + u + h_2)H_z + (au + h_3)H_u &= aH + h_3''. \end{aligned} \tag{28}$$

- Case: $a \neq 0$

Introducing the variables

$$y = \bar{y} - \frac{h_1}{a} + \frac{h_2}{a^2} - \frac{h_3}{a^3}, \quad z = \bar{z} - \frac{h_2}{a} + \frac{h_3}{a^2}, \quad u = \bar{u} - \frac{h_3}{a},$$

$$F = \bar{F} - \frac{h_1''}{a} + \frac{h_2''}{a^2} - \frac{h_3''}{a^3}, \quad G = \bar{G} - \frac{h_2''}{a} + \frac{h_3''}{a^2}, \quad H = \bar{H} - \frac{h_3''}{a}$$

equations (28) become

$$\begin{aligned} (a\bar{y} + \bar{z})\bar{F}_{\bar{y}} + (a\bar{z} + \bar{u})\bar{F}_{\bar{z}} + a\bar{u}\bar{F}_{\bar{u}} &= a\bar{F} + \bar{G}, \\ (a\bar{y} + \bar{z})\bar{G}_{\bar{y}} + (a\bar{z} + \bar{u})\bar{G}_{\bar{z}} + a\bar{u}\bar{G}_{\bar{u}} &= a\bar{G} + \bar{H}, \\ (a\bar{y} + \bar{z})\bar{H}_{\bar{y}} + (a\bar{z} + \bar{u})\bar{H}_{\bar{z}} + a\bar{u}\bar{H}_{\bar{u}} &= a\bar{H}. \end{aligned}$$

In the variables

$$\bar{y} = e^{as}\left(\frac{ws^2}{2} + v\right), \quad \bar{z} = e^{as}(ws + v), \quad \bar{u} = e^{as}w$$

these equations are

$$\bar{F}_s = a\bar{F} + \bar{G}, \quad \bar{G}_s = a\bar{G} + \bar{H}, \quad \bar{H}_s = a\bar{H}.$$

The general solution of the last set of equations is

$$\bar{F}(x, y, z, u) = e^{as}\left(h(x, v, w)\frac{s^2}{2} + g(x, v, w)s + f(x, v, w)\right),$$

$$\bar{G}(x, y, z, u) = e^{as}(h(x, v, w)s + g(x, v, w)),$$

$$\bar{H}(x, y, z, u) = e^{as}h(x, v, w)$$

where $f(x, v, w)$, $g(x, v, w)$ and $h(x, v, w)$ are arbitrary functions.

- Case: $a = 0$

In this case (28) become

$$\begin{aligned} (z + h_1)F_y + (u + h_2)F_z + h_3F_u &= G + h_1'', \\ (z + h_1)G_y + (u + h_2)G_z + h_3G_u &= H + h_2'', \\ (z + h_1)H_y + (u + h_2)H_z + h_3H_u &= h_3''. \end{aligned} \tag{29}$$

Introducing the variables

$$y = \bar{y}, \quad z = \bar{z} - h_1, \quad u = \bar{u} - h_2,$$

$$F = \bar{F}, \quad G = \bar{G} - h_1'', \quad H = \bar{H} - h_2''$$

equations (29) become

$$\begin{aligned} \bar{z}\bar{F}_{\bar{y}} + \bar{u}\bar{F}_{\bar{z}} + h_3\bar{F}_{\bar{u}} &= \bar{G}, \\ \bar{z}\bar{G}_{\bar{y}} + \bar{u}\bar{G}_{\bar{z}} + h_3\bar{G}_{\bar{u}} &= \bar{H}, \\ \bar{z}\bar{H}_{\bar{y}} + \bar{u}\bar{H}_{\bar{z}} + h_3\bar{H}_{\bar{u}} &= h_3''. \end{aligned} \tag{30}$$

– If $h_3 \neq 0$, then in the variables

$$\bar{y} = \frac{h_3 s^3}{6} + vs + w, \quad \bar{z} = \frac{h_3 s^2}{2} + v, \quad \bar{u} = h_3 s$$

equations (30) become

$$\bar{F}_s = \bar{G}, \quad \bar{G}_s = \bar{H}, \quad \bar{H}_s = h_3''$$

with general solution

$$\bar{F}(x, y, z, u) = \frac{h_3 s^3}{6} + h(x, v, w) \frac{s^2}{2} + g(x, v, w)s + f(x, v, w),$$

$$\bar{G}(x, y, z, u) = \frac{h_3 s^2}{2} + h(x, v, w)s + g(x, v, w),$$

$$\bar{H}(x, y, z, u) = h_3 s + h(x, v, w).$$

– If $h_3 = 0$, then in the variables

$$\bar{y} = \frac{ws^2}{2} + vs, \quad \bar{z} = ws + v, \quad \bar{u} = w$$

equations (30) become

$$\bar{F}_s = \bar{G}, \quad \bar{G}_s = \bar{H}, \quad \bar{H}_s = 0$$

with general solution

$$\bar{F}(x, y, z, u) = h(x, v, w) \frac{s^2}{2} + g(x, v, w)s + f(x, v, w),$$

$$\bar{G}(x, y, z, u) = h(x, v, w)s + g(x, v, w),$$

$$\bar{H}(x, y, z, u) = h(x, v, w).$$

3. Systems of linear equations

Linear second-order ordinary differential equations have the following form,

$$\mathbf{y}'' = A(x)\mathbf{y}' + B(x)\mathbf{y} + f(x), \quad (31)$$

where $A(x)$ and $B(x)$ are matrices, and $f(x)$ is a vector. Using a particular solution $\mathbf{y}_p(x)$ and the change

$$\mathbf{y} = \tilde{\mathbf{y}} + \mathbf{y}_p,$$

we can, without loss of generality assume that $f(x) = 0$. The matrix $A(x)$ or $B(x)$ can also be assumed to be zero if we use the change, $\mathbf{y} = C(x)\tilde{\mathbf{y}}$, where $C = C(x)$ is a nonsingular matrix. In the present paper the matrix $A(x)$ is reduced to zero. In this case the linear equations (3) are linear functions of y, z and u :

$$\begin{aligned} F(x, y, z, u) &= c_{11}(x)y + c_{12}(x)z + c_{13}(x)u, \\ G(x, y, z, u) &= c_{21}(x)y + c_{22}(x)z + c_{23}(x)u, \\ H(x, y, z, u) &= c_{31}(x)y + c_{32}(x)z + c_{33}(x)u. \end{aligned} \quad (32)$$

In matrix form one can write as

$$\mathbf{F}(x, \mathbf{y}) = C(x)\mathbf{y} \quad (33)$$

Any linear system of equations admits the following generators

$$y\partial_y + z\partial_z + u\partial_u, \quad (34)$$

$$\zeta_1(x)\partial_y + \zeta_2(x)\partial_z + \zeta_3(x)\partial_u \quad (35)$$

where $\zeta_1(x)$, $\zeta_2(x)$ and $\zeta_3(x)$ are solutions of the equations:

$$\zeta_1'' = c_{11}\zeta_1 + c_{12}\zeta_2 + c_{13}\zeta_3,$$

$$\zeta_2'' = c_{21}\zeta_1 + c_{22}\zeta_2 + c_{23}\zeta_3,$$

$$\zeta_3'' = c_{31}\zeta_1 + c_{32}\zeta_2 + c_{33}\zeta_3.$$

For the classification problem one needs to study equations which admit generators different from (34) and (35).

3.1. Equivalence transformations

Similar to systems of two second-order ordinary differential equations calculations show that the equivalence transformations are defined by the following two types of transformations. The first type corresponds to the linear change of the dependent variables $\tilde{y} = Py$ with a constant nonsingular matrix P . The second type is

$$\tilde{x} = \varphi(x), \quad \tilde{y} = y\psi(x), \quad \tilde{z} = z\psi(x), \quad \tilde{u} = u\psi(x),$$

where the functions $\varphi(x)$ and $\psi(x)$ satisfy the condition

$$\frac{\varphi''}{\varphi'} = 2\frac{\psi'}{\psi}.$$

Remark. In [29] it is shown that using the equivalence transformations of the presented above form, one can reduce any system of homogeneous linear second-order ordinary differential equations to a system with a matrix C such that¹ $\text{trac } C = 0$. In this case one can obtain that the coefficient $\xi(x)$ of the admitted generator related with the independent variable x has the form

$$\xi = k_2x^2 + k_1x + k_0,$$

where k_i , ($i = 0, 1, 2$) are constant. This means that the admitted generators are defined up to constants. This property allows one to apply an algebraic approach for group classification. In algebraic approach the constants are defined from the algebraic properties of admitted Lie algebras. Substituting the constants into determining equations one obtains systems of ordinary differential equations for the entries of the matrix $C(x)$. Solving this system of equations one finds forms of the entries. For systems of two linear second-order ordinary differential equations it shown in [29] that these systems of ordinary differential equations for the entries can be solved. However, the complete study of the problem of group classification using the algebraic approach led to the study of more cases than those found in [18], where Ovsiannikov's approach was applied.

3.2. Case $\xi \neq 0$

Using the obtained general forms of equations admitting an infinitesimal generator with $\xi \neq 0$ linear systems of equations (9), (11), (13) and (15) have the following form:

$$\begin{aligned} f(s, v, w) &= \alpha_{11}s + \alpha_{12}v + \alpha_{13}w, \\ g(s, v, w) &= \alpha_{21}s + \alpha_{22}v + \alpha_{23}w, \\ h(s, v, w) &= \alpha_{31}s + \alpha_{32}v + \alpha_{33}w. \end{aligned} \quad (36)$$

¹For a system with two equations this property was obtained in [12].

3.2.1. Case $A = J_1$

Substituting (36) into (32), the equations involving F, G and H become

$$\begin{aligned} F &= \alpha_{11}y + e^{\alpha x}\alpha_{12}z + e^{\beta x}\alpha_{13}u, \\ G &= e^{-\alpha x}\alpha_{21}y + \alpha_{22}z + e^{-(\alpha-\beta)x}\alpha_{23}u, \\ H &= e^{-\beta x}\alpha_{31}y + e^{(\alpha-\beta)x}\alpha_{32}z + \alpha_{33}u \end{aligned}$$

where $\alpha = a - b$ and $\beta = a - d$.

Without loss of generality one can assume that $\alpha_{12} = 1$. We can assume that $\alpha_{12} \neq 0$ because for $\alpha_{12} = 0$ we have $\alpha_{13} \neq 0$ otherwise the system is degenerate. Applying the scaling of y we conclude that $\alpha_{12} = 1$. Thus the studied linear system of equations becomes

$$\begin{aligned} F &= \alpha_{11}y + e^{\alpha x}z + e^{\beta x}\alpha_{13}u, \\ G &= e^{-\alpha x}\alpha_{21}y + \alpha_{22}z + e^{-(\alpha-\beta)x}\alpha_{23}u, \\ H &= e^{-\beta x}\alpha_{31}y + e^{(\alpha-\beta)x}\alpha_{32}z + \alpha_{33}u. \end{aligned} \tag{37}$$

3.2.2. Case $A = J_2$

Substituting (36) into the functions F, G and H one finds

$$\begin{aligned} F &= \alpha_{11}y + e^{(a-b)x}(\alpha_{12}\cos(cx) + \alpha_{13}\sin(cx))z + e^{(a-b)x}(\alpha_{13}\cos(cx) \\ &\quad - \alpha_{12}\sin(cx))u, \\ G &= e^{-(a-b)x}(\alpha_{21}\cos(cx) + \alpha_{31}\sin(cx))y + (\alpha_{22}\cos^2(cx) \\ &\quad + (\alpha_{23} + \alpha_{32})\cos(cx)\sin(cx) + \alpha_{33}\sin^2(cx))z \\ &\quad + (\alpha_{23}\cos^2(cx) + (\alpha_{33} - \alpha_{22})\cos(cx)\sin(cx) - \alpha_{32}\sin^2(cx))u, \\ H &= e^{-(a-b)x}(\alpha_{31}\cos(cx) - \alpha_{21}\sin(cx))y + (\alpha_{32}\cos^2(cx) \\ &\quad + (\alpha_{33} - \alpha_{22})\cos(cx)\sin(cx) - \alpha_{23}\sin^2(cx))z \\ &\quad + (\alpha_{33}\cos^2(cx) - (\alpha_{32} + \alpha_{23})\cos(cx)\sin(cx) + \alpha_{22}\sin^2(cx))u. \end{aligned}$$

Using trigonometry formulae and introducing the constants β, γ, c_1 and c_2 :

$$\begin{pmatrix} \alpha_{22} & \alpha_{23} \\ \alpha_{32} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} \beta + c_2 & \gamma - c_1 \\ \gamma + c_1 & -\beta + c_2 \end{pmatrix},$$

one can rewrite functions F, G and H in the form

$$\begin{aligned} F &= \alpha_{11}y + e^{\alpha x}(\cos(cx)\alpha_{12} + \sin(cx)\alpha_{13})z + e^{\alpha x}(\cos(cx)\alpha_{13} \\ &\quad - \sin(cx)\alpha_{12})u, \\ G &= e^{-\alpha x}(\cos(cx)\alpha_{21} + \sin(cx)\alpha_{31})y + (\cos(2cx)\beta \\ &\quad + \sin(2cx)\gamma + c_2)z + (\cos(2cx)\gamma - \sin(2cx)\beta - c_1)u, \\ H &= e^{-\alpha x}(\cos(cx)\alpha_{31} - \sin(cx)\alpha_{21})y + (\cos(2cx)\gamma \\ &\quad - \sin(2cx)\beta + c_1)z - (\cos(2cx)\beta + \sin(2cx)\gamma - c_2)u, \end{aligned}$$

where $\alpha = a - b$.

Applying rotation of the dependent variables z and u , and their dilation, we can assume that $\alpha_{12} = 1$ and $\alpha_{13} = 0$. Hence the studied linear system of equations becomes

$$\begin{aligned} F &= \alpha_{11}y + e^{\alpha x} \cos(cx) z - e^{\alpha x} \sin(cx) u, \\ G &= e^{-\alpha x} (\cos(cx) \alpha_{21} + \sin(cx) \alpha_{31}) y + (\cos(2cx) \beta \\ &\quad + \sin(2cx) \gamma + c_2) z + (\cos(2cx) \gamma - \sin(2cx) \beta - c_1) u, \\ H &= e^{-\alpha x} (\cos(cx) \alpha_{31} - \sin(cx) \alpha_{21}) y + (\cos(2cx) \gamma \\ &\quad - \sin(2cx) \beta + c_1) z - (\cos(2cx) \beta + \sin(2cx) \gamma - c_2) u. \end{aligned} \quad (38)$$

3.2.3. Case $A = J_3$

Substituting (36) into the functions F, G and H one finds

$$\begin{aligned} F &= \alpha_{11}y + e^{\alpha x} \alpha_{12}z + e^{\alpha x} (-\alpha_{12}x + \alpha_{13}) u, \\ G &= e^{-\alpha x} (\alpha_{21} + \alpha_{31}x) y + (\alpha_{22} + \alpha_{32}x) z \\ &\quad + (\alpha_{23} + (\alpha_{33} - \alpha_{22}) x - \alpha_{32}x^2) u, \\ H &= e^{-\alpha x} \alpha_{31}y + \alpha_{32}z + (\alpha_{33} - \alpha_{32}x) u \end{aligned} \quad (39)$$

where $\alpha = a - b$.

3.2.4. Case $A = J_4$

Substituting (36) into the functions F, G and H one finds

$$\begin{aligned} F &= (\lambda + \beta x + \frac{1}{2}\gamma x^2) (y - xz + \frac{1}{2}x^2u), \\ G &= e^{-\alpha x} (\beta + \gamma x) (y - xz + \frac{1}{2}x^2u), \\ H &= e^{-\alpha x} \gamma (y - xz + \frac{1}{2}x^2u) \end{aligned} \quad (40)$$

where $\lambda = \alpha_{11} + \alpha_{12} + \alpha_{13}$, $\beta = \alpha_{21} + \alpha_{22} + \alpha_{23}$, $\gamma = \alpha_{31} + \alpha_{32} + \alpha_{33}$.

Since for $\gamma = 0$ the linear system of equations is reduced to the degenerate case $H = 0$, one has to consider $\gamma \neq 0$.

3.3. Case $\xi = 0$

Substituting (33) into (17) and splitting it with respect to y, z and u one has

$$CA - AC = 0. \quad (41)$$

A nontrivial admitted generator is of the form

$$X_o = (Ay) \cdot \nabla. \quad (42)$$

Equations (41) can be simplified by using the Jordan form of the matrix A .

In particular, if $A = J_1$, then equations (41) become

$$\begin{aligned} c_{12}(b - a) = 0, \quad c_{13}(d - a) = 0, \quad c_{21}(a - b) = 0, \\ c_{23}(b - d) = 0, \quad c_{31}(a - d) = 0, \quad c_{32}(b - d) = 0. \end{aligned} \quad (43)$$

If $(a - b)^2 + (a - d)^2 + (b - d)^2 = 0$, then we find that $a = b = d$ and the generator (42) is also trivial. Hence we can assume that $a \neq b$. Using this assumption we see from equations (43) that

$$c_{12} = 0, \quad c_{21} = 0.$$

For a non-degenerate linear system one has to assume that $c_{13} \neq 0$, which leads to $a = d$ and

$$c_{23} = 0, \quad c_{32} = 0.$$

The linear system of equations with $c_{21} = 0$ and $c_{23} = 0$ is degenerate.

Similar results are obtained for the for other cases of the matrix A . This means that in contrast to a linear system with two equations there is no linear systems with three dependent variables such that all nontrivial admitted generators have the form (42).

4. Solutions of the determining equations

Solving the determining equations leads to the following solutions:

- For system (37) there is the only nontrivial admitted generator

$$\partial_x - \alpha z \partial_z - \beta u \partial_u. \quad (44)$$

- In the case of system (38) there is the only nontrivial admitted generator

$$\partial_x + \alpha y \partial_y + c(u \partial_z - z \partial_u). \quad (45)$$

- Similarly for system (39) there is the only nontrivial admitted generator

$$\partial_x + \alpha y \partial_y + u \partial_z. \quad (46)$$

- Finally for system (40) there is the only nontrivial admitted generator

$$\partial_x + z \partial_y + u \partial_z. \quad (47)$$

Theorem. All non-degenerate linear systems of three second-order ordinary differential equations admitting a non-trivial generator are equivalent to one of the cases:

- system (37) with the generator (44),
- system (38) with the generator (45),
- system (39) with the generator (46),
- system (40) with the generator (47).

5. Conclusion

The results of the current paper are obtained using Ovsiannikov's approach [4], which involves simplifying one generator and finding the associated functions. These functions are then used to solve the determining equations. We found all forms of nonlinear systems $\mathbf{y}'' = \mathbf{F}(x, \mathbf{y})$ admitting at least one generator. Finding their forms is reduced to solving a linear system of first-order ordinary differential equations with constant coefficients. Methods for solving these systems are well-known in the theory of ordinary differential equations; their solution is given by studying purely algebraic properties of some constant matrix A . Since the matrix A is considered in the Jordan form, these properties are simple in each particular case.

Any normal linear system of second-order ordinary differential equations can be reduced to the form

$$\mathbf{y}'' = C\mathbf{y},$$

where $C(x)$ is a square matrix. These systems admit the set of trivial generators $\mathbf{y} \cdot \nabla$, $\mathbf{h}(x) \cdot \nabla$, where $\mathbf{h}'' = C\mathbf{h}$. Complete group classification of linear systems containing three equations and admitting a nontrivial generator is given; it is shown that all such systems have one of the four forms. We have also listed an explicit form of the non-trivial admitted generators for each of the respective cases. The obtained results are summarized in the theorem.

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