# Chapter 3

# Group classification of systems of three second-order ordinary differential equations

## 3.1 Introduction

Systems of ordinary differential equations appear in the modeling of natural phenomena and have generated sufficient interest in theoretical studies of these equations and their symmetry properties. For instance, one of the popular problems in the analysis of differential equations in the 19th century was that of group classification of ordinary differential equations (see the works by S. Lie [1, 2]). Group classification of differential equations here means to classify given differential equations with respect to arbitrary elements.

In this paper we consider the complete group classification of systems of three linear second-order ordinary differential equations. Systems of second-order ordinary differential equations appear in the study of many applications. The study of their symmetry structure constitutes an important field of application of the group symmetry analysis method. Earlier studies in this area initiated by Lie [3] gave a complete group classification of a scalar ordinary differential equation of the form $y'' = f(x, y)$. Later on L.V.Ovsiannikov [4] did the group classification using a different approach. In this approach the classification was obtained by directly solving the determining equations and exploiting the equivalence transformations. In more recent works the same approach was used in [5] for the group classification of more general types of equations. We note here that in the general case of a scalar ordinary differential equation, $y'' = f(x, y, y')$, the application of the method that involves directly solving the determining equations gives rise to some difficulties. The group classification of such equations [6] is based on the enumeration of all possible Lie algebras of operators acting on the plane $(x, y)$. In the works by Lie [3] is the classification of all non similar Lie algebras (under complex change of variables) in two complex variables. In 1992, Gonzalez-Lopez et al. ordered the Lie classification of realizations of complex Lie algebras and extended it to the real case [7]. In the literature and references therein is a large amount of results on the dimension and structure of symmetry algebras of linearizable ordinary differential equations (see [6, 8, 9, 10, 11, 12]).

It is apparent from these sources that there is a significant number of studies on symmetry properties of scalar ordinary differential equations but equally not as much on the group classification of systems of three linear second-order equations. In recent works [13, 14, 15, 16, 17] the authors focused on the study of systems of second-order ordinary differential equations with constant coefficients of the form

$$\mathbf{y}'' = M\mathbf{y}, \tag{3.1}$$

where $M$ is a matrix with constant entries.

In the general case of systems of two linear second-order ordinary differential equations the more advanced results are obtained in [12], where the canonical form

$$\mathbf{y}'' = \begin{pmatrix} a(x) & b(x) \\ c(x) & -a(x) \end{pmatrix} \mathbf{y}, \tag{3.2}$$

was used by the authors to obtain several admitted Lie groups. We note that the list of all distinguished representatives of systems of two linear second-order ordinary differential equations found in [12] was not exhaustive and hence this formed the basis of the paper [18] where the complete group classification of two linear second-order ordinary differential equations using Ovsiannikov's approach was performed. It is also worth to note here that the form (3.2) allows one to apply an algebraic approach for group classification. The algebraic approach takes into account algebraic properties of an admitted Lie group and the knowledge of the algebraic structure of admitted Lie algebras that can significantly simplify the group classification. In particular, the group classification of a single second-order ordinary differential equation, done by the founder of the group analysis method, S.Lie [1, 3], cannot be performed without using the algebraic structure of admitted Lie groups. Recently the algebraic properties for group classification was applied in [19, 20, 21, 22, 23, 24, 25, 26]. We also note that the use of the algebraic structure of admitted Lie groups completely simplified the group classification of equations describing behavior of fluids with internal inertia in [27, 28]. The study of [29] showed that the problem of classification of systems of two linear second-order ordinary differential equations using the algebraic approach leads to the study of more cases than those found in [18], where Ovsiannikov's approach was applied.

The system considered in the current paper is a generalization; it is a system of three linear second-order ordinary differential equations. We exclude from our consideration the study of systems of second-order ordinary differential equations with constant coefficients and the degenerate case given as follows:

$$y'' = F(x, y, z, u), \;\; z'' = G(x, y, z, u), \;\; u'' = 0,$$

or

$$y'' = F(x, y, z, u), \;\; z'' = G(x, y, z), \;\; u'' = H(x, y, z) = 0.$$

The results found here are new and have not been reported in the literature as far as we are aware.

## 3.2 Preliminary study of systems of nonlinear equations

A system of three second-order nonlinear differential equations of the form

$$y'' = F(x, y, z, u), \quad z'' = G(x, y, z, u), \quad u'' = H(x, y, z, u) \tag{3.3}$$

is considered in this section. In matrix form equations (3.3) are given by

$$\mathbf{y}'' = \mathbf{F}(x, \mathbf{y}), \tag{3.4}$$

where

$$\mathbf{y} = \begin{pmatrix} y \\ z \\ u \end{pmatrix}, \;\; \mathbf{F} = \begin{pmatrix} F(x, y, z, u) \\ G(x, y, z, u) \\ H(x, y, z, u) \end{pmatrix}.$$

We consider a system of nonlinear equations here as it will later allow us to separate equations given into their respective classes. We exclude from the study the degenerate systems which are equivalent with respect to change of the dependent and independent variables to one of the classes: (a) the class where $F_z = 0, \;\; F_u = 0$; (b) the class where $F_u = 0, \;\; G_u = 0$. The class (a) is characterized by the property that one of the equations has the form $y'' = F(x, y)$. For the class (b) two equations are of the form

$$y'' = F(x, y, z), \quad z'' = G(x, y, z).$$

Hence for these classes the group classification is reduced to the study of a single equation or a system of two equations. It is noted that group classifications of linear equations with the number of equations $n \leq 2$ is complete.

### 3.2.1 Equivalence transformations

Calculations show that the equivalence Lie group is defined by the generators:

$$\begin{gathered}
X_1^e = y\partial_y + F\partial_F,\quad X_2^e = z\partial_y + G\partial_F,\quad X_3^e = u\partial_y + H\partial_F,\\
X_4^e = y\partial_z + F\partial_G,\quad X_5^e = z\partial_z + G\partial_G,\quad X_6^e = u\partial_z + H\partial_G,\\
X_7^e = y\partial_u + F\partial_H,\quad X_8^e = z\partial_u + G\partial_H,\quad X_9^e = u\partial_u + H\partial_H,\\
X_{10}^e = \phi_1(x)\partial_y + \phi_1''(x)\partial_F,\quad X_{11}^e = \phi_2(x)\partial_z + \phi_2''(x)\partial_G,\\
X_{12}^e = \phi_3(x)\partial_u + \phi_3''(x)\partial_H,\\
X_{13}^e = 2\xi(x)\partial_x + \xi'(x)(y\partial_y + z\partial_z + u\partial_u\\
-3F\partial_F - 3G\partial_G - 3H\partial_H) + \xi'''(x)(y\partial_F + z\partial_G + u\partial_H),
\end{gathered}$$

where $\xi(x)$ and $\phi_i(x)$, $(i = 1, 2, 3)$ are arbitrary functions.

The transformations related with the generators $X_i^e$, $(i = 1, 2, ..., 9)$ correspond to the linear change of the dependent variables $\widetilde{\mathbf{y}} = P\mathbf{y}$ with a constant nonsingular matrix $P$. The transformations corresponding to the generators $X_i^e$, $(i = 10, 11, 12)$ define the change

$$\widetilde{y} = y + \varphi_1(x),\quad \widetilde{z} = z + \varphi_2(x),\quad \widetilde{u} = u + \varphi_3(x).$$

The equivalence transformation related with the generator $X_{13}^e$ is

$$\widetilde{x} = \varphi(x),\ \widetilde{y} = y\psi(x),\ \widetilde{z} = z\psi(x),\ \widetilde{u} = u\psi(x),$$

where the functions $\varphi(x)$ and $\psi(x)$ satisfy the condition

$$\frac{\varphi''}{\varphi'} = 2\frac{\psi'}{\psi}. \tag{3.5}$$

### 3.2.2 Determining equations

Consider the generator

$$X = \xi(x,y,z,u)\frac{\partial}{\partial x} + \eta_1(x,y,z,u)\frac{\partial}{\partial y} + \eta_2(x,y,z,u)\frac{\partial}{\partial z} + \eta_3(x,y,z,u)\frac{\partial}{\partial u}.$$

According to the Lie algorithm [11], $X$ is admitted by system (3.3) if it satisfies the associated determining equations. The first part of the determining equation is given by

$$3F\xi_1 + G\xi_2 + H\xi_3 = 3\xi_1''y + \xi_2''z + \xi_3''u - \xi_0'' - 2h_1',$$

$$F\xi_1 + 3G\xi_2 + H\xi_3 = \xi y + 3\xi_2''z + \xi_3''u - \xi_0'' - 2h_2',$$

$$F\xi_1 + G\xi_2 + 3H\xi_3 = \xi_1''y + \xi_2''z + 3\xi_3''u - \xi_0'' - 2h_3',$$

$$F\xi_2 = \xi_2''y + h_{12}',\quad F\xi_3 = \xi_3''y + h_{13}',$$

$$G\xi_1 = \xi_1''z + h_{21}', \quad G\xi_3 = \xi_3''z + h_{23}',$$
$$H\xi_1 = \xi_1''u + h_{31}', \quad H\xi_2 = \xi_2''u + h_{32}'$$

where

$$\xi(x,y,z,u) = \xi_1(x)y + \xi_2(x)z + \xi_3(x)u + \xi_0(x)$$

and the functions $h_i = h_i(x)$, and $h_{ij} = h_{ij}(x)$.

From these equations one can conclude that $\xi_1^2 + \xi_2^2 + \xi_3^2 \neq 0$ only for the case where two of the equations are equivalent to the free particle equation, for instance,

$$G = 0, \quad H = 0.$$

Hence we consider the case where

$$\xi_1 = 0, \quad \xi_2 = 0, \quad \xi_3 = 0.$$

The determining equations in this case are given by

$$\begin{gathered} F_y(y(\xi' + k_1) + zk_2 + uk_3 + \zeta_1) + F_z(yk_4 + (\xi' + k_5)z + uk_6 + \zeta_2) \\ +F_u(yk_7 + zk_8 + (\xi' + k_9)u + \zeta_3) \\ +2F_x\xi - \xi'''y + (3\xi' - k_1)F - k_2G - k_3H - \zeta_1'' = 0, \end{gathered}$$

$$\begin{gathered} G_y(y(\xi' + k_1) + zk_2 + uk_3 + \zeta_1) + G_z(yk_4 + (\xi' + k_5)z + uk_6 + \zeta_2) \\ +G_u(yk_7 + zk_8 + (\xi' + k_9)u + \zeta_3) \\ +2G_x\xi - \xi'''z - k_4F + (3\xi' - k_5)G - k_6H - \zeta_2'' = 0, \end{gathered}$$

$$\begin{gathered} H_y(y(\xi' + k_1) + zk_2 + uk_3 + \zeta_1) + H_z(yk_4 + (\xi' + k_5)z + uk_6 + \zeta_2) \\ +H_u(yk_7 + zk_8 + (\xi' + k_9)u + \zeta_3) \\ +2H_x\xi - \xi'''u - k_7F - k_8G + (3\xi' - k_9)H - \zeta_3'' = 0, \end{gathered}$$

where an admitted generator has the form

$$\begin{aligned} X = {} & 2\xi\frac{\partial}{\partial x} + ((\xi' + k_1)y + k_2z + k_3u + \zeta_1)\frac{\partial}{\partial y} \\ & +(k_4y + (\xi' + k_5)z + k_6u + \zeta_2)\frac{\partial}{\partial z} \\ & +(k_7y + k_8z + (\xi' + k_9)u + \zeta_3)\frac{\partial}{\partial u} \end{aligned}$$

with $\xi = \xi(x)$, $\zeta_i = \zeta_i(x)$ and $k_i$, $i = 1, 2, 3$ constant.

For further analysis the study of the determining equations is separated into the cases:

(a) the case in which there is at least one admitted generator with $\xi \neq 0$;

(b) the case in which for all admitted generators $\xi = 0$.

### 3.2.3 Case $\xi \neq 0$

We consider the generator $X_o$ for which $\xi \neq 0$. Using the equivalence transformation:

$$y_1 = y + \varphi_1(x), \quad z_1 = z + \varphi_2, \quad u_1 = u + \varphi_3(x),$$

the generator $X_o$ becomes

$$\begin{array}{l} X_o = 2\xi\frac{\partial}{\partial x} + ((\xi' + k_1)y_1 + k_2 z_1 + k_3 u_1 + \widetilde{\zeta}_1)\frac{\partial}{\partial y_1} \\ \quad +(k_4 y_1 + (\xi' + k_5)z_1 + k_6 u_1 + \widetilde{\zeta}_2)\frac{\partial}{\partial z_1} \\ \quad +(k_7 y_1 + k_8 z_1 + (\xi' + k_9)u_1 + \widetilde{\zeta}_3)\frac{\partial}{\partial u_1}, \end{array}$$

where

$$\widetilde{\zeta}_1 = 2\xi\varphi_1' - (\xi' + k_1)\varphi_1 - k_2\varphi_2 - k_3\varphi_3 + \zeta_1,$$

$$\widetilde{\zeta}_2 = 2\xi\varphi_2' - k_4\varphi_1 - (\xi' + k_5)\varphi_2 - k_6\varphi_3 + \zeta_2,$$

$$\widetilde{\zeta}_3 = 2\xi\varphi_3' - k_7\varphi_1 - k_8\varphi_2 - (\xi' + k_9)\varphi_3 + \zeta_3.$$

The functions $\varphi_i(x)$ $(i = 1, 2, 3)$ can be chosen such that

$$\widetilde{\zeta}_1 = 0, \quad \widetilde{\zeta}_2 = 0, \quad \widetilde{\zeta}_3 = 0.$$

Then without loss of generality one can assume that the generator $X_o$ has the form

$$\begin{array}{l} X_o = 2\xi\frac{\partial}{\partial x} + ((\xi' + k_1)y + k_2 z + k_3 u)\frac{\partial}{\partial y} + (k_4 y + (\xi' + k_5)z + k_6 u)\frac{\partial}{\partial z} \\ \quad +(k_7 y + k_8 z + (\xi' + k_9)u)\frac{\partial}{\partial u}. \end{array}$$

The equivalence transformation

$$x_1 = \alpha(x), \quad y_1 = y\beta(x), \quad z_1 = z\beta(x), \quad u_1 = u\beta(x),$$

where

$$\alpha''\beta = 2\alpha'\beta', \quad (\alpha'\beta \neq 0),$$

reduces the generator $X_o$ to

$$\begin{array}{l} X_o = 2\alpha'\xi\frac{\partial}{\partial x_1} + ((2\xi\beta'/\beta + \xi' + k_1)y_1 + k_2 z_1 + k_3 u_1)\frac{\partial}{\partial y_1} \\ \quad +(k_4 y_1 + (2\xi\beta'/\beta + \xi' + k_5)z_1 + k_6 u_1)\frac{\partial}{\partial z_1} \\ \quad +(k_7 y_1 + k_8 z_1 + (2\xi\beta'/\beta + \xi' + k_9)u_1)\frac{\partial}{\partial u_1}. \end{array}$$

Choosing $\beta(x)$ such that $2\xi\beta'/\beta + \xi' = 0$ makes the generator $X_o$ become

$$\begin{array}{l} X_o = 2\alpha'\xi\frac{\partial}{\partial x_1} + (k_1 y_1 + k_2 z_1 + k_3 u_1)\frac{\partial}{\partial y_1} + (k_4 y_1 + k_5 z_1 + k_6 u_1)\frac{\partial}{\partial z_1} \\ \quad +(k_7 y_1 + k_8 z_1 + k_9 u_1)\frac{\partial}{\partial u_1}. \end{array}$$

Note that in this case

$$\frac{d(\alpha'\xi)}{dx_1} = 0.$$

This means

$$\frac{d(\alpha'\xi)}{dx_1} = \frac{(\alpha'\xi)'}{\alpha'} = \xi' + \frac{\alpha''}{\alpha'}\xi = -2\xi\frac{\beta'}{\beta} + 2\frac{\beta'}{\beta}\xi = 0.$$

Hence without loss of generality we can assume that the generator $X_o$ has the form

$$X_o = k\tfrac{\partial}{\partial x} + (k_1 y + k_2 z + k_3 u)\tfrac{\partial}{\partial y} + (k_4 y + k_5 z + k_6 u)\tfrac{\partial}{\partial z} \\ +(k_7 y + k_8 z + k_9 u)\tfrac{\partial}{\partial u},$$

where $k = 2\alpha'\xi \neq 0$ is constant. We rewrite the generator $X_o$ in the form

$$X_o = \partial_x + (a_{11} y + a_{12} z + a_{13} u)\tfrac{\partial}{\partial y} + (a_{21} y + a_{22} z + a_{23} u)\tfrac{\partial}{\partial z} \\ +(a_{31} y + a_{32} z + a_{33} u)\tfrac{\partial}{\partial u}.$$

The determining equations become

$$\begin{aligned}
&(a_{11}y + a_{12}z + a_{13}u)\,F_y + (a_{21}y + a_{22}z + a_{23}u)\,F_z \\
&+ (a_{31}y + a_{32}z + a_{33}u)\,F_u + F_x = a_{11}F + a_{12}G + a_{13}H, \\
&(a_{11}y + a_{12}z + a_{13}u)\,G_y + (a_{21}y + a_{22}z + a_{23}u)\,G_z \\
&+ (a_{31}y + a_{32}z + a_{33}u)\,G_u + G_x = a_{21}F + a_{22}G + a_{23}H, \\
&(a_{11}y + a_{12}z + a_{13}u)\,H_y + (a_{21}y + a_{22}z + a_{23}u)\,H_z \\
&+ (a_{31}y + a_{32}z + a_{33}u)\,H_u + H_x = a_{31}F + a_{32}G + a_{33}H.
\end{aligned} \tag{3.6}$$

Here $a_{ij}$, $(i, j = 1, 2, 3)$ are constant. In matrix form these equations are rewritten as

$$\mathbf{F}_x + ((A\mathbf{y}) \cdot \nabla)\,\mathbf{F} - A\mathbf{F} = 0, \tag{3.7}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \nabla = \begin{pmatrix} \partial_y \\ \partial_z \\ \partial_u \end{pmatrix}.$$

Here "·" means the scalar product

$$\mathbf{b} \cdot \nabla = b_i \partial_{y_i},$$

where a vector $\mathbf{b} = (b_1, b_2, b_3)$, $\mathbf{y} = (y_1, y_2, y_3)$, and it is also used standard agreement: summation with respect to a repeat index.

Further simplifications are related with simplifications of the matrix $A$.

We apply the change $\widetilde{\mathbf{y}} = P\mathbf{y}$ where $P$ is a nonsingular matrix with constant entries

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}.$$

Equations (3.3) become

$$\tilde{\mathbf{y}}'' = \tilde{\mathbf{F}}(x, \tilde{\mathbf{y}})$$

where

$$\tilde{\mathbf{F}}(x, \tilde{\mathbf{y}}) = P\mathbf{F}(x, P^{-1}\tilde{\mathbf{y}}).$$

The partial derivatives $\partial_y, \partial_z$ and $\partial_u$ are changed as follows:

$$\nabla = P^t\tilde{\nabla}.$$

Hence equations (3.7) become

$$\begin{aligned}
&\left((AP^{-1}\tilde{\mathbf{y}}) \cdot P^t\tilde{\nabla}\right)(P^{-1}\tilde{\mathbf{F}}) + P^{-1}\tilde{\mathbf{F}}_x - AP^{-1}\tilde{\mathbf{F}} \\
&= P^{-1}\left(\left((PAP^{-1}\tilde{\mathbf{y}}) \cdot \tilde{\nabla}\right)\tilde{\mathbf{F}} + \tilde{\mathbf{F}}_x - PAP^{-1}\tilde{\mathbf{F}}\right) \\
&= P^{-1}\left(\left((\tilde{A}\tilde{\mathbf{y}})^t\tilde{\nabla}\right) \cdot \tilde{\mathbf{F}} + \tilde{\mathbf{F}}_x - \tilde{A}\tilde{\mathbf{F}}\right) = 0
\end{aligned}$$

where

$$\tilde{A} = PAP^{-1}.$$

This means that the change $\tilde{\mathbf{y}} = P\mathbf{y}$ reduces equation (3.7) to the same form with the matrix $A$ changed. The infinitesimal generator is also changed as

$$X_o = \partial_x + (\tilde{A}\tilde{\mathbf{y}}) \cdot \tilde{\nabla}.$$

Using this change matrix $A$ can be presented in the Jordan form. For a real-valued $3 \times 3$ matrix $A$, if the matrix $P$ also has real-valued entries, then the Jordan matrix is one of the following four types:

$$J_1 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & d \end{pmatrix}, \quad J_2 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & -c & b \end{pmatrix}, \quad J_3 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 1 \\ 0 & 0 & b \end{pmatrix}, \quad J_4 = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}, \tag{3.8}$$

where $a$, $b$, $c$ and $d > 0$ are real numbers.

**Case $A = J_1$**

We assume that

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & d \end{pmatrix}.$$

In this case the equations for the functions F, G and H are

$$\begin{aligned} ayF_y + bzF_z + duF_u + F_x &= aF, \\ ayG_y + bzG_z + duG_u + G_x &= bG, \\ ayH_y + bzH_z + duH_u + H_x &= dH. \end{aligned}$$

The general solution of these equations is

$$\begin{gathered} F(x,y,z,u) = e^{ax} f(s,v,w), \; G(x,y,z,u) = e^{bx} g(s,v,w), \\ H(x,y,z,u) = e^{dx} h(s,v,w) \end{gathered} \tag{3.9}$$

where

$$s = ye^{-ax}, \; v = ze^{-bx}, \; w = ue^{-dx}.$$

The admitted generator is

$$X_0 = \frac{\partial}{\partial x} + ay\frac{\partial}{\partial y} + bz\frac{\partial}{\partial z} + du\frac{\partial}{\partial u}.$$

**Case** $A = J_2$

We assume that

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & -c & b \end{pmatrix}.$$

In this case equations (3.6) become

$$\begin{aligned} ayF_y + (bz + cu) F_z + (-cz + bu) F_u + F_x &= aF, \\ ayG_y + (bz + cu) G_z + (-cz + bu) G_u + G_x &= bG + cH, \\ ayH_y + (bz + cu) H_z + (-cz + bu) H_u + H_x &= cG + bH. \end{aligned} \tag{3.10}$$

Here again we introduce the variables

$$\begin{gathered} s = ye^{-ax}, \; v = e^{-bx}(z\cos(cx) - u\sin(cx)), \\ w = e^{-bx}(z\sin(cx) + u\cos(cx)), \end{gathered}$$

equations (3.10) become

$$F_x - aF = 0, \; G_x - bG - cH = 0, \; H_x + cG - bH = 0.$$

The general solution of these equations is

$$\begin{aligned} F(x,y,z,u) &= e^{ax} f(s,v,w), \\ G(x,y,z,u) &= e^{bx}(\cos(cx)\, g(s,v,w) + \sin(cx)\, h(s,v,w)), \\ H(x,y,z,u) &= e^{bx}(-\sin(cx)\, g(s,v,w) + \cos(cx)\, h(s,v,w)) \end{aligned} \tag{3.11}$$

where $f(s,v,w), g(s,v,w)$ and $h(s,v,w)$ are arbitrary functions. The admitted generator is

$$X_0 = \frac{\partial}{\partial x} + ay\frac{\partial}{\partial y} + (bz + cu)\frac{\partial}{\partial z} + (-cz + bu)\frac{\partial}{\partial u}.$$

**Case $A = J_3$**

We assume that

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 1 \\ 0 & 0 & b \end{pmatrix}.$$

In this case equations (3.6) become

$$\begin{aligned} &ayF_y + (bz + cu) F_z + buF_u + F_x = aF, \\ &ayG_y + (bz + cu) G_z + buG_u + G_x = bG + H, \\ &ayH_y + (bz + cu) H_z + buH_u + H_x = bH. \end{aligned} \tag{3.12}$$

As in the previous case we introduce the variables

$$s = ye^{-ax}, \ v = e^{-bx} (z - ux), \ w = e^{-bx}u$$

so that equations (3.12) become

$$F_x - aF = 0, \ G_x - bG - H = 0, \ H_x - bH = 0.$$

The general solution of these equations is

$$\begin{aligned} &F(x, y, z, u) = e^{ax} f(s, v, w), \\ &G(x, y, z, u) = e^{bx} (h(s, v, w) x + g(s, v, w)), \\ &H(x, y, z, u) = e^{bx} h(s, v, w), \end{aligned} \tag{3.13}$$

where $f(s, v, w), g(s, v, w)$ and $h(s, v, w)$ are arbitrary functions. The admitted generator is

$$X_0 = \frac{\partial}{\partial x} + ay\frac{\partial}{\partial y} + (bz + cu) \frac{\partial}{\partial z} + bu\frac{\partial}{\partial u}.$$

**Case $A = J_4$**

We assume that

$$A = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}.$$

In this case equations (3.6) become

$$\begin{aligned} &(ay + z) F_y + (az + u) F_z + auF_u + F_x = aF, \\ &(ay + z) G_y + (az + u) G_z + auG_u + G_x = aG + H, \\ &(ay + z) H_y + (az + u) H_z + auH_u + H_x = aH. \end{aligned} \tag{3.14}$$

Introducing the variables

$$s = e^{-ax}\left(y - xz + \frac{1}{2}x^2u\right), \; v = e^{-ax}\left(z - xu\right), \; w = e^{-ax}u$$

so that equations (3.14) become

$$F_x - aF - G = 0, \; G_x - aG - H = 0, \; H_x - aH = 0.$$

The general solution of these equations is

$$\begin{aligned} &F\left(x,y,z,u\right) = e^{ax}\left(h\left(s,v,w\right)\tfrac{x^2}{2} + g\left(s,v,w\right) + f\left(s,v,w\right)\right),\\ &G\left(x,y,z,u\right) = e^{ax}\left(h\left(s,v,w\right)x + g\left(s,v,w\right)\right),\\ &H\left(x,y,z,u\right) = e^{bx}h\left(s,v,w\right) \end{aligned} \tag{3.15}$$

where $f(s,v,w), g(s,v,w)$ and $h(s,v,w)$ are arbitrary functions. The admitted generator is

$$X_0 = \frac{\partial}{\partial x} + (ay+z)\frac{\partial}{\partial y} + (az+u)\frac{\partial}{\partial z} + au\frac{\partial}{\partial u}.$$

### 3.2.4 Case $\xi = 0$

Substituting $\xi = 0$ into the determining equations we find that

$$\begin{aligned} &(a_{11}y + a_{12}z + a_{13}u + \zeta_1)F_y + (a_{21}y + a_{22}z + a_{23}u + \zeta_2)F_z\\ &+(a_{31}y + a_{32}z + a_{33}u + \zeta_3)F_u = a_{11}F + a_{12}G + a_{13}H + \zeta_1'',\\ &(a_{11}y + a_{12}z + a_{13}u + \zeta_1)G_y + (a_{21}y + a_{22}z + a_{23}u + \zeta_2)G_z\\ &+(a_{31}y + a_{32}z + a_{33}u + \zeta_3)G_u = a_{21}F + a_{22}G + a_{23}H + \zeta_2'',\\ &(a_{11}y + a_{12}z + a_{13}u + \zeta_1)H_y + (a_{21}y + a_{22}z + a_{23}u + \zeta_2)H_z\\ &+(a_{31}y + a_{32}z + a_{33}u + \zeta_3)H_u = a_{31}F + a_{32}G + a_{33}H + \zeta_3'', \end{aligned} \tag{3.16}$$

or in matrix form

$$\left(\left(A\mathbf{y} + \mathbf{h}\right)\cdot\nabla\right)\mathbf{F} = A\mathbf{F} + \mathbf{h}'' \tag{3.17}$$

where

$$A = \begin{pmatrix} k_1 & k_2 & k_3\\ k_4 & k_5 & k_6\\ k_7 & k_8 & k_9 \end{pmatrix}, \quad \mathbf{h}(x) = \begin{pmatrix} \zeta_1(x)\\ \zeta_2(x)\\ \zeta_3(x) \end{pmatrix}.$$

Similarly to the case where $\xi \neq 0$ we use the Jordan forms (3.8) of the matrix $A$. The admitted generator has the form

$$\begin{aligned} X_o = &(k_1y + k_2z + k_3u + \zeta_1(x))\partial_y + (k_4y + k_5z + k_6u + \zeta_2(x))\partial_z\\ &+(k_7y + k_8z + k_9u + \zeta_3(x))\partial_u. \end{aligned}$$

**Case $A = J_1$**

Assuming that $A = J_1$, equations (3.16) for the functions $F$ and $G$ are

$$(ay+h_1)F_y + (bz+h_2)F_z + (du+h_3)F_u = aF + h_1'',$$
$$(ay+h_1)G_y + (bz+h_2)G_z + (du+h_3)G_u = bG + h_2'',$$
$$(ay+h_1)H_y + (bz+h_2)H_z + (du+h_3)H_u = dH + h_3''.$$

The general solution of these equations depends on a value of $a, b$ and $d$:

- Case: $a \neq 0,\ b \neq 0,\ d \neq 0$

$$aF + h_1'' = (ay+h_1)f(x,s,v), \quad bG + h_2'' = (bz+h_2)g(x,s,v),$$
$$dH + h_3'' = (du+h_3)h(x,s,v), \quad s = (bz+h_2)^a(ay+h_1)^{-b},$$
$$v = (du+h_3)^b(bz+h_2)^{-d}.$$

- Case: $a \neq 0,\ b \neq 0,\ d = 0$

$$aF + h_1'' = (ay+h_1)f(x,s,v), \quad bG + h_2'' = (bz+h_2)g(x,s,v),$$
$$H = \tfrac{h_3''}{a}\ln(ay+h_1) + h(x,s,v), \quad s = (bz+h_2)^a(ay+h_1)^{-b},$$
$$v = u - \tfrac{h_3}{b}\ln(bz+h_2).$$

- Case: $a \neq 0,\ b = 0,\ d = 0$

$$aF + h_1'' = (ay+h_1)f(x,s,v), \quad G = \tfrac{h_2''}{a}\ln(ay+h_1) + g(x,s,v),$$
$$H = \tfrac{h_3''}{a}\ln(ay+h_1) + h(x,s,v), \quad s = z - \tfrac{h_2}{a}\ln(ay+h_1),$$
$$v = u - \tfrac{h_3}{a}\ln(ay+h_1).$$

- Case: $a = 0,\ b = 0,\ d = 0, h_1 \neq 0$

$$F = \tfrac{h_1''}{h_1}y + f(x,s,v), \quad G = \tfrac{h_2''}{h_1}y + g(x,s,v),$$
$$H = \tfrac{h_3''}{h_1}y + h(x,s,v), \quad s = z - \tfrac{h_2}{h_1}y, \quad v = u - \tfrac{h_3}{h_1}y.$$

- Case: $a = 0,\ b = 0,\ d = 0, h_1 = 0, h_2 \neq 0$

$$F = f(x,s), \quad G = \tfrac{h_2''}{h_2}z + g(x,s), \quad H = \tfrac{h_3''}{h_2}z + h(x,s), \quad s = u - \tfrac{h_3}{h_2}z.$$

Here $f(x,s,v), g(x,s,v)$ and $h(x,s,v)$ are arbitrary functions.

### 3.2.5 Case $A = J_2$

In this case equations (3.16) become

$$\begin{aligned}&(ay+h_1)F_y+(bz+cu+h_2)F_z+(-cz+bu+h_3)F_u=aF+h_1'',\\&(ay+h_1)G_y+(bz+cu+h_2)G_z+(-cz+bu+h_3)G_u=bG+cH+h_2'',\\&(ay+h_1)H_y+(bz+cu+h_2)H_z+(-cz+bu+h_3)H_u=-cG+bH+h_3''.\end{aligned}\tag{3.18}$$

- Case: $a \neq 0$

  Introducing the variables

$$y=\overline{y}-\tfrac{h_1}{a},\quad z=\overline{z}-(b^2+c^2)^{-1}(bh_2-ch_3),$$
$$u=\overline{u}-(b^2+c^2)^{-1}(ch_2+bh_3),$$

$$F=\overline{F}-\tfrac{h_1''}{a},\quad G=\overline{G}-(b^2+c^2)^{-1}(bh_2''-ch_3''),$$
$$H=\overline{H}-(b^2+c^2)^{-1}(ch_2''+bh_3'')$$

  equations (3.4) become

$$\begin{aligned}&a\overline{y}\overline{F}_{\overline{y}}+(b\overline{z}+c\overline{u})\overline{F}_{\overline{z}}+(-c\overline{z}+b\overline{u})\overline{F}_{\overline{u}}=a\overline{F},\\&a\overline{y}\overline{G}_{\overline{y}}+(b\overline{z}+c\overline{u})\overline{G}_{\overline{z}}+(-c\overline{z}+b\overline{u})\overline{G}_{\overline{u}}=b\overline{G}+c\overline{H},\\&a\overline{y}\overline{H}_{\overline{y}}+(b\overline{z}+c\overline{u})\overline{H}_{\overline{z}}+(-c\overline{z}+b\overline{u})\overline{H}_{\overline{u}}=-c\overline{G}+b\overline{H}.\end{aligned}$$

  In the variables

$$\overline{y}=ve^{as},\quad \overline{z}=we^{bs}\sin(cs),\quad \overline{u}=we^{bs}\cos(cs)$$

  these equations are

$$\overline{F}_s=a\overline{F},\quad \overline{G}_s=b\overline{G}+c\overline{H},\quad \overline{H}_s=-c\overline{G}+b\overline{H}.$$

  The general solution of the last set of equations is

$$\overline{F}(x,y,z,u)=e^{as}f(x,v,w),$$

$$\overline{G}(x,y,z,u)=e^{as}\left(\cos(cs)g(x,v,w)+\sin(cs)h(x,v,w)\right),$$

$$\overline{H}(x,y,z,u)=e^{bs}\left(-\sin(cs)g(x,v,w)+\cos(cs)h(x,v,w)\right)$$

  where $f(x,v,w), g(x,v,w)$ and $h(x,v,w)$ are arbitrary functions.

- Case: $a = 0$

  In this case equations (3.4) become

$$\begin{aligned}&h_1F_y+(bz+cu+h_2)F_z+(-cz+bu+h_3)F_u=h_1'',\\&h_1G_y+(bz+cu+h_2)G_z+(-cz+bu+h_3)G_u=bG+cH+h_2'',\\&h_1H_y+(bz+cu+h_2)H_z+(-cz+bu+h_3)H_u=-cG+bH+h_3''.\end{aligned}\tag{3.19}$$

– If $h_1 \neq 0$, introducing the variables

$$y = \overline{y}h_1, \quad z = \overline{z} - (b^2+c^2)^{-1}(bh_2 - ch_3), \quad u = \overline{u} - (b^2+c^2)^{-1}(ch_2 + bh_3),$$

$$F = \overline{F}, \quad G = \overline{G} - (b^2+c^2)^{-1}(bh_2'' - ch_3''), \quad H = \overline{H} - (b^2+c^2)^{-1}(ch_2'' + bh_3'')$$

equations (3.19) become

$$\begin{aligned} &\overline{F}_{\overline{y}} + (b\overline{z} + c\overline{u})\overline{F}_{\overline{z}} + (-c\overline{z} + b\overline{u})\overline{F}_{\overline{u}} = h_1'', \\ &\overline{G}_{\overline{y}} + (b\overline{z} + c\overline{u})\overline{G}_{\overline{z}} + (-c\overline{z} + b\overline{u})\overline{G}_{\overline{u}} = b\overline{G} + c\overline{H}, \\ &\overline{H}_{\overline{y}} + (b\overline{z} + c\overline{u})\overline{H}_{\overline{z}} + (-c\overline{z} + b\overline{u})\overline{H}_{\overline{u}} = -c\overline{G} + b\overline{H}. \end{aligned}$$

In the variables

$$s = \overline{y}, \quad v = e^{-b\overline{y}}\left(\overline{z}\cos(c\overline{y}) - \overline{u}\sin(c\overline{y})\right), \quad w = e^{-b\overline{y}}\left(\overline{z}\sin(c\overline{y}) + \overline{u}\cos(c\overline{y})\right)$$

these equations become

$$\overline{F}_s = h_1'', \quad \overline{G}_s = b\overline{G} + c\overline{H}, \quad \overline{H}_s = -c\overline{G} + b\overline{H}.$$

The general solution of the last set of equations is

$$\overline{F}(x,y,z,u) = h_1''s + f(x,v,w),$$

$$\overline{G}(x,y,z,u) = e^{bs}\left(\cos(cs)g(x,v,w) + \sin(cs)h(x,v,w)\right),$$

$$\overline{H}(x,y,z,u) = e^{bs}\left(-\sin(cs)g(x,v,w) + \cos(cs)h(x,v,w)\right).$$

– If $h_1 = 0$, in this case equations (3.19) become

$$\begin{aligned} &(bz + cu + h_2)F_z + (-cz + bu + h_3)F_u = 0, \\ &(bz + cu + h_2)G_z + (-cz + bu + h_3)G_u = bG + cH + h_2'', \\ &(bz + cu + h_2)H_z + (-cz + bu + h_3)H_u = -cG + bH + h_3''. \end{aligned} \tag{3.20}$$

Introducing the variables

$$z = \overline{z} - (b^2+c^2)^{-1}(bh_2 - ch_3), \quad u = \overline{u} - (b^2+c^2)^{-1}(ch_2 + bh_3),$$

$$F = \overline{F}, \quad G = \overline{G} - (b^2+c^2)^{-1}(bh_2'' - ch_3''), \quad H = \overline{H} - (b^2+c^2)^{-1}(ch_2'' + bh_3'')$$

equations (3.20) become

$$\begin{aligned}
(b\overline{z}+c\overline{u})\overline{F}_{\overline{z}}+(-c\overline{z}+b\overline{u})\overline{F}_{\overline{u}}&=0,\\
(b\overline{z}+c\overline{u})\overline{G}_{\overline{z}}+(-c\overline{z}+b\overline{u})\overline{G}_{\overline{u}}&=b\overline{G}+c\overline{H},\\
(b\overline{z}+c\overline{u})\overline{H}_{\overline{z}}+(-c\overline{z}+b\overline{u})\overline{H}_{\overline{u}}&=-c\overline{G}+b\overline{H}.
\end{aligned}$$

In the variables

$$\overline{y}=w,\quad \overline{z}=ve^{bs}\sin(cs),\quad \overline{u}=ve^{bs}\cos(cs)$$

these equations become

$$\overline{F}_s=0,\quad \overline{G}_s=b\overline{G}+c\overline{H},\quad \overline{H}_s=-c\overline{G}+b\overline{H}$$

and their general solution is

$$\begin{aligned}
\overline{F}(x,y,z,u)&=f(x,v,w),\\
\overline{G}(x,y,z,u)&=e^{bs}\left(\cos(cs)g(x,v,w)+\sin(cs)h(x,v,w)\right),\\
\overline{H}(x,y,z,u)&=e^{bs}\left(-\sin(cs)g(x,v,w)+\cos(cs)h(x,v,w)\right).
\end{aligned}$$

**Case** $A=J_3$

In this case equations (3.16) become

$$\begin{aligned}
(ay+h_1)F_y+(bz+u+h_2)F_z+(bu+h_3)F_u&=aF+h_1'',\\
(ay+h_1)G_y+(bz+u+h_2)G_z+(bu+h_3)G_u&=bG+H+h_2'',\\
(ay+h_1)H_y+(bz+u+h_2)H_z+(bu+h_3)H_u&=bH+h_3''.
\end{aligned}\tag{3.21}$$

- Case: $a\neq 0, b\neq 0$

  Introducing the variables

$$y=\overline{y}-\frac{h_1}{a},\quad z=\overline{z}-\frac{h_2}{b}+\frac{h_3}{b^2},\quad u=\overline{u}-\frac{h_3}{b},$$

$$F=\overline{F}-\frac{h_1''}{a},\quad G=\overline{G}+\frac{h_3''}{b^2}-\frac{h_2''}{b},\quad H=\overline{H}-\frac{h_3''}{b}$$

  equations (3.21) become

$$\begin{aligned}
a\overline{y}\overline{F}_{\overline{y}}+(b\overline{z}+\overline{u})\overline{F}_{\overline{z}}+b\overline{u}\overline{F}_{\overline{u}}&=a\overline{F},\\
a\overline{y}\overline{G}_{\overline{y}}+(b\overline{z}+\overline{u})\overline{G}_{\overline{z}}+b\overline{u}\overline{G}_{\overline{u}}&=b\overline{G}+\overline{H},\\
a\overline{y}\overline{H}_{\overline{y}}+(b\overline{z}+\overline{u})\overline{H}_{\overline{z}}+b\overline{u}\overline{H}_{\overline{u}}&=b\overline{H}.
\end{aligned}$$

In the variables

$$\overline{y} = e^{as}, \quad \overline{z} = e^{bs}(sw + v), \quad \overline{u} = e^{bs}w$$

these equations are

$$\overline{F}_s = a\overline{F}, \quad \overline{G}_s = b\overline{G} + c\overline{H}, \quad \overline{H}_s = b\overline{H}$$

with general solution

$$\begin{aligned} \overline{F}(x,y,z,u) &= e^{as} f(x,v,w), \\ \overline{G}(x,y,z,u) &= e^{bs}(h(x,v,w)s + g(x,v,w)), \\ \overline{H}(x,y,z,u) &= e^{bs} h(x,v,w) \end{aligned}$$

where $f(x,v,w), g(x,v,w)$ and $h(x,v,w)$ are arbitrary functions.

- Case: $a \neq 0, b = 0$

  In this case equations (3.21) become

$$\begin{aligned} (ay + h_1)F_y + (u + h_2)F_z + h_3 F_u &= aF + h_1'', \\ (ay + h_1)G_y + (u + h_2)G_z + h_3 G_u &= H + h_2'', \\ (ay + h_1)H_y + (u + h_2)H_z + h_3 H_u &= h_3''. \end{aligned} \tag{3.22}$$

  Introducing the variables

$$y = \overline{y} - \frac{h_1}{a}, \quad z = \overline{z}, \quad u = \overline{u} - h_2,$$

$$F = \overline{F} - \frac{h_1''}{a}, \quad G = \overline{G}, \quad H = \overline{H} - h_2''$$

  equations (3.22) become

$$\begin{aligned} a\overline{y}\overline{F}_{\overline{y}} + \overline{u}\overline{F}_{\overline{z}} + h_3\overline{F}_{\overline{u}} &= a\overline{F}, \\ a\overline{y}\overline{G}_{\overline{y}} + \overline{u}\overline{G}_{\overline{z}} + h_3\overline{G}_{\overline{u}} &= \overline{H}, \\ a\overline{y}\overline{H}_{\overline{y}} + \overline{u}\overline{H}_{\overline{z}} + h_3\overline{H}_{\overline{u}} &= h_3''. \end{aligned} \tag{3.23}$$

  - If $h_3 \neq 0$, then in the variables

$$\overline{y} = e^{as}, \quad \overline{z} = \frac{h_3 s^2}{2} + ws + v, \quad \overline{u} = h_3 s + w$$

    equations (3.23) become

$$\overline{F}_s = a\overline{F}, \quad \overline{G}_s = \overline{H}, \quad \overline{H}_s = h_3''$$

    with general solution

$$\begin{aligned} \overline{F}(x,y,z,u) &= e^{as} f(x,v,w), \\ \overline{G}(x,y,z,u) &= \frac{h_3 s^2}{2} + h(x,v,w)s + g(x,v,w), \\ \overline{H}(x,y,z,u) &= h_3'' s + h(x,v,w). \end{aligned}$$

– If $h_3 = 0$ then in the variables

$$\overline{y} = e^{as}, \quad \overline{z} = ws + v, \quad \overline{u} = w$$

equations (3.23) become

$$\overline{F}_s = a\overline{F}, \quad \overline{G}_s = \overline{H}, \quad \overline{H}_s = 0$$

and their general solution is

$$\begin{aligned} \overline{F}(x,y,z,u) &= e^{as} f(x,v,w), \\ \overline{G}(x,y,z,u) &= h(x,v,w)s + g(x,v,w), \\ \overline{H}(x,y,z,u) &= h(x,v,w). \end{aligned}$$

- Case: $a = 0, b \neq 0$

In this case equations (3.21) become

$$\begin{aligned} &h_1 F_y + (bz + u + h_2)F_z + (bu + h_3)F_u = h_1'', \\ &h_1 G_y + (bz + u + h_2)G_z + (bu + h_3)G_u = bG + H + h_2'', \\ &h_1 H_y + (bz + u + h_2)H_z + (bu + h_3)H_u = bH + h_3''. \end{aligned} \tag{3.24}$$

Introducing the variables

$$\begin{aligned} &y = \overline{y}, \quad z = \overline{z} - \frac{h_2}{b} + \frac{h_3}{b^2}, \quad u = \overline{u} - \frac{h_3}{b}, \\ &F = \overline{F}, \quad G = \overline{G} - \frac{h_2''}{b} + \frac{h_3''}{b^2}, \quad H = \overline{H} - \frac{h_3''}{b} \end{aligned}$$

equations (3.24) become

$$\begin{aligned} &h_1 \overline{F}_{\overline{y}} + (b\overline{z} + \overline{u})\overline{F}_{\overline{z}} + b\overline{u}\overline{F}_{\overline{u}} = h_1'', \\ &h_1 \overline{G}_{\overline{y}} + (b\overline{z} + \overline{u})\overline{G}_{\overline{z}} + b\overline{u}\overline{G}_{\overline{u}} = b\overline{G} + \overline{H}, \\ &h_1 \overline{H}_{\overline{y}} + (b\overline{z} + \overline{u})\overline{H}_{\overline{z}} + b\overline{u}\overline{H}_{\overline{u}} = b\overline{H}. \end{aligned} \tag{3.25}$$

– If $h_1 \neq 0$, then in the variables

$$\overline{y} = h_1 s, \quad \overline{z} = e^{bs}(ws + v), \quad \overline{u} = e^{bs} w$$

equations (3.25) become

$$\overline{F}_s = h_1'', \quad \overline{G}_s = b\overline{G} + \overline{H}, \quad \overline{H}_s = b\overline{H}.$$

The general solution of the last set of equations is

$$\begin{aligned} \overline{F}(x,y,z,u) &= h_1'' s + f(x,v,w), \\ \overline{G}(x,y,z,u) &= e^{bs}(h(x,v,w)s + g(x,v,w)), \\ \overline{H}(x,y,z,u) &= e^{bs} h(x,v,w). \end{aligned}$$

– If $h_1 = 0$, then in the variables

$$\overline{y} = w, \quad \overline{z} = e^{bs}(w+v), \quad \overline{u} = e^{bs}w$$

equations (3.25) become

$$\overline{F}_s = 0, \quad \overline{G}_s = b\overline{G} + \overline{H}, \quad \overline{H}_s = b\overline{H}$$

and their general solution is

$$\begin{aligned} \overline{F}(x,y,z,u) &= f(x,v,w), \\ \overline{G}(x,y,z,u) &= e^{bs}(h(x,v,w)s + g(x,v,w)), \\ \overline{H}(x,y,z,u) &= e^{bs}h(x,v,w). \end{aligned}$$

- Case: $a = 0, b = 0$

In this case equations (3.21) become

$$\begin{aligned} &h_1F_y + (u+h_2)F_z + h_3F_u = h_1'', \\ &h_1G_y + (u+h_2)G_z + h_3G_u = H + h_2'', \\ &h_1H_y + (u+h_2)H_z + h_3H_u = h_3''. \end{aligned} \tag{3.26}$$

Introducing the variables

$$\begin{aligned} y = \overline{y}, \quad z = \overline{z}, \quad u = \overline{u} - h_2, \\ F = \overline{F}, \quad G = \overline{G}, \quad H = \overline{H} - h_2'' \end{aligned}$$

equations (3.26) become

$$\begin{aligned} &h_1\overline{F}_{\overline{y}} + \overline{u}\overline{F}_{\overline{z}} + h_3\overline{F}_{\overline{u}} = h_1'', \\ &h_1\overline{G}_{\overline{y}} + \overline{u}\overline{G}_{\overline{z}} + h_3\overline{G}_{\overline{u}} = \overline{H}, \\ &h_1\overline{H}_{\overline{y}} + \overline{u}\overline{H}_{\overline{z}} + h_3\overline{H}_{\overline{u}} = h_3''. \end{aligned} \tag{3.27}$$

– If $h_1 \neq 0, h_3 \neq 0$, then in the variables

$$\overline{y} = h_1 s, \quad \overline{z} = \frac{h_3 s^2}{2} + ws + v, \quad \overline{u} = h_3 s + w$$

equations (3.27) become

$$\overline{F}_s = h_1'', \quad \overline{G}_s = \overline{H}, \quad \overline{H}_s = h_3''.$$

The general solution of the last set of equations is

$$\begin{aligned} \overline{F}(x,y,z,u) &= h_1''s + f(x,v,w), \\ \overline{G}(x,y,z,u) &= \frac{h_3''s^2}{2} + h(x,v,w)s + g(x,v,w), \\ \overline{H}(x,y,z,u) &= h_3''s + h(x,v,w). \end{aligned}$$

– If $h_1 \neq 0, h_3 = 0$, then in the variables

$$\overline{y} = h_1 s, \quad \overline{z} = ws + v, \quad \overline{u} = w$$

equations (3.27) become

$$\overline{F}_s = h_1'', \quad \overline{G}_s = \overline{H}, \quad \overline{H}_s = 0.$$

The general solution of the last set of equations is

$$\overline{F}(x,y,z,u) = h_1'' s + f(x,v,w),$$

$$\overline{G}(x,y,z,u) = h(x,v,w)s + g(x,v,w),$$
$$\overline{H}(x,y,z,u) = h(x,v,w).$$

– If $h_1 = 0, h_3 \neq 0$, then in the variables

$$\overline{y} = w, \quad \overline{z} = \frac{h_3 s^2}{2} + ws + v, \quad \overline{u} = h_3 s + w$$

equations (3.27) become

$$\overline{F}_s = 0, \quad \overline{G}_s = \overline{H}, \quad \overline{H}_s = h_3''$$

with general solution

$$\overline{F}(x,y,z,u) = f(x,v,w),$$
$$\overline{G}(x,y,z,u) = \frac{h_3'' s^2}{2} + h(x,v,w)s + g(x,v,w),$$
$$\overline{H}(x,y,z,u) = h_3'' s + h(x,v,w).$$

– If $h_1 = 0, h_3 = 0$, then in the variables

$$\overline{y} = v, \quad \overline{z} = ws + v, \quad \overline{u} = w$$

equations (3.27) become

$$\overline{F}_s = 0, \quad \overline{G}_s = \overline{H}, \quad \overline{H}_s = 0$$

and their general solution is

$$\overline{F}(x,y,z,u) = f(x,v,w),$$

$$\overline{G}(x,y,z,u) = h(x,v,w)s + g(x,v,w),$$
$$\overline{H}(x,y,z,u) = h(x,v,w).$$

**Case $A = J_4$**

In this case equations (3.16) become

$$\begin{aligned}(ay+z+h_1)F_y+(az+u+h_2)F_z+(au+h_3)F_u &= aF+G+h_1'',\\ (ay+z+h_1)G_y+(az+u+h_2)G_z+(au+h_3)G_u &= aG+H+h_2'',\\ (ay+z+h_1)H_y+(az+u+h_2)H_z+(au+h_3)H_u &= aH+h_3''.\end{aligned} \tag{3.28}$$

- Case: $a \neq 0$

  Introducing the variables

$$y=\overline{y}-\frac{h_1}{a}+\frac{h_2}{a^2}-\frac{h_3}{a^3},\quad z=\overline{z}-\frac{h_2}{a}+\frac{h_3}{a^2},\quad u=\overline{u}-\frac{h_3}{a},$$
$$F=\overline{F}-\frac{h_1''}{a}+\frac{h_2''}{a^2}-\frac{h_3''}{a^3},\quad G=\overline{G}-\frac{h_2''}{a}+\frac{h_3''}{a^2},\quad H=\overline{H}-\frac{h_3''}{a}$$

  equations (3.28) become

$$\begin{aligned}(a\overline{y}+\overline{z})\overline{F}_{\overline{y}}+(a\overline{z}+\overline{u})\overline{F}_{\overline{z}}+a\overline{u}\overline{F}_{\overline{u}} &= a\overline{F}+\overline{G},\\ (a\overline{y}+\overline{z})\overline{G}_{\overline{y}}+(a\overline{z}+\overline{u})\overline{G}_{\overline{z}}+a\overline{u}\overline{G}_{\overline{u}} &= a\overline{G}+\overline{H},\\ (a\overline{y}+\overline{z})\overline{H}_{\overline{y}}+(a\overline{z}+\overline{u})\overline{H}_{\overline{z}}+a\overline{u}\overline{H}_{\overline{u}} &= a\overline{H}.\end{aligned}$$

  In the variables

$$\overline{y}=e^{as}(\frac{ws^2}{2}+v),\quad \overline{z}=e^{as}(ws+v),\quad \overline{u}=e^{as}w$$

  these equations are

$$\overline{F}_s=a\overline{F}+\overline{G},\quad \overline{G}_s=a\overline{G}+\overline{H},\quad \overline{H}_s=a\overline{H}.$$

  The general solution of the last set of equations is

$$\begin{aligned}\overline{F}(x,y,z,u) &= e^{as}(h(x,v,w)\frac{s^2}{2}+g(x,v,w)s+f(x,v,w)),\\ \overline{G}(x,y,z,u) &= e^{as}(h(x,v,w)s+g(x,v,w)),\\ \overline{H}(x,y,z,u) &= e^{as}h(x,v,w)\end{aligned}$$

  where $f(x,v,w), g(x,v,w)$ and $h(x,v,w)$ are arbitrary functions.

- Case: $a = 0$

  In this case (3.28) become

$$\begin{aligned}(z+h_1)F_y+(u+h_2)F_z+h_3F_u &= G+h_1'',\\ (z+h_1)G_y+(u+h_2)G_z+h_3G_u &= H+h_2'',\\ (z+h_1)H_y+(u+h_2)H_z+h_3H_u &= h_3''.\end{aligned} \tag{3.29}$$

Introducing the variables

$$y = \overline{y}, \quad z = \overline{z} - h_1, \quad u = \overline{u} - h_2,$$

$$F = \overline{F}, \quad G = \overline{G} - h_1'', \quad H = \overline{H} - h_2''$$

equations (3.29) become

$$\begin{aligned} &\overline{z}\overline{F}_{\overline{y}} + \overline{u}\overline{F}_{\overline{z}} + h_3\overline{F}_{\overline{u}} = \overline{G}, \\ &\overline{z}\overline{G}_{\overline{y}} + \overline{u}\overline{G}_{\overline{z}} + h_3\overline{G}_{\overline{u}} = \overline{H}, \\ &\overline{z}\overline{H}_{\overline{y}} + \overline{u}\overline{H}_{\overline{z}} + h_3\overline{H}_{\overline{u}} = h_3''. \end{aligned} \tag{3.30}$$

– If $h_3 \neq 0$, then in the variables

$$\overline{y} = \frac{h_3 s^3}{6} + vs + w, \quad \overline{z} = \frac{h_3 s^2}{2} + v, \quad \overline{u} = h_3 s$$

equations (3.30) become

$$\overline{F}_s = \overline{G}, \quad \overline{G}_s = \overline{H}, \quad \overline{H}_s = h_3''$$

with general solution

$$\overline{F}(x, y, z, u) = \frac{h_3 s^3}{6} + h(x, v, w)\frac{s^2}{2} + g(x, v, w)s + f(x, v, w),$$

$$\overline{G}(x, y, z, u) = \frac{h_3 s^2}{2} + h(x, v, w)s + g(x, v, w),$$

$$\overline{H}(x, y, z, u) = h_3 s + h(x, v, w).$$

– If $h_3 = 0$, then in the variables

$$\overline{y} = \frac{ws^2}{2} + vs, \quad \overline{z} = ws + v, \quad \overline{u} = w$$

equations (3.30) become

$$\overline{F}_s = \overline{G}, \quad \overline{G}_s = \overline{H}, \quad \overline{H}_s = 0$$

with general solution

$$\overline{F}(x, y, z, u) = h(x, v, w)\frac{s^2}{2} + g(x, v, w)s + f(x, v, w),$$

$$\overline{G}(x, y, z, u) = h(x, v, w)s + g(x, v, w),$$

$$\overline{H}(x, y, z, u) = h(x, v, w).$$

## 3.3 Systems of linear equations

Linear second-order ordinary differential equations have the following form,

$$\mathbf{y}'' = A(x)\mathbf{y}' + B(x)\mathbf{y} + f(x), \tag{3.31}$$

where $A(x)$ and $B(x)$ are matrices, and $f(x)$ is a vector. Using a particular solution $\mathbf{y}_p(x)$ and the change

$$\mathbf{y} = \tilde{\mathbf{y}} + \mathbf{y}_p,$$

we can, without loss of generality assume that $f(x) = 0$. The matrix $A(x)$ or $B(x)$ can also be assumed to be zero if we use the change, $\mathbf{y} = C(x)\tilde{\mathbf{y}}$, where $C = C(x)$ is a nonsingular matrix. In the present paper the matrix $A(x)$ is reduced to zero. In this case the linear equations (3.3) are linear functions of $y, z$ and $u$:

$$\begin{aligned}F(x,y,z,u) &= c_{11}(x)y + c_{12}(x)z + c_{13}(x)u,\\ G(x,y,z,u) &= c_{21}(x)y + c_{22}(x)z + c_{23}(x)u,\\ H(x,y,z,u) &= c_{31}(x)y + c_{32}(x)z + c_{33}(x)u.\end{aligned} \tag{3.32}$$

In matrix form one can write as

$$\mathbf{F}(x,\mathbf{y}) = C(x)\mathbf{y} \tag{3.33}$$

Any linear system of equations admits the following generators

$$y\partial_y + z\partial_z + u\partial_u, \tag{3.34}$$

$$\zeta_1(x)\partial_y + \zeta_2(x)\partial_z + \zeta_3(x)\partial_u \tag{3.35}$$

where $\zeta_1(x), \zeta_2(x)$ and $\zeta_3(x)$ are solutions of the equations:

$$\zeta_1'' = c_{11}\zeta_1 + c_{12}\zeta_2 + c_{13}\zeta_3,$$

$$\zeta_2'' = c_{21}\zeta_1 + c_{22}\zeta_2 + c_{23}\zeta_3,$$

$$\zeta_3'' = c_{31}\zeta_1 + c_{32}\zeta_2 + c_{33}\zeta_3.$$

For the classification problem one needs to study equations which admit generators different from (3.34) and (3.35).

### 3.3.1 Equivalence transformations

Similar to systems of two second-order ordinary differential equations calculations show that the equivalence transformations are defined by the following two types of transformations. The first type corresponds to the linear change of the dependent variables $\tilde{\mathbf{y}} = P\mathbf{y}$ with a constant nonsingular matrix $P$. The second type is

$$\tilde{x} = \varphi(x),\ \tilde{y} = y\psi(x),\ \tilde{z} = z\psi(x),\ \tilde{u} = u\psi(x),$$

where the functions $\varphi(x)$ and $\psi(x)$ satisfy the condition

$$\frac{\varphi''}{\varphi'} = 2\frac{\psi'}{\psi}.$$

**Remark.** In [29] it is shown that using the equivalence transformations of the presented above form, one can reduce any system of homogeneous linear second-order ordinary differential equations to a system with a matrix $C$ such that[1] $trac\ C = 0$. In this case one can obtain that the coefficient $\xi(x)$ of the admitted generator related with the independent variable $x$ has the form

$$\xi = k_2x^2 + k_1x + k_0,$$

where $k_i$, $(i = 0, 1, 2)$ are constant. This means that the admitted generators are defined up to constants. This property allows one to apply an algebraic approach for group classification. In algebraic approach the constants are defined from the algebraic properties of admitted Lie algebras. Substituting the constants into determining equations one obtains systems of ordinary differential equations for the entries of the matrix $C(x)$. Solving this system of equations one finds forms of the entries. For systems of two linear second-order ordinary differential equations it shown in [29] that these systems of ordinary differential equations for the entries can be solved. However, the complete study of the problem of group classification using the algebraic approach led to the study of more cases than those found in [18], where Ovsiannikov's approach was applied.

### 3.3.2 Case $\xi \neq 0$

Using the obtained general forms of equations admitting an infinitesimal generator with $\xi \neq 0$ linear systems of equations (3.9), (3.11), (3.13) and (3.15) have the following form:

$$\begin{aligned} f(s,v,w) &= \alpha_{11}s + \alpha_{12}v + \alpha_{13}w, \\ g(s,v,w) &= \alpha_{21}s + \alpha_{22}v + \alpha_{23}w, \\ h(s,v,w) &= \alpha_{31}s + \alpha_{32}v + \alpha_{33}w. \end{aligned} \tag{3.36}$$

[1]For a system with two equations this property was obtained in [12].

**Case $A = J_1$**

Substituting (3.36) into (3.32), the equations involving $F, G$ and $H$ become

$$\begin{aligned} F &= \alpha_{11}y + e^{\alpha x}\alpha_{12}z + e^{\beta x}\alpha_{13}u, \\ G &= e^{-\alpha x}\alpha_{21}y + \alpha_{22}z + e^{-(\alpha-\beta)x}\alpha_{23}u, \\ H &= e^{-\beta x}\alpha_{31}y + e^{(\alpha-\beta)x}\alpha_{32}z + \alpha_{33}u \end{aligned}$$

where $\alpha = a - b$ and $\beta = a - d$.

Without loss of generality one can assume that $\alpha_{12} = 1$. We can assume that $\alpha_{12} \neq 0$ because for $\alpha_{12} = 0$ we have $\alpha_{13} \neq 0$ otherwise the system is degenerate. Applying the scaling of $y$ we conclude that $\alpha_{12} = 1$. Thus the studied linear system of equations becomes

$$\begin{aligned} F &= \alpha_{11}y + e^{\alpha x}z + e^{\beta x}\alpha_{13}u, \\ G &= e^{-\alpha x}\alpha_{21}y + \alpha_{22}z + e^{-(\alpha-\beta)x}\alpha_{23}u, \\ H &= e^{-\beta x}\alpha_{31}y + e^{(\alpha-\beta)x}\alpha_{32}z + \alpha_{33}u. \end{aligned} \tag{3.37}$$

**Case $A = J_2$**

Substituting (3.36) into the functions $F, G$ and $H$ one finds

$$\begin{aligned} F &= \alpha_{11}y + e^{(a-b)x}\left(\alpha_{12}\cos(cx) + \alpha_{13}\sin(cx)\right)z + e^{(a-b)x}(\alpha_{13}\cos(cx) \\ &\quad -\alpha_{12}\sin(cx))u, \\ G &= e^{-(a-b)x}(\alpha_{21}\cos(cx) + \alpha_{31}\sin(cx))y + (\alpha_{22}\cos^2(cx) \\ &\quad +(\alpha_{23}+\alpha_{32})\cos(cx)\sin(cx) + \alpha_{33}\sin^2(cx))z \\ &\quad +(\alpha_{23}\cos^2(cx) + (\alpha_{33}-\alpha_{22})\cos(cx)\sin(cx) - \alpha_{32}\sin^2(cx))u, \end{aligned}$$

$$\begin{aligned} H &= e^{-(a-b)x}(\alpha_{31}\cos(cx) - \alpha_{21}\sin(cx))y + (\alpha_{32}\cos^2(cx) \\ &\quad +(\alpha_{33}-\alpha_{22})\cos(cx)\sin(cx) - \alpha_{23}\sin^2(cx))z \\ &\quad +(\alpha_{33}\cos^2(cx) - (\alpha_{32}+\alpha_{23})\cos(cx)\sin(cx) + \alpha_{22}\sin^2(cx))u. \end{aligned}$$

Using trigonometry formulae and introducing the constants $\beta$, $\gamma$, $c_1$ and $c_2$:

$$\begin{pmatrix} \alpha_{22} & \alpha_{23} \\ \alpha_{32} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} \beta + c_2 & \gamma - c_1 \\ \gamma + c_1 & -\beta + c_2 \end{pmatrix},$$

one can rewrite functions $F$, $G$ and $H$ in the form

$$\begin{aligned}
F &= \alpha_{11}y + e^{\alpha x}(\cos(cx)\,\alpha_{12} + \sin(cx)\,\alpha_{13})\,z + e^{\alpha x}(\cos(cx)\,\alpha_{13} \\
&\quad - \sin(cx)\,\alpha_{12})u, \\
G &= e^{-\alpha x}(\cos(cx)\,\alpha_{21} + \sin(cx)\,\alpha_{31})\,y + (\cos(2cx)\,\beta \\
&\quad + \sin(2cx)\,\gamma + c_2)z + (\cos(2cx)\,\gamma - \sin(2cx)\,\beta - c_1)\,u, \\
H &= e^{-\alpha x}(\cos(cx)\,\alpha_{31} - \sin(cx)\,\alpha_{21})\,y + (\cos(2cx)\,\gamma \\
&\quad - \sin(2cx)\,\beta + c_1)z - (\cos(2cx)\,\beta + \sin(2cx)\,\gamma - c_2)\,u,
\end{aligned}$$

where $\alpha = a - b$.

Applying rotation of the dependent variables $z$ and $u$, and their dilation, we can assume that $\alpha_{12} = 1$ and $\alpha_{13} = 0$. Hence the studied linear system of equations becomes

$$\begin{aligned}
&F = \alpha_{11}y + e^{\alpha x}\cos(cx)\,z - e^{\alpha x}\sin(cx)\,u, \\
&G = e^{-\alpha x}(\cos(cx)\,\alpha_{21} + \sin(cx)\,\alpha_{31})\,y + (\cos(2cx)\,\beta \\
&\quad + \sin(2cx)\,\gamma + c_2)z + (\cos(2cx)\,\gamma - \sin(2cx)\,\beta - c_1)\,u, \\
&H = e^{-\alpha x}(\cos(cx)\,\alpha_{31} - \sin(cx)\,\alpha_{21})\,y + (\cos(2cx)\,\gamma \\
&\quad - \sin(2cx)\,\beta + c_1)z - (\cos(2cx)\,\beta + \sin(2cx)\,\gamma - c_2)\,u.
\end{aligned} \tag{3.38}$$

**Case $A = J_3$**

Substituting (3.36) into the functions $F, G$ and $H$ one finds

$$\begin{aligned}
&F = \alpha_{11}y + e^{\alpha x}\alpha_{12}z + e^{\alpha x}(-\alpha_{12}x + \alpha_{13})\,u, \\
&G = e^{-\alpha x}(\alpha_{21} + \alpha_{31}x)\,y + (\alpha_{22} + \alpha_{32}x)\,z \\
&\quad + (\alpha_{23} + (\alpha_{33} - \alpha_{22})\,x - \alpha_{32}x^2)\,u, \\
&H = e^{-\alpha x}\alpha_{31}y + \alpha_{32}z + (\alpha_{33} - \alpha_{32}x)\,u
\end{aligned} \tag{3.39}$$

where $\alpha = a - b$.

**Case $A = J_4$**

Substituting (3.36) into the functions $F, G$ and $H$ one finds

$$\begin{aligned}
&F = \left(\lambda + \beta x + \tfrac{1}{2}\gamma x^2\right)\left(y - xz + \tfrac{1}{2}x^2u\right), \\
&G = e^{-\alpha x}(\beta + \gamma x)\left(y - xz + \tfrac{1}{2}x^2u\right), \\
&H = e^{-\alpha x}\gamma\left(y - xz + \tfrac{1}{2}x^2u\right)
\end{aligned} \tag{3.40}$$

where $\lambda = \alpha_{11} + \alpha_{12} + \alpha_{13}$, $\beta = \alpha_{21} + \alpha_{22} + \alpha_{23}$, $\gamma = \alpha_{31} + \alpha_{32} + \alpha_{33}$.

Since for $\gamma = 0$ the linear system of equations is reduced to the degenerate case $H = 0$, one has to consider $\gamma \neq 0$.

### 3.3.3 Case $\xi = 0$

Substituting (3.33) into (3.17) and splitting it with respect to $y, z$ and $u$ one has

$$CA - AC = 0. \tag{3.41}$$

A nontrivial admitted generator is of the form

$$X_o = (A\mathrm{y}) \cdot \nabla. \tag{3.42}$$

Equations (3.41) can be simplified by using the Jordan form of the matrix $A$. In particular, if $A = J_1$, then equations (3.41) become

$$\begin{array}{lll} c_{12}(b-a) = 0, & c_{13}(d-a) = 0, & c_{21}(a-b) = 0, \\ c_{23}(b-d) = 0, & c_{31}(a-d) = 0, & c_{32}(b-d) = 0. \end{array} \tag{3.43}$$

If $(a-b)^2 + (a-d)^2 + (b-d)^2 = 0$, then we find that $a = b = d$ and the generator (3.42) is also trivial. Hence we can assume that $a \neq b$. Using this assumption we see from equations (3.43) that

$$c_{12} = 0, \quad c_{21} = 0.$$

For a non-degenerate linear system one has to assume that $c_{13} \neq 0$, which leads to $a = d$ and

$$c_{23} = 0, \; c_{32} = 0.$$

The linear system of equations with $c_{21} = 0$ and $c_{23} = 0$ is degenerate.

Similar results are obtained for the for other cases of the matrix $A$. This means that in contrast to a linear system with two equations there is no linear systems with three dependent variables such that all nontrivial admitted generators have the form (3.42).

## 3.4 Solutions of the determining equations

Solving the determining equations leads to the following solutions:

- For system (3.37) there is the only nontrivial admitted generator

$$\partial_x - \alpha z \partial_z - \beta u \partial_u. \tag{3.44}$$

- In the case of system (3.38) there is the only nontrivial admitted generator

$$\partial_x + \alpha y \partial_y + c\left(u \partial_z - z \partial_u\right). \tag{3.45}$$

- Similarly for system (3.39) there is the only nontrivial admitted generator

$$\partial_x + \alpha y \partial_y + u \partial_z. \tag{3.46}$$

- Finally for system (3.40) there is the only nontrivial admitted generator

$$\partial_x + z \partial_y + u \partial_z. \tag{3.47}$$

**Theorem.** All non-degenerate linear systems of three second-order ordinary differential equations admitting a non-trivial generator are equivalent to one of the cases:
(a) system (3.37) with the generator (3.44),
(b) system (3.38) with the generator (3.45),
(c) system (3.39) with the generator (3.46),
(d) system (3.40) with the generator (3.47).

## 3.5 Conclusion

The results of the current paper are obtained using Ovsiannikov's approach [4], which involves simplifying one generator and finding the associated functions. These functions are then used to solve the determining equations. We found all forms of nonlinear systems $\mathbf{y}'' = \mathbf{F}(x, \mathbf{y})$ admitting at least one generator. Finding their forms is reduced to solving a linear system of first-order ordinary differential equations with constant coefficients. Methods for solving these systems are well-known in the theory of ordinary differential equations; their solution is given by studying purely algebraic properties of some constant matrix $A$. Since the matrix $A$ is considered in the Jordan form, these properties are simple in each particular case.

Any normal linear system of second-order ordinary differential equations can be reduced to the form

$$\mathbf{y}'' = C\mathbf{y},$$

where $C(x)$ is a square matrix. These systems admit the set of trivial generators $\mathbf{y} \cdot \nabla$, $\mathbf{h}(x) \cdot \nabla$, where $\mathbf{h}'' = C\mathbf{h}$. Complete group classification of linear systems containing three equations and admitting a nontrivial generator is given; it is shown that all such systems have one of the four forms . We have also listed an explicit form of the non-trivial admitted generators for each of the respective cases. The obtained results are summarized in the theorem.