Chapter 2

Linearization of system of three second-order ordinary differential equations via point transformations

2.1 Introduction

In general, physical applications of differential equations are in the form of nonlinear equations, which are very difficult to solve explicitly. Most of the solutions are approximate solutions. In solving of nonlinear ordinary differential equations, one of the solving methods is reducing a nonlinear ordinary differential equations to be a linear ordinary differential equations. Which make the solving easier and then we have exact solution of original equation.

2.1.1 Single second-order ordinary differential equation

The first linearization problem for ordinary differential equations was solved by S. Lie [1]. He found the general form of all ordinary differential equations of second order that can be reduced to a linear equation by changing the independent and dependent variables. He showed that any linearizable second-order equation should be at most cubic in the first-order derivative and provided a linearization test in terms of its coefficients. The linearization criterion is written through relative invariants of the equivalence group. A.M. Tresse [2] and R. Liouville [3] treated the equivalence problem for second-order ordinary differential equations in terms of relative invariants of the equivalence group of point transformations. In [4], an infinitesimal technique for obtaining relative invariants was applied to the linearization problem.

A different approach for tackling the equivalence problem of second order ordinary differential equations was developed by E.Cartan [5]. The idea of his approach was to associate with every differential equation a uniquely defined geometric structure of a certain form.

2.1.2 System of two second-order ordinary differential equations

In [6], Wafo and Mahomed found the criteria for linearization of a system of two second-order ordinary differential equations which are related with the existence of an admitted four-dimensional Lie algebra. In [7], Aminova and Aminov gave the necessary and sufficient conditions for a system of second-order ordinary differential equations to be equivalent to the free particle equations. Particular class of systems of two (n = 2) second-order ordinary differential equations were considered by Mahomed and Qadir [8] and they also provided the construction of the linearizing point transformation by using complex variables. Some first-order and second-order relative invariants with respect to point transformations for a system of two ordinary differential equations were obtained in [9]. In [10], Sookmee and Meleshko proposed a new method of linearizing a system of equations, where a given system of equations is reduced to a single equation to which the Lie theorem on linearization is applied. In [11], necessary and sufficient conditions for a system of two second-order ordinary differential equations to be equivalent to the simplest equations were obtained by using the implementation of Cartan's method. Linearization criteria for a system of two second-order ordinary under general point transformation were obtained in [12]. In [13], linearization criteria for a system of two second-order ordinary differential equations to be equivalent to the linear system with constant coefficients matrix via fiber preserving point transformations were achieved.

Nowadays, the linearization problem of a system of three second-order ordinary differential equations to be equivalent to linear system via point transformations is open. Hence, it's worth to solve this problem as essential part of a study of differential equations.

2.2 Necessary conditions for linearization

We begin with investigating the necessary conditions for linearization. We consider a system of three second-order ordinary differential equations

$$\begin{aligned} y_1'' =& \ F_1\left(x, y_1, y_2, y_3, y_1', y_2', y_3'\right), \\ y_2'' =& \ F_2\left(x, y_1, y_2, y_3, y_1', y_2', y_3'\right), \\ y_3'' =& \ F_3\left(x, y_1, y_2, y_3, y_1', y_2', y_3'\right), \end{aligned} \tag{2.1}$$

which can be transformed to a linear system

$$u_1'' = 0,\ u_2'' = 0,\ u_3'' = 0, \tag{2.2}$$

under the point transformation

$$\begin{aligned} t =& \ \varphi(x, y_1, y_2, y_3), \\ u_1 =& \ \psi_1(x, y_1, y_2, y_3), \\ u_2 =& \ \psi_2(x, y_1, y_2, y_3), \\ u_3 =& \ \psi_3(x, y_1, y_2, y_3). \end{aligned} \tag{2.3}$$

So that we arrive at the following theorem.

Theorem 2.2.1. *Any system of three second-order ordinary differential equations (2.1) obtained from a linear system (2.2) by a point transformation (2.3) has to be the form*

$$\begin{aligned} y_1'' =& \ a_{11}{y_1'}^3 + (a_{12}y_2' + a_{13}y_3' + a_{14}){y_1'}^2 \\ &+(a_{15}{y_2'}^2 + (a_{16}y_3' + a_{17})y_2' + a_{18}{y_3'}^2 + a_{19}y_3' + a_{110})y_1' \\ &+a_{111}{y_2'}^2 + (a_{112}y_3' + a_{113})y_2' + a_{114}{y_3'}^2 + a_{115}y_3' + a_{116}, \\ y_2'' =& \ a_{15}{y_2'}^3 + (a_{12}y_1' + a_{16}y_3' + a_{24}){y_2'}^2 \\ &+(a_{11}{y_1'}^2 + (a_{13}y_3' + a_{27})y_1' + a_{18}{y_3'}^2 + a_{29}y_3' + a_{210})y_2' \\ &+a_{211}{y_1'}^2 + (a_{212}y_3' + a_{213})y_1' + a_{214}{y_3'}^2 + a_{215}y_3' + a_{216}, \\ y_3'' =& \ a_{18}{y_3'}^3 + (a_{13}y_1' + a_{16}y_2' + a_{34}){y_3'}^2 \\ &+(a_{11}{y_1'}^2 + (a_{12}y_2' + a_{37})y_1' + a_{15}{y_2'}^2 + a_{39}y_2' + a_{310})y_3' \\ &+a_{311}{y_1'}^2 + (a_{312}y_2' + a_{313})y_1' + a_{314}{y_2'}^2 + a_{315}y_2' + a_{316} \end{aligned} \tag{2.4}$$

where

$$a_{11} = (h_1\varphi_{y_1y_1} + h_2\psi_{1y_1y_1} + h_3\psi_{2y_1y_1} + h_4\psi_{3y_1y_1})/\Delta, \tag{2.5}$$

$$a_{12} = (2h_1\varphi_{y_1y_2} + 2h_2\psi_{1y_1y_2} + 2h_3\psi_{2y_1y_2} + 2h_4\psi_{3y_1y_2})/\Delta, \tag{2.6}$$

$$a_{13} = (2h_1\varphi_{y_1y_3} + 2h_2\psi_{1y_1y_3} + 2h_3\psi_{2y_1y_3} + 2h_4\psi_{3y_1y_3})/\Delta, \tag{2.7}$$

$$\begin{aligned} a_{14} =& (2h_1\varphi_{xy_1} + h_5\varphi_{y_1y_1} + 2h_2\psi_{1xy_1} + h_6\psi_{1y_1y_1} + 2h_3\psi_{2xy_1} \\ &+ h_7\psi_{2y_1y_1} + 2h_4\psi_{3xy_1} + h_8\psi_{3y_1y_1})/\Delta, \end{aligned} \tag{2.8}$$

$$a_{15} = (h_1\varphi_{y_2y_2} + h_2\psi_{1y_2y_2} + h_3\psi_{2y_2y_2} + h_4\psi_{3y_2y_2})/\Delta, \tag{2.9}$$

$$a_{16} = (2h_1\varphi_{y_2y_3} + 2h_2\psi_{1y_2y_3} + 2h_3\psi_{2y_2y_3} + 2h_4\psi_{3y_2y_3})/\Delta, \tag{2.10}$$

$$\begin{aligned} a_{17} =& (2h_1\varphi_{xy_2} + 2h_5\varphi_{y_1y_2} + 2h_2\psi_{1xy_2} + 2h_6\psi_{1y_1y_2} + 2h_3\psi_{2xy_2} \\ &+ 2h_7\psi_{2y_1y_2} + 2h_4\psi_{3xy_2} + 2h_8\psi_{3y_1y_2})/\Delta, \end{aligned} \tag{2.11}$$

$$a_{18} =(h_1\varphi_{y_3y_3} + h_2\psi_{1y_3y_3} + h_3\psi_{2y_3y_3} + h_4\psi_{3y_3y_3})/\Delta, \tag{2.12}$$

$$a_{19} =(2h_1\varphi_{xy_3} + 2h_5\varphi_{y_1y_3} + 2h_2\psi_{1xy_3} + 2h_6\psi_{1y_1y_3} + 2h_3\psi_{2xy_3} + 2h_7\psi_{2y_1y_3} + 2h_4\psi_{3xy_3} + 2h_8\psi_{3y_1y_3})/\Delta, \tag{2.13}$$

$$a_{110} =(2h_5\varphi_{xy_1} + h_1\varphi_{xx} + 2h_6\psi_{1xy_1} + h_2\psi_{1xx} + 2h_7\psi_{2xy_1} + h_3\psi_{2xx} + 2h_8\psi_{3xy_1} + h_4\psi_{3xx})/\Delta, \tag{2.14}$$

$$a_{111} =(h_5\varphi_{y_2y_2} + h_6\psi_{1y_2y_2} + h_7\psi_{2y_2y_2} + h_8\psi_{3y_2y_2})/\Delta, \tag{2.15}$$

$$a_{112} =(2h_5\varphi_{y_2y_3} + 2h_6\psi_{1y_2y_3} + 2h_7\psi_{2y_2y_3} + 2h_8\psi_{3y_2y_3})/\Delta, \tag{2.16}$$

$$a_{113} =(2h_5\varphi_{xy_2} + 2h_6\psi_{1xy_2} + 2h_7\psi_{2xy_2} + 2h_8\psi_{3xy_2})/\Delta, \tag{2.17}$$

$$a_{114} =(h_5\varphi_{y_3y_3} + h_6\psi_{1y_3y_3} + h_7\psi_{2y_3y_3} + h_8\psi_{3y_3y_3})/\Delta, \tag{2.18}$$

$$a_{115} =(2h_5\varphi_{xy_3} + 2h_6\psi_{1xy_3} + 2h_7\psi_{2xy_3} + 2h_8\psi_{3xy_3})/\Delta, \tag{2.19}$$

$$a_{116} =(h_5\varphi_{xx} + h_6\psi_{1xx} + h_7\psi_{2xx} + h_8\psi_{3xx})/\Delta, \tag{2.20}$$

$$a_{24} =(2h_1\varphi_{xy_2} + h_9\varphi_{y_2y_2} + 2h_2\psi_{1xy_2} + h_{10}\psi_{1y_2y_2} + 2h_3\psi_{2xy_2} + h_{11}\psi_{2y_2y_2} + 2h_4\psi_{3xy_2} + h_{12}\psi_{3y_2y_2})/\Delta, \tag{2.21}$$

$$a_{27} =(2h_1\varphi_{xy_1} + 2h_9\varphi_{y_1y_2} + 2h_2\psi_{1xy_1} + 2h_{10}\psi_{1y_1y_2} + 2h_3\psi_{2xy_1} + 2h_{11}\psi_{2y_1y_2} + 2h_4\psi_{3xy_1} + 2h_{12}\psi_{3y_1y_2})/\Delta, \tag{2.22}$$

$$a_{29} =(2h_1\varphi_{xy_3} + 2h_9\varphi_{y_2y_3} + 2h_2\psi_{1xy_3} + 2h_{10}\psi_{1y_2y_3} + 2h_3\psi_{2xy_3} + 2h_{11}\psi_{2y_2y_3} + 2h_4\psi_{3xy_3} + 2h_{12}\psi_{3y_2y_3})/\Delta, \tag{2.23}$$

$$a_{210} =(2h_9\varphi_{xy_2} + h_1\varphi_{xx} + 2h_{10}\psi_{1xy_2} + h_2\psi_{1xx} + 2h_{11}\psi_{2xy_2} + h_3\psi_{2xx} + 2h_{12}\psi_{3xy_2} + h_4\psi_{3xx})/\Delta, \tag{2.24}$$

$$a_{211} =(h_9\varphi_{y_1y_1} + h_{10}\psi_{1y_1y_1} + h_{11}\psi_{2y_1y_1} + h_{12}\psi_{3y_1y_1})/\Delta, \tag{2.25}$$

$$a_{212} =(2h_9\varphi_{y_1y_3} + 2h_{10}\psi_{1y_1y_3} + 2h_{11}\psi_{2y_1y_3} + 2h_{12}\psi_{3y_1y_3})/\Delta, \tag{2.26}$$

$$a_{213} =(2h_9\varphi_{xy_1} + 2h_{10}\psi_{1xy_1} + 2h_{11}\psi_{2xy_1} + 2h_{12}\psi_{3xy_1})/\Delta, \tag{2.27}$$

$$a_{214} =(h_9\varphi_{y_3y_3} + h_{10}\psi_{1y_3y_3} + h_{11}\psi_{2y_3y_3} + h_{12}\psi_{3y_3y_3})/\Delta, \tag{2.28}$$

$$a_{215} =(2h_9\varphi_{xy_3} + 2h_{10}\psi_{1xy_3} + 2h_{11}\psi_{2xy_3} + 2h_{12}\psi_{3xy_3})/\Delta, \tag{2.29}$$

$$a_{216} =(h_9\varphi_{xx} + h_{10}\psi_{1xx} + h_{11}\psi_{2xx} + h_{12}\psi_{3xx})/\Delta, \tag{2.30}$$

$$a_{34} =(2h_1\varphi_{xy_3} + h_{13}\varphi_{y_3y_3} + 2h_2\psi_{1xy_3} + h_{14}\psi_{1y_3y_3} + 2h_3\psi_{2xy_3} + h_{15}\psi_{2y_3y_3} + 2h_4\psi_{3xy_3} + h_{16}\psi_{3y_3y_3})/\Delta, \tag{2.31}$$

$$a_{37} =(2h_1\varphi_{xy_1} + 2h_{13}\varphi_{y_1y_3} + 2h_2\psi_{1xy_1} + 2h_{14}\psi_{1y_1y_3} + 2h_3\psi_{2xy_1} + 2h_{15}\psi_{2y_1y_3} + 2h_4\psi_{3xy_1} + 2h_{16}\psi_{3y_1y_3})/\Delta, \tag{2.32}$$

$$a_{39} =(2h_1\varphi_{xy_2} + 2h_{13}\varphi_{y_2y_3} + 2h_2\psi_{1xy_2} + 2h_{14}\psi_{1y_2y_3} + 2h_3\psi_{2xy_2} + 2h_{15}\psi_{2y_2y_3} + 2h_4\psi_{3xy_2} + 2h_{16}\psi_{3y_2y_3})/\Delta, \quad (2.33)$$

$$a_{310} =(2h_{13}\varphi_{xy_3} + h_1\varphi_{xx} + 2h_{14}\psi_{1xy_3} + h_2\psi_{1xx} + 2h_{15}\psi_{2xy_3} + h_3\psi_{2xx} + 2h_{16}\psi_{3xy_3} + h_4\psi_{3xx})/\Delta, \quad (2.34)$$

$$a_{311} =(h_{13}\varphi_{y_1y_1} + h_{14}\psi_{1y_1y_1} + h_{15}\psi_{2y_1y_1} + h_{16}\psi_{3y_1y_1})/\Delta, \quad (2.35)$$

$$a_{312} =(2h_{13}\varphi_{y_1y_2} + 2h_{14}\psi_{1y_1y_2} + 2h_{15}\psi_{2y_1y_2} + 2h_{16}\psi_{3y_1y_2})/\Delta, \quad (2.36)$$

$$a_{313} =(2h_{13}\varphi_{xy_1} + 2h_{14}\psi_{1xy_1} + 2h_{15}\psi_{2xy_1} + 2h_{16}\psi_{3xy_1})/\Delta, \quad (2.37)$$

$$a_{314} =(h_{13}\varphi_{y_2y_2} + h_{14}\psi_{1y_2y_2} + h_{15}\psi_{2y_2y_2} + h_{16}\psi_{3y_2y_2})/\Delta, \quad (2.38)$$

$$a_{315} =(2h_{13}\varphi_{xy_2} + 2h_{14}\psi_{1xy_2} + 2h_{15}\psi_{2xy_2} + 2h_{16}\psi_{3xy_2})/\Delta, \quad (2.39)$$

$$a_{316} =(h_{13}\varphi_{xx} + h_{14}\psi_{1xx} + h_{15}\psi_{2xx} + h_{16}\psi_{3xx})/\Delta \quad (2.40)$$

$$h_1 = (\psi_{1y_1}\psi_{2y_2}\psi_{3y_3} - \psi_{1y_1}\psi_{2y_3}\psi_{3y_2} - \psi_{1y_2}\psi_{2y_1}\psi_{3y_3} + \psi_{1y_2}\psi_{2y_3}\psi_{3y_1} + \psi_{1y_3}\psi_{2y_1}\psi_{3y_2} - \psi_{1y_3}\psi_{2y_2}\psi_{3y_1}),$$

$$h_2 = (-\varphi_{y_1}\psi_{2y_2}\psi_{3y_3} + \varphi_{y_1}\psi_{2y_3}\psi_{3y_2} + \varphi_{y_2}\psi_{2y_1}\psi_{3y_3} - \varphi_{y_2}\psi_{2y_3}\psi_{3y_1} - \varphi_{y_3}\psi_{2y_1}\psi_{3y_2} + \varphi_{y_3}\psi_{2y_2}\psi_{3y_1}),$$

$$h_3 = (\varphi_{y_1}\psi_{1y_2}\psi_{3y_3} - \varphi_{y_1}\psi_{1y_3}\psi_{3y_2} - \varphi_{y_2}\psi_{1y_1}\psi_{3y_3} + \varphi_{y_2}\psi_{1y_3}\psi_{3y_1} + \varphi_{y_3}\psi_{1y_1}\psi_{3y_2} - \varphi_{y_3}\psi_{1y_2}\psi_{3y_1}),$$

$$h_4 = (-\varphi_{y_1}\psi_{1y_2}\psi_{2y_3} + \varphi_{y_1}\psi_{1y_3}\psi_{2y_2} + \varphi_{y_2}\psi_{1y_1}\psi_{2y_3} - \varphi_{y_2}\psi_{1y_3}\psi_{2y_1} - \varphi_{y_3}\psi_{1y_1}\psi_{2y_2} + \varphi_{y_3}\psi_{1y_2}\psi_{2y_1}),$$

$$h_5 = (\psi_{1x}\psi_{2y_2}\psi_{3y_3} - \psi_{1x}\psi_{2y_3}\psi_{3y_2} - \psi_{1y_2}\psi_{2x}\psi_{3y_3} + \psi_{1y_2}\psi_{2y_3}\psi_{3x} + \psi_{1y_3}\psi_{2x}\psi_{3y_2} - \psi_{1y_3}\psi_{2y_2}\psi_{3x}),$$

$$h_6 = (-\varphi_x\psi_{2y_2}\psi_{3y_3} + \varphi_x\psi_{2y_3}\psi_{3y_2} + \varphi_{y_2}\psi_{2x}\psi_{3y_3} - \varphi_{y_2}\psi_{2y_3}\psi_{3x} - \varphi_{y_3}\psi_{2x}\psi_{3y_2} + \varphi_{y_3}\psi_{2y_2}\psi_{3x}),$$

$$h_7 = (\varphi_x\psi_{1y_2}\psi_{3y_3} - \varphi_x\psi_{1y_3}\psi_{3y_2} - \varphi_{y_2}\psi_{1x}\psi_{3y_3} + \varphi_{y_2}\psi_{1y_3}\psi_{3x} + \varphi_{y_3}\psi_{1x}\psi_{3y_2} - \varphi_{y_3}\psi_{1y_2}\psi_{3x}),$$

$$h_8 = (-\varphi_x\psi_{1y_2}\psi_{2y_3} + \varphi_x\psi_{1y_3}\psi_{2y_2} + \varphi_{y_2}\psi_{1x}\psi_{2y_3} - \varphi_{y_2}\psi_{1y_3}\psi_{2x} - \varphi_{y_3}\psi_{1x}\psi_{2y_2} + \varphi_{y_3}\psi_{1y_2}\psi_{2x}),$$

$$h_9 = (-\psi_{1x}\psi_{2y_1}\psi_{3y_3} + \psi_{1x}\psi_{2y_3}\psi_{3y_1} + \psi_{1y_1}\psi_{2x}\psi_{3y_3} - \psi_{1y_1}\psi_{2y_3}\psi_{3x} - \psi_{1y_3}\psi_{2x}\psi_{3y_1} + \psi_{1y_3}\psi_{2y_1}\psi_{3x}),$$

$$h_{10} = (\varphi_x\psi_{2y_1}\psi_{3y_3} - \varphi_x\psi_{2y_3}\psi_{3y_1} - \varphi_{y_1}\psi_{2x}\psi_{3y_3} + \varphi_{y_1}\psi_{2y_3}\psi_{3x} + \varphi_{y_3}\psi_{2x}\psi_{3y_1} - \varphi_{y_3}\psi_{2y_1}\psi_{3x}),$$

$$
\begin{aligned}
h_{11} = \;& (-\varphi_x\psi_{1y_1}\psi_{3y_3} + \varphi_x\psi_{1y_3}\psi_{3y_1} + \varphi_{y_1}\psi_{1x}\psi_{3y_3} - \varphi_{y_1}\psi_{1y_3}\psi_{3x} - \varphi_{y_3}\psi_{1x}\psi_{3y_1} \\
& +\varphi_{y_3}\psi_{1y_1}\psi_{3x}), \\
h_{12} = \;& (\varphi_x\psi_{1y_1}\psi_{2y_3} - \varphi_x\psi_{1y_3}\psi_{2y_1} - \varphi_{y_1}\psi_{1x}\psi_{2y_3} + \varphi_{y_1}\psi_{1y_3}\psi_{2x} + \varphi_{y_3}\psi_{1x}\psi_{2y_1} \\
& -\varphi_{y_3}\psi_{1y_1}\psi_{2x}), \\
h_{13} = \;& (\psi_{1x}\psi_{2y_1}\psi_{3y_2} - \psi_{1x}\psi_{2y_2}\psi_{3y_1} - \psi_{1y_1}\psi_{2x}\psi_{3y_2} + \psi_{1y_1}\psi_{2y_2}\psi_{3x} + \psi_{1y_2}\psi_{2x}\psi_{3y_1} \\
& -\psi_{1y_2}\psi_{2y_1}\psi_{3x}), \\
h_{14} = \;& (-\varphi_x\psi_{2y_1}\psi_{3y_2} + \varphi_x\psi_{2y_2}\psi_{3y_1} + \varphi_{y_1}\psi_{2x}\psi_{3y_2} - \varphi_{y_1}\psi_{2y_2}\psi_{3x} - \varphi_{y_2}\psi_{2x}\psi_{3y_1} \\
& +\varphi_{y_2}\psi_{2y_1}\psi_{3x}), \\
h_{15} = \;& (\varphi_x\psi_{1y_1}\psi_{3y_2} - \varphi_x\psi_{1y_2}\psi_{3y_1} - \varphi_{y_1}\psi_{1x}\psi_{3y_2} + \varphi_{y_1}\psi_{1y_2}\psi_{3x} + \varphi_{y_2}\psi_{1x}\psi_{3y_1} \\
& -\varphi_{y_2}\psi_{1y_1}\psi_{3x}), \\
h_{16} = \;& (-\varphi_x\psi_{1y_1}\psi_{2y_2} + \varphi_x\psi_{1y_2}\psi_{2y_1} + \varphi_{y_1}\psi_{1x}\psi_{2y_2} - \varphi_{y_1}\psi_{1y_2}\psi_{2x} - \varphi_{y_2}\psi_{1x}\psi_{2y_1} \\
& +\varphi_{y_2}\psi_{1y_1}\psi_{2x})
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta = \;& (\varphi_x\psi_{1y_1}\psi_{2y_2}\psi_{3y_3} - \varphi_x\psi_{1y_1}\psi_{2y_3}\psi_{3y_2} - \varphi_x\psi_{1y_2}\psi_{2y_1}\psi_{3y_3} + \varphi_x\psi_{1y_2}\psi_{2y_3}\psi_{3y_1} \\
& +\varphi_x\psi_{1y_3}\psi_{2y_1}\psi_{3y_2} - \varphi_x\psi_{1y_3}\psi_{2y_2}\psi_{3y_1} - \varphi_{y_1}\psi_{1x}\psi_{2y_2}\psi_{3y_3} + \varphi_{y_1}\psi_{1x}\psi_{2y_3}\psi_{3y_2} \\
& +\varphi_{y_1}\psi_{1y_2}\psi_{2x}\psi_{3y_3} - \varphi_{y_1}\psi_{1y_2}\psi_{2y_3}\psi_{3x} - \varphi_{y_1}\psi_{1y_3}\psi_{2x}\psi_{3y_2} + \varphi_{y_1}\psi_{1y_3}\psi_{2y_2}\psi_{3x} \\
& +\varphi_{y_2}\psi_{1x}\psi_{2y_1}\psi_{3y_3} - \varphi_{y_2}\psi_{1x}\psi_{2y_3}\psi_{3y_1} - \varphi_{y_2}\psi_{1y_1}\psi_{2x}\psi_{3y_3} + \varphi_{y_2}\psi_{1y_1}\psi_{2y_3}\psi_{3x} \\
& +\varphi_{y_2}\psi_{1y_3}\psi_{2x}\psi_{3y_1} - \varphi_{y_2}\psi_{1y_3}\psi_{2y_1}\psi_{3x} - \varphi_{y_3}\psi_{1x}\psi_{2y_1}\psi_{3y_2} + \varphi_{y_3}\psi_{1x}\psi_{2y_2}\psi_{3y_1} \\
& +\varphi_{y_3}\psi_{1y_1}\psi_{2x}\psi_{3y_2} - \varphi_{y_3}\psi_{1y_1}\psi_{2y_2}\psi_{3x} - \varphi_{y_3}\psi_{1y_2}\psi_{2x}\psi_{3y_1} + \varphi_{y_3}\psi_{1y_2}\psi_{2y_1}\psi_{3x}) \neq 0.
\end{aligned}
$$

Proof. Applying a point transformation (2.3), one obtains the following transformation of the first-order derivatives

$$
\begin{aligned}
u_1'(t) &= \frac{D_x\psi_1}{D_x\varphi} \\
&= \frac{\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2} + y_3'\psi_{1y_3}}{\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3}} \\
&= g_1(x, y_1, y_2, y_3, y_1', y_2', y_3'),
\end{aligned}
$$

$$
\begin{aligned}
u_2'(t) &= \frac{D_x\psi_2}{D_x\varphi} \\
&= \frac{\psi_{2x} + y_1'\psi_{2y_1} + y_2'\psi_{2y_2} + y_3'\psi_{2y_3}}{\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3}} \\
&= g_2(x, y_1, y_2, y_3, y_1', y_2', y_3'),
\end{aligned}
$$

$$\begin{aligned} u_3'(t) &= \frac{D_x\psi_3}{D_x\varphi} \\ &= \frac{\psi_{3x} + y_1'\psi_{3y_1} + y_2'\psi_{3y_2} + y_3'\psi_{3y_3}}{\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3}} \\ &= g_3(x, y_1, y_2, y_3, y_1', y_2', y_3'). \end{aligned}$$

The transformed second-order derivatives are

$$\begin{aligned} u_1''(t) &= \frac{d^2u_1}{dt^2} \\ &= \frac{D_xg_1}{D_x\varphi} \\ &= \frac{g_{1x} + y_1'g_{1y_1} + y_2'g_{1y_2} + y_3'g_{1y_3} + y_1''g_{1y_1'} + y_2''g_{1y_2'} + y_3''g_{1y_3'}}{\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3}}, \end{aligned} \tag{2.41}$$

$$\begin{aligned} u_2''(t) &= \frac{d^2u_2}{dt^2} \\ &= \frac{D_xg_2}{D_x\varphi} \\ &= \frac{g_{2x} + y_1'g_{2y_1} + y_2'g_{2y_2} + y_3'g_{2y_3} + y_1''g_{2y_1'} + y_2''g_{2y_2'} + y_3''g_{2y_3'}}{\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3}}, \end{aligned} \tag{2.42}$$

$$\begin{aligned} u_3''(t) &= \frac{d^2u_3}{dt^2} \\ &= \frac{D_xg_3}{D_x\varphi} \\ &= \frac{g_{3x} + y_1'g_{3y_1} + y_2'g_{3y_2} + y_3'g_{3y_3} + y_1''g_{3y_1'} + y_2''g_{3y_2'} + y_3''g_{3y_3'}}{\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3}}, \end{aligned} \tag{2.43}$$

where

$$\begin{aligned} g_{1x} &= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})\frac{\partial}{\partial x}(\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2} + y_3'\psi_{1y_3}) \\ &\quad -(\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2} + y_3'\psi_{1y_3})\frac{\partial}{\partial x}(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} \\ &\quad +y_3'\varphi_{y_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\ &= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{1xx} + y_1'\psi_{1xy_1} + \psi_{1y_1}(0) + y_2'\psi_{1xy_2} \\ &\quad +\psi_{1y_2}(0) + y_3'\psi_{1xy_3} + \psi_{1y_3}(0)) - (\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2} \\ &\quad +y_3'\psi_{1y_3})(\varphi_{xx} + y_1'\varphi_{xy_1} + \varphi_{y_1}(0) + y_2'\varphi_{xy_2} + \varphi_{y_2}(0) + y_3'\varphi_{xy_3} \\ &\quad +\varphi_{y_3}(0)))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\ &= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{1xx} + y_1'\psi_{1xy_1} + y_2'\psi_{1xy_2} + y_3'\psi_{1xy_3}) \\ &\quad -(\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2} + y_3'\psi_{1y_3})(\varphi_{xx} + y_1'\varphi_{xy_1} + y_2'\varphi_{xy_2} \\ &\quad +y_3'\varphi_{xy_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2, \end{aligned}$$

$$\begin{aligned}
g_{1y_1} &= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})\frac{\partial}{\partial y_1}(\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2} + y_3'\psi_{1y_3}) \\
&\quad -(\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2} + y_3'\psi_{1y_3})\frac{\partial}{\partial y_1}(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} \\
&\quad +y_3'\varphi_{y_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{1xy_1} + y_1'\psi_{1y_1y_1} + \psi_{1y_1}(0) + y_2'\psi_{1y_1y_2} \\
&\quad +\psi_{1y_2}(0) + y_3'\psi_{1y_1y_3} + \psi_{1y_3}(0)) - (\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2} \\
&\quad +y_3'\psi_{1y_3})(\varphi_{xy_1} + y_1'\varphi_{y_1y_1} + \varphi_{y_1}(0) + y_2'\varphi_{y_1y_2} + \varphi_{y_2}(0) + y_3'\varphi_{y_1y_3} \\
&\quad +\varphi_{y_3}(0)))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{1xy_1} + y_1'\psi_{1y_1y_1} + y_2'\psi_{1y_1y_2} + y_3'\psi_{1y_1y_3}) \\
&\quad -(\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2} + y_3'\psi_{1y_3})(\varphi_{xy_1} + y_1'\varphi_{y_1y_1} + y_2'\varphi_{y_1y_2} \\
&\quad +y_3'\varphi_{y_1y_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2,
\end{aligned}$$

$$\begin{aligned}
g_{1y_2} &= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})\frac{\partial}{\partial y_2}(\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2} + y_3'\psi_{1y_3}) \\
&\quad -(\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2} + y_3'\psi_{1y_3})\frac{\partial}{\partial y_2}(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} \\
&\quad +y_3'\varphi_{y_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{1xy_2} + y_1'\psi_{1y_1y_2} + \psi_{1y_1}(0) + y_2'\psi_{1y_2y_2} \\
&\quad +\psi_{1y_2}(0) + y_3'\psi_{1y_2y_3} + \psi_{1y_3}(0)) - (\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2} \\
&\quad +y_3'\psi_{1y_3})(\varphi_{xy_2} + y_1'\varphi_{y_1y_2} + \varphi_{y_1}(0) + y_2'\varphi_{y_2y_2} + \varphi_{y_2}(0) \\
&\quad +y_3'\varphi_{y_2y_3} + \varphi_{y_3}(0)))(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{1xy_2} + y_1'\psi_{1y_1y_2} + y_2'\psi_{1y_2y_2} + y_3'\psi_{1y_2y_3}) \\
&\quad -(\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2} + y_3'\psi_{1y_3})(\varphi_{xy_2} + y_1'\varphi_{y_1y_2} + y_2'\varphi_{y_2y_2} \\
&\quad +y_3'\varphi_{y_2y_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2,
\end{aligned}$$

$$\begin{aligned}
g_{1y_3} &= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})\frac{\partial}{\partial y_3}(\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2} + y_3'\psi_{1y_3}) \\
&\quad -(\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2} + y_3'\psi_{1y_3})\frac{\partial}{\partial y_3}(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} \\
&\quad + y_3'\varphi_{y_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{1xy_3} + y_1'\psi_{1y_1y_3} + \psi_{1y_1}(0) + y_2'\psi_{1y_2y_3} \\
&\quad + \psi_{1y_2}(0) + y_3'\psi_{1y_3y_3} + \psi_{1y_3}(0)) - (\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2} \\
&\quad + y_3'\psi_{1y_3})(\varphi_{xy_3} + y_1'\varphi_{y_1y_3} + \varphi_{y_1}(0) + y_2'\varphi_{y_2y_3} + \varphi_{y_2}(0) \\
&\quad + y_3'\varphi_{y_3y_3} + \varphi_{y_3}(0)))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{1xy_3} + y_1'\psi_{1y_1y_3} + y_2'\psi_{1y_2y_3} + y_3'\psi_{1y_3y_3}) \\
&\quad -(\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2} + y_3'\psi_{1y_3})(\varphi_{xy_3} + y_1'\varphi_{y_1y_3} + y_2'\varphi_{y_2y_3} \\
&\quad + y_3'\varphi_{y_3y_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2,
\end{aligned}$$

$$\begin{aligned}
g_{1y_1'} &= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})\frac{\partial}{\partial y_1'}(\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2} + y_3'\psi_{1y_3}) \\
&\quad -(\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2} + y_3'\psi_{1y_3})\frac{\partial}{\partial y_1'}(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} \\
&\quad + y_3'\varphi_{y_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})((0) + y_1'(0) + \psi_{1y_1}(1) + y_2'(0) \\
&\quad + \psi_{1y_2}(0) + y_3'(0) + \psi_{1y_3}(0)) - (\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2} \\
&\quad + y_3'\psi_{1y_3})((0) + y_1'(0) + \varphi_{y_1}(1) + y_2'(0) + \varphi_{y_2}(0) \\
&\quad + y_3'(0) + \varphi_{y_3}(0)))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{1y_1}) - (\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2} \\
&\quad + y_3'\psi_{1y_3})(\varphi_{y_1}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2,
\end{aligned}$$

$$\begin{aligned}
g_{1y_2'} &= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})\frac{\partial}{\partial y_2'}(\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2} + y_3'\psi_{1y_3}) \\
&\quad -(\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2} + y_3'\psi_{1y_3})\frac{\partial}{\partial y_2'}(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} \\
&\quad + y_3'\varphi_{y_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})((0) + y_1'(0) + \psi_{1y_1}(0) + y_2'(0) \\
&\quad + \psi_{1y_2}(1) + y_3'(0) + \psi_{1y_3}(0)) - (\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2} \\
&\quad + y_3'\psi_{1y_3})((0) + y_1'(0) + \varphi_{y_1}(0) + y_2'(0) + \varphi_{y_2}(1) \\
&\quad + y_3'(0) + \varphi_{y_3}(0)))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{1y_2}) - (\psi_{1x} + y_1'\psi_{1y_1} + y_2'\psi_{1y_2} \\
&\quad + y_3'\psi_{1y_3})(\varphi_{y_2}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2,
\end{aligned}$$

$$\begin{aligned}
g_{1y'_3} &= ((\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})\frac{\partial}{\partial y'_{31}}(\psi_{1x} + y'_1\psi_{1y_1} + y'_2\psi_{1y_2} + y'_3\psi_{1y_3}) \\
&\quad -(\psi_{1x} + y'_1\psi_{1y_1} + y'_2\psi_{1y_2} + y'_3\psi_{1y_3})\frac{\partial}{\partial y'_3}(\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} \\
&\quad +y'_3\varphi_{y_3}))/(\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})^2 \\
&= ((\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})((0) + y'_1(0) + \psi_{1y_1}(0) + y'_2(0) \\
&\quad +\psi_{1y_2}(0) + y'_3(0) + \psi_{1y_3}(1)) - (\psi_{1x} + y'_1\psi_{1y_1} + y'_2\psi_{1y_2} \\
&\quad +y'_3\psi_{1y_3})((0) + y'_1(0) + \varphi_{y_1}(0) + y'_2(0) + \varphi_{y_2}(0) \\
&\quad +y'_3(0) + \varphi_{y_3}(1)))/(\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})^2 \\
&= ((\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})(\psi_{1y_3}) - (\psi_{1x} + y'_1\psi_{1y_1} + y'_2\psi_{1y_2} \\
&\quad +y'_3\psi_{1y_3})(\varphi_{y_3}))/(\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})^2,
\end{aligned}$$

$$\begin{aligned}
g_{2x} &= ((\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})\frac{\partial}{\partial x}(\psi_{2x} + y'_1\psi_{2y_1} + y'_2\psi_{2y_2} + y'_3\psi_{2y_3}) \\
&\quad -(\psi_{2x} + y'_1\psi_{2y_1} + y'_2\psi_{2y_2} + y'_3\psi_{2y_3})\frac{\partial}{\partial x}(\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} \\
&\quad +y'_3\varphi_{y_3}))/(\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})^2 \\
&= ((\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})(\psi_{2xx} + y'_1\psi_{2xy_1} + \psi_{2y_1}(0) + y'_2\psi_{2xy_2} \\
&\quad +\psi_{2y_2}(0) + y'_3\psi_{2xy_3} + \psi_{2y_3}(0)) - (\psi_{2x} + y'_1\psi_{2y_1} + y'_2\psi_{2y_2} \\
&\quad +y'_3\psi_{2y_3})(\varphi_{xx} + y'_1\varphi_{xy_1} + \varphi_{y_1}(0) + y'_2\varphi_{xy_2} + \varphi_{y_2}(0) \\
&\quad +y'_3\varphi_{xy_3} + \varphi_{y_3}(0)))/(\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})^2 \\
&= ((\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})(\psi_{2xx} + y'_1\psi_{2xy_1} + y'_2\psi_{2xy_2} + y'_3\psi_{2xy_3}) \\
&\quad -(\psi_{2x} + y'_1\psi_{2y_1} + y'_2\psi_{2y_2} + y'_3\psi_{2y_3})(\varphi_{xx} + y'_1\varphi_{xy_1} + y'_2\varphi_{xy_2} \\
&\quad +y'_3\varphi_{xy_3}))/(\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})^2,
\end{aligned}$$

$$
\begin{aligned}
g_{2y_1} &= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})\frac{\partial}{\partial y_1}(\psi_{2x} + y_1'\psi_{2y_1} + y_2'\psi_{2y_2} + y_3'\psi_{2y_3}) \\
&\quad -(\psi_{2x} + y_1'\psi_{2y_1} + y_2'\psi_{2y_2} + y_3'\psi_{2y_3})\frac{\partial}{\partial y_1}(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} \\
&\quad +y_3'\varphi_{y_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{2xy_1} + y_1'\psi_{2y_1y_1} + \psi_{2y_1}(0) + y_2'\psi_{2y_1y_2} \\
&\quad +\psi_{2y_2}(0) + y_3'\psi_{2y_1y_3} + \psi_{2y_3}(0)) - (\psi_{2x} + y_1'\psi_{2y_1} + y_2'\psi_{2y_2} \\
&\quad +y_3'\psi_{2y_3})(\varphi_{xy_1} + y_1'\varphi_{y_1y_1} + \varphi_{y_1}(0) + y_2'\varphi_{y_1y_2} + \varphi_{y_2}(0) \\
&\quad +y_3'\varphi_{y_1y_3} + \varphi_{y_3}(0)))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{2xy_1} + y_1'\psi_{2y_1y_1} + y_2'\psi_{2y_1y_2} + y_3'\psi_{2y_1y_3}) \\
&\quad -(\psi_{2x} + y_1'\psi_{2y_1} + y_2'\psi_{2y_2} + y_3'\psi_{2y_3})(\varphi_{xy_1} + y_1'\varphi_{y_1y_1} + y_2'\varphi_{y_1y_2} \\
&\quad +y_3'\varphi_{y_1y_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2,
\end{aligned}
$$

$$
\begin{aligned}
g_{2y_2} &= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})\frac{\partial}{\partial y_2}(\psi_{2x} + y_1'\psi_{2y_1} + y_2'\psi_{2y_2} + y_3'\psi_{2y_3}) \\
&\quad -(\psi_{2x} + y_1'\psi_{2y_1} + y_2'\psi_{2y_2} + y_3'\psi_{2y_3})\frac{\partial}{\partial y_2}(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} \\
&\quad +y_3'\varphi_{y_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{2xy_2} + y_1'\psi_{2y_1y_2} + \psi_{2y_1}(0) + y_2'\psi_{2y_2y_2} \\
&\quad +\psi_{2y_2}(0) + y_3'\psi_{2y_2y_3} + \psi_{2y_3}(0)) - (\psi_{2x} + y_1'\psi_{2y_1} + y_2'\psi_{2y_2} \\
&\quad +y_3'\psi_{2y_3})(\varphi_{xy_2} + y_1'\varphi_{y_1y_2} + \varphi_{y_1}(0) + y_2'\varphi_{y_2y_2} + \varphi_{y_2}(0) \\
&\quad +y_3'\varphi_{y_2y_3} + \varphi_{y_3}(0)))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{2xy_2} + y_1'\psi_{2y_1y_2} + y_2'\psi_{2y_2y_2} + y_3'\psi_{2y_2y_3}) \\
&\quad -(\psi_{2x} + y_1'\psi_{2y_1} + y_2'\psi_{2y_2} + y_3'\psi_{2y_3})(\varphi_{xy_2} + y_1'\varphi_{y_1y_2} + y_2'\varphi_{y_2y_2} \\
&\quad +y_3'\varphi_{y_2y_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2,
\end{aligned}
$$

$$\begin{aligned}
g_{2y_3} &= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})\frac{\partial}{\partial y_3}(\psi_{2x} + y_1'\psi_{2y_1} + y_2'\psi_{2y_2} + y_3'\psi_{2y_3}) \\
&\quad -(\psi_{2x} + y_1'\psi_{2y_1} + y_2'\psi_{2y_2} + y_3'\psi_{2y_3})\frac{\partial}{\partial y_3}(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} \\
&\quad +y_3'\varphi_{y_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{2xy_3} + y_1'\psi_{2y_1y_3} + \psi_{2y_1}(0) + y_2'\psi_{2y_2y_3} \\
&\quad +\psi_{2y_2}(0) + y_3'\psi_{2y_3y_3} + \psi_{2y_3}(0)) - (\psi_{2x} + y_1'\psi_{2y_1} + y_2'\psi_{2y_2} \\
&\quad +y_3'\psi_{2y_3})(\varphi_{xy_3} + y_1'\varphi_{y_1y_3} + \varphi_{y_1}(0) + y_2'\varphi_{y_2y_3} + \varphi_{y_2}(0) \\
&\quad +y_3'\varphi_{y_3y_3} + \varphi_{y_3}(0)))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{2xy_3} + y_1'\psi_{2y_1y_3} + y_2'\psi_{2y_2y_3} + y_3'\psi_{2y_3y_3}) \\
&\quad -(\psi_{2x} + y_1'\psi_{2y_1} + y_2'\psi_{2y_2} + y_3'\psi_{2y_3})(\varphi_{xy_3} + y_1'\varphi_{y_1y_3} + y_2'\varphi_{y_2y_3} \\
&\quad +y_3'\varphi_{y_3y_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2,
\end{aligned}$$

$$\begin{aligned}
g_{2y_1'} &= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})\frac{\partial}{\partial y_1'}(\psi_{2x} + y_1'\psi_{2y_1} + y_2'\psi_{2y_2} + y_3'\psi_{2y_3}) \\
&\quad -(\psi_{2x} + y_1'\psi_{2y_1} + y_2'\psi_{2y_2} + y_3'\psi_{2y_3})\frac{\partial}{\partial y_1'}(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} \\
&\quad +y_3'\varphi_{y_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})((0) + y_1'(0) + \psi_{2y_1}(1) + y_2'(0) \\
&\quad +\psi_{2y_2}(0) + y_3'(0) + \psi_{2y_3}(0)) - (\psi_{2x} + y_1'\psi_{2y_1} + y_2'\psi_{2y_2} \\
&\quad +y_3'\psi_{2y_3})((0) + y_1'(0) + \varphi_{y_1}(1) + y_2'(0) + \varphi_{y_2}(0) \\
&\quad +y_3'(0) + \varphi_{y_3}(0)))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{2y_1}) - (\psi_{2x} + y_1'\psi_{2y_1} + y_2'\psi_{2y_2} \\
&\quad +y_3'\psi_{2y_3})(\varphi_{y_1}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2,
\end{aligned}$$

$$\begin{aligned}
g_{2y_2'} &= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})\frac{\partial}{\partial y_2'}(\psi_{2x} + y_1'\psi_{2y_1} + y_2'\psi_{2y_2} + y_3'\psi_{2y_3}) \\
&\quad -(\psi_{2x} + y_1'\psi_{2y_1} + y_2'\psi_{2y_2} + y_3'\psi_{2y_3})\frac{\partial}{\partial y_2'}(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} \\
&\quad +y_3'\varphi_{y_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})((0) + y_1'(0) + \psi_{2y_1}(0) + y_2'(0) \\
&\quad +\psi_{2y_2}(1) + y_3'(0) + \psi_{2y_3}(0)) - (\psi_{2x} + y_1'\psi_{2y_1} + y_2'\psi_{2y_2} \\
&\quad +y_3'\psi_{2y_3})((0) + y_1'(0) + \varphi_{y_1}(0) + y_2'(0) + \varphi_{y_2}(1) \\
&\quad +y_3'(0) + \varphi_{y_3}(0)))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{2y_2}) - (\psi_{2x} + y_1'\psi_{2y_1} + y_2'\psi_{2y_2} \\
&\quad +y_3'\psi_{2y_3})(\varphi_{y_2}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2,
\end{aligned}$$

$$\begin{aligned}
g_{2y'_3} &= ((\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})\frac{\partial}{\partial y'_{31}}(\psi_{2x} + y'_1\psi_{2y_1} + y'_2\psi_{2y_2} + y'_3\psi_{2y_3}) \\
&\quad -(\psi_{2x} + y'_1\psi_{2y_1} + y'_2\psi_{2y_2} + y'_3\psi_{2y_3})\frac{\partial}{\partial y'_3}(\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} \\
&\quad +y'_3\varphi_{y_3}))/(\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})^2 \\
&= ((\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})((0) + y'_1(0) + \psi_{2y_1}(0) + y'_2(0) \\
&\quad +\psi_{2y_2}(0) + y'_3(0) + \psi_{2y_3}(1)) - (\psi_{2x} + y'_1\psi_{2y_1} + y'_2\psi_{2y_2} \\
&\quad +y'_3\psi_{2y_3})((0) + y'_1(0) + \varphi_{y_1}(0) + y'_2(0) + \varphi_{y_2}(0) \\
&\quad +y'_3(0) + \varphi_{y_3}(1)))/(\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})^2 \\
&= ((\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})(\psi_{2y_3}) - (\psi_{2x} + y'_1\psi_{2y_1} + y'_2\psi_{2y_2} \\
&\quad +y'_3\psi_{2y_3})(\varphi_{y_3}))/(\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})^2,
\end{aligned}$$

$$\begin{aligned}
g_{3x} &= ((\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})\frac{\partial}{\partial x}(\psi_{3x} + y'_1\psi_{3y_1} + y'_2\psi_{3y_2} + y'_3\psi_{3y_3}) \\
&\quad -(\psi_{3x} + y'_1\psi_{3y_1} + y'_2\psi_{3y_2} + y'_3\psi_{3y_3})\frac{\partial}{\partial x}(\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} \\
&\quad +y'_3\varphi_{y_3}))/(\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})^2 \\
&= ((\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})(\psi_{3xx} + y'_1\psi_{3xy_1} + \psi_{3y_1}(0) + y'_2\psi_{3xy_2} \\
&\quad +\psi_{3y_2}(0) + y'_3\psi_{3xy_3} + \psi_{3y_3}(0)) - (\psi_{3x} + y'_1\psi_{3y_1} + y'_2\psi_{3y_2} \\
&\quad +y'_3\psi_{3y_3})(\varphi_{xx} + y'_1\varphi_{xy_1} + \varphi_{y_1}(0) + y'_2\varphi_{xy_2} + \varphi_{y_2}(0) \\
&\quad +y'_3\varphi_{xy_3} + \varphi_{y_3}(0)))/(\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})^2 \\
&= ((\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})(\psi_{3xx} + y'_1\psi_{3xy_1} + y'_2\psi_{3xy_2} + y'_3\psi_{3xy_3}) \\
&\quad -(\psi_{3x} + y'_1\psi_{3y_1} + y'_2\psi_{3y_2} + y'_3\psi_{3y_3})(\varphi_{xx} + y'_1\varphi_{xy_1} + y'_2\varphi_{xy_2} \\
&\quad +y'_3\varphi_{xy_3}))/(\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})^2,
\end{aligned}$$

$$\begin{aligned}
g_{3y_1} &= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})\frac{\partial}{\partial y_1}(\psi_{3x} + y_1'\psi_{3y_1} + y_2'\psi_{3y_2} + y_3'\psi_{3y_3}) \\
&\quad -(\psi_{3x} + y_1'\psi_{3y_1} + y_2'\psi_{3y_2} + y_3'\psi_{3y_3})\frac{\partial}{\partial y_1}(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} \\
&\quad +y_3'\varphi_{y_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{3xy_1} + y_1'\psi_{3y_1y_1} + \psi_{3y_1}(0) + y_2'\psi_{3y_1y_2} \\
&\quad +\psi_{3y_2}(0) + y_3'\psi_{3y_1y_3} + \psi_{3y_3}(0)) - (\psi_{3x} + y_1'\psi_{3y_1} + y_2'\psi_{3y_2} \\
&\quad +y_3'\psi_{3y_3})(\varphi_{xy_1} + y_1'\varphi_{y_1y_1} + \varphi_{y_1}(0) + y_2'\varphi_{y_1y_2} + \varphi_{y_2}(0) \\
&\quad +y_3'\varphi_{y_1y_3} + \varphi_{y_3}(0)))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{3xy_1} + y_1'\psi_{3y_1y_1} + y_2'\psi_{3y_1y_2} + y_3'\psi_{3y_1y_3}) \\
&\quad -(\psi_{3x} + y_1'\psi_{3y_1} + y_2'\psi_{3y_2} + y_3'\psi_{3y_3})(\varphi_{xy_1} + y_1'\varphi_{y_1y_1} + y_2'\varphi_{y_1y_2} \\
&\quad +y_3'\varphi_{y_1y_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2,
\end{aligned}$$

$$\begin{aligned}
g_{3y_2} &= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})\frac{\partial}{\partial y_2}(\psi_{3x} + y_1'\psi_{3y_1} + y_2'\psi_{3y_2} + y_3'\psi_{3y_3}) \\
&\quad -(\psi_{3x} + y_1'\psi_{3y_1} + y_2'\psi_{3y_2} + y_3'\psi_{3y_3})\frac{\partial}{\partial y_2}(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} \\
&\quad +y_3'\varphi_{y_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{3xy_2} + y_1'\psi_{3y_1y_2} + \psi_{3y_1}(0) + y_2'\psi_{3y_2y_2} \\
&\quad +\psi_{3y_2}(0) + y_3'\psi_{3y_2y_3} + \psi_{3y_3}(0)) - (\psi_{3x} + y_1'\psi_{3y_1} + y_2'\psi_{3y_2} \\
&\quad +y_3'\psi_{3y_3})(\varphi_{xy_2} + y_1'\varphi_{y_1y_2} + \varphi_{y_1}(0) + y_2'\varphi_{y_2y_2} + \varphi_{y_2}(0) \\
&\quad +y_3'\varphi_{y_2y_3} + \varphi_{y_3}(0)))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{3xy_2} + y_1'\psi_{3y_1y_2} + y_2'\psi_{3y_2y_2} + y_3'\psi_{3y_2y_3}) \\
&\quad -(\psi_{3x} + y_1'\psi_{3y_1} + y_2'\psi_{3y_2} + y_3'\psi_{3y_3})(\varphi_{xy_2} + y_1'\varphi_{y_1y_2} + y_2'\varphi_{y_2y_2} \\
&\quad +y_3'\varphi_{y_2y_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2,
\end{aligned}$$

$$\begin{aligned}
g_{3y_3} &= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})\frac{\partial}{\partial y_3}(\psi_{3x} + y_1'\psi_{3y_1} + y_2'\psi_{3y_2} + y_3'\psi_{3y_3}) \\
&\quad -(\psi_{3x} + y_1'\psi_{3y_1} + y_2'\psi_{3y_2} + y_3'\psi_{3y_3})\frac{\partial}{\partial y_3}(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} \\
&\quad +y_3'\varphi_{y_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{3xy_3} + y_1'\psi_{3y_1y_3} + \psi_{3y_1}(0) + y_2'\psi_{3y_2y_3} \\
&\quad +\psi_{3y_2}(0) + y_3'\psi_{3y_3y_3} + \psi_{3y_3}(0)) - (\psi_{3x} + y_1'\psi_{3y_1} + y_2'\psi_{3y_2} \\
&\quad +y_3'\psi_{3y_3})(\varphi_{xy_3} + y_1'\varphi_{y_1y_3} + \varphi_{y_1}(0) + y_2'\varphi_{y_2y_3} + \varphi_{y_2}(0) \\
&\quad +y_3'\varphi_{y_3y_3} + \varphi_{y_3}(0)))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{3xy_3} + y_1'\psi_{3y_1y_3} + y_2'\psi_{3y_2y_3} + y_3'\psi_{3y_3y_3}) \\
&\quad -(\psi_{3x} + y_1'\psi_{3y_1} + y_2'\psi_{3y_2} + y_3'\psi_{3y_3})(\varphi_{xy_3} + y_1'\varphi_{y_1y_3} + y_2'\varphi_{y_2y_3} \\
&\quad +y_3'\varphi_{y_3y_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2,
\end{aligned}$$

$$\begin{aligned}
g_{3y_1'} &= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})\frac{\partial}{\partial y_1'}(\psi_{3x} + y_1'\psi_{3y_1} + y_2'\psi_{3y_2} + y_3'\psi_{3y_3}) \\
&\quad -(\psi_{3x} + y_1'\psi_{3y_1} + y_2'\psi_{3y_2} + y_3'\psi_{3y_3})\frac{\partial}{\partial y_1'}(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} \\
&\quad +y_3'\varphi_{y_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})((0) + y_1'(0) + \psi_{3y_1}(1) + y_2'(0) \\
&\quad +\psi_{3y_2}(0) + y_3'(0) + \psi_{3y_3}(0)) - (\psi_{3x} + y_1'\psi_{3y_1} + y_2'\psi_{3y_2} \\
&\quad +y_3'\psi_{3y_3})((0) + y_1'(0) + \varphi_{y_1}(1) + y_2'(0) + \varphi_{y_2}(0) \\
&\quad +y_3'(0) + \varphi_{y_3}(0)))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{3y_1}) - (\psi_{3x} + y_1'\psi_{3y_1} + y_2'\psi_{3y_2} \\
&\quad +y_3'\psi_{3y_3})(\varphi_{y_1}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2,
\end{aligned}$$

$$\begin{aligned}
g_{3y_2'} &= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})\frac{\partial}{\partial y_2'}(\psi_{3x} + y_1'\psi_{3y_1} + y_2'\psi_{3y_2} + y_3'\psi_{3y_3}) \\
&\quad -(\psi_{3x} + y_1'\psi_{3y_1} + y_2'\psi_{3y_2} + y_3'\psi_{3y_3})\frac{\partial}{\partial y_2'}(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} \\
&\quad +y_3'\varphi_{y_3}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})((0) + y_1'(0) + \psi_{3y_1}(0) + y_2'(0) \\
&\quad +\psi_{3y_2}(1) + y_3'(0) + \psi_{3y_3}(0)) - (\psi_{3x} + y_1'\psi_{3y_1} + y_2'\psi_{3y_2} \\
&\quad +y_3'\psi_{3y_3})((0) + y_1'(0) + \varphi_{y_1}(0) + y_2'(0) + \varphi_{y_2}(1) \\
&\quad +y_3'(0) + \varphi_{y_3}(0)))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2 \\
&= ((\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})(\psi_{3y_2}) - (\psi_{3x} + y_1'\psi_{3y_1} + y_2'\psi_{3y_2} \\
&\quad +y_3'\psi_{3y_3})(\varphi_{y_2}))/(\varphi_x + y_1'\varphi_{y_1} + y_2'\varphi_{y_2} + y_3'\varphi_{y_3})^2,
\end{aligned}$$

$$\begin{aligned}
g_{3y'_3} &= ((\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})\frac{\partial}{\partial y'_{31}}(\psi_{3x} + y'_1\psi_{3y_1} + y'_2\psi_{3y_2} + y'_3\psi_{3y_3}) \\
&\quad -(\psi_{3x} + y'_1\psi_{3y_1} + y'_2\psi_{3y_2} + y'_3\psi_{3y_3})\frac{\partial}{\partial y'_3}(\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} \\
&\quad +y'_3\varphi_{y_3}))/(\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})^2 \\
&= ((\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})((0) + y'_1(0) + \psi_{3y_1}(0) + y'_2(0) \\
&\quad +\psi_{3y_2}(0) + y'_3(0) + \psi_{3y_3}(1)) - (\psi_{3x} + y'_1\psi_{3y_1} + y'_2\psi_{3y_2} \\
&\quad +y'_3\psi_{3y_3})((0) + y'_1(0) + \varphi_{y_1}(0) + y'_2(0) + \varphi_{y_2}(0) \\
&\quad +y'_3(0) + \varphi_{y_3}(1)))/(\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})^2 \\
&= ((\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})(\psi_{3y_3}) - (\psi_{3x} + y'_1\psi_{3y_1} + y'_2\psi_{3y_2} \\
&\quad +y'_3\psi_{3y_3})(\varphi_{y_3}))/(\varphi_x + y'_1\varphi_{y_1} + y'_2\varphi_{y_2} + y'_3\varphi_{y_3})^2,
\end{aligned}$$

and

$D_x = \frac{\partial}{\partial x} + y'_1\frac{\partial}{\partial y_1} + y'_2\frac{\partial}{\partial y_2} + y'_3\frac{\partial}{\partial y_3} + y''_1\frac{\partial}{\partial y'_1} + y''_2\frac{\partial}{\partial y'_2} + y''_3\frac{\partial}{\partial y'_4} + ...$ is a total derivatives.

Replacing $g_{1x}, g_{1y_1}, g_{1y_2}, g_{1y_3}, g_{1y'_1}, g_{1y'_2}, g_{1y'_3}, g_{2x}, g_{2y_1}, g_{2y_2}, g_{2y_3}, g_{2y'_1}, g_{2y'_2}, g_{2y'_3}$, $g_{3x}, g_{3y_1}, g_{3y_2}, g_{3y_3}, g_{3y'_1}, g_{3y'_2}, g_{3y'_3}$ into equations (2.41) - (2.43), one gets the system (2.4). □

2.3 Sufficient conditions for linearization

We have shown in the previous section that every linearizable system of three second-order ordinary differential equations belongs to the class of systems (2.4). In this section, we formulate the theorem containing sufficient conditions for linearization under the restrict class of point transformation

$$\begin{aligned}
t &= \varphi(x), \\
u_1 &= \psi_1(x, y_1), \\
u_2 &= \psi_2(x, y_1, y_2), \\
u_3 &= \psi_3(x, y_1, y_2, y_3).
\end{aligned} \tag{2.44}$$

We arrive at the following theorem.

Theorem 2.3.1. *System (2.4) is linearizable by restrict class of point transformation (2.44) if and only if its coefficients satisfied the following equations*

$$\begin{aligned}
a_{11} &= 0,\ a_{12} = 0,\ a_{13} = 0,\ a_{15} = 0,\ a_{16} = 0,\ a_{17} = 0,\ a_{18} = 0, \\
a_{19} &= 0,\ a_{111} = 0,\ a_{112} = 0,\ a_{113} = 0,\ a_{114} = 0,\ a_{115} = 0,\ a_{29} = 0, \\
a_{212} &= 0,\ a_{214} = 0,\ a_{215} = 0,
\end{aligned} \tag{2.45}$$

$$\begin{aligned} a_{14y_2} = &\ 0,\ a_{14y_3} = 0,\ a_{110y_2} = 0,\ a_{110y_3} = 0,\ a_{116y_2} = 0,\ a_{116y_3} = 0, \\ a_{24y_3} = &\ 0,\ a_{27y_3} = 0,\ a_{210y_3} = 0,\ a_{211y_3} = 0,\ a_{213y_3} = 0,\ a_{216y_3} = 0, \end{aligned} \tag{2.46}$$

$$\begin{aligned} a_{110x} = &(4a_{116y_1} + 2a_{310x} - 4a_{316y_3} + a_{110}^2 - 4a_{116}a_{14} + 2a_{116}a_{37} \\ &+ 2a_{216}a_{39} - a_{310}^2 + 4a_{316}a_{34})/2, \end{aligned} \tag{2.47}$$

$$a_{110y_1} = 2a_{14x}, \tag{2.48}$$

$$a_{116y_1y_1} = a_{116y_1}a_{14} + a_{14xx} - a_{14x}a_{110} + a_{14y_1}a_{116}, \tag{2.49}$$

$$a_{24y_1} = a_{27y_2}/2, \tag{2.50}$$

$$\begin{aligned} a_{210x} = &(4a_{216y_2} + 2a_{310x} - 4a_{316y_3} - 2a_{116}a_{27} + 2a_{116}a_{37} + a_{210}^2 - 4a_{216}a_{24} \\ &+ 2a_{216}a_{39} - a_{310}^2 + 4a_{316}a_{34})/2, \end{aligned} \tag{2.51}$$

$$a_{210y_1} = a_{27x}, \tag{2.52}$$

$$a_{210y_2} = 2a_{24x}, \tag{2.53}$$

$$a_{211x} = (2a_{213y_1} - 2a_{110}a_{211} + 2a_{14}a_{213} + 2a_{210}a_{211} - a_{213}a_{27})/4, \tag{2.54}$$

$$a_{211y_2} = (2a_{27y_1} + 2a_{14}a_{27} + 4a_{211}a_{24} - a_{27}^2)/4, \tag{2.55}$$

$$a_{213x} = (4a_{216y_1} + a_{110}a_{213} - 4a_{116}a_{211} + a_{210}a_{213} - 2a_{216}a_{27})/2, \tag{2.56}$$

$$a_{213y_2} = (2a_{27x} + a_{110}a_{27} - a_{210}a_{27} + 2a_{213}a_{24})/2, \tag{2.57}$$

$$\begin{aligned} a_{216y_1y_2} = &(a_{116y_1}a_{27} + 2a_{216y_1}a_{24} + a_{27xx} - a_{27x}a_{210} + a_{27y_1}a_{116} \\ &+ a_{27y_2}a_{216})/2, \end{aligned} \tag{2.58}$$

$$a_{216y_2y_2} = (2a_{216y_2}a_{24} + 2a_{24xx} - 2a_{24x}a_{210} + 2a_{24y_2}a_{216} + a_{27y_2}a_{116})/2, \tag{2.59}$$

$$a_{34y_1} = a_{37y_3}/2, \tag{2.60}$$

$$a_{34y_2} = a_{39y_3}/2, \tag{2.61}$$

$$a_{37y_2} = a_{39y_1}, \tag{2.62}$$

$$a_{310y_1} = a_{37x}, \tag{2.63}$$

$$a_{310y_2} = a_{39x}, \tag{2.64}$$

$$a_{310y_3} = 2a_{34x}, \tag{2.65}$$

$$\begin{aligned} a_{311x} = &(2a_{313y_1} - 2a_{110}a_{311} + 2a_{14}a_{313} + 2a_{211}a_{315} - a_{213}a_{312} \\ &+ 2a_{310}a_{311} - a_{313}a_{37})/4, \end{aligned} \tag{2.66}$$

$$a_{311y_2} = (2a_{312y_1} + 2a_{14}a_{312} + 4a_{211}a_{314} - a_{27}a_{312} + 2a_{311}a_{39} - a_{312}a_{37})/4, \tag{2.67}$$

$$a_{311y_3} = (2a_{37y_1} + 2a_{14}a_{37} + 2a_{211}a_{39} + 4a_{311}a_{34} - a_{37}^2)/4, \tag{2.68}$$

$$a_{312x} = (2a_{315y_1} - a_{210}a_{312} + a_{27}a_{315} + a_{310}a_{312} - a_{315}a_{37})/2, \tag{2.69}$$

$$a_{312y_2} = (4a_{314y_1} - 2a_{24}a_{312} + 2a_{27}a_{314} + a_{312}a_{39} - 2a_{314}a_{37})/2, \tag{2.70}$$

$$a_{312y_3} = (2a_{39y_1} + a_{27}a_{39} + 2a_{312}a_{34} - a_{37}a_{39})/2, \tag{2.71}$$

$$a_{313x} = (4a_{316y_1} + a_{110}a_{313} - 4a_{116}a_{311} + a_{213}a_{315} - 2a_{216}a_{312} + a_{310}a_{313} - 2a_{316}a_{37})/2, \tag{2.72}$$

$$a_{313y_2} = (2a_{315y_1} + a_{110}a_{312} - a_{210}a_{312} + 2a_{213}a_{314} + a_{313}a_{39} - a_{315}a_{37})/2, \tag{2.73}$$

$$a_{313y_3} = (2a_{37x} + a_{110}a_{37} + a_{213}a_{39} - a_{310}a_{37} + 2a_{313}a_{34})/2, \tag{2.74}$$

$$a_{314x} = (2a_{315y_2} - 2a_{210}a_{314} + 2a_{24}a_{315} + 2a_{310}a_{314} - a_{315}a_{39})/4, \tag{2.75}$$

$$a_{314y_3} = (2a_{39y_2} + 2a_{24}a_{39} + 4a_{314}a_{34} - a_{39}^2)/4, \tag{2.76}$$

$$a_{315x} = (4a_{316y_2} - 2a_{116}a_{312} + a_{210}a_{315} - 4a_{216}a_{314} + a_{310}a_{315} - 2a_{316}a_{39})/2, \tag{2.77}$$

$$a_{315y_3} = (2a_{39x} + a_{210}a_{39} - a_{310}a_{39} + 2a_{315}a_{34})/2, \tag{2.78}$$

$$a_{316y_1y_3} = (a_{116y_1}a_{37} + a_{216y_1}a_{39} + 2a_{316y_1}a_{34} + a_{37xx} - a_{37x}a_{310} + a_{37y_1}a_{116} + a_{37y_3}a_{316} + a_{39y_1}a_{216})/2, \tag{2.79}$$

$$a_{316y_2y_3} = (a_{216y_2}a_{39} + 2a_{316y_2}a_{34} + a_{39xx} - a_{39x}a_{310} + a_{39y_1}a_{116} + a_{39y_2}a_{216} + a_{39y_3}a_{316})/2, \tag{2.80}$$

$$a_{316y_3y_3} = (2a_{316y_3}a_{34} + 2a_{34x} - 2a_{34x}a_{310} + 2a_{34y_3}a_{316} + a_{37y_3}a_{116} + a_{39y_3}a_{216})/2. \tag{2.81}$$

Proof. Since $\varphi_{y_1}, \varphi_{y_2}, \varphi_{y_3}, \psi_{1y_2}, \psi_{1y_3}$ and ψ_{2y_3} are 0, then $\Delta = \varphi_x \psi_{3y_3} v \neq 0$ where $v = \psi_{1y_1}\psi_{2y_2}$.

Considering equation $(\psi_{2y_2})_{y_3} = 0$ one obtains

$$\frac{v_{y_3}}{\psi_{1y_1}} = 0$$
$$v_{y_3} = 0.$$

From equations (2.5) - (2.7), (2.9) - (2.13), (2.15) - (2.19), (2.23), (2.26) and (2.28) - (2.29) one gets the conditions

$$a_{11} = a_{12} = a_{13} = a_{15} = a_{16} = a_{17} = a_{18} = a_{19} = a_{111} = 0,$$

$$a_{112} = a_{113} = a_{114} = a_{115} = a_{29} = a_{212} = a_{214} = a_{215} = 0.$$

From equations (2.31) - (2.34), (2.8), (2.21), (2.22), (2.25), (2.27), (2.38), (2.36), (2.37), (2.14), (2.20), (2.24), (2.37), (2.39), (2.30) and (2.40) we have

$$\begin{aligned}
\psi_{3y_3y_3} &= -\psi_{3y_3}a_{34}, \\
\psi_{3y_1y_3} &= -\frac{\psi_{3y_3}a_{37}}{2}, \\
\psi_{3y_2y_3} &= -\frac{\psi_{3y_3}a_{39}}{2}, \\
\varphi_{xx} &= \frac{\varphi_x(2\psi_{3xy_3} + \psi_{3y_3}a_{310})}{\psi_{3y_3}}, \\
\psi_{1y_1y_1} &= -\psi_{1y1}a_{14}, \\
v_{y_2} &= -a_{24}v, \\
v_{y_1} &= \frac{v(-a_{27} - 2a_{14})}{2}, \\
\psi_{2y_1y_1} &= \frac{-(\psi_{1y_1}\psi_{2y_1}a_{14} + a_{211}v)}{\psi_{1y_1}},
\end{aligned}$$

$$\begin{aligned}
\psi_{2xy_1} &= \frac{2\psi_{2y_1}\psi_{1xy_1} - a_{213}v}{2\psi_{1y_1}}, \\
\psi_{3y_2y_2} &= -(\psi_{3y_2}a_{24} + \psi_{3y_3}a_{314}), \\
\psi_{3y_1y_2} &= \frac{-(\psi_{3y_2}a_{27} + \psi_{3y_3}a_{312})}{2}, \\
\psi_{3y_1y_1} &= -(\psi_{3y_1}a_{14} + \psi_{3y_2}a_{211} + \psi_{3y_3}a_{311}), \\
\psi_{1xy_1} &= \frac{\psi_{1y_1}(2\psi_{3xy_3} - \psi_{3y_3}a_{110} + \psi_{3y_3}a_{310})}{2\psi_{3y_3}}, \\
\psi_{1xx} &= \frac{2\psi_{1x}\psi_{3xy_3} + \psi_{1x}\psi_{3y_3}a_{310} - \psi_{1y_1}\psi_{3y_3}a_{116}}{\psi_{3y_3}}, \\
\psi_{3xy_3} &= \frac{\psi_{3y_3}(2v_x + a_{110}v + a_{210}v - 2a_{310}v)}{4v}, \\
\psi_{3xy_1} &= \frac{2\psi_{3y_1}v_x - \psi_{3y_1}a_{110}v + \psi_{3y_1}a_{210}v - 2\psi_{3y_2}a_{213}v - 2\psi_{3y_3}a_{313}v}{4v}, \\
\psi_{3xy_2} &= \frac{2\psi_{3y_2}v_x + \psi_{3y_2}a_{110}v - \psi_{3y_2}a_{210}v - 2\psi_{3y_3}a_{315}v}{4v}, \\
\psi_{2xx} &= \frac{2\psi_{1y_1}\psi_{2x}v_x + \psi_{1y_1}\psi_{2x}a_{110}v + \psi_{1y_1}\psi_{2x}a_{210}v - 2\psi_{1y_1}\psi_{2y_1}a_{116}v - 2a_{216}v^2}{2\psi_{1y_1}v}, \\
\psi_{3xx} &= 2\psi_{3x}v_x + \psi_{3x}a_{110}v + \psi_{3x}a_{210}v - 2\psi_{3y_1}a_{116}v - 2\psi_{3y_2}a_{216}v - 2\psi_{3y_3}a_{316}v/2v
\end{aligned}$$

respectively. After all calculations, one can rewrite some results as

$$
\begin{aligned}
\varphi_{xx} = & \ (\varphi_x(2v_x + a_{110}v + a_{210}v))/(2v), \\
\psi_{1xx} = & \ (2\psi_{1x}v_x + \psi_{1x}a_{110}v + \psi_{1x}a_{210}v - 2\psi_{1y_1}a_{116}v)/(2v), \\
\psi_{1xy_1} = & \ (\psi_{1y_1}(2v_x - a_{110}v + a_{210}v))/(4v), \\
\psi_{1y_1y_1} = & \ -\psi_{1y_1}a_{14}, \\
\psi_{2xx} = & \ (2\psi_{1y_1}\psi_{2x}v_x + \psi_{1y_1}\psi_{2x}a_{110}v + \psi_{1y_1}\psi_{2x}a_{210}v - 2\psi_{1y_1}\psi_{2y_1}a_{116}v \\
& -2a_{216}v^2)/(2\psi_{1y_1}v), \\
\psi_{2xy_1} = & \ (2\psi_{1y_1}\psi_{2y_1}v_x - \psi_{1y_1}\psi_{2y_1}a_{110}v + \psi_{1y_1}\psi_{2y_1}a_{210}v - 2a_{213}v^2)/(4\psi_{1y_1}v), \\
\psi_{2y_1y_1} = & \ (-(\psi_{1y_1}\psi_{2y_1}a_{14} + a_{211}v))/\psi_{1y_1}, \\
\psi_{3xx} = & \ (2\psi_{3x}v_x + \psi_{3x}a_{110}v + \psi_{3x}a_{210}v - 2\psi_{3y_1}a_{116}v - 2\psi_{3y_2}a_{216}v \\
& -2\psi_{3y_3}a_{316}v)/(2v), \\
\psi_{3xy_1} = & \ (2\psi_{3y_1}v_x - \psi_{3y_1}a_{110}v + \psi_{3y_1}a_{210}v - 2\psi_{3y_2}a_{213}v - 2\psi_{3y_3}a_{313}v)/(4v), \\
\psi_{3xy_2} = & \ (2\psi_{3y_2}v_x + \psi_{3y_2}a_{110}v - \psi_{3y_2}a_{210}v - 2\psi_{3y_3}a_{315}v)/(4v), \\
\psi_{3xy_3} = & \ (\psi_{3y_3}(2v_x + a_{110}v + a_{210}v - 2a_{310}v))/(4v), \\
\psi_{3y_1y_1} = & \ -(\psi_{3y_1}a_{14} + \psi_{3y_2}a_{211} + \psi_{3y_3}a_{311}), \\
\psi_{3y_1y_2} = & \ (-(\psi_{3y_2}a_{27} + \psi_{3y_3}a_{312}))/2, \\
\psi_{3y_1y_3} = & \ (-\psi_{3y_3}a_{37})/2, \\
\psi_{3y_2y_2} = & \ -(\psi_{3y_2}a_{24} + \psi_{3y_3}a_{314}), \\
\psi_{3y_2y_3} = & \ (-\psi_{3y_3}a_{39})/2,
\end{aligned}
$$

$$
\begin{aligned}
\psi_{3y_3y_3} = & \ -\psi_{3y_3}a_{34}, \\
v_{y_1} = & \ (v(-2a_{14} - a_{27}))/2, \\
v_{y_2} = & \ -a_{24}v.
\end{aligned}
$$

Mixing the derivatives :

- $(v_{y_2})_{y_3} = 0$

$$-a_{24}(0) - a_{24y_3}v = 0$$

$$a_{24y_3} = 0$$

- $(\psi_{1y_1y_1})_{y_2} = 0$

$$-a_{14y_2}\psi_{1y_1} + a_{14}(0) = 0$$

$$a_{14y_2} = 0$$

- $(\psi_{1y_1y_1})_{y_3} = 0$

$$-a_{14y_3}\psi_{1y_1} + a_{14}(0) = 0$$
$$a_{14y_3} = 0$$

- $(\psi_{2y_1y_1})_{y_3} = 0$

$$-\frac{va_{211y_3}}{\psi_{1y_1}} = 0$$
$$a_{211y_3} = 0$$

- $(v_{y_1})_{y_3} = 0$

$$-\frac{va_{27y_3}}{2} = 0$$
$$a_{27y_3} = 0$$

- $(\varphi_{xx})_{y_3} = 0$

$$\frac{\varphi_x(a_{110y_3} + a_{210y_3})}{2} = 0$$
$$a_{110y_3} = -a_{210y_3}$$

- $(\psi_{1xy_1})_{y_3} = 0$

$$\frac{\psi_{1y_1}(-a_{110y_3} + a_{210y_3})}{4} = 0$$
$$a_{210y_3} + a_{210y_3} = 0$$
$$a_{210y_3} = 0$$

- $(\psi_{1xx})_{y_3} = 0$

$$\frac{a_{110y_3}\psi_{1x} - 2a_{116y_3}\psi_{1y_1} + a_{210y_3}\psi_{1x}}{2} = 0$$
$$-a_{116y_3}\psi_{1y_1} = 0$$
$$a_{116y_3} = 0$$

- $(\psi_{2xy_1})_{y_3} = 0$

$$\frac{-a_{110y_3}\psi_{1y_1}\psi_{2y_1} + a_{210y_3}\psi_{1y_1}\psi_{2y_1} - 2a_{213y_3}v}{4\psi_{1y_1}} = 0$$
$$\frac{-a_{213y_3}v}{2\psi_{1y_1}} = 0$$
$$a_{213y_3} = 0$$

- $(\psi_{2xx})_{y_3} = 0$

$$\frac{a_{110y_3}\psi_{1y_1}\psi_{2x} - 2a_{116y_3}\psi_{1y_1}\psi_{2y_1} + a_{210y_3}\psi_{1y_1}\psi_{2x} - 2a_{216y_3}v}{2\psi_{1y_1}} = 0$$
$$-\frac{a_{216y_3}v}{\psi_{1y_1}} = 0$$
$$a_{216y_3} = 0$$

- $(\psi_{3xy_3})_{y_3} = (\psi_{3y_3y_3})_x$

$$\psi_{3y_3}a_{310y_3} = \frac{\psi_{3y_3}(a_{110y_3} + a_{210y_3} + 4a_{34x})}{2}$$
$$a_{310y_3} = 2a_{34x}$$

- $(v_{y_1})_{y_2} = (v_{y_2})_{y_1}$

$$-va_{24y_1} = \frac{v(-2a_{14y_2} - a_{27y_2})}{2}$$
$$a_{24y_1} = \frac{a_{27y_2}}{2}$$

- $(\psi_{3y_2y_3})_{y_3} = (\psi_{3y_3y_3})_{y_2}$

$$\psi_{3y_3}a_{34y_2} = \frac{\psi_{3y_3}a_{39y_3}}{2}$$
$$a_{34y_2} = \frac{a_{39y_3}}{2}$$

- $(\psi_{3y_1y_3})_{y_2} = (\psi_{3y_2y_3})_{y_1}$

$$\frac{\psi_{3y_3}a_{37y_2}}{2} = \frac{\psi_{3y_3}a_{39y_1}}{2}$$
$$a_{37y_2} = a_{39y_1}$$

- $(\psi_{3y_1y_3})_{y_3} = (\psi_{3y_3y_3})_{y_1}$

$$\psi_{3y_3}a_{34y_1} = \frac{\psi_{3y_3}a_{37y_3}}{2}$$
$$a_{34y_1} = \frac{a_{37y_3}}{2}$$

- $(\varphi_{xx})_{y_2} = 0$

$$\frac{\varphi_x a_{110y_2}}{2} = \frac{\varphi_x(-a_{210y_2} + 2a_{24x})}{2}$$
$$a_{110y_2} = -a_{210y_2} + 2a_{24x}$$

- $(\varphi_{xx})_{y_1} = 0$

$$\frac{\varphi_x a_{110y_1}}{2} = \frac{\varphi_x(2a_{14x} - a_{210y_1} + a_{27x})}{2}$$
$$a_{110y_1} = 2a_{14x} - a_{210y_1} + a_{27x}$$

- $(\psi_{1xy_1})_{y_2} = 0$

$$\frac{\psi_{1y_1} a_{210y_2}}{4} = \frac{\psi_{1y_1} a_{24x}}{2}$$
$$a_{210y_2} = 2a_{24x}$$

- $(\psi_{1xx})_{y_2} = 0$

$$\psi_{1y_1} a_{116y_2} = \frac{a_{110y_2}\psi_{1x} + a_{210y_2}\psi_{1x} - 2a_{24x}\psi_{1x}}{2}$$
$$a_{116y_2} = 0$$

- $(\psi_{1xy_1})_{y_1} = (\psi_{1y_1y_1})_x$

$$\frac{\psi_{1y_1} a_{210y_1}}{4} = \frac{\psi_{1y_1}(a_{110y_1} - 2a_{14x} + a_{27x})}{4}$$
$$a_{210y_1} = -a_{210y_1} + 2a_{27x}$$
$$a_{210y_1} = a_{27x}$$

- $(\psi_{3xy_3})_{y_1} = (\psi_{3y_1y_3})_x$

$$\frac{\psi_{3y_3} a_{310y_1}}{2} = \frac{\psi_{3y_3}(a_{110y_1} - 2a_{14x} + a_{210y_1} - a_{27x} + 2a_{37x})}{4}$$
$$a_{310y_1} = a_{37x}$$

- $(\psi_{3xy_3})_{y_2} = (\psi_{3y_2y_3})_x$

$$\frac{\psi_{3y_3} a_{310y_2}}{2} = \frac{\psi_{3y_3}(a_{110y_2} + a_{210y_2} - 2a_{24x} + 2a_{39x})}{4}$$
$$a_{310y_2} = a_{39x}$$

- $(\psi_{2y_2})_{xy_1} = (\psi_{2xy_1})_{y_2}$

$$\begin{aligned} -\frac{a_{213y_2}v}{2\psi_{1y_1}} &= (-2a_{27x} - a_{110}a_{27} + a_{210}a_{27} - 2a_{213}a_{24})/4\psi_{1y_1} \\ a_{213y_2} &= (2a_{27x} + a_{110}a_{27} - a_{210}a_{27} + 2a_{213}a_{24})/2 \end{aligned}$$

- $(\psi_{2y_2})_{y_1y_1} = (\psi_{2y_1y_1})_{y_2}$

$$-\frac{a_{211y_2}v}{\psi_{1y_1}} = \frac{v(-2a_{27y_1} - 2a_{14}a_{27} - 4a_{211}a_{24} + a_{27}^2)}{4\psi_{1y_1}}$$

$$a_{211y_2} = \frac{2a_{27y_1} + 2a_{14}a_{27} + 4a_{211}a_{24}v - a_{27}^2}{4}$$

- $(\psi_{2xy_1})_{y_1} = (\psi_{2y_1y_1})_x$

$$\frac{a_{211x}v}{\psi_{1y_1}} = \frac{v(2a_{213y_1} - 2a_{110}a_{211} + 2a_{14}a_{213} + 2a_{210}a_{211} - a_{213}a_{27})}{4\psi_{1y_1}}$$

$$a_{211x} = \frac{2a_{213y_1} - 2a_{110}a_{211} + 2a_{14}a_{213} + 2a_{210}a_{211} - a_{213}a_{27}}{4}$$

- $(\psi_{3xy_1})_{y_3} = (\psi_{3xy_3})_{y_1}$

$$\frac{\psi_{3y_3}a_{313y_3}}{2} = \frac{\psi_{3y_3}(2a_{37x} + a_{110}a_{37} + a_{213}a_{39} - a_{310}a_{37} + 2a_{313}a_{34})}{4}$$

$$a_{313y_3} = \frac{2a_{37x} + a_{110}a_{37} + a_{213}a_{39} - a_{310}a_{37} + 2a_{313}a_{34}}{2}$$

- $(\psi_{3xy_2})_{y_3} = (\psi_{3xy_3})_{y_2}$

$$\frac{\psi_{3y_3}a_{315y_3}}{2} = \frac{\psi_{3y_3}(2a_{39x} + a_{210}a_{39} - a_{310}a_{39} + 2a_{315}a_{34})}{4}$$

$$a_{315y_3} = \frac{2a_{39x} + a_{210}a_{39} - a_{310}a_{39} + 2a_{315}a_{34}}{2}$$

- $(\psi_{3xy_2})_{y_1} = (\psi_{3y_1y_2})_x$

$$\frac{\psi_{3y_3}a_{312x}}{2} = \frac{\psi_{3y_3}(2a_{315y_1} - a_{210}a_{312} + a_{27}a_{315} + a_{310}a_{312} - a_{315}a_{37})}{4}$$
$$a_{312x} = \frac{2a_{315y_1} - a_{210}a_{312} + a_{27}a_{315} + a_{310}a_{312} - a_{315}a_{37}}{2}$$

- $(\psi_{3xy_2})_{y_2} = (\psi_{3y_2y_2})_x$

$$\psi_{3y_3}a_{314x} = \frac{\psi_{3y_3}(2a_{315y_2} - 2a_{210}a_{314} + 2a_{24}a_{315} + 2a_{310}a_{314} - a_{315}a_{39})}{4}$$
$$a_{314x} = \frac{2a_{315y_2} - 2a_{210}a_{314} + 2a_{24}a_{315} + 2a_{310}a_{314} - a_{315}a_{39}}{4}$$

- $(\psi_{3y_1y_1})_{y_3} = (\psi_{3y_1y_3})_{y_1}$

$$\psi_{3y_3}a_{311y_3} = \frac{\psi_{3y_3}(2a_{37y_1} + 2a_{14}a_{37} + 2a_{211}a_{39} + 4a_{311}a_{34} - a_{37}^2)}{4}$$
$$a_{311y_3} = \frac{2a_{37y_1} + 2a_{14}a_{37} + 2a_{211}a_{39} + 4a_{311}a_{34} - a_{37}^2}{4}$$

- $(\psi_{3y_1y_2})_{y_3} = (\psi_{3y_1y_3})_{y_2}$

$$\frac{a_{312y_3}\psi_{3y_3}}{2} = \frac{\psi_{3y_3}(2a_{39y_1} + a_{27}a_{39} + 2a_{312}a_{34} - a_{37}a_{39})}{4}$$
$$a_{312y_3} = \frac{2a_{39y_1} + a_{27}a_{39} + 2a_{312}a_{34} - a_{37}a_{39}}{2}$$

- $(\psi_{3y_1y_2})_{y_2} = (\psi_{3y_2y_2})_{y_1}$

$$\frac{\psi_{3y_3}a_{312y_2}}{2} = \frac{\psi_{3y_3}(4a_{314y_1} - 2a_{24}a_{312} + 2a_{27}a_{314} + a_{312}a_{39} - 2a_{314}a_{37})}{4}$$
$$a_{312y_2} = \frac{4a_{314y_1} - 2a_{24}a_{312} + 2a_{27}a_{314} + a_{312}a_{39} - 2a_{314}a_{37}}{2}$$

- $(\psi_{3y_2y_2})_{y_3} = (\psi_{3y_2y_3})_{y_2}$

$$a_{314y_3}\psi_{3y_3} = \frac{\psi_{3y_3}(2a_{39y_2} + 2a_{24}a_{39} + 4a_{314}a_{34} - a_{39}^2)}{4}$$
$$a_{314y_3} = \frac{2a_{39y_2} + 2a_{24}a_{39} + 4a_{314}a_{34} - a_{39}^2}{4}$$

- $(\psi_{3xy_1})_{y_2} = (\psi_{3xy_2})_{y_1}$

$$\frac{\psi_{3y_3}a_{313y_2}}{2} = \frac{\psi_{3y_3}(2a_{315y_1} + a_{110}a_{312} - a_{210}a_{312} + 2a_{213}a_{314} + a_{313}a_{39} - a_{315}a_{37})}{4}$$
$$a_{313y_2} = \frac{2a_{315y_1} + a_{110}a_{312} - a_{210}a_{312} + 2a_{213}a_{314} + a_{313}a_{39} - a_{315}a_{37}}{2}$$

- $(\psi_{3xy_1})_{y_1} = (\psi_{3y_1y_1})_x$

$$\begin{aligned}\psi_{3y_3}a_{311x} &= \psi_{3y_3}(2a_{313y_1} - 2a_{110}a_{311} + 2a_{14}a_{313} + 2a_{211}a_{315} - a_{213}a_{312} \\ &\quad +2a_{310}a_{311} - a_{313}a_{37})/4 \\ a_{311x} &= (2a_{313y_1} - 2a_{110}a_{311} + 2a_{14}a_{313} + 2a_{211}a_{315} - a_{213}a_{312} \\ &\quad +2a_{310}a_{311} - a_{313}a_{37})/4\end{aligned}$$

- $(\psi_{3y_1y_1})_{y_2} = (\psi_{3y_1y_2})_{y_1}$

$$\psi_{3y_3}a_{311y_2} = \frac{\psi_{3y_3}(2a_{312y_1} + 2a_{14}a_{312} + 4a_{211}a_{314} - a_{27}a_{312} + 2a_{311}a_{39} - a_{312}a_{37})}{4}$$
$$a_{311y_2} = \frac{2a_{312y_1} + 2a_{14}a_{312} + 4a_{211}a_{314} - a_{27}a_{312} + 2a_{311}a_{39} - a_{312}a_{37}}{4}$$

- $(\psi_{2y_2})_{xx} = (\psi_{2xx})_{y_2}$

$$\begin{aligned}\frac{v_{xx}}{2\psi_{1y_1}} &= (12v_x^2 + 4v_xv(a_{110} + a_{210}) - v^2(4a_{110x} - 4a_{210x} + 16a_{216y_2} \\ &\quad -a_{110}^2 - 2a_{110}a_{210} - 8a_{116}a_{27} + 3a_{210}^2 - 16a_{216}a_{24}))/(16v\psi_{1y_1}) \\ v_{xx} &= (12v_x^2 + 4v_xv(a_{110} + a_{210}) - v^2(4a_{110x} - 4a_{210x} + 16a_{216y_2} \\ &\quad -a_{110}^2 - 2a_{110}a_{210} - 8a_{116}a_{27} + 3a_{210}^2 - 16a_{216}a_{24}))/(8v)\end{aligned}$$

- $(\psi_{1xx})_{y_1} = (\psi_{2xy_1})_x$

$$\begin{aligned}
\frac{\psi_{1y_1}a_{110x}}{2} &= \psi_{1y_1}(4a_{116y_1} + 2a_{210x} - 4a_{216y_2} + a_{110}^2 - 4a_{116}a_{14} \\
&\quad +2a_{116}a_{27} - a_{210}^2 + 4a_{216}a_{24})/4 \\
a_{110x} &= (4a_{116y_1} + 2a_{210x} - 4a_{216y_2} + a_{110}^2 - 4a_{116}a_{14} \\
&\quad +2a_{116}a_{27} - a_{210}^2 + 4a_{216}a_{24})/2
\end{aligned}$$

- $(\psi_{1xx})_{y_1y_1} = (\psi_{2y_1y_1})_{xx}$

$$\psi_{1y_1}a_{116y_1y_1} = \psi_{1y_1}(a_{116y_1}a_{14} + a_{14xx} - a_{14x}a_{110} + a_{14y_1}a_{116})$$

$$a_{116y_1y_1} = a_{116y_1}a_{14} + a_{14xx} - a_{14x}a_{110} + a_{14y_1}a_{116}$$

- $(\psi_{3xx})_{y_2} = (\psi_{3xy_2})_x$

$$\begin{aligned}
\frac{\psi_{3y_3}a_{315x}}{2} &= \psi_{3y_3}(4a_{316y_2} - 2a_{116}a_{312} + a_{210}a_{315} - 4a_{216}a_{314} \\
&\quad +a_{310}a_{315} - 2a_{316}a_{39})/4 \\
a_{315x} &= (4a_{316y_2} - 2a_{116}a_{312} + a_{210}a_{315} - 4a_{216}a_{314} \\
&\quad +a_{310}a_{315} - 2a_{316}a_{39})/2
\end{aligned}$$

- $(\psi_{3xx})_{y_3} = (\psi_{3xy_3})_x$

$$\begin{aligned}
\frac{\psi_{3y_3}a_{210x}}{2} &= \psi_{3y_3}(4a_{216y_2} + 2a_{310x} - 4a_{316y_3} - 2a_{116}a_{27} + 2a_{116}a_{37} \\
&\quad +a_{210}^2 - 4a_{216}a_{24} + 2a_{216}a_{39} - a_{310}^2 + 4a_{316}a_{34})/4 \\
a_{210x} &= (4a_{216y_2} + 2a_{310x} - 4a_{316y_3} - 2a_{116}a_{27} + 2a_{116}a_{37} \\
&\quad +a_{210}^2 - 4a_{216}a_{24} + 2a_{216}a_{39} - a_{310}^2 + 4a_{316}a_{34})/2
\end{aligned}$$

- $(\psi_{2xx})_{y_1} = (\psi_{2xy_1})_x$

$$\frac{va_{213x}}{2\psi_{1y_1}} = \frac{v(4a_{216y_1} + a_{110}a_{213} - 4a_{116}a_{211} + a_{210}a_{213} - 2a_{216}a_{27})}{4\psi_{1y_1}}$$

$$a_{213x} = \frac{4a_{216y_1} + a_{110}a_{213} - 4a_{116}a_{211} + a_{210}a_{213} - 2a_{216}a_{27}}{2}$$

- $(\psi_{3xx})_{y_1} = (\psi_{3xy_1})_x$

$$\begin{aligned}\frac{\psi_{3y_3}a_{313x}}{2} &= \psi_{3y_3}(4a_{316y_1} + a_{110}a_{313} - 4a_{116}a_{311} + a_{213}a_{315} \\ &\quad -2a_{216}a_{312} + a_{310}a_{313} - 2a_{316}a_{37})/4 \\ a_{313x} &= (4a_{316y_1} + a_{110}a_{313} - 4a_{116}a_{311} + a_{213}a_{315} \\ &\quad -2a_{216}a_{312} + a_{310}a_{313} - 2a_{316}a_{37})/2\end{aligned}$$

- $(\psi_{3xx})_{y_1y_3} = (\psi_{3y_1y_3})_{xx}$

$$\begin{aligned}\psi_{3y_3}a_{316y_1y_3} &= \psi_{3y_3}(a_{116y_1}a_{37} + a_{216y_1}a_{39} + 2a_{316y_1}a_{34} + a_{37xx} \\ &\quad -a_{37x}a_{310} + a_{37y_1}a_{116} + a_{37y_3}a_{316} + a_{39y_1}a_{216})/2 \\ a_{316y_1y_3} &= a_{116y_1}a_{37} + a_{216y_1}a_{39} + 2a_{316y_1}a_{34} + a_{37xx} \\ &\quad -a_{37x}a_{310} + a_{37y_1}a_{116} + a_{37y_3}a_{316} + a_{39y_1}a_{216}\end{aligned}$$

- $(\psi_{3xx})_{y_2y_3} = (\psi_{3y_2y_3})_{xx}$

$$\begin{aligned}\psi_{3y_3}a_{316y_2y_3} &= \psi_{3y_3}(a_{216y_2}a_{39} + 2a_{316y_2}a_{34} + a_{39x} - a_{39x}a_{310} + a_{39y_1}a_{116} \\ &\quad +a_{39y_2}a_{216} + a_{39y_3}a_{316})/2 \\ a_{316y_2y_3} &= (a_{216y_2}a_{39} + 2a_{316y_2}a_{34} + a_{39x} - a_{39x}a_{310} + a_{39y_1}a_{116} \\ &\quad +a_{39y_2}a_{216} + a_{39y_3}a_{316})/2\end{aligned}$$

- $(\psi_{3xx})_{y_3y_3} = (\psi_{3y_3y_3})_{xx}$

$$\begin{aligned}\psi_{3y_3}a_{316y_3y_3} &= \psi_{3y_3}(2a_{316y_3}a_{34} + 2a_{34xx} - 2a_{34x}a_{310} + 2a_{34y_3}a_{316} \\ &\quad +a_{37y_3}a_{116} + a_{39y_3}a_{216})/2 \\ a_{316y_3y_3} &= (2a_{316y_3}a_{34} + 2a_{34xx} - 2a_{34x}a_{310} + 2a_{34y_3}a_{316} \\ &\quad +a_{37y_3}a_{116} + a_{39y_3}a_{216})/2.\end{aligned}$$

Mixing the derivatives again :

- $(v_{y_1})_{xx} = (v_{xx})_{y_1}$

$$va_{216y_1y_2} = \frac{v(a_{116y_1}a_{27} + 2a_{216y_1}a_{24} + a_{27xx} - a_{27x}a_{210} + a_{27y_1}a_{116} + a_{27y_2}a_{216})}{2}$$

$$a_{216y_1y_2} = \frac{a_{116y_1}a_{27} + 2a_{216y_1}a_{24} + a_{27xx} - a_{27x}a_{210} + a_{27y_1}a_{116} + a_{27y_2}a_{216}}{2}$$

- $(v_{y_2})_{xx} = (v_{xx})_{y_2}$

$$va_{216y_2y_2} = \frac{v(2a_{216y_2}a_{24} + 2a_{24xx} - 2a_{24x}a_{210} + 2a_{24y_2}a_{216} + a_{27y_2}a_{116})}{2}$$

$$a_{216y_2y_2} = \frac{2a_{216y_2}a_{24} + 2a_{24xx} - 2a_{24x}a_{210} + 2a_{24y_2}a_{216} + a_{27y_2}a_{116}}{2}.$$

After all calculations, we can rewrite some results as (2.46) - (2.81) and satisfied the equations $(\psi_{2y_2})_{y_3} = 0$, $(\psi_{3xx})_{y_1y_1} = (\psi_{3y_1y_1})_{xx}$, $(\psi_{3xx})_{y_1y_2} = (\psi_{3y_1y_2})_{xx}$, $(\psi_{3xx})_{y_2y_2} = (\psi_{3y_2y_2})_{xx}$, $(\psi_{3xy_1})_{y_2} = (\psi_{3y_1y_2})_{x}$, $(\psi_{3xy_1})_{y_3} = (\psi_{3y_1y_3})_{x}$, $(\psi_{3xy_1})_{y_2y_2} = (\psi_{3y_2y_2})_{xy_1}$, $(\psi_{3xy_1})_{y_2y_3} = (\psi_{3y_2y_3})_{xy_1}$, $(\psi_{3xy_1})_{y_3y_3} = (\psi_{3y_3y_3})_{xy_1}$, $(\psi_{3xy_2})_{y_1y_3} = (\psi_{3y_1y_3})_{xy_2}$, $(\psi_{3xy_2})_{y_3} = (\psi_{3y_2y_3})_{x}$, $(\psi_{3xy_2})_{y_3y_3} = (\psi_{3y_3y_3})_{xy_2}$, $(\psi_{3xy_3})_{y_1y_1} = (\psi_{3y_1y_1})_{xy_3}$, $(\psi_{3xy_3})_{y_1y_2} = (\psi_{3y_1y_2})_{xy_3}$, $(\psi_{3xy_3})_{y_2y_2} = (\psi_{3y_2y_2})_{xy_3}$, $(\psi_{3y_1y_1})_{y_2y_2} = (\psi_{3y_2y_2})_{y_1y_1}$, $(\psi_{3y_1y_1})_{y_2y_3} = (\psi_{3y_2y_3})_{y_1y_1}$, $(\psi_{3y_1y_1})_{y_3y_3} = (\psi_{3y_3y_3})_{y_1y_1}$, $(\psi_{3y_1y_2})_{y_3} = (\psi_{3y_2y_3})_{y_1}$, $(\psi_{3y_1y_2})_{y_3y_3} = (\psi_{3y_3y_3})_{y_1y_2}$, $(\psi_{3y_1y_3})_{y_2y_2} = (\psi_{3y_2y_2})_{y_1y_3}$, $(\psi_{3y_2y_2})_{y_3y_3} = (\psi_{3y_3y_3})_{y_2y_2}$, $(\psi_{3y_2y_3})_{y_3} = (\psi_{3y_3y_3})_{y_2}$, and $(v_{xx})_{y_3} = 0$. □

2.4 Linearizing transformation

By the prove of Theorem 2.3.1, we arrive at the following corollary.

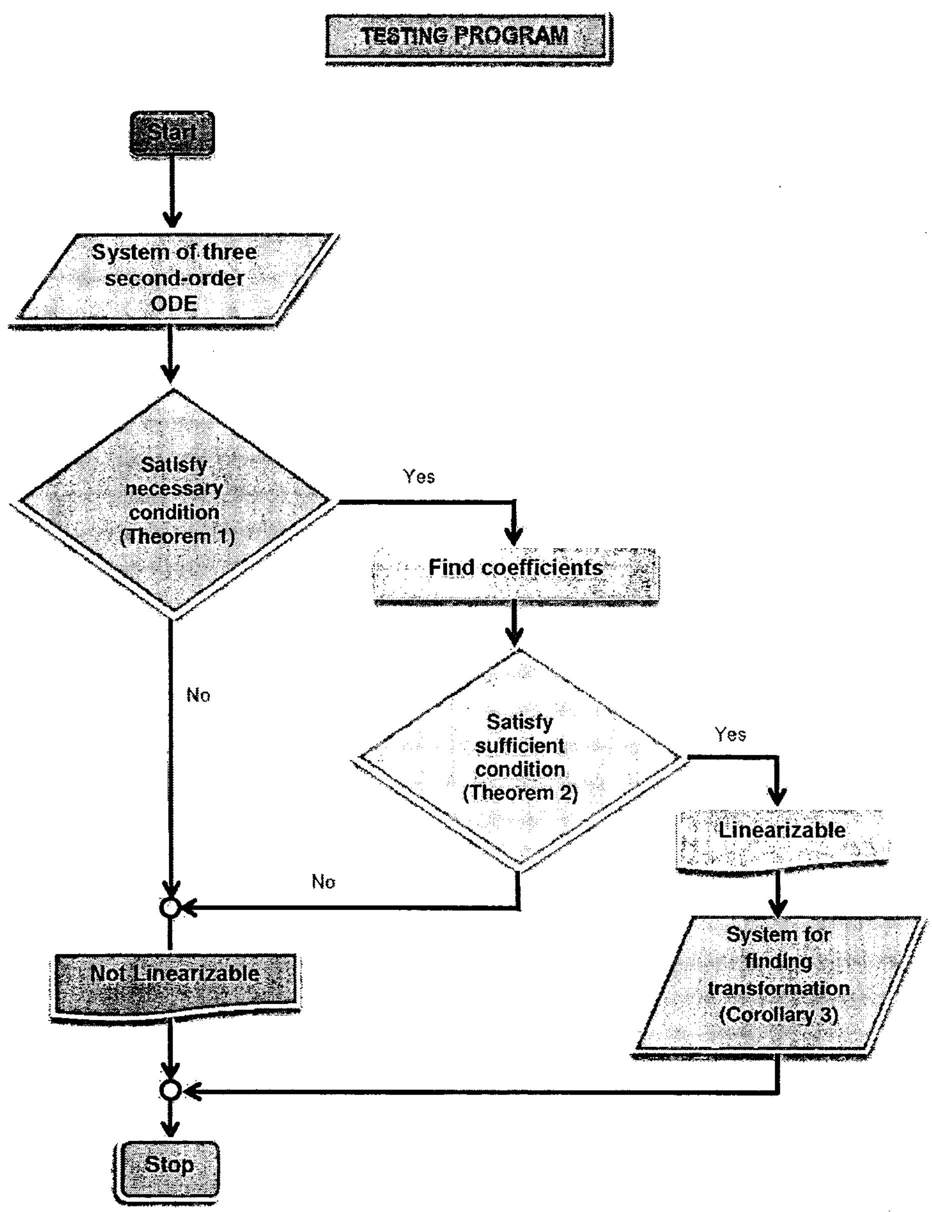
Corollary 2.4.1. *Provided that the sufficient conditions in Theorem 2.3.1 are satisfied, the transformation (2.44) mapping system (2.4) to a linear system (2.2) is obtained by solving the following compatible system of equations for the functions* $\varphi(x), \psi_1(x, y_1), \psi_2(x, y_1, y_2)$ *and* $\psi_3(x, y_1, y_2, y_3)$*:*

$$\begin{aligned}
v_{y_1} &= (v(-2a_{14} - a_{27}))/2, && (2.82)\\
v_{y_2} &= -va24, && (2.83)\\
v_{xx} &= (12v_x^2 + 4v_xv(a_{110} + a_{210}) + v^2(-8a_{116y_1} - 8a_{216y_2} - a_{110}^2 \\
&\quad +2a_{110}a_{210} + 8a_{116}a_{14} + 4a_{116}a_{27} - a_{210}^2 + 8a_{216}a_{24}))/(8v), && (2.84)\\
\varphi_{xx} &= (2\varphi_xv_x + \varphi_xv(a_{110} + a_{210}))/(2v), && (2.85)\\
\psi_{1xx} &= (2\psi_{1x}v_x + \psi_{1x}v(a_{110} + a_{210}) - 2v\psi_{1y_1}a_{116})/(2v), && (2.86)\\
\psi_{1xy_1} &= (2v_x\psi_{1y_1} + v\psi_{1y_1}(-a_{110} + a_{210}))/(4v), && (2.87)\\
\psi_{1y_1y_1} &= -\psi_{1y_1}a_{14}, && (2.88)\\
\psi_{2y_2} &= v/\psi_{1y_1}, && (2.89)
\end{aligned}$$

$$\begin{aligned}
\psi_{2xx} &= (2\psi_{2x}v_x\psi_{1y_1} + \psi_{2x}v\psi_{1y_1}(a_{110} + a_{210}) - 2\psi_{2y_1}v\psi_{1y_1}a_{116} \\
&\quad -2v^2a_{216})/(2v\psi_{1y_1}), && (2.90)\\
\psi_{2xy_1} &= (2\psi_{2y_1}v_x\psi_{1y_1} + \psi_{2y_1}v\psi_{1y_1}(-a_{110} + a_{210}) - 2v^2a_{213})/(4v\psi_{1y_1}), && (2.91)\\
\psi_{2y_1y_1} &= (-\psi_{2y_1}\psi_{1y_1}a_{14} - va_{211})/\psi_{1y_1}, && (2.92)\\
\psi_{3xx} &= (2\psi_{3x}v_x + \psi_{3x}v(a_{110} + a_{210}) - 2\psi_{3y_1}va_{116} - 2\psi_{3y_2}va_{216} \\
&\quad -2\psi_{3y_3}va_{316})/(2v), && (2.93)\\
\psi_{3xy_1} &= (2\psi_{3y_1}v_x + \psi_{3y_1}v(-a_{110} + a_{210}) - 2\psi_{3y_2}va_{213} \\
&\quad -2\psi_{3y_3}va_{313})/(4v), && (2.94)\\
\psi_{3xy_2} &= (2\psi_{3y_2}v_x + \psi_{3y_2}v(a_{110} - a_{210}) - 2\psi_{3y_3}va_{315})/(4v), && (2.95)\\
\psi_{3xy_3} &= (2\psi_{3y_3}v_x + \psi_{3y_3}v(a_{110} + a_{210} - 2a_{310}))/(4v), && (2.96)\\
\psi_{3y_1y_1} &= -\psi_{3y_1}a_{14} - \psi_{3y_2}a_{211} - \psi_{3y_3}a_{311}, && (2.97)\\
\psi_{3y_1y_2} &= (-\psi_{3y_2}a_{27} - \psi_{3y_3}a_{312})/2, && (2.98)\\
\psi_{3y_1y_3} &= (-\psi_{3y_3}a_{37})/2, && (2.99)\\
\psi_{3y_2y_2} &= -\psi_{3y_2}a_{24} - \psi_{3y_3}a_{314}, && (2.100)\\
\psi_{3y_2y_3} &= (-\psi_{3y_3}a_{39})/2, && (2.101)\\
\psi_{3y_3y_3} &= -\psi_{3y_3}a_{34}. && (2.102)
\end{aligned}$$

2.5 Flowchart of testing program

This flowchart shows demonstration of how to use the obtained theorems.



2.6 Examples

Example 2.6.1. Consider the system of nonlinear ordinary differential equation

$$\begin{aligned} &y_1'' + \tfrac{1}{2}y_1'^2 + y_1' = 0, \\ &y_2'' + \tfrac{1}{2}y_2'^2 + y_1'y_2' + y_2' = 0, \\ &y_3'' + y_3'^2 + y_1'y_3' + y_2'y_3' + y_3' = 0. \end{aligned} \tag{2.103}$$

It is a system of the form (2.4) with the coefficients

$$\begin{aligned} a_{11} =&\ 0,\ a_{12} = 0,\ a_{13} = 0,\ a_{14} = (-1)/2,\ a_{15} = 0,\ a_{16} = 0,\ a_{17} = 0, \\ a_{18} =&\ 0,\ a_{19} = 0,\ a_{110} = -1,\ a_{111} = 0,\ a_{112} = 0,\ a_{113} = 0,\ a_{114} = 0, \\ a_{115} =&\ 0,\ a_{116} = 0,\ a_{24} = (-1)/2,\ a_{27} = -1,\ a_{29} = 0,\ a_{210} = -1, \\ a_{211} =&\ 0, a_{212} = 0,\ a_{213} = 0,\ a_{214} = 0,\ a_{215} = 0,\ a_{216} = 0,\ a_{34} = -1, \\ a_{37} =&\ -1,\ a_{39} = -1,\ a_{310} = -1,\ a_{311} = 0,\ a_{312} = 0,\ a_{313} = 0,\ a_{314} = 0, \\ a_{315} =&\ 0,\ a_{316} = 0. \end{aligned} \tag{2.104}$$

One can check that the coefficients (2.104) obey the conditions (2.45) - (2.81). Thus, the equation (2.103) is linearizable. We have

$$v_{y_1} = v, \tag{2.105}$$
$$v_{y_2} = v/2, \tag{2.106}$$
$$v_{xx} = (v_x(3v_x - 2v))/(2v), \tag{2.107}$$
$$\varphi_{xx} = (\varphi_x(v_x - v))/v, \tag{2.108}$$
$$\psi_{1xx} = (\psi_{1x}(v_x - v))/v, \tag{2.109}$$
$$\psi_{1xy_1} = (\psi_{1y_1}v_x)/(2v), \tag{2.110}$$
$$\psi_{1y_1y_1} = \psi_{1y_1}/2, \tag{2.111}$$
$$\psi_{2y_2} = v/\psi_{1y_1}, \tag{2.112}$$
$$\psi_{2xx} = (\psi_{2x}(v_x - v))/v, \tag{2.113}$$
$$\psi_{2xy_1} = (\psi_{2y_1}v_x)/(2v), \tag{2.114}$$
$$\psi_{2y_1y_1} = \psi_{2y_1}/2, \tag{2.115}$$
$$\psi_{3xx} = (\psi_{3x}(v_x - v))/(v), \tag{2.116}$$
$$\psi_{3xy_1} = (\psi_{3y_1}v_x)/(2v), \tag{2.117}$$

$$\psi_{3xy_2} = (\psi_{3y_2}v_x)/(2v), \tag{2.118}$$
$$\psi_{3xy_3} = (\psi_{3y_3}v_x)/(2v), \tag{2.119}$$
$$\psi_{3y_1y_1} = \psi_{3y_1}/2, \tag{2.120}$$
$$\psi_{3y_1y_2} = \psi_{3y_2}/2, \tag{2.121}$$
$$\psi_{3y_1y_3} = \psi_{3y_3}/2, \tag{2.122}$$
$$\psi_{3y_2y_2} = \psi_{3y_2}/2, \tag{2.123}$$
$$\psi_{3y_2y_3} = \psi_{3y_3}/2, \tag{2.124}$$
$$\psi_{3y_3y_3} = \psi_{3y_3}. \tag{2.125}$$

From (2.105) and (2.106) we get

$$v = e^{y_1+c_1(x,y_2)}$$

and

$$v = e^{\frac{y_2}{2}+c_2(x,y_1)},$$

respectively. Since one can use any particular solution, we can take

$$v = e^{2x+y_1+\frac{y_2}{2}},$$

and this solution satisfies (2.107). Now the equation (2.108) becomes

$$\varphi_{xx} = \varphi_x$$

and yields

$$\varphi_x = e^{x+c_3}, \qquad c_3 = const.$$

Therefore

$$\varphi = e^{x+c_3} + c_4.$$

Since one can use any particular solution, we set $c_3 = 0$, $c_4 = 0$ and take

$$\varphi = e^x.$$

Now the equations (2.109) - (2.111) are written

$$\psi_{1xx} = \psi_{1x}, \tag{2.126}$$
$$\psi_{1xy_1} = \psi_{1y_1}, \tag{2.127}$$
$$\psi_{1y_1y_1} = \frac{\psi_{1y_1}}{2}. \tag{2.128}$$

To consider (2.127) and (2.128), one takes

$$\psi_{1y_1} = e^{x+\frac{y_1}{2}},$$

then

$$\psi_1 = 2e^{x+\frac{y_1}{2}} + c_5(x).$$

Since one can use any particular solution, we set $c_5(x) = 0$ and take

$$\psi_1 = 2e^{x+\frac{y_1}{2}},$$

this solution satisfies (2.126). Now the equations (2.112) - (2.115) are written as

$$\psi_{2y_2} = \frac{e^{x+y_1+\frac{y_2}{2}}}{e^{\frac{y_1}{2}}}, \tag{2.129}$$

$$\psi_{2xx} = \psi_{2x}, \tag{2.130}$$

$$\psi_{2xy_1} = \psi_{2y_1}, \tag{2.131}$$

$$\psi_{2y_1y_1} = \frac{\psi_{2y_1}}{2}. \tag{2.132}$$

To consider (2.129), after integration, one finds

$$\psi_2 = 2e^{x+\frac{y_1}{2}+\frac{y_2}{2}} + c_6(x, y_1).$$

Since one can use any particular solution, we set $c_6(x, y_1) = 0$ and take

$$\psi_2 = 2e^{x+\frac{y_1}{2}+\frac{y_2}{2}},$$

this solution satisfies (2.130) - (2.132). Now the equations (2.116) - (2.125) are written

$$\psi_{3xx} = \psi_{3x}, \tag{2.133}$$

$$\psi_{3xy_1} = \psi_{3y_1}, \tag{2.134}$$

$$\psi_{3xy_2} = \psi_{3y_2}, \tag{2.135}$$

$$\psi_{3xy_3} = \psi_{3y_3}, \tag{2.136}$$

$$\psi_{3y_1y_1} = \frac{\psi_{3y_1}}{2}, \tag{2.137}$$

$$\psi_{3y_1y_2} = \frac{\psi_{3y_2}}{2}, \tag{2.138}$$

$$\psi_{3y_1y_3} = \frac{\psi_{3y_3}}{2}, \tag{2.139}$$

$$\psi_{3y_2y_2} = \frac{\psi_{3y_2}}{2}, \tag{2.140}$$

$$\psi_{3y_2y_3} = \frac{\psi_{3y_3}}{2}, \tag{2.141}$$

$$\psi_{3y_3y_3} = \psi_{3y_3}. \tag{2.142}$$

To consider (2.136), (2.139), (2.141) and (2.142), one obtains

$$\psi_{3y_3} = e^{x+\frac{y_1}{2}+\frac{y_2}{2}+y_3}$$

so that

$$\psi_3 = e^{x+\frac{y_1}{2}+\frac{y_2}{2}+y_3} + c_7(x, y_1, y_2).$$

Since one can use any particular solution, we set $c_7(x, y_1, y_2) = 0$ and take

$$\psi_3 = e^{x+\frac{y_1}{2}+\frac{y_2}{2}+y_3},$$

this solution satisfies (2.133) - (2.135), (2.137) - (2.138), (2.140).
Then, one obtains the following transformations

$$t = e^x,\ u_1 = 2e^{x+\frac{y_1}{2}},\ u_2 = 2e^{x+\frac{y_1}{2}+\frac{y_2}{2}},\ u_3 = e^{x+\frac{y_1}{2}+\frac{y_2}{2}+y_3}. \tag{2.143}$$

Hence, the system (2.103) is mapped by the transformations (2.143) to the linear system

$$u_1'' = 0,\ u_2'' = 0,\ u_3'' = 0.$$

The solution of this linear system is

$$\begin{aligned} u_1(t) &= c_1 t + c_2, \\ u_2(t) &= c_3 t + c_4, \\ u_3(t) &= c_5 t + c_6 \end{aligned}$$

where $c_i, (i = 1, 2, ..., 6)$ are arbitrary constants. By using the transformation (2.143), one finds

$$\begin{aligned} 2e^{x+\frac{y_1}{2}} &= c_1 e^x + c_2, \\ 2e^{x+\frac{y_1}{2}+\frac{y_2}{2}} &= c_3 e^x + c_4, \\ e^{x+\frac{y_1}{2}+\frac{y_2}{2}+y_3} &= c_5 e^x + c_6. \end{aligned}$$

Hence, the solution of the system (2.103) is

$$y_1 = 2\ln\left[\frac{c_1+c_2e^{-x}}{2}\right],$$

$$y_2 = 2\ln\left[\frac{c_3+c_4e^{-x}}{c_1+c_2e^{-x}}\right],$$

$$y_3 = \ln\left[\frac{2(c_5+c_6e^{-x})}{c_3+c_4e^{-x}}\right].$$

Example 2.6.2. Consider the system of nonlinear ordinary differential equation

$$\begin{aligned} &y_1'' + \tfrac{1}{2}y_1'^2 + y_1' = 0, \\ &y_2'' + y_2'^2 + y_1'y_2' + y_2' = 0, \\ &y_3'' + y_3'^2 + y_1'y_3' + y_3' = 0. \end{aligned} \tag{2.144}$$

It is a system of the form (2.4) with the coefficients

$$\begin{aligned} a_{11} =\ & 0,\ a_{12} = 0,\ a_{13} = 0,\ a_{14} = (-1)/2,\ a_{15} = 0,\ a_{16} = 0,\ a_{17} = 0, \\ a_{18} =\ & 0,\ a_{19} = 0,\ a_{110} = -1,\ a_{111} = 0,\ a_{112} = 0,\ a_{113} = 0,\ a_{114} = 0, \\ a_{115} =\ & 0,\ a_{116} = 0,\ a_{24} = -1,\ a_{27} = -1,\ a_{29} = 0,\ a_{210} = -1, \\ a_{211} =\ & 0, a_{212} = 0,\ a_{213} = 0,\ a_{214} = 0,\ a_{215} = 0,\ a_{216} = 0,\ a_{34} = -1, \\ a_{37} =\ & -1,\ a_{39} = 0,\ a_{310} = -1,\ a_{311} = 0,\ a_{312} = 0,\ a_{313} = 0,\ a_{314} = 0, \\ a_{315} =\ & 0,\ a_{316} = 0. \end{aligned} \tag{2.145}$$

One can check that the coefficients (2.145) obey the conditions (2.45) - (2.81). Thus, the equation (2.144) is linearizable. We have

$$v_{y_1} = v, \tag{2.146}$$

$$v_{y_2} = v, \tag{2.147}$$

$$v_{xx} = (v_x(3v_x - 2v))/(2v), \tag{2.148}$$

$$\varphi_{xx} = (\varphi_x(v_x - v))/v, \tag{2.149}$$

$$\psi_{1xx} = (\psi_{1x}(v_x - v))/v, \tag{2.150}$$

$$\psi_{1xy_1} = (\psi_{1y_1}v_x)/(2v), \tag{2.151}$$

$$\psi_{1y_1y_1} = \psi_{1y_1}/2, \tag{2.152}$$

$$\psi_{2y_2} = v/\psi_{1y_1}, \tag{2.153}$$

$$\psi_{2xx} = (\psi_{2x}(v_x - v))/v, \tag{2.154}$$

$$\psi_{2xy_1} = (\psi_{2y_1}v_x)/(2v), \tag{2.155}$$

$$\psi_{2y_1y_1} = \psi_{2y_1}/2, \tag{2.156}$$

$$\psi_{3xx} = (\psi_{3x}(v_x - v))/(v), \tag{2.157}$$

$$\psi_{3xy_1} = (\psi_{3y_1}v_x)/(2v), \tag{2.158}$$

$$\psi_{3xy_2} = (\psi_{3y_2} v_x)/(2v), \quad (2.159)$$
$$\psi_{3xy_3} = (\psi_{3y_3} v_x)/(2v), \quad (2.160)$$
$$\psi_{3y_1y_1} = \psi_{3y_1}/2, \quad (2.161)$$
$$\psi_{3y_1y_2} = \psi_{3y_2}/2, \quad (2.162)$$
$$\psi_{3y_1y_3} = \psi_{3y_3}/2, \quad (2.163)$$
$$\psi_{3y_2y_2} = \psi_{3y_2}, \quad (2.164)$$
$$\psi_{3y_2y_3} = 0, \quad (2.165)$$
$$\psi_{3y_3y_3} = \psi_{3y_3}. \quad (2.166)$$

From (2.146) and (2.147) we get

$$v = e^{y_1 + c_1(x,y_2)}$$

and

$$v = e^{y_2 + c_2(x,y_1)},$$

respectively. Since one can use any particular solution, we can take

$$v = e^{2x + y_1 + y_2}$$

and this solution satisfies (2.148). Now the equation (2.149) becomes

$$\varphi_{xx} = \varphi_x$$

and yields

$$\varphi_x = e^{x + c_3}, \qquad c_3 = const.$$

Thus

$$\varphi = e^{x + c_3} + c_4.$$

Since one can use any particular solution, we set $c_3 = 0$, $c_4 = 0$ and take

$$\varphi = e^x.$$

Now the equations (2.150) - (2.152) are written

$$\psi_{1xx} = \psi_{1x}, \quad (2.167)$$
$$\psi_{1xy_1} = \psi_{1y_1}, \quad (2.168)$$
$$\psi_{1y_1y_1} = \frac{\psi_{1y_1}}{2}. \quad (2.169)$$

To consider (2.168) and (2.169), one gets

$$\psi_{1y_1} = e^{x + \frac{y_1}{2}},$$

then

$$\psi_1 = 2e^{x+\frac{y_1}{2}} + c_5(x).$$

Since one can use any particular solution, we set $c_5(x) = 0$ and take

$$\psi_1 = 2e^{x+\frac{y_1}{2}},$$

this solution satisfies (2.167). Now the equations (2.153) - (2.156) are written as

$$\psi_{2y_2} = \frac{e^{x+y_1+y_2}}{e^{\frac{y_1}{2}}}, \tag{2.170}$$
$$\psi_{2xx} = \psi_{2x}, \tag{2.171}$$
$$\psi_{2xy_1} = \psi_{2y_1}, \tag{2.172}$$
$$\psi_{2y_1y_1} = \frac{\psi_{2y_1}}{2}. \tag{2.173}$$

To consider (2.170), after integration, one finds

$$\psi_2 = e^{x+\frac{y_1}{2}+y_2} + c_6(x, y_1).$$

Since one can use any particular solution, we set $c_6(x, y_1) = 0$ and take

$$\psi_2 = e^{x+\frac{y_1}{2}+y_2},$$

this solution satisfies (2.171) - (2.173). Now the equations (2.157) - (2.166) are written

$$\psi_{3xx} = \psi_{3x}, \tag{2.174}$$
$$\psi_{3xy_1} = \psi_{3y_1}, \tag{2.175}$$
$$\psi_{3xy_2} = \psi_{3y_2}, \tag{2.176}$$
$$\psi_{3xy_3} = \psi_{3y_3}, \tag{2.177}$$
$$\psi_{3y_1y_1} = \frac{\psi_{3y_1}}{2}, \tag{2.178}$$
$$\psi_{3y_1y_2} = \frac{\psi_{3y_2}}{2}, \tag{2.179}$$
$$\psi_{3y_1y_3} = \frac{\psi_{3y_3}}{2}, \tag{2.180}$$
$$\psi_{3y_2y_2} = \psi_{3y_2}, \tag{2.181}$$
$$\psi_{3y_2y_3} = 0, \tag{2.182}$$
$$\psi_{3y_3y_3} = \psi_{3y_3}. \tag{2.183}$$

To consider (2.177), (2.180) and (2.183), one takes

$$\psi_{3y_3} = e^{x+\frac{y_1}{2}+y_3}$$

so that

$$\psi_3 = e^{x+\frac{y_1}{2}+y_3} + c_7(x, y_1, y_2).$$

Since one can use any particular solution, we set $c_7(x, y_1, y_2) = 0$ and take

$$\psi_3 = e^{x+\frac{y_1}{2}+y_3},$$

this solution satisfies (2.174) - (2.176), (2.178) - (2.179), (2.181) - (2.182).
Then, one obtains the following transformations

$$t = e^x,\ u_1 = 2e^{x+\frac{y_1}{2}},\ u_2 = e^{x+\frac{y_1}{2}+y_2},\ u_3 = e^{x+\frac{y_1}{2}+y_3}. \tag{2.184}$$

Hence, the system (2.144) is mapped by the transformations (2.184) to the linear system

$$u_1'' = 0,\ u_2'' = 0,\ u_3'' = 0.$$

The solution of this linear system is

$$\begin{aligned} u_1(t) &= c_1 t + c_2, \\ u_2(t) &= c_3 t + c_4, \\ u_3(t) &= c_5 t + c_6 \end{aligned}$$

where $c_i, (i = 1, 2, ..., 6)$ are arbitrary constants. By using the transformation (2.184), one finds

$$\begin{aligned} 2e^{x+\frac{y_1}{2}} &= c_1 e^x + c_2, \\ e^{x+\frac{y_1}{2}+y_2} &= c_3 e^x + c_4, \\ e^{x+\frac{y_1}{2}+y_3} &= c_5 e^x + c_6. \end{aligned}$$

Hence, the solution of the system (2.144) is

$$y_1 = 2\ln\left[\frac{c_1+c_2e^{-x}}{2}\right],$$

$$y_2 = \ln\left[\frac{2(c_3+c_4e^{-x})}{c_1+c_2e^{-x}}\right],$$

$$y_3 = \ln\left[\frac{2(c_5+c_6e^{-x})}{c_1+c_2e^{-x}}\right].$$

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