Chapter 1

Basic concepts

1.1 Definitions and theorems used in the research

Definition 1.1.1. A transformation

$$\begin{aligned} t &= \varphi(x,y), \\ u &= \psi(x,y), \end{aligned} \tag{1.1}$$

where φ and ψ are sufficiently smooth functions is called a *point transformation.* If $\varphi_y = 0$, a transformation (1.1) is called a fiber preserving transformation.

Definition 1.1.2. Two equations are call *equivalent* if there is an invertible transformation which transforms one equation into another.

Definition 1.1.3. The problem of finding all equations, which are equivalent to a given equation is called an *equivalence problem.* If the given equation is a linear equation, then the equivalence problem is called a *linearization problem.*

Definition 1.1.4. If $F(u,v)$ and $G(u,v)$ are differentiable in a region, the *Jacobian determinant*, or briefly the *Jacobian*, of F and G with respect to u and v is the second-order functional determinant defined by

$$\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix} = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}.$$

Similarly, the third-order determinant

$$\frac{\partial(F,G,H)}{\partial(u,v,w)} = \begin{vmatrix} F_u & F_v & F_w \\ G_u & G_v & G_w \\ H_u & H_v & H_w \end{vmatrix}$$

is called the Jacobian of F, G and H with respect to u, v and w.

Theorem 1.1.5. *(Inverse Function Theorem) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable on some open set containing a, and suppose Jacobian of f(a) not equal to zero. Then there is some open set V containing a and an open W containing $f(a)$ such that $f : V \to W$ has a continuous inverse $f^{-1} : W \to V$ which is differentiable for all $y \in W$.*

Theorem 1.1.6. *(Clairaut's theorem on equality of mixed partials)*

- **Statement for second-order mixed partial of function of two variables**
  Suppose f is a real-valued function of two variables x, y and $f(x, y)$ is defined on an open subset U of $\mathbb{R}^2$. Suppose further that both the second-order mixed partial derivatives $f_{xy}(x, y)$ and $f_{yx}(x, y)$ exist and are continuous on U. Then, we have:

$$f_{xy} = f_{yx}$$

  on all of U.

- **General statement**
  The statement can be generalized in two ways:
  - *We can generalize it to higher-order partial derivatives.*
  - *We can generalize it to functions of more than two variables.*

  The general version states the following. Suppose f is a function of n variables defined on an open subset U of $\mathbb{R}^n$. Suppose all mixed partials with a certain number of differentiations in each input variable exist and are continuous on U. Then, all the mixed partials are continuous. Some examples are given below:
  - *Suppose f is a function of two variables x and y, and the three mixed partials $f_{xxy}, f_{xyx}, f_{yxx}$ exist and are continuous on an open subset U of $\mathbb{R}^2$. Then, all three of them are equal on U. (Note that these mixed partials all involve differentiating twice with respect to x and once with respect to y.)*
  - *Suppose f is a function of three variables x, y, z, and the six mixed partials f_{xyz}, f_{xzy}, f_{yxz}, f_{yzx}, f_{zxy}, f_{zyx} exist and are continuous on an open subset U of $\mathbb{R}^3$. Then, all six of them are equal on U.*

Theorem 1.1.7. *(Laguerre-Forsyth canonical form) Any second-order linear ordinary differential equations*

$$y'' + a_1(x)y' + a_0(x)y = 0$$

can be transformed to the form

$$u'' = 0$$

by point transformation (1.1).

Theorem 1.1.8. *(Laguerre-Forsyth canonical form) Any kth-order linear ordinary differential equations*

$$y^{(k)} + \sum_{i=0}^{k-1} a_i(x) y^{(i)} = 0, \qquad k \geq 3$$

can be transformed to the form

$$u^{(k)} + \sum_{i=0}^{k-3} a_i(x) u^{(i)} = 0 \tag{1.2}$$

by point transformation (1.1).

Note that (1.2) is called the *Laguerre-Forsyth canonical form* of kth-order linear ordinary differential equations.

1.2 Transformation of derivatives

For simplicity of understanding, let us consider second-order ordinary differential equation

$$y'' = F(x, y, y'). \tag{1.3}$$

An invertible change of the independent and dependent variables (1.1) leads equation (1.3) into the equation

$$u'' = f(t, u, u'). \tag{1.4}$$

Notice that we require the Jacobian

$$\Delta = \frac{\partial(t, u)}{\partial(x, y)} = \frac{\partial(\varphi, \psi)}{\partial(x, y)} = \varphi_x \psi_y - \varphi_y \psi_x \neq 0.$$

in the neighberhood of a.

First of all, we have to change $y(x)$ to $u(t)$. Assume that we know a solution of equation (1.3), i.e.

$$y = y(x).$$

The transformed function $u(t)$ is found from equation

$$t = \varphi(x, y(x)).$$

Since $\varphi'(x,y(x)) = \varphi_x + y'\varphi_y \in C$ (i.e. $\varphi \in C^1$) and $\Delta\ (\varphi(x,y(x))) = \varphi_x + y'\varphi_y \neq 0$ then by the virtue of Inverse Function Theorem one finds

$$x = \alpha(t).$$

Thus, one obtains

$$u(t) = \psi(\alpha(t), y(\alpha(t))). \tag{1.5}$$

The transformation of the first derivatives can be found as follows. Let us differentiate (1.5) with respect to t

$$u'(t) = \frac{du}{dt} = \frac{\partial\psi}{\partial x}\frac{d\alpha}{dt} + \frac{\partial\psi}{\partial y}\frac{dy}{dx}\frac{d\alpha}{dt} = (\psi_x + y'\psi_y)\frac{d\alpha}{dt}. \tag{1.6}$$

Since $t = \varphi(\alpha(t), y(\alpha(t)))$ then

$$\frac{dt}{dt} = \frac{\partial\varphi}{\partial x}\frac{d\alpha}{dt} + \frac{\partial\varphi}{\partial y}\frac{dy}{dx}\frac{d\alpha}{dt}$$

or

$$\frac{d\alpha}{dt} = \frac{1}{(\varphi_x + y'\varphi_y)}. \tag{1.7}$$

Substituting equation (1.7) into equation (1.6), one obtains

$$u'(t) = \frac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y} = \frac{D_x\psi}{D_x\varphi} = g(x, y(x), y'(x)).$$

where $D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \ldots$ is the *total derivative*. So that the first prolongation of transformation (1.1) is $u' = g(x,y,y')$.

Next, to find the transformation of second-order derivative. Consider

$$\begin{aligned} u''(t) &= \frac{\partial g}{\partial x}\frac{d\alpha}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dx}\frac{d\alpha}{dt} + \frac{\partial g}{\partial y'}\frac{dy'}{dx}\frac{d\alpha}{dt} \\ &= (g_x + y'g_y + y''g_{y'})\frac{d\alpha}{dt} \\ &= \frac{g_x + y'g_y + y''g_{y'}}{\varphi_x + y'\varphi_y} \\ &= \frac{D_x g}{D_x\varphi} \\ &= h(x, y(x), y'(x), y''(x)). \end{aligned}$$

So that the second prolongation of transformation (1.1) is $u'' = h(x,y,y',y'')$.

Moreover, we can rewrite equation (1.4) as

$$h(x,y,y',y'') = f(\varphi(x,y), \psi(x,y), g(x,y,y')). \tag{1.8}$$

We see that if we find u from equation (1.4), then we can find y from equation (1.8). Hence $u'' = f(t,u,u')$ and $y'' = F(x,y,y')$ are equivalent.

1.3 Completely integrable systems

One class of overdetermined systems, for which the problem of compatibility is solved, is the class of completely integrable systems. The theory of completely integrable systems is developed in the general case.

Definition 1.3.1. A system

$$\frac{\partial z^i}{\partial a^j} = f^i_j(a, z),\ (i = 1, 2, ..., N;\ j = 1, 2, ..., r) \tag{1.9}$$

is called *completely integrable* if it has a solution for any initial values a_0, z_0 in some open domain D.

Theorem 1.3.2. *A system of the type (1.9) is completely integrable if and only if all of the mixed derivatives equalities*

$$\frac{\partial f^i_j}{\partial a^\beta} + \sum_{\gamma=1}^{N} f^\gamma_\beta \frac{\partial f^i_j}{\partial z^\gamma} = \frac{\partial f^i_\beta}{\partial a^j} + \sum_{\gamma=1}^{N} f^\gamma_j \frac{\partial f^i_\beta}{\partial z^\gamma},\ (i = 1, 2, ..., N;\ \beta, j = 1, 2, ..., r) \tag{1.10}$$

are identically satisfied with respect to the variable $(a, z) \in D$.

In practice, sometimes it is enough to use a particular case of the compatibility theorem:

Corollary 1.3.3. *If in an overdetermined system of partial differential equations all derivatives of order n are defined and comparison of all mixed derivatives of order n + 1 does not produre new equations of order less or equal to n, then this system is compatible.*

1.4 The Lie linearization test

Theorem 1.4.1. ([1], S. Lie) *Any second-order ordinary differential equations (1.3) obtained from a linear equation*

$$u'' = 0 \tag{1.11}$$

by a point transformation (1.1) has to be either to the form

$$y'' + a(x, y)\, y'^3 + b(x, y)\, y'^2 + c(x, y)\, y' + d(x, y) = 0, \tag{1.12}$$

where

$$\begin{aligned} a =&\ \Delta^{-1}(\varphi_y\psi_{yy} - \varphi_{yy}\psi_y), \\ b =&\ \Delta^{-1}(\varphi_x\psi_{yy} - \varphi_{yy}\psi_x + 2(\varphi_y\psi_{xy} - \varphi_{xy}\psi_y)), \\ c =&\ \Delta^{-1}(\varphi_y\psi_{xx} - \varphi_{xx}\psi_y + 2(\varphi_x\psi_{xy} - \varphi_{xy}\psi_x)), \\ d =&\ \Delta^{-1}(\varphi_x\psi_{xx} - \varphi_{xx}\psi_x) \end{aligned} \tag{1.13}$$

and $\Delta = \varphi_x\psi_y - \varphi_y\psi_x \neq 0$.

Proof.

Since $u = \psi(x, y)$, thus

$$\begin{aligned} u'(t) &= \frac{D_x\psi}{D_x\varphi} \\ &= \frac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y} \\ &= g(x, y, y'), \\ u''(t) &= \frac{D_x g}{D_x \varphi} \\ &= \frac{g_x + y' g_y + y'' g_{y'}}{\varphi_x + y'\varphi_y} \\ &= P(x, y, y') \end{aligned} \tag{1.14}$$

where

$$\begin{aligned} g_x &= \frac{(\varphi_x + y'\varphi_y)\frac{\partial}{\partial x}(\psi_x + y'\psi_y) - (\psi_x + y'\psi_y)\frac{\partial}{\partial x}(\varphi_x + y'\varphi_y)}{(\varphi_x + y'\varphi_y)^2} \\ &= \frac{(\varphi_x + y'\varphi_y)\left(\psi_{xx} + \left(y'\frac{\partial}{\partial x}\psi_y + \psi_y\frac{\partial}{\partial x}y'\right)\right) - (\psi_x + y'\psi_y)\left(\varphi_{xx} + \left(y'\frac{\partial}{\partial x}\varphi_y + \varphi_y\frac{\partial}{\partial x}y'\right)\right)}{(\varphi_x + y'\varphi_y)^2} \\ &= \frac{(\varphi_x + y'\varphi_y)(\psi_{xx} + y'\psi_{xy} + \psi_y(0)) - (\psi_x + y'\psi_y)(\varphi_{xx} + y'\varphi_{xy} + \varphi_y(0))}{(\varphi_x + y'\varphi_y)^2} \\ &= \frac{(\varphi_x + y'\varphi_y)(\psi_{xx} + y'\psi_{xy}) - (\psi_x + y'\psi_y)(\varphi_{xx} + y'\varphi_{xy})}{(\varphi_x + y'\varphi_y)^2}, \end{aligned}$$

$$\begin{aligned} g_y &= \frac{(\varphi_x + y'\varphi_y)\frac{\partial}{\partial y}(\psi_x + y'\psi_y) - (\psi_x + y'\psi_y)\frac{\partial}{\partial y}(\varphi_x + y'\varphi_y)}{(\varphi_x + y'\varphi_y)^2} \\ &= \frac{(\varphi_x + y'\varphi_y)\left(\psi_{xy} + \left(y'\frac{\partial}{\partial y}\psi_y + \psi_y\frac{\partial}{\partial y}y'\right)\right) - (\psi_x + y'\psi_y)\left(\varphi_{xy} + \left(y'\frac{\partial}{\partial y}\varphi_y + \varphi_y\frac{\partial}{\partial y}y'\right)\right)}{(\varphi_x + y'\varphi_y)^2} \\ &= \frac{(\varphi_x + y'\varphi_y)(\psi_{xy} + y'\psi_{yy} + \psi_y(0)) - (\psi_x + y'\psi_y)(\varphi_{xy} + y'\varphi_{yy} + \varphi_y(0))}{(\varphi_x + y'\varphi_y)^2} \\ &= \frac{(\varphi_x + y'\varphi_y)(\psi_{xy} + y'\psi_{yy}) - (\psi_x + y'\psi_y)(\varphi_{xy} + y'\varphi_{yy})}{(\varphi_x + y'\varphi_y)^2}, \end{aligned}$$

$$\begin{aligned} g_{y'} &= \frac{(\varphi_x + y'\varphi_y)\frac{\partial}{\partial y'}(\psi_x + y'\psi_y) - (\psi_x + y'\psi_y)\frac{\partial}{\partial y'}(\varphi_x + y'\varphi_y)}{(\varphi_x + y'\varphi_y)^2} \\ &= \frac{(\varphi_x + y'\varphi_y)\left(0 + \left(y'\frac{\partial}{\partial y'}\psi_y + \psi_y\frac{\partial}{\partial y'}y'\right)\right) - (\psi_x + y'\psi_y)\left(0 + \left(y'\frac{\partial}{\partial y'}\varphi_y + \varphi_y\frac{\partial}{\partial y'}y'\right)\right)}{(\varphi_x + y'\varphi_y)^2} \\ &= \frac{(\varphi_x + y'\varphi_y)(0 + y'(0) + \psi_y(1)) - (\psi_x + y'\psi_y)(0 + y'(0) + \varphi_y(1))}{(\varphi_x + y'\varphi_y)^2} \\ &= \frac{(\varphi_x + y'\varphi_y)(\psi_y) - (\psi_x + y'\psi_y)(\varphi_y)}{(\varphi_x + y'\varphi_y)^2} \end{aligned}$$

and $D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + ...$ is total derivatives.
Replacing $g_x, g_y, g_{y'}$ into equation (1.14) one gets

$$\begin{aligned} u'' = \; & (y''(\varphi_x\psi_y - \varphi_y\psi_x) + y'^3(\varphi_y\psi_{yy} - \varphi_{yy}\psi_y) + y'^2(\varphi_x\psi_{yy} - \varphi_{yy}\psi_x \\ & +2(\varphi_y\psi_{xy} - \varphi_{xy}\psi_y)) + y'(\varphi_y\psi_{xx} - \varphi_{xx}\psi_y \\ & +2(\varphi_x\psi_{xy} - \varphi_{xy}\psi_x)) + \varphi_x\psi_{xx} - \varphi_{xx}\psi_x)/(\varphi_x + y'\varphi_y)^3. \end{aligned} \tag{1.15}$$

Since the jacobian $\Delta \neq 0$, then after replacing u'' into equation (1.11) one gets equation (1.12). □

Theorem 1.4.2. ([1], S. Lie) *Equation (1.12) is linearizable by point transformation (1.1) if and only if its coefficients satisfied the follows.*

(a) *If $\varphi_y = 0$ then the conditions are*

$$a = 0, c_y = 2b_x, d_{yy} - b_{xx} - b_x c + b_y d + d_y b = 0. \tag{1.16}$$

(b) *If $\varphi_y \neq 0$ then the conditions are*

$$\begin{aligned} & 3a_{xx} - 2b_{xy} + c_{yy} - 3a_x c + 3a_y d + 2b_x b - 3c_x a - c_y b + 6d_y a = 0, \\ & b_{xx} - 2c_{xy} + 3d_{yy} - 6a_x d + b_x c + 3b_y d - 2c_y c - 3d_x a + 3d_y b = 0. \end{aligned} \tag{1.17}$$

Proof. We will find conditions from system of equation (1.13).
Case 1 : $\varphi_y = 0$. That is φ does not depend on y. From (1.13) one gets that

$$\begin{aligned} a &= 0, \\ b &= (\varphi_x\psi_y)^{-1}(\varphi_x\psi_{yy}) = \frac{\varphi_x\psi_{yy}}{\varphi_x\psi_y}, \\ c &= (\varphi_x\psi_y)^{-1}(-\varphi_{xx}\psi_y + 2\varphi_x\psi_{xy}) = \left(-\varphi_x{}^{-1}\varphi_{xx} + 2\psi_y{}^{-1}\psi_{xy}\right), \\ d &= (\varphi_x\psi_y)^{-1}(\varphi_x\psi_{xx} - \varphi_{xx}\psi_x) = \frac{\psi_{xx}}{\psi_y} - \frac{\varphi_{xx}\psi_x}{\varphi_x\psi_y} \end{aligned}$$

thus

$$a = 0, \psi_{yy} = \psi_y b, \psi_{xy} = \frac{1}{2}\left(\varphi_x{}^{-1}\psi_y\varphi_{xx} + \psi_y c\right), \psi_{xx} = \varphi_x{}^{-1}\psi_x\varphi_{xx} + \psi_y d. \tag{1.18}$$

Mixing the derivatives:

- $(\psi_{xy})_y = (\psi_{yy})_x$

$$\frac{1}{2}\left[\frac{\varphi_{xx}\psi_{yy}}{\varphi_x} + c\psi_{yy} + c_y\psi_y\right] = \psi_{xy}b + \psi_y b_x$$

$$\frac{\varphi_{xx}\psi_{yy}}{\varphi_x} + c\psi_{yy} + c_y\psi_y = 2\left(\psi_{xy}b + \psi_y b_x\right)$$

replacing (1.18), one gets

$$\begin{aligned} \frac{\varphi_{xx}\psi_y b}{\varphi_x} + c\psi_y b + c_y\psi_y &= 2\left(\frac{1}{2}\left(\frac{\psi_y\varphi_{xx}}{\varphi_x} + \psi_y c\right) b + \psi_y b_x\right) \\ &= \frac{\psi_y\varphi_{xx}b}{\varphi_x} + c\psi_y b + 2\psi_y b_x \\ c_y\psi_y &= 2\psi_y b_x \\ c_y &= 2b_x \end{aligned}$$

- $(\psi_{xy})_x = (\psi_{xx})_y$

$$\varphi_x{}^{-2}\left(2\varphi_x\varphi_{xxx} - 3\varphi_{xx}^2\right) = 4\left(d_y + bd\right) - \left(2c_x + c^2\right). \quad (1.19)$$

Since $\varphi_y = 0$, then differentiating (1.19) respect to y one arrives at

$$d_{yy} - b_{xx} - b_x c + b_y d + d_y b = 0.$$

Hence a second-order ordinary differential equations in the form (1.12) can be linearizable by function $\varphi = \varphi(x)$ if and only if it's coefficients satisfied (1.16).

Case 2 : $\varphi_y \neq 0$.
Consider (1.13).

From $a = \Delta^{-1}\left(\varphi_y\psi_{yy} - \varphi_{yy}\psi_y\right)$, one gets

$$\psi_{yy} = \frac{\left(\varphi_{yy}\psi_y + a\Delta\right)}{\varphi_y}.$$

From $b = \Delta^{-1}\left(\varphi_x\psi_{yy} - \varphi_{yy}\psi_x + 2\left(\varphi_y\psi_{xy} - \varphi_{xy}\psi_y\right)\right)$, one gets

$$\psi_{xy} = \frac{2\varphi_{xy}\varphi_y\psi_y - \varphi_{yy}\Delta - \left(a\varphi_x - b\varphi_y\right)\Delta}{2\varphi_y^2}.$$

From $c = \Delta^{-1}\left(\varphi_y\psi_{xx} - \varphi_{xx}\psi_y + 2\left(\varphi_x\psi_{xy} - \varphi_{xy}\psi_x\right)\right)$, one gets

$$\psi_{xx} = \frac{2\varphi_{xy}\varphi_y\psi_x - \varphi_x\varphi_{yy}\psi_x - \varphi_x^2\psi_x a + \varphi_x\varphi_y\psi_x b + \varphi_y^2\left(\psi_y d - \psi_x c\right)}{\varphi_y^2}.$$

From $d = \Delta^{-1}\left(\varphi_x\psi_{xx} - \varphi_{xx}\psi_x\right)$, one gets

$$\varphi_{xx} = \left(\frac{2\varphi_{xy}\varphi_x\varphi_y - \varphi_x^2\varphi_{yy} - \varphi_x^3 a + \varphi_x^2\varphi_y b - \varphi_x\varphi_y^2 c + \varphi_y^3 d}{\varphi_y^2}\right).$$

Therefore, by the relation (1.13) we have

$$\begin{aligned}
\psi_{yy} &= \tfrac{(\varphi_{yy}\psi_y + a\Delta)}{\varphi_y}, \\
\psi_{xy} &= \tfrac{2\varphi_{xy}\varphi_y\psi_y - \varphi_{yy}\Delta - (a\varphi_x - b\varphi_y)\Delta}{2\varphi_y^2}, \\
\psi_{xx} &= \tfrac{2\varphi_{xy}\varphi_y\psi_x - \varphi_x\varphi_{yy}\psi_x - \varphi_x^2\psi_x a + \varphi_x\varphi_y\psi_x b + \varphi_y^2(\psi_y d - \psi_x c)}{\varphi_y^2}, \\
\varphi_{xx} &= \left(\tfrac{2\varphi_{xy}\varphi_x\varphi_y - \varphi_x^2\varphi_{yy} - \varphi_x^3 a + \varphi_x^2\varphi_y b - \varphi_x\varphi_y^2 c + \varphi_y^3 d}{\varphi_y^2}\right).
\end{aligned} \tag{1.20}$$

Mixing the derivatives :

- $(\psi_{xy})_y = (\psi_{yy})_x$

$$\begin{aligned}
2\varphi_y\varphi_{yyy} &= 3\left(\varphi_{yy}^2 - 2\varphi_{xy}\varphi_y a + 2\varphi_x\varphi_{yy} a + \varphi_x^2 a^2\right) - 2\varphi_x\varphi_y\left(a_y + ab\right) \\
&\quad + \varphi_y^2\left(2b_y - 4a_x + 4ac - b^2\right) \\
\varphi_{yyy} &= (3\left(\varphi_{yy}^2 - 2\varphi_{xy}\varphi_y a + 2\varphi_x\varphi_{yy} a + \varphi_x^2 a^2\right) - 2\varphi_x\varphi_y\left(a_y + ab\right) \\
&\quad + \varphi_y^2\left(2b_y - 4a_x + 4ac - b^2\right))/(2\varphi_y)
\end{aligned}$$

- $(\psi_{xy})_x = (\psi_{xx})_y$

$$\begin{aligned}
6\varphi_y^2\varphi_{xyy} &= 3\left(4\varphi_{xy}\varphi_{yy}\varphi_y - \varphi_x\varphi_{yy}^2 + 2\varphi_x\varphi_{yy}\varphi_y b - 2\varphi_{xy}\varphi_y^2 b\right) \\
&\quad + 3\varphi_x^3 a^2 + 3\varphi_x\varphi_y^2\left(-2a_x + 2ac - b^2\right) + 2\varphi_y^3\left(-b_x + 2c_y + 3ad\right) \\
\varphi_{xyy} &= (3\left(4\varphi_{xy}\varphi_{yy}\varphi_y - \varphi_x\varphi_{yy}^2 + 2\varphi_x\varphi_{yy}\varphi_y b - 2\varphi_{xy}\varphi_y^2 b\right) + 3\varphi_x^3 a^2 \\
&\quad + 3\varphi_x\varphi_y^2\left(-2a_x + 2ac - b^2\right) + 2\varphi_y^3\left(-b_x + 2c_y + 3ad\right))/(6\varphi_y^2).
\end{aligned}$$

Mixing the derivative again :

- $(\varphi_{xyy})_y = (\varphi_{yyy})_x$ and $(\varphi_{xx})_{yy} = (\varphi_{xyy})_x$ one obtains (1.17). □