Appendix

1. S. Lekkoksung. *On characterizations of regular ordered semigroups in terms of Q-fuzzy subsets.* Far East Journal of Mathematic Science (FJMS) Volume 76, Number 2, May 2013: 273-290.

2. S. Lekkoksung. *On Q-fuzzy ideals in ordered semigroups.* International Journal of pure and applied mathematics Volume 92 No. 3 2014: 369-379.

ON CHARACTERIZATIONS OF REGULAR ORDERED SEMIGROUPS IN TERMS OF Q-FUZZY SUBSETS

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Abstract

In this paper, the concept of Q-fuzzy generalized bi-ideals of an ordered semigroup is introduced. Regular ordered semigroups are characterized by means of Q-fuzzy left ideals, Q-fuzzy right ideals and Q-fuzzy (generalized) bi-ideals.

1. Introduction

A fuzzy set theory was conceptualized by Professor L. A. Zadeh at the University of California in 1965 [14] as a generalization of abstract set theory. Zadeh's initiation is virtually a complete paradigm shift that initially gained popularity in the Far East and its successful applications have gained further ground almost round the globe. Rosenfeld [11] used the ideal of fuzzy set to introduce the notions of fuzzy subgroups. The ideal of fuzzy

subsemigroup was also introduced by Kuroki [7, 9]. In [8], Kuroki characterized several classes of semigroups in terms of fuzzy left, fuzzy right and fuzzy bi-ideals. Xie [12] introduced the idea of extensions of fuzzy ideals in semigroups. The concept of fuzzy generalized bi-ideals of an ordered semigroup is introduced by Xie and Tang [13] and characterized regular ordered semigroups by means of fuzzy left ideals, fuzzy right ideals and fuzzy (generalized) bi-ideals. In [10], Majumder introduced the concept of Q-fuzzification of ideals of Γ-semigroups and some important properties have been investigated. In this paper, the concept of Q-fuzzy generalized bi-ideals of an ordered semigroup is introduced. Regular ordered semigroups are characterized by means of Q-fuzzy left ideals, Q-fuzzy right ideals and Q-fuzzy (generalized) bi-ideals.

2. Preliminaries

Throughout this paper, unless stated otherwise, S stands for an ordered semigroup. A function f from $S \times Q$ to the real closed interval $[0, 1]$ is called Q-fuzzy subset of S, where Q is a non-empty set. The ordered semigroup S itself is a Q-fuzzy subset of S, its characteristic function, also denoted by S, is defined as follows:

$$S : S \times Q \to [0, 1] | (x, q) \mapsto S(x, q) := 1,$$

for all $x \in S$ and $q \in Q$.

Let f and g be two Q-fuzzy subsets of S. Then the inclusion relation $f \subseteq g$ means that

$$f(x, q) \le g(x, q),$$

for all $x \in S$ and $q \in Q$, $f \cap g$ and $f \cup g$ are defined by

$$(f \cap g)(x, q) := \min\{f(x, q), g(x, q)\},$$

$$(f \cup g)(x, q) := \max\{f(x, q), g(x, q)\},$$

for all $x \in S$ and $q \in Q$.

Let S and Q be an ordered semigroup and a non-empty set, respectively. For $x \in S$, we define $A_x := \{(y, z) \in S \times S \mid x \leq yz\}$. The product $f \circ g$ of f and g is defined by

$$(\forall x \in S, \forall q \in Q)(f \circ g)(x, q) := \begin{cases} \bigvee_{(y,z) \in A_x} \min\{f(y,q), g(z,q)\} & \text{if } A_x \neq \varnothing, \\ 0 & \text{if } A_x = \varnothing. \end{cases}$$

Lemma 2.1. *Let S and Q be an ordered semigroup and a non-empty set, respectively. Let f_1, f_2, g_1, g_2 be Q-fuzzy subsets of S such that $f_1 \subseteq g_1$ and $f_2 \subseteq g_2$. Then $f_1 \circ f_2 \subseteq g_1 \circ g_2$.*

Proof. Let $a \in S$ and $q \in Q$. Then, if $A_a = \varnothing$, we have $(f_1 \circ f_2)(a, q) = 0 \leq (g_1 \circ g_2)(a, q)$. Thus, $f_1 \circ f_2 \subseteq g_1 \circ g_2$. If $A_a \neq \varnothing$, then we have

$$(f_1 \circ f_2)(a, q) = \bigvee_{(b,c) \in A_a} \min\{f_1(b, q), f_2(c, q)\}$$

$$\leq \bigvee_{(b,c) \in A_a} \min\{g_1(b, q), g_2(c, q)\}$$

$$= (g_1 \circ g_2)(a, q).$$

Hence $f_1 \circ f_2 \subseteq g_1 \circ g_2$. $\square$

We denote by $f_{A \times Q}$ the characteristic function of $A \times Q$, that is, the mapping of $S \times Q$ into $[0, 1]$ is defined by

$$f_{A \times Q}(x, q) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

for all $q \in Q$.

Let S be an ordered semigroup. For $H \subseteq S$, we define

$$(H] := \{t \in S \mid t \leq h, \text{ for some } h \in H\}.$$

For $H = \{a\}$, we write $(a]$ instead of $(\{a\}]$.

A subsemigroup B of an ordered semigroup S is called a *bi-ideal* of S if

(1) $BSB \subseteq B$.

(2) If $a \in B$ and $S \ni b \leq a$, then $b \in B$.

Definition 2.2. Let S and Q be an ordered semigroup and a non-empty set, respectively. A Q-fuzzy subset f of S is called a *Q-fuzzy subsemigroup* of S if

(1) $x \leq y$ implies $f(x, q) \geq f(y, q)$, and

(2) $f(xy, q) \geq \min\{f(x, q), f(y, q)\}$,

for all $x, y \in S$, $q \in Q$.

Lemma 2.3. *Let S and Q be an ordered semigroup and a non-empty set, respectively. f is a Q-fuzzy subset of S such that*

$$x \leq y \Rightarrow f(x, q) \geq f(y, q), \forall x, y \in S, \forall q \in Q.$$

If f is a Q-fuzzy subsemigroup of S, then $f \circ f \subseteq f$. Conversely, if $f \circ f \subseteq f$, then f is a Q-fuzzy subsemigroup of S.

Proof. Let $x \in S$ and $q \in Q$. If $A_x = \varnothing$, then $(f \circ f)(x, q) = 0 \leq f(x, q)$. If $A_x \neq \varnothing$, then

$$(f \circ f)(x, q) = \bigvee_{(b,c) \in A_x} \min\{f(b, q), f(c, q)\}$$

$$\leq \bigvee_{(b,c) \in A_x} f(bc, q)$$

$$\leq \bigvee_{(b,c) \in A_x} f(x, q)$$

$$= f(x, q).$$

Thus, we have $f \circ f \subseteq f$.

Conversely, if $f \circ f \subseteq f$, then for all $x, y \in S$ and for all $q \in Q$,

$$f(xy, q) \geq (f \circ f)(xy, q)$$

$$= \bigvee_{(b, c) \in A_{xy}} \min\{f(b, q), f(c, q)\}$$

$$\geq \min\{f(x, q), f(y, q)\}. \qquad \square$$

Definition 2.4. Let S and Q be an ordered semigroup and a non-empty set, respectively. A Q-fuzzy subset f of S is called a *Q-fuzzy left ideal* of S if

(1) $x \leq y \Rightarrow f(x, q) \geq f(y, q)$,

(2) $f(xy, q) \geq f(y, q)$,

for all $x, y \in S$ and for all $q \in Q$.

A Q-fuzzy subset f of S is called a *Q-fuzzy right ideal* of S if

(1) $x \leq y \Rightarrow f(x, q) \geq f(y, q)$,

(2) $f(xy, q) \geq f(x, q)$,

for all $x, y \in S$ and for all $q \in Q$.

A Q-fuzzy subset f of S is called a *Q-fuzzy ideal* of S if it is both a Q-fuzzy left ideal and a Q-fuzzy right ideal of S.

Lemma 2.5. *Let S and Q be an ordered semigroup and a non-empty set, respectively. Let A be a non-empty subset of S. Then A is a left ideal of S if and only if the characteristic function $f_{A \times Q}$ of $A \times Q$ is a Q-fuzzy left ideal of S.*

Proof. Let A be a left ideal of an ordered semigroup S and let $x, y \in S$, $q \in Q$ be such that $x \leq y$. If $y \in A$, then since A is a left (resp. right) ideal of S, we have $x \in A$ and then $f_{A \times Q}(x, q) = 1 \geq f_{A \times Q}(y, q)$. If $y \notin A$, then $f_{A \times Q}(y, q) = 0 \leq f_{A \times Q}(x, q)$. Next, if $y \in A$, then $xy \in A$ and we have $f_{A \times Q}(xy, q) = 1 \geq f(y, q)$. If $y \notin A$, then $f_{A \times Q}(xy, q) \geq 0 = f_{A \times Q}(y, q)$.

Conversely, let $f_{A \times Q}$ be a Q-fuzzy left ideal of an ordered semigroup S. Let $x, y \in S$, $q \in Q$ be such that $x \leq y$. If $y \in A$, then $f_{A \times Q}(x, q) \geq 1 = f_{A \times Q}(y, q)$ and, we have $f_{A \times Q}(x, q) = 1$, it follows that $x \in A$. Next, let $y \in A$. Then, since $f_{A \times Q}$ is a Q-fuzzy left ideal of S, we have $f_{A \times Q}(xy, q) \geq 1 = f_{A \times Q}(y, q)$, and then $f_{A \times Q}(xy, q) = 1$. Thus, $xy \in A$.   □

Similar to Lemma 2.5, we have the following lemma. The proof is similar to that of Lemma 2.5.

Lemma 2.6. *Let S and Q be an ordered semigroup and a non-empty set, respectively. Let A be a non-empty subset of S. Then A is a right ideal of S if and only if the characteristic function $f_{A \times Q}$ of $A \times Q$ is a Q-fuzzy right ideal of S.*

By Lemma 2.5 and Lemma 2.6, we have the following lemma:

Lemma 2.7. *Let S and Q be an ordered semigroup and a non-empty set, respectively. Let A be a non-empty subset of S. Then A is an ideal of S if and only if the characteristic function $f_{A \times Q}$ of $A \times Q$ is a Q-fuzzy ideal of S.*

Definition 2.8. Let S and Q be an ordered semigroup and a non-empty set, respectively. A Q-fuzzy subsemigroup f of S is called a *Q-fuzzy bi-ideal* of S if

(1) $x \leq y \Rightarrow f(x, q) \geq f(y, q)$,

(2) $f(xyz, q) \geq \min\{f(x, q), f(z, q)\}$,

for all $x, y, z \in S$ and for all $q \in Q$.

Lemma 2.9. *Let S and Q be an ordered semigroup and a non-empty set, respectively. Let B be a non-empty subset of S. Then B is a bi-ideal of S if and only if the characteristic function $f_{B \times Q}$ of $B \times Q$ is a Q-fuzzy bi-ideal of S.*

Lemma 2.10. *Let S and Q be an ordered semigroup and a non-empty set, respectively. Let f be a Q-fuzzy subset of S. Then f is a Q-fuzzy left ideal of S if and only if f satisfies that*

(1) $x \leq y \Rightarrow f(x, q) \geq f(y, q)$, *for all* $x, y \in S$, $q \in Q$.

(2) $S \circ f \subseteq f$.

Proof. Let f be a Q-fuzzy left ideal of an ordered semigroup S. Let $x, y \in S$ and $q \in Q$ be such that $x \leq y$. Then, since f is a Q-fuzzy left ideal of S, we have $f(x, q) \geq f(y, q)$. Next, let $x \in S$, $q \in Q$. If $A_x = \varnothing$, then $(S \circ f)(x, q) = 0 \leq f(x, q)$. If $A_x \neq \varnothing$, then

$$(S \circ f)(x, q) = \bigvee_{(b, c) \in A_x} \min\{S(b, q), f(c, q)\}$$

$$\leq \bigvee_{(b, c) \in A_x} f(bc, q)$$

$$= f(bc, q)$$

$$\leq f(x, q).$$

Conversely, assume that (2) holds. Let $x, y \in S$ and $q \in Q$. Then, we put $a = xy$. By (2), we have

$$f(xy, q) = f(a, q) \geq (S \circ f)(a, q)$$

$$= \bigvee_{(b, c) \in A_a} \min\{S(b, q), f(c, q)\}$$

$$\geq \min\{S(x, q), f(y, q)\}$$

$$= f(y, q).$$

By hypothesis (1) and Definition 2.4, f is a Q-fuzzy left ideal of S. $\square$

Similar to Lemma 2.10, we have the following lemmas:

Lemma 2.11. *Let S and Q be an ordered semigroup and a non-empty set, respectively. Let f be a Q-fuzzy subset of S. Then f is a Q-fuzzy right ideal*

of S if and only if f satisfies that

(1) $x \le y \Rightarrow f(x, q) \ge f(y, q)$, *for all* $x, y \in S$, $q \in Q$.

(2) $f \circ S \subseteq f$.

Lemma 2.12. *Let S and Q be an ordered semigroup and a non-empty set, respectively. Let f be a Q-fuzzy subset of S. Then f is a Q-fuzzy ideal of S if and only if f satisfies that*

(1) $x \le y \Rightarrow f(x, q) \ge f(y, q)$, *for all* $x, y \in S$, $q \in Q$,

(2) $S \circ f \subseteq f$ *and* $f \circ S \subseteq f$.

3. *Q*-fuzzy Bi-ideals and *Q*-fuzzy Generalized Bi-ideals of Ordered Semigroups

Definition 3.1 [13]. A non-empty subset B of an ordered semigroup S is called a *generalized bi-ideal* of S if

(1) $BSB \subseteq B$,

(2) if $a \in B$ and $S \ni b \le a$, then $b \in B$.

Furthermore, we now introduce the analogous definition by means of *Q*-fuzzy subsets as follows:

Definition 3.2. Let S and Q be an ordered semigroup and a non-empty set, respectively. A Q-fuzzy subset f of S is called a *Q-fuzzy generalized bi-ideal* of S if

(1) $x \le y \Rightarrow f(x, q) \ge f(y, q)$,

(2) $f(xyz, q) \ge \min\{f(x, q), f(z, q)\}$,

for all $x, y, z \in S$ and for all $q \in Q$.

By Definition 3.2, we have that every Q-fuzzy bi-ideal is a generalized bi-ideal.

Proposition 3.3. *Let S and Q be an ordered semigroup and a non-empty set, respectively. If a Q-fuzzy subset f of S is a Q-fuzzy generalized bi-ideal of S, then $f \circ S \circ f \subseteq f$.*

Proof. Let f be a Q-fuzzy generalized bi-ideal of S. Then, since f is a Q-fuzzy subset of S, we have $f(a, q) \geq 0$, for all $a \in S$, $q \in Q$. If $A_a = \emptyset$, then $(f \circ S \circ f)(a, q) = 0 \leq f(a, q)$. If $A_a \neq \emptyset$, then there exist $x, y, s, t \in S$ such that $(x, y) \in A_a$, $(s, t) \in A_x$, that is, $a \leq xy$, $x \leq st$. Since f is a Q-fuzzy generalized bi-ideal of S, we have $f(sty, q) \geq \min\{f(s, q), f(y, q)\}$, for all $q \in Q$. Thus,

$$(f \circ S \circ f)(a, q) = \bigvee_{(x, y) \in A_a} \min\{(f \circ S)(x, q), f(y, q)\}$$

$$= \bigvee_{(x, y) \in A_a} \min\{\bigvee_{(s, t) \in A_x} \min\{f(s, q), S(t, q)\}, f(y, q)\}$$

$$= \bigvee_{(x, y) \in A_a} \bigvee_{(s, t) \in A_x} \min\{f(s, q), S(t, q), f(y, q)\}$$

$$= \bigvee_{(x, y) \in A_a} \bigvee_{(s, t) \in A_x} \min\{f(s, q), f(y, q)\}$$

$$\leq \bigvee_{(x, y) \in A_a} \bigvee_{(s, t) \in A_x} f(sty, q)$$

$$\leq \bigvee_{(s, t) \in A_a} f(xy, q) \leq f(a, q),$$

which means that $f \circ S \circ f \subseteq f$. $\square$

In a similar way as in the previous proposition, we can show the following result:

Proposition 3.4. *Let S and Q be an ordered semigroup and a non-empty set, respectively. If a Q-fuzzy subset f of S is a Q-fuzzy bi-ideal of S, then $f \circ f \subseteq f$ and $f \circ S \circ f \subseteq f$.*

Lemma 3.5. *Let S and Q be an ordered semigroup and a non-empty set, respectively. If a Q-fuzzy subset f of S and $f \circ S \circ f \subseteq f$, then*

$$(\forall x, y, z \in S, \forall q \in Q), \ f(xyz, q) \geq \min\{f(x, q), f(z, q)\}.$$

Proof. For any $x, y, z \in S$ and $q \in Q$, put $a = xyz$. Since $f \circ S \circ f$ $\subseteq f$, we have

$$f(xyz, q) = f(a, q) \geq (f \circ S \circ f)(a, q)$$

$$= \bigvee_{(s,t) \in A_a} \min\{(f \circ S)(s, q), f(t, q)\}$$

$$\geq \min\{(f \circ S)(xy, q), f(z, q)\}$$

$$= \min\{\bigvee_{(u,v) \in A_{xy}} \min\{f(u, q), S(v, q)\}, f(z, q)\}$$

$$\geq \min\{\min\{f(x, q), S(y, q)\}, f(z, q)\}$$

$$= \min\{f(x, q), S(y, q), f(z, q)\}$$

$$= \min\{f(x, q), f(z, q)\}. \qquad \square$$

Proposition 3.6. *Let S and Q be an ordered semigroup and a non-empty set, respectively. Then the product of two Q-fuzzy generalized bi-ideals of S is a Q-fuzzy bi-ideal of S.*

Proof. Let f, g be Q-fuzzy generalized bi-ideals of an ordered semigroup S. Then $f \circ S \circ f \subseteq f$ and $g \circ S \circ g \subseteq g$. Thus,

$$(f \circ g) \circ (f \circ g) \subseteq f \circ (g \circ S \circ g) \subseteq f \circ g.$$

Therefore, by Lemma 2.3, $f \circ g$ is a Q-fuzzy subsemigroup of S. Furthermore, by Lemma 2.1, we have

$$(f \circ g) \circ S \circ (f \circ g) = f \circ g \circ (S \circ f) \circ g \subseteq f \circ (g \circ S \circ g) \subseteq f \circ g.$$

By Lemma 3.5,

$$(\forall x, y, z \in S, \forall q \in Q), \ (f \circ g)(xyz, q)$$

$$\geq \min\{(f \circ g)(x, q), (f \circ g)(z, q)\}.$$

Furthermore, if $x \leq y$, then, if $A_x = \varnothing$, then $(f \circ g)(y, q) = 0$. Since $f \circ g$ is a Q-fuzzy subset of S, we have $(f \circ g)(x, q) \geq 0 = (f \circ g)(y, q)$. If $A_y \neq \varnothing$, then, since $x \leq y$, we have $A_y \subseteq A_x$. Thus, we have

$$(f \circ g)(y, q) = \bigvee_{(u, v) \in A_y} \min\{f(u, q), g(v, q)\}$$

$$\leq \bigvee_{(u, v) \in A_x} \min\{f(u, q), g(v, q)\}$$

$$= (f \circ g)(x, q). \qquad \square$$

An ordered semigroup S is called *regular* if, for each element a of S, there exists an element x in S such that $a \leq axa$. Equivalent definition:

(1) $A \subseteq (ASA]$, $\forall A \subseteq A$.

(2) $a \in (aSa]$, $\forall a \in S$.

Theorem 3.7. *Let S and Q be a regular ordered semigroup and a non-empty set, respectively. Then every Q-fuzzy generalized bi-ideal of S is a Q-fuzzy bi-ideal of S.*

Proof. Let f be any Q-fuzzy generalized bi-ideal of an ordered semigroup S and $a, b \in S$, $q \in Q$. Then, since S is regular, there exists $x \in S$ such that $b \leq bxb$. Then

$$f(ab, q) \geq f(a(bxb), q)$$

$$= f(a(bx)b, q)$$

$$\geq \min\{f(a, q), f(b, q)\}.$$

Thus, f is a Q-fuzzy subsemigroup of S, and so it is a Q-fuzzy bi-ideal of S. $\qquad \square$

4. Characterizations of Regular Ordered Semigroups

In this section, we give characterizations of regular ordered semigroups in terms of Q-fuzzy subsets.

Lemma 4.1 [13]. *An ordered semigroup S is regular if and only if* $B = (BSB]$ *for any bi-ideal of S.*

Now we give characterizations of regular semigroups by Q-fuzzy (generalized) bi-ideals.

Theorem 4.2. *Let S and Q be an ordered semigroup and a non-empty set, respectively. Then the following statements are equivalent*:

(1) *S is regular*,

(2) $f = f \circ S \circ f$ *for any Q-fuzzy bi-ideal f of S.*

Proof. (1) $\Rightarrow$ (2) Let f be any Q-fuzzy bi-ideal of S and $a \in S$, $q \in Q$. Then, since S is regular, there exists $x \in S$ such that $a \le axa$, and $(ax, a) \in A_a$. Thus,

$$(f \circ S \circ f)(a, q) = \bigvee_{(y,z) \in A_a} \min\{(f \circ S)(y, q), f(z, q)\}$$

$$\ge \min\{(f \circ S)(ax, q), f(a, q)\}$$

$$= \min\{ \bigvee_{(s,t) \in A_{ax}} \min\{f(s, q), S(t, q)\}, f(a, q)\}$$

$$\ge \min\{\min\{f(a, q), S(x, q)\}, f(a, q)\}$$

$$= \min\{\min\{f(a, q), 1\}, f(a, q)\}$$

$$= f(a, q),$$

and so we have $f \circ S \circ f \supseteq f$. Since f is a Q fuzzy bi-ideal of S, we have $f \circ S \circ f \subseteq f$. Thus, $f \circ S \circ f = f$.

(2) $\Rightarrow$ (1) Let B be any bi-ideal of S and $q \in Q$. Then f_B is a Q-fuzzy bi-ideal of S, and for each $a \in B$,

$$\bigvee_{(y,z) \in A_a} \min\{(f_B \circ S)(y, q), f_B(z, q)\} = ((f_B \circ S) \circ f_B)(a, q)$$

$$= f_B(a, q) = 1,$$

which implies that $A_a \neq \emptyset$, and there exist $b, c \in S$ such that $a \leq bc$, $(f_B \circ S)(b, q) = 1$ and $f_B(c, q) = 1$. Then $c \in B$, and

$$\bigvee_{(s,t) \in A_b} \min\{f_B(s, q), S(t, q)\} = (f_B \circ S)(b, q) = 1,$$

which implies that $A_b \neq \emptyset$, and there exist $u, v \in S$ such that $b \leq uv$, $f_B(u, q) = 1$, and $S(v, q) = 1$. Then $u \in B$, and

$$a \leq bc \leq uvc \in BSB,$$

and so $B \subseteq (BSB]$. On the other hand, since B is a bi-ideal of S, we have $(BSB] \subseteq (B] = B$. Therefore, $B = (BSB]$, and then S is regular. $\square$

In a similar way, the following theorem can be proved.

Theorem 4.3. *Let S and Q be an ordered semigroup and a non-empty set, respectively. Then the following statements are equivalent:*

(1) *S is regular.*

(2) *$f = f \circ S \circ f$ for any Q-fuzzy generalized bi-ideal f of S.*

Theorem 4.4. *Let S and Q be an ordered semigroup and a non-empty set, respectively. Then the following statements are equivalent:*

(1) *S is regular.*

(2) *$f \circ g \circ f = f \cap g$ for any Q-fuzzy bi-ideal f and any Q-fuzzy ideal g of S.*

Proof. (1) $\Rightarrow$ (2) Let f, g be a Q-fuzzy bi-ideal and a Q-fuzzy ideal of S, respectively. Then, we have $f \circ g \circ f \subseteq f \circ S \circ f \subseteq f$. Since g is a Q-fuzzy ideal of S, we have

$$f \circ g \circ f \subseteq S \circ g \circ S \subseteq S \circ g \subseteq g.$$

Thus, $f \circ g \circ f \subseteq f \cap g$. On the other hand, let $a \in S$ and $q \in Q$. Then, since S is regular, there exists $x \in S$ such that $a \leq axa \leq axaxa$, so $(a, xaxa) \in A_a$. Since g is a Q-fuzzy ideal of S, we have $g(xax, q) \geq$

$g(ax, q) \geq g(a, q)$. Thus,

$$(f \circ g \circ f)(a, q) = \bigvee_{(y,z) \in A_a} \min\{f(y, q), (g \circ f)(z, q)\}$$

$$\geq \min\{f(a, q), (g \circ f)(xaxa, q)\}$$

$$= \min\{f(a, q), \bigvee_{(s,t) \in A_{xaxa}} \min\{g(s, q), g(t, q)\}\}$$

$$\geq \min\{f(a, q), \min\{g(xax, q), f(a, q)\}\}$$

$$\geq \min\{f(a, q), \min\{g(a, q), f(a, q)\}\}$$

$$= \min\{f(a, q), g(a, q)\}$$

$$= (f \cap g)(a, q),$$

which implies that $f \circ g \circ f \supseteq f \cap g$. Therefore, $f \circ g \circ f = f \cap g$.

$(2) \Rightarrow (1)$ Since S itself is a Q-fuzzy ideal of S, by hypothesis, we have

$$f = f \cap g = f \circ g \circ f.$$

By Theorem 4.2, S is regular. $\square$

In a similar way, we can show the following theorem:

Theorem 4.5. *Let S and Q be an ordered semigroup and a non-empty set, respectively. Then the following statements are equivalent:*

(1) *S is regular.*

(2) *$f \circ g \circ f = f \cap g$ for any Q-fuzzy generalized bi-ideal f and any Q-fuzzy ideal g of S.*

Lemma 4.6 [13]. *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) *S is regular.*

(2) *$B \cap L \subseteq (BL]$, for every bi-ideal B and every left ideal L of S.*

(3) $R \cap B \cap L \subseteq (RBL]$, *for every bi-ideal B, every right ideal R and every left ideal L of S.*

Now we give characterizations of regular ordered semigroups by Q-fuzzy left ideals, Q-fuzzy right ideals and Q-fuzzy (generalized) bi-ideals.

Theorem 4.7. *Let S and Q be an ordered semigroup and a non-empty set, respectively. Then the following statements are equivalent:*

(1) *S is regular.*

(2) $f \cap g \subseteq f \circ g$, *for every Q-fuzzy generalized bi-ideal f and every Q-fuzzy left ideal g of S.*

(3) $f \cap g \subseteq f \circ g$, *for every Q-fuzzy bi-ideal and every Q-fuzzy left ideal g of S.*

(4) $h \cap f \cap g \subseteq h \circ f \circ g$, *for every Q-fuzzy generalized bi-ideal f, every Q-fuzzy left ideal g and every Q-fuzzy right ideal h of S.*

(5) $h \cap f \cap g \subseteq h \circ f \circ g$, *for every Q-fuzzy bi-ideal f, every Q-fuzzy left ideal g and every Q-fuzzy right ideal h of S.*

Proof. (1) $\Rightarrow$ (2) Let f and g be a Q-fuzzy generalized bi-ideal and a Q-fuzzy left ideal of S, respectively, let $a \in S$ and $q \in Q$. Then, since S is regular, there exists $x \in S$ such that $a \leq axa$. Since $(a, xa) \in A_a$, we have

$$(f \circ g)(a, q) = \bigvee_{(y, z) \in A_a} \{f(y, q), g(z, q)\}$$

$$\geq \min\{f(a, q), g(xa, q)\}$$

$$\geq \min\{f(a, q), g(a, q)\}$$

$$= (f \cap g)(a, q),$$

which means that $f \cap g \subseteq f \circ g$.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (1) Let B and L be a bi-ideal and a left ideal of S, respectively. Let $a \in B \cap L$ and $q \in Q$. Then f_B is a Q-fuzzy bi-ideal of S and f_L is a Q-fuzzy left ideal of S. Thus, by hypothesis, $f_B \cap f_L \subseteq f_B \circ f_L$, and

$$(f_B \circ f_L)(a, q) \geq (f_B \cap f_L)(a, q) = \min\{f_B(a, q), f_L(a, q)\} = 1,$$

for all $a \in B \cap L$, $q \in Q$, and $A_a \neq \varnothing$. Since $f_B \circ f_L$ is a Q-fuzzy subset of S, we have $(f_B \circ f_L)(a, q) \leq 1$, for all $a \in S$, $q \in Q$. Then

$$\bigvee_{(y, z) \in A_a} \min\{f_B(y, q), f_L(z, q)\} = (f_B \circ f_L)(a, q) = 1,$$

which implies that there exist $b, c \in S$ such that $a \leq bc$, $f_B(b, q) = 1$ and $f_L(c, q) = 1$. Then $a \leq bc \in BL$, and so $B \cap L \subseteq (BL]$. By Lemma 4.6, S is regular.

(1) $\Rightarrow$ (4) Let f, g and h be a Q-fuzzy generalized bi-ideal, a Q-fuzzy left ideal and a Q-fuzzy right ideal of S, respectively. Let $a \in S$ and $q \in Q$. Then, since S is regular, there exists $x \in S$ such that $a \leq axa$. Thus, $(a, xa) \in A_a$, and

$$
\begin{aligned}
(h \circ f \circ g)(a, q) &= \bigvee_{(y, z) \in A_a} \min\{h(y, q), (f \circ g)(z, q)\} \\
&\geq \min\{h(ax, q), (f \circ g)(a, q)\} \\
&\geq \min\{h(a, q), \bigvee_{(s, t) \in A_a} \min\{f(s, q), g(t, q)\}\} \\
&\geq \min\{h(a, q), \min\{f(a, q), g(xa, q)\}\} \\
&\geq \min\{h(a, q), \min\{f(a, q), g(a, q)\}\} \\
&= \min\{h(a, q), f(a, q), g(a, q)\} \\
&= (h \cap f \cap g)(a, q).
\end{aligned}
$$

Therefore, $h \cap f \cap g \subseteq h \circ f \circ g$.

(4) $\Rightarrow$ (5) Clear.

(5) $\Rightarrow$ (1) Let B, L and R be a bi-ideal, a left ideal and a right ideal of S, respectively. Let $a \in R \cap B \cap L$ and $q \in Q$. Then, we have f_B, f_L and f_R be a Q-fuzzy bi-ideal, a Q-fuzzy left ideal and a Q-fuzzy right ideal of S, respectively. Thus, by hypothesis, $f_R \cap f_B \cap f_L \subseteq f_R \circ f_B \circ f_L$ and then

$$(f_R \circ f_B \circ f_L)(a, q) \geq (f_R \cap f_B \cap f_L)(a, q)$$

$$= \min\{f_R(a, q), f_B(a, q), f_L(a, q)\} = 1.$$

Thus, $A_a \neq \varnothing$. Furthermore, since $f_R \circ f_B \circ f_L$ is a Q-fuzzy subset of S, we have $(f_R \circ f_B \circ f_L)(a, q) \leq 1$, for all $a \in S$, $q \in Q$. Thus,

$$\bigvee_{(y, z) \in A_a} \min\{(f_R \circ f_B)(y, q), f_L(z, q)\} = (f_R \circ f_B \circ f_L)(a, q) = 1,$$

which implies that there exist $b, c \in S$ such that $a \leq bc$, and $(f_R \circ f_B)(b, q) = 1$ and $f_L(c, q) = 1$. Then $A_b \neq \varnothing$,

$$\bigvee_{(s, t) \in A_b} \min\{f_R(s, q), f_B(t, q)\} = (f_R \circ f_B)(b, q) = 1,$$

which implies that there exist $d, e \in S$ such that $b \leq de$, and $f_R(d, q) = f_B(e, q) = 1$. Then $a \leq bc \leq dec \in RBL$, that is, $a \in (RBL]$. Thus, $R \cap B \cap L \subseteq (RBL]$. By Lemma 4.6, S is regular. $\square$

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ON Q-FUZZY IDEALS IN ORDERED SEMIGROUPS

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Abstract: In this paper we shows that in ordered groupoids the Q-fuzzy right (resp. Q-fuzzy left) ideals are Q-fuzzy quasi-ideals, in ordered semigroups the Q-fuzzy quasi-ideals are Q-fuzzy bi-ideals, and in regular ordered semigroups the Q-fuzzy quasi-ideals and the Q-fuzzy bi-ideals coincide and show that if S is an ordered semigroup, then a Q-fuzzy subset f is a Q-fuzzy quasi-ideal of S if and only if there exist a Q-fuzzy right ideal g and a Q-fuzzy left ideal h of S such that $f = g \cap h$.

1. Introduction

A fuzzy set theory was conceptualized by Professor L. A. Zadeh at the University of California in 1965, [14] as a generalization of abstract set theory. Zadehs initiation is virtually a complete paradigm shift that initially gained popularity in the Far East and its successful applications has gained further ground almost round the globe. Rosenfeld [11] used the ideal of fuzzy set to introduce the no-

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tions of fuzzy subgroups. The ideal of fuzzy subsemigroup was also introduced by Kuroki [7], [9]. In [8], Kuroki characterized several classes of semigroups in terms of fuzzy left, fuzzy right and fuzzy bi-ideals. Xie [12] introduced the idea of extensions of fuzzy ideals in semigroups. The concept of fuzzy generalized bi-ideals of an ordered semigroup is introduced by Xie and Tang [13] and characteried regular ordered semigroups by means of fuzzy left ideals, fuzzy right ideals and fuzzy (generalized) bi-ideals. In [10], Majumder introduced the concept of Q-fuzzification of ideals of Γ-semigroups and some important properties have been investigated. In this paper we shows that in ordered groupoids the Q-fuzzy right (resp. Q-fuzzy left) ideals are Q-fuzzy quasi-ideals, in ordered semigroups the Q-fuzzy quasi-ideals are Q-fuzzy bi-ideals, and in regular ordered semigroups the Q-fuzzy quasi-ideals and the Q-fuzzy bi-ideals coincide and show that if S is an ordered semigroup, then a Q-fuzzy subset f is a Q-fuzzy quasi-ideal of S if and only if there exist a Q-fuzzy right ideal g and a Q-fuzzy left ideal h of S such that $f = g \cap h$.

2. Preliminaries

Throughout this paper, unless stated otherwise, S stands for an ordered semigroup. A function f from $S \times Q$ to the real closed interval $[0, 1]$ is called Q-*fuzzy subset* of S, where Q is a non-empty set. The ordered semigroup S itself is a Q-fuzzy subset of S, its characteristic function, also denoted by S, is defined as follows:

$$S : S \times Q \longrightarrow [0, 1] \mid (x, q) \mapsto S(x, q) := 1,$$

for all $x \in S$ and $q \in Q$.

Let f and g be two Q-fuzzy subsets of S. Then the inclusion relation $f \subseteq g$ means that

$$f(x, q) \leq g(x, q),$$

for all $x \in S$ and $q \in Q$, $f \cap g$ and $f \cup g$ are defined by

$$(f \cap g)(x, q) := \min\{f(x, q), g(x, q)\},$$

$$(f \cup g)(x, q) := \max\{f(x, q), g(x, q)\},$$

for all $x \in S$ and $q \in Q$.

Let S and Q be an ordered semigroup and a non-empty set, respectively. For $x \in S$, we define $A_x := \{(y, z) \in S \times S \mid x \leq yz\}$. The product $f \circ g$ of f

and g is defined by

$$(\forall x \in S, \forall q \in Q)(f \circ g)(x,q) := \begin{cases} \bigvee_{(y,z)\in A_x} \min\{f(y,q), g(z,q)\} & \text{if } A_x \neq \emptyset, \\ 0 & \text{if } A_x = \emptyset, \end{cases}$$

We denote by $f_{A\times Q}$ the characteristic function of $A \times Q$, that is, the mapping of $S \times Q$ into $[0,1]$ defined by

$$f_{A\times Q}(x,q) := \begin{cases} 1 & \text{if } x \in A \times Q, \\ 0 & \text{if } x \notin A \times Q, \end{cases}$$

for all $(x,q) \in A \times Q$.

3. Main Results

In this section, we introduced the notion of Q-fuzzy right (resp. Q-fuzzy left) ideals, Q-fuzzy quasi-ideals, Q-fuzzy bi-ideals of ordered semigroups, and investigate related properties.

Definition 3.1. Let S and Q be an ordered groupoid and a non-empty set, respectively. A Q-fuzzy subset f of S is called a Q-fuzzy right ideal (resp. Q-fuzzy left) ideal of S if:

(1) $x \leq y$ implies $f(x,q) \geq f(y,q)$, and

(2) $f(xy,q) \geq f(x,q)$(resp. $f(xy,q) \geq f(y,q)$),

for all $x, y \in S$ and for all $q \in Q$.

Definition 3.2. Let S and Q be an ordered groupoid and a non-empty set, respectively. A Q-fuzzy subset f of S is called a Q-fuzzy quasi-ideal of S if:

(1) $x \leq y \Rightarrow f(x,q) \geq f(y,q)$,

(2) $(f \circ S) \cap (S \circ f) \subseteq f$,

for all $x, y \in S$ and for all $q \in Q$.

Definition 3.3. Let S and Q be an ordered semigroup and a non-empty set, respectively. A Q-fuzzy subsemigroup f of S is called a Q-fuzzy bi-ideal of S if:

(1) $x \leq y \Rightarrow f(x,q) \geq f(y,q)$,

(2) $f(xyz,q) \geq \min\{f(x,q), f(z,q)\}$,

for all $x, y, z \in S$ and for all $q \in Q$.

Theorem 3.4. *If S is an ordered groupoid and Q is a non-empty set, then the Q-fuzzy right (resp. left) ideals of S are Q-fuzzy quasi-ideals of S.*

Proof. Let f be a Q-fuzzy right ideal of S and $x \in S, q \in Q$. First of all,

$$((f \circ S) \cap (S \circ f))(x, q) = \min\{(f \circ S)(x, q), (S \circ f)(x, q)\}.$$

If $A_x = \emptyset$, then we have $(f \circ S)(x, q) = 0 = (S \circ f)(x, q)$ and, since f is a Q-fuzzy right ideal of S, we have $\min\{(f \circ S)(x, q), (S \circ f)(x, q)\} = 0 \leq f(x, q)$.

If $A_x \neq \emptyset$, then

$$(f \circ S)(x, q) = \bigvee_{(u,v) \in A_x} \{\min\{f(u, q), S(v, q)\}\}.$$

On the other hand, if $(u, v) \in A_x$, then $x \leq uv$ and $f(x, q) \geq f(uv, q) \geq f(u, q) = \min\{f(u, q), S(v, q)\}$. Hence, we have

$$
\begin{aligned}
f(x, q) &\geq \bigvee_{(u,v) \in A_x} \{\min\{f(u, q)\}\} \\
&\geq \min\{(f \circ S)(x, q), (S \circ f)(x, q)\} \\
&= ((f \circ S) \cap (S \circ f))(x, q).
\end{aligned}
$$

Therefore f is a Q-fuzzy quasi-ideal of S. $\square$

Theorem 3.5. *If S is an ordered semigroup and Q is a non-empty set, then the Q-fuzzy quasi-ideals are Q-fuzzy bi-ideals of S.*

Proof. Let f be a Q-fuzzy quasi-ideal of S and $x, y, z \in S, q \in Q$. Then we have

$$f(xyz, q) \geq ((f \circ S) \cap (S \circ f))(xyz, q) = \min\{(f \circ S)(xyz, q), (S \circ f)(xyz, q)\}.$$

Since $(x, yz) \in A_{xyz}$, we have

$$
\begin{aligned}
(f \circ S)(xyz, q) &= \bigvee_{(u,v) \in A_{xyz}} \{\min\{f(u, q), S(v, q)\}\} \\
&\geq \min\{f(x, q), S(yz, q)\} \\
&= f(x, q).
\end{aligned}
$$

Since $(xy, z) \in A_{xyz}$, we have

$$(S \circ f)(xyz, q) = \bigvee_{(u,v) \in A_{xyz}} \{\min\{S(u, q), f(v, q)\}\}$$

$$\geq \min\{S(xy,q), f(z,q)\}$$
$$= f(z,q).$$

Thus we have

$$f(xyz,q) \geq \min\{(f \circ S)(xyz,q), (S \circ f)(xyz,q)\} \geq \min\{f(x,q), f(z,q)\}.$$

Hence f is a Q-fuzzy bi-ideal of S.   □

An ordered semigroup S is called regular if for any $a \in S$ there exists $x \in S$ such that $a \leq axa$.

Theorem 3.6. *If S is a regular ordered semigroup and Q is a non-empty set, then the Q-fuzzy quasi-ideals and the Q-fuzzy bi-ideals coincide.*

Proof. Let f be a Q-fuzzy bi-ideal of S and $x \in S, q \in Q$. We will prove that

$$((f \circ S) \cap (S \circ f))(x,q) \leq f(x,q). \tag{1}$$

First of all, we have

$$((f \circ S) \cap (S \circ f))(x,q) = \min\{(f \circ S)(x,q), (S \circ f)(x,q)\}.$$

If $A_x = \emptyset$, then as we have already seen in Theorem 3.4, condition (1) is satisfied. If $A_x \neq \emptyset$, then

$$(f \circ S)(x,q) = \bigvee_{(z,w) \in A_x} \{\min\{f(z,q), S(w,q)\}\}, \tag{2}$$

$$(S \circ f)(x,q) = \bigvee_{(u,v) \in A_x} \{\min\{S(u,q), f(v,q)\}\}. \tag{3}$$

Let $(f \circ S)(x,q) \leq f(x,q)$. Then, we have

$$f(x,q) \geq (f \circ S)(x,q)$$
$$\geq \min\{(f \circ S)(x,q), (S \circ f)(x,q)\}$$
$$= ((f \circ S) \cap (S \circ f))(x,q),$$

and condition (1) is satisfied.

Let $(f \circ S)(x,q) > f(x,q)$. Then, by (2), there exists $(z,w) \in A_x$ such that

$$\min\{f(z,q), S(w,q)\} > f(x,q) \tag{4}$$

(otherwise $f(x,q) \leq (f \circ S)(x,q)$, which is impossible). Since $(z,w) \in A_x$, we have $z, w \in S$ and $x \leq zw$. Similarly, from $\min\{f(z,q), S(w,q)\} = f(z,q)$, by (4), we obtain

$$f(z,q) > f(x,q). \tag{5}$$

We will prove that $(S \circ f)(x,q) \leq f(x,q)$, then

$$\min\{(f \circ S)(x,q), (S \circ f)(x,q)\} \leq (S \circ f)(x,q) \leq f(x,q),$$

so that $((f \circ S) \cap (S \circ f))(x,q) \leq f(x,q)$, and condition (1) is satisfied.

By (3), it is enough to prove that

$$\min\{S(u,q), f(v,q)\} \leq f(x,q), \forall (u,v) \in A_x.$$

Let $(u,v) \in A_x$. Then $x \leq uv$ for some $u, v \in S$. Since S is regular, there exists $s \in S$ such that $x \leq xsx$. It follows that $x \leq zwsuv$. Since f is a Q-fuzzy bi-ideal of S, we have

$$\min\{S(u,q), f(v,q)\} \leq f(x,q), \forall (u,v) \in A_x,$$

and, we have

$$f(x,q) \geq f(zwsuv,q) \geq \min\{f(z,q), f(v,q)\}.$$

If $\min\{f(z,q), f(v,q)\} = f(z,q)$, then $f(z,q) \leq f(x,q)$ which is impossible by (5). Thus we have $\min\{f(z,q), f(v,q)\} = f(v,q)$, then $f(x,q) \geq f(v,q) = \min\{S(u,q), f(v,q)\}$. $\square$

In the following, using the usual definitions of ideals mentioned above, we show that the Q-fuzzy quasi-ideals of an ordered semigroup are just intersections of Q-fuzzy right and Q-fuzzy left ideals.

Lemma 3.7. *Let S and Q be an ordered semigroup and a non-empty set respectively. Let f be a Q-fuzzy subset of S. Then we have the following:*
(1) $(S \circ f)(xy,q) \geq f(y,q)$ for all $x, y \in S, q \in Q$,
(2) $(S \circ f)(xy,q) \geq (S \circ f)(y,q)$ for all $x, y \in S, q \in Q$.

Proof. (1) Let $x, y \in S$ and $q \in Q$. Since $(x,y) \in A_{xy}$, we have

$$(S \circ f)(xy,q) = \bigvee_{(w,z) \in A_{xy}} \{\min\{S(w,q), f(z,q)\}\} \geq \min\{S(x,q), f(y,q)\} = f(y,q).$$

(2) Let $x, y \in S$ and $q \in Q$. If $A_y = \emptyset$, then $(S \circ f)(y, q) = 0$. Since $(S \circ f)$ is a Q-fuzzy subset of S, we have $(S \circ f)(xy, q) \geq 0 = (S \circ f)(y, q)$. If $A_x \neq \emptyset$, then

$$(S \circ f)(y, q) = \bigvee_{(w,z) \in A_y} \{\min\{S(w, q), f(z, q)\}\}.$$

On the other hand,

$$(S \circ f)(xy, q) \geq \min\{S(w, q), f(z, q)\}, \forall (w, z) \in A_y. \tag{6}$$

Indeed, let $(w, z) \in A_y$. Since $(x, y) \in A_{xy}$, we have

$$(S \circ f)(xy, q) = \bigvee_{(s,t) \in A_{xy}} \{\min\{S(s, q), f(t, q)\}\}.$$

Since $(w, z) \in A_y$, we have $y \leq wz$, then $xy \leq xwz$, and $(xw, z) \in A_{xy}$. Hence we have

$$(S \circ f)(xy, q) \geq \min\{S(xw, q), f(z, q)\} = f(z, q) = \min\{S(w, q), f(z, q)\}.$$

By (6), we have

$$(S \circ f)(xy, q) \geq \bigvee_{(w,z) \in A_y} \{\min\{S(w, q), f(z, q)\}\} = (S \circ f)(y, q). \quad \square$$

In a similar way we can prove the following lemmas:

Lemma 3.8. *Let S and Q be an ordered semigroup and a non-empty set, respectively. Let f be a Q-fuzzy subset of S. Then we have the following:*
(1) $(S \circ f)(xy, q) \geq f(x, q)$ *for all* $x, y \in S, q \in Q$,
(2) $(S \circ f)(xy, q) \geq (S \circ f)(x, q)$ *for all* $x, y \in S, q \in Q$.

Lemma 3.9. *Let S and Q be an ordered semigroup and a non-empty set, respectively. Let f be a Q-fuzzy subset of S and $x \leq y$. Then we have $(S \circ f)(x, q) \geq (S \circ f)(y, q)$, for all $q \in Q$.*

Proof. Let $x, y \in S$ and $q \in Q$. Then, if $A_y = \emptyset$, then $(S \circ f)(y, q) = 0$. Since $S \circ f$ is a Q-fuzzy subset of S, we have $(S \circ f)(x, q) \geq 0$, then $(S \circ f)(x, q) \geq (S \circ f)(y, q)$. If $A_y \neq \emptyset$, then

$$(S \circ f)(y, q) = \bigvee_{(w,z) \in A_y} \{\min\{S(w, q), f(z, q)\}\} = \bigvee_{(w,z) \in A_y} \{f(z, q)\}.$$

On the other hand,

$$(S \circ f)(x, q) \geq f(z, q), \forall (w, z) \in A_y. \tag{7}$$

Indeed, let $(w, z) \in A_y$. Since $x \leq y \leq wz$, we have $(w, z) \in A_x$. Then

$$(S \circ f)(xy, q) = \bigvee_{(s,t) \in A_{xy}} \{\min\{S(s, q), f(t, q)\}\} \geq \min\{S(w, q), f(z, q)\} = f(z, q).$$

Thus, by (7), we have

$$(S \circ f)(x, q) \geq \bigvee_{(w,z) \in A_y} \{f(z, q)\} = (S \circ f)(y, q).$$

The proof is completed. $\qquad\square$

Lemma 3.10. *Let S and Q be an ordered semigroup and a non-empty set, respectively. Let f be a Q-fuzzy subset of S and $x \leq y$. Then we have $(f \circ S)(x, q) \geq (f \circ S)(y, q)$, for all $q \in Q$.*

Lemma 3.11. *Let S and Q be an ordered semigroup and a non-empty set, respectively. Let f be a Q-fuzzy subset of S such that $x \leq y$, we have $f(x, q) \geq f(y, q)$ for all $x, y \in S, q \in Q$. Then the Q-fuzzy subset $f \cup (S \circ f)$ is a Q-fuzzy left ideal of S.*

Proof. Let $x, y \in S$ and $q \in Q$. By, Theorem 3.9, we have $(f \cup (S \circ f))(xy, q) \geq (f \cup (S \circ f))(y, q)$. Let now $x \leq y$. Then $(f \cup (S \circ f))(x, q) \geq (f \cup (S \circ f))(y, q)$. Indeed: Since f is a Q-fuzzy subset of S and $x \leq y$, by Lemma 3.7, we get $(S \circ f)(x, q) \geq (S \circ f)(y, q)$ and, by hypothesis, $f(x, q) \geq f(y, q)$. Then

$$\begin{aligned}
(f \cup (S \circ f))(x, q) &= \max\{f(x, q), (S \circ f)(x, q)\} \\
&\geq \max\{f(y, q), (S \circ f)(y, q)\} \\
&= (f \cup (S \circ f))(y, q). \qquad\square
\end{aligned}$$

In a similar way we can prove the following:

Lemma 3.12. *Let S and Q be an ordered semigroup and a non-empty set, respectively. Let f be a Q-fuzzy subset of S such that $x \leq y$, we have $f(x, q) \geq f(y, q)$ for all $x, y \in S, q \in Q$. Then the Q-fuzzy subset $f \cup (f \circ S)$ is a Q-fuzzy right ideal of S.*

Lemma 3.13. Let S and f, g, h be an ordered semigroup and Q-fuzzy subsets of S, respectively. Then

$$f \cap (g \cup h) = (f \cap g) \cup (f \cap h).$$

Proof. Let $x \in S$ and $q \in Q$. Then we have

$$
\begin{aligned}
(f \cap (g \cup h))(x,q) &= \min\{f(x,q), (g \cup h)(x,q)\} \\
&= \min\{f(x,q), \max\{g(x,q), h(x,q)\}\} \\
&= \max\{\min\{f(x,q), g(x,q)\}, \min\{f(x,q), h(x,q)\} \\
&= \max\{(f \cap g)(x,q), (f \cap h)(x,q)\} \\
&= ((f \cap g) \cup (f \cap h))(x,q). \qquad \square
\end{aligned}
$$

Corollary 3.14. . Let S and Q be an ordered semigroup. Then the set of all Q-fuzzy subsets of S is a distributive lattice.

Theorem 3.15. Let S and Q be an ordered semigroup and a non-empty set, respectively. Then a Q-fuzzy subset f of S is a Q-fuzzy quasi-ideal of S if and only if there exist a Q-fuzzy right ideal g and a Q-fuzzy left ideal h of S such that $f = g \cap h$.

Proof. ($\Rightarrow$). By Lemma 3.11 and Lemma 3.12, $f \cup (S \circ f)$ is a Q-fuzzy left ideal and $f \cup (f \circ S)$ is a Q-fuzzy right ideal of S. Moreover, we have

$$f = (f \cup (S \circ f)) \cap (f \cup (f \circ S)).$$

In fact, by Corolary 3.14, we have

$$
\begin{aligned}
(f \cup (S \circ f)) \cap (f \cup (f \circ S)) &= ((f \cup (S \circ f)) \cap f) \cup ((f \cup (S \circ f)) \cap (f \circ S)) \\
&= (f \cap f) \cup ((S \circ f) \cap f) \cup (f \cap (f \circ S)) \cup ((S \circ f) \\
&\quad \cap (f \circ S)) \\
&= f \cup ((S \circ f) \cap f) \cup (f \cap (f \circ S)) \cup ((S \circ f) \\
&\quad \cap (f \circ S)).
\end{aligned}
$$

Since f is a Q-fuzzy quasi-ideal of S, we have $(f \circ S) \cap (S \circ f) \subseteq f$. Besides, $(S \circ f) \cap f \subseteq f$ and $f \cap (f \circ S) \subseteq f$. Hence

$$(f \cup (S \circ f)) \cap (f \cup (f \circ S)) = f.$$

($\Leftarrow$). Let $x \in S$ and $q \in Q$. Then

$$((f \circ S) \cap (S \circ f))(x,q) \leq f(x,q) \qquad (8)$$

In fact, $((f \circ S) \cap (S \circ f))(x,q) = \min\{(f \circ S)(x,q), (S \circ f)(x,q)\}$. If $A_x = \emptyset$, then $(f \circ S)(x,q) = 0 = (S \circ f)(x,q)$. Thus, in this case condition (8) is satisfied. If $A_x \neq \emptyset$, then

$$(f \circ S)(x,q) = \bigvee_{(y,z) \in A_x} \{\min\{f(y,q), S(z,q)\}\} = \bigvee_{(y,z) \in A_x} \{f(y,q)\}. \quad (9)$$

We have

$$f(y,q) \leq h(x,q), \forall (y,z) \in A_x. \quad (10)$$

Indeed, for $(y,z) \in A_x$, we have $x \leq yz$ and $h(x,q) \geq h(yz,q) \geq h(y,q)$ because h is a Q-fuzzy left ideal of S. Thus, applying (10) to (9), we obtain

$$(f \circ S)(x,q) = \bigvee_{(y,z) \in A_x} \{f(y,q)\} \leq h(x,q).$$

In a similar way, we get $(S \circ f)(x,q) \leq g(x,q)$. Hence

$$
\begin{aligned}
((f \circ S) \cap (S \circ f))(x,q) &= \min\{(f \circ S)(x,q), (S \circ f)(x,q)\} \\
&\leq \min\{h(x,q), g(xq)\} \\
&= (h \cap g)(x,q) \\
&= f(x,q),
\end{aligned}
$$

which completes the proof of (8). $\qquad \square$

Acknowledgments

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