CHAPTER V

CONCLUSIONS

5.1 Conclusions

In this research, we introduce an iterative scheme by the shrinking projection method for finding the common element of the set of common fixed points for nonexpansive semigroups, the set of common fixed points for an infinite family of a ξ-strict pseudo-contraction, the set of solutions of a systems of mixed equilibrium problems and the set of solutions of the variational inclusions problem.

The following results are all main theorems of this research:

(1). Let C be a nonempty closed convex subset of a real Hilbert space H, let $\{F_k : C \times C \to \mathcal{R},\ \ k = 1, 2, \ldots, N\}$ be a finite family of mixed equilibrium functions satisfying conditions (H1)-(H3). Let $\mathcal{S} = \{S(s) : 0 \le s < \infty\}$ be a nonexpansive semigroup on C and let $\{t_n\}$ be a positive real divergent sequence. Let $\{V_i : C \to C\}_{i=1}^{\infty}$ be a countable family of uniformly ξ-strict pseudo-contractions, $\{T_i : C \to C\}_{i=1}^{\infty}$ be the countable family of nonexpansive mappings defined by $T_i x = tx + (1-t)V_i x, \forall x \in C, \forall i \ge 1, t \in [\xi, 1)$, W_n be the W-mapping defined by (4.38) and W be a mapping defined by (4.39) with $F(W) \neq \emptyset$. Let $A, B : C \to H$ be γ, β-inverse-strongly monotone mappings and $M_1, M_2 : H \longrightarrow 2^H$ be maximal monotone mappings such that

$$\Theta := F(\mathcal{S}) \cap F(W) \cap \left(\cap_{k=1}^{N} SMEP(F_k)\right) \cap I(A, M_1) \cap I(B, M_2) \neq \emptyset.$$

Let $r_k > 0, k = 1, 2, \ldots, N$, which are constants. Let $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, $\{z_n\}$ and

$\{u_n\}$ be sequences generated by $x_0 \in C$, $C_1 = C$, $x_1 = P_{C_1}x_0$, $u_n \in C$ and

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrary}, \\ u_n = K^{F_N}_{r_{N,n}} K^{F_{N-1}}_{r_{N-1,n}} K^{F_{N-2}}_{r_{N-2,n}} \cdots K^{F_2}_{r_{2,n}} K^{F_1}_{r_{1,n}} x_n, \\ y_n = J_{M_2,\delta_n}(u_n - \delta_n B u_n), \\ v_n = J_{M_1,\lambda_n}(y_n - \lambda_n A y_n), \\ z_n = \alpha_n v_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} S(s)W_n v_n ds, \\ C_{n+1} = \Big\{ z \in C_n : \|z_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n(1-\alpha_n)\Big\| v_n \\ \quad - \frac{1}{t_n}\int_0^{t_n} S(s)W_n v_n ds \Big\|^2 \Big\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases}$$

where $K^{F_k}_{r_k} : C \to C$, $k = 1, 2, \ldots, N$ is the mapping defined by (2.16) and $\{\alpha_n\}$ be a sequence in $(0,1)$ for all $n \in \mathbb{N}$. Assume the following conditions are satisfied:

(C1) $\eta_k : C \times C \to H$ is L_k-Lipschitz continuous with constant $k = 1, 2, \ldots, N$ such that

   (a) $\eta_k(x,y) + \eta_k(y,x) = 0, \quad \forall x, y \in C$,

   (b) $x \mapsto \eta_k(x,y)$ is affine,

   (c) for each fixed $y \in C$, $y \mapsto \eta_k(x,y)$ is sequentially continuous from the weak topology to the weak topology;

(C2) $\mathcal{K}_k : C \to \mathcal{R}$ is η_k-strongly convex with constant $\sigma_k > 0$ and its derivative $\mathcal{K}'_k$ is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with a Lipschitz constant $\nu_k > 0$ such that $\sigma_k > L_k \nu_k$;

(C3) For each $k \in \{1, 2, \ldots, N\}$ and for all $x \in C$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that for any $y \in C \backslash D_x$,

$$F_k(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r_k}\Big\langle \mathcal{K}'(y) - \mathcal{K}'(x), \eta(z_x, y) \Big\rangle < 0;$$

(C4) $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (\xi, 1)$;

(C5) $\{\lambda_n\} \subset [a_1, b_1]$ for some $a_1, b_1 \in (0, 2\gamma]$;

(C6) $\{\delta_n\} \subset [a_2, b_2]$ for some $a_2, b_2 \in (0, 2\beta]$;

(C7) $\liminf_{n\to\infty} r_{k,n} > 0$ for each $k \in 1, 2, 3, ..., N$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_\Theta x_0$.

(2). Let C be a nonempty closed convex subset of a real Hilbert space H, let $\{F_k : C \times C \to \mathcal{R},\ \ k = 1, 2, \ldots, N\}$ be a finite family of mixed equilibrium functions satisfying conditions (H1)-(H3). Let $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C and let $\{t_n\}$ be a positive real divergent sequence. Let $\{V_i : C \to C\}_{i=1}^{\infty}$ be a countable family of uniformly ξ-strict pseudo-contractions, $\{T_i : C \to C\}_{i=1}^{\infty}$ be the countable family of nonexpansive mappings defined by $T_i x = tx + (1-t)V_i x, \forall x \in C, \forall i \geq 1, t \in [\xi, 1)$, W_n be the W-mapping defined by (3.38) and W be a mapping defined by (3.39) with $F(W) \neq \emptyset$. Let $A, B : C \to H$ be γ, β-inverse-strongly monotone mapping. Such that

$$\Theta := F(\mathcal{S}) \cap F(W) \cap (\cap_{k=1}^{N} SMEP(F_k)) \cap VI(C, A) \cap VI(C, B) \neq \emptyset.$$

Let $r_k > 0, k = 1, 2, \ldots, N$, which are constants. Let $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, $\{z_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in C$, $C_1 = C$, $x_1 = P_{C_1} x_0$, $u_n \in C$ and

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrary}, \\ u_n = K_{r_{N,n}}^{F_N} K_{r_{N-1,n}}^{F_{N-1}} K_{r_{N-2,n}}^{F_{N-2}} \cdots K_{r_{2,n}}^{F_2} K_{r_{1,n}}^{F_1} x_n, \\ y_n = P_C(u_n - \delta_n B u_n), \\ v_n = P_C(y_n - \lambda_n A y_n), \\ z_n = \alpha_n v_n + (1-\alpha_n) \dfrac{1}{t_n} \displaystyle\int_0^{t_n} S(s) W_n v_n ds, \\ C_{n+1} = \Big\{ z \in C_n : \|z_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n(1-\alpha_n) \Big\| v_n \\ \quad - \dfrac{1}{t_n} \displaystyle\int_0^{t_n} S(s) W_n v_n ds \Big\|^2 \Big\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases}$$

where $K_{r_k}^{F_k} : C \to C, \quad k = 1, 2, \ldots, N$ is the mapping defined by (2.16) and $\{\alpha_n\}$ be a sequence in $(0,1)$ for all $n \in \mathbb{N}$. Assume the following conditions are satisfied:

(C1) $\eta_k : C \times C \to H$ is L_k-Lipschitz continuous with constant $k = 1, 2, \ldots, N$ such that

(a) $\eta_k(x,y) + \eta_k(y,x) = 0, \quad \forall x, y \in C,$

(b) $x \mapsto \eta_k(x,y)$ is affine,

(c) for each fixed $y \in C$, $y \mapsto \eta_k(x,y)$ is sequentially continuous from the weak topology to the weak topology;

(C2) $\mathcal{K}_k : C \to \mathcal{R}$ is η_k-strongly convex with constant $\sigma_k > 0$ and its derivative $\mathcal{K}_k'$ is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with a Lipschitz constant $\nu_k > 0$ such that $\sigma_k > L_k \nu_k$;

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$$F_k(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r_k}\Big\langle \mathcal{K}'(y) - \mathcal{K}'(x), \eta(z_x, y)\Big\rangle < 0;$$

(C4) $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (\xi, 1)$;

(C5) $\{\lambda_n\} \subset [a_1, b_1]$ for some $a_1, b_1 \in (0, 2\gamma]$;

(C6) $\{\delta_n\} \subset [a_2, b_2]$ for some $a_2, b_2 \in (0, 2\beta]$;

(C7) $\liminf_{n\to\infty} r_{k,n} > 0$ for each $k \in 1, 2, 3, \ldots, N$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_\Theta x_0$.

(3). Let C be a nonempty closed convex subset of a real Hilbert space H, let $\{F_k : C \times C \to \mathcal{R}, \quad k = 1, 2, \ldots, N\}$ be a finite family of mixed equilibrium functions satisfying conditions (H1)-(H3). Let $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C and let $\{t_n\}$ be a positive real divergent sequence. Let $\{V_i : C \to C\}_{i=1}^{\infty}$ be a countable family of uniformly ξ-strict pseudo-contractions, $\{T_i : C \to C\}_{i=1}^{\infty}$ be the countable family of nonexpansive mappings defined by $T_i x = tx + (1-t)V_i x, \forall x \in C, \forall i \geq 1, t \in [\xi, 1)$, W_n be the W-mapping defined by (3.38) and W be a mapping defined by (3.39) with $F(W) \neq \emptyset$. Let $A, B : C \to H$ be γ, β-inverse-strongly monotone mapping and S_A, S_B be $\kappa_\gamma, \kappa_\beta$-strictly pseudo-contraction mapping of C into C for some $0 \leq \kappa_\gamma < 1,\ 0 \leq \kappa_\beta < 1$ such that

$$\Theta := F(\mathcal{S}) \cap F(W) \cap (\cap_{k=1}^{N} SMEP(F_k)) \cap F(S_A) \cap F(S_B) \neq \emptyset.$$

Let $r_k > 0, k = 1, 2, \ldots, N$, which are constants. Let $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, $\{z_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in C$, $C_1 = C$, $x_1 = P_{C_1} x_0$, $u_n \in C$ and

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrary}, \\ u_n = K_{r_{N,n}}^{F_N} K_{r_{N-1,n}}^{F_{N-1}} K_{r_{N-2,n}}^{F_{N-2}} \cdots K_{r_{2,n}}^{F_2} K_{r_{1,n}}^{F_1} x_n, \\ y_n = (1-\delta_n) u_n + \delta_n S_B u_n, \\ v_n = (1-\lambda_n) y_n + \lambda_n S_A y_n, \\ z_n = \alpha_n v_n + (1-\alpha_n) \frac{1}{t_n} \int_0^{t_n} S(s) W_n v_n ds, \\ C_{n+1} = \Big\{ z \in C_n : \|z_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n(1-\alpha_n) \Big\| v_n \\ \quad - \frac{1}{t_n} \int_0^{t_n} S(s) W_n v_n ds \Big\|^2 \Big\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases}$$

where $K_{r_k}^{F_k} : C \to C,\ \ k = 1, 2, \ldots, N$ is the mapping defined by (2.16) and $\{\alpha_n\}$ be a sequence in $(0, 1)$ for all $n \in \mathbb{N}$. Assume the following conditions are satisfied:

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(C7) $\liminf_{n\to\infty} r_{k,n} > 0$ for each $k \in 1, 2, 3, ..., N$

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_\Theta x_0$.

5.2 Outputs

International Conference (Oral Presentations)

(1) **C. Jaiboon**, 2011, Approximate analytical solutions of system of variational inclusion problem and fixed point problems of nonexpansive semigroup, The seventh international conference on Nonlinear Analysis and Convex Analysis August 2–5, 2011, Busan, KOREA