CHAPTER IV

MAIN RESULTS

In this section, we will introduce an iterative scheme by using a shrinking projection method for finding the common element of the set of common fixed points for nonexpansive semigroups, the set of common fixed points for an infinite family of ξ-strict pseudo-contraction, the set of solutions of a systems of mixed equilibrium problems and the set of solutions of the variational inclusions problem in a real Hilbert space.

In order to prove our main results, we need the following lemmas.

Lemma 4.43. [45] *Let $V : C \to H$ be a ξ-strict pseudo-contraction, then*
(1) the fixed point set $F(V)$ of V is closed convex so that the projection $P_{F(V)}$ is well defined;
(2) define a mapping $T : C \to H$ by

$$Tx = tx + (1-t)Vx, \forall x \in C \tag{4.36}$$

If $t \in [\xi, 1)$, then T is a nonexpansive mapping such that $F(V) = F(T)$.

A family of mappings $\{V_i : C \to H\}_{i=1}^{\infty}$ is called *a family of uniformly ξ-strict pseudo-contractions*, if there exists a constant $\xi \in [0,1)$ such that

$$\|V_ix - V_iy\|^2 \leq \|x-y\|^2 + \xi\|(I-V_i)x - (I-V_i)y\|^2, \ \forall x, y \in C, \forall i \geq 1.$$

Let $\{V_i : C \to C\}_{i=1}^{\infty}$ be a countable family of uniformly ξ-strict pseudo-contractions. Let
$\{T_i : C \to C\}_{i=1}^{\infty}$ be the sequence of nonexpansive mappings defined by (4.36), i.e.,

$$T_ix = tx + (1-t)V_ix, \forall x \in C, \forall i \geq 1, t \in [\xi, 1) \tag{4.37}$$

Let $\{T_i\}$ be a sequence of nonexpansive mappings of C into itself defined by (4.37) and let $\{\mu_i\}$ be a sequence of nonnegative numbers in [0,1]. For each $n \geq 1$, define a mapping W_n of C into itself as follows:

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= \mu_n T_n U_{n,n+1} + (1-\mu_n)I, \\ U_{n,n-1} &= \mu_{n-1} T_{n-1} U_{n,n} + (1-\mu_{n-1})I, \\ &\vdots \\ U_{n,k} &= \mu_k T_k U_{n,k+1} + (1-\mu_k)I, \\ U_{n,k-1} &= \mu_{k-1} T_{k-1} U_{n,k} + (1-\mu_{k-1})I, \\ &\vdots \\ U_{n,2} &= \mu_2 T_2 U_{n,3} + (1-\mu_2)I, \\ W_n = U_{n,1} &= \mu_1 T_1 U_{n,2} + (1-\mu_1)I. \end{aligned} \tag{4.38}$$

Such a mapping W_n is nonexpansive from C to C and it is called the W-mapping generated by $T_1, T_2, ..., T_n$ and $\mu_1, \mu_2, ..., \mu_n$. For each $n, k \in \mathbb{N}$, let the mapping $U_{n,k}$ be defined by (4.38). Then we can have the following crucial conclusions concerning W_n.

Lemma 4.44. *[33, 44] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, ...$ be an infinite family of nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$, let $\mu_1, \mu_2, ...$ be real numbers such that $0 \leq \mu_n \leq b < 1$ for every $n \geq 1$. Then,*

(1) for every $x \in C$ and $k \in \mathbb{N}$, the limit $\lim_{n \longrightarrow \infty} U_{n,k}x$ exists;

(2) the mapping W of C into itself as follows:

$$Wx = \lim_{n \longrightarrow \infty} W_n x = \lim_{n \longrightarrow \infty} U_{n,1}x, \quad x \in C, \tag{4.39}$$

is a nonexpansive mapping satisfying $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$, which it is called the W-mapping generated by $T_1, T_2, ...$ and $\mu_1, \mu_2, ...$

(3) $F(W_n) = \cap_{n=1}^{\infty} F(T_n)$, for each $n \geq 1$;

(4) If E is any bounded subset of C, then $\lim_{n\to\infty} \sup_{x\in E} \|Wx - W_n x\| = 0$.

Theorem 4.45. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $\{F_k : C \times C \to \mathcal{R},\quad k = 1, 2, \dots, N\}$ be a finite family of mixed equilibrium functions satisfying conditions (H1)-(H3). Let $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C and let $\{t_n\}$ be a positive real divergent sequence. Let $\{V_i : C \to C\}_{i=1}^{\infty}$ be a countable family of uniformly ξ-strict pseudo-contractions, $\{T_i : C \to C\}_{i=1}^{\infty}$ be the countable family of nonexpansive mappings defined by $T_i x = tx + (1-t)V_i x, \forall x \in C, \forall i \geq 1, t \in [\xi, 1)$, W_n be the W-mapping defined by (4.38) and W be a mapping defined by (4.39) with $F(W) \neq \emptyset$. Let $A, B : C \to H$ be γ, β-inverse-strongly monotone mappings and $M_1, M_2 : H \longrightarrow 2^H$ be maximal monotone mappings such that*

$$\Theta := F(\mathcal{S}) \cap F(W) \cap (\cap_{k=1}^{N} SMEP(F_k)) \cap I(A, M_1) \cap I(B, M_2) \neq \emptyset.$$

Let $r_k > 0, k = 1, 2, \dots, N$, which are constants. Let $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, $\{z_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in C$, $C_1 = C$, $x_1 = P_{C_1} x_0$, $u_n \in C$ and

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrary,} \\ u_n = K_{r_{N,n}}^{F_N} K_{r_{N-1,n}}^{F_{N-1}} K_{r_{N-2,n}}^{F_{N-2}} \cdots K_{r_{2,n}}^{F_2} K_{r_{1,n}}^{F_1} x_n, \\ y_n = J_{M_2,\delta_n}(u_n - \delta_n B u_n), \\ v_n = J_{M_1,\lambda_n}(y_n - \lambda_n A y_n), \\ z_n = \alpha_n v_n + (1-\alpha_n)\dfrac{1}{t_n}\displaystyle\int_0^{t_n} S(s) W_n v_n ds, \\ C_{n+1} = \Big\{ z \in C_n : \|z_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n(1-\alpha_n)\Big\| v_n \\ \quad - \dfrac{1}{t_n}\displaystyle\int_0^{t_n} S(s) W_n v_n ds \Big\|^2 \Big\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases} \tag{4.40}$$

where $K_{r_k}^{F_k} : C \to C,\quad k = 1, 2, \dots, N$ is the mapping defined by (2.16) and $\{\alpha_n\}$ be a sequence in $(0, 1)$ for all $n \in \mathbb{N}$. Assume the following conditions are satisfied:

(C1) $\eta_k : C \times C \to H$ is L_k-Lipschitz continuous with constant $k = 1, 2, \dots, N$ such that

(a) $\eta_k(x,y)+\eta_k(y,x)=0, \quad \forall x,y\in C,$

(b) $x\mapsto \eta_k(x,y)$ *is affine,*

(c) for each fixed $y\in C$, $y\mapsto \eta_k(x,y)$ *is sequentially continuous from the weak topology to the weak topology;*

(C2) $\mathcal{K}_k : C\to \mathcal{R}$ *is* η_k*-strongly convex with constant* $\sigma_k>0$ *and its derivative* $\mathcal{K}'_k$ *is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with a Lipschitz constant* $\nu_k>0$ *such that* $\sigma_k > L_k\nu_k$*;*

(C3) For each $k\in\{1,2,\ldots,N\}$ *and for all* $x\in C$*, there exist a bounded subset* $D_x\subset C$ *and* $z_x\in C$ *such that for any* $y\in C\backslash D_x$*,*

$$F_k(y,z_x)+\varphi(z_x)-\varphi(y)+\frac{1}{r_k}\Big\langle \mathcal{K}'(y)-\mathcal{K}'(x),\eta(z_x,y)\Big\rangle<0;$$

(C4) $\{\alpha_n\}\subset[c,d]$ *for some* $c,d\in(\xi,1)$*;*

(C5) $\{\lambda_n\}\subset[a_1,b_1]$ *for some* $a_1,b_1\in(0,2\gamma]$*;*

(C6) $\{\delta_n\}\subset[a_2,b_2]$ *for some* $a_2,b_2\in(0,2\beta]$*;*

(C7) $\liminf_{n\to\infty} r_{k,n}>0$ *for each* $k\in 1,2,3,...,N$.

Then, $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $z=P_\Theta x_0$.

Proof. Pick any $p\in\Theta$. Taking $\Im_n^k = K_{r_{k,n}}^{F_k}K_{r_{k-1,n}}^{F_{k-1}}K_{r_{k-2,n}}^{F_{k-2}}\cdots K_{r_{2,n}}^{F_2}K_{r_{1,n}}^{F_1}$ for $k\in\{1,2,3,\ldots,N\}$ and $\Im_n^0=I$ for all $n\in\mathbb{N}$. From the definition of $K_{r_{k,n}}^{F_k}$ is non-expansive for each $k=1,2,3,\ldots,N$, then $\Im_n^k$ also and $p=\Im_{r_{k,n}}^{F_k}p$, we note that $u_n=\Im_n^N x_n$. If follows that

$$\|u_n-p\|=\|\Im_n^N x_n-\Im_n^N p\|\le\|x_n-p\|.$$

Next, we will divide the proof into eight steps.

Step 1. We first show by induction that $\Theta \subset C_n$ for each $n \geq 1$.

Taking $p \in \Theta$, we get that $p = J_{M_1,\lambda_k}(p - \lambda_k Ap) = J_{M_2,\delta_k}(p - \delta_k Bp)$. Since J_{M_1,λ_k}, J_{M_2,δ_k} are nonexpansive. From the assumption, we see that $\Theta \subset C = C_1$. Suppose $\Theta \subset C_k$ for some $k \geq 1$. For any $p \in \Theta = C_k$, we have

$$\begin{aligned}\|v_k - p\| &= \|J_{M_1,\lambda_k}(y_k - \lambda_k Ay_k) - J_{M_1,\lambda_k}(p - \lambda_k Ap)\| \\ &\leq \|(y_k - \lambda_k Ay_k) - (p - \lambda_k Ap)\| \\ &\leq \|(I - \lambda_k A)y_k - (I - \lambda_k A)p\| \\ &\leq \|y_k - p\|, \end{aligned} \tag{4.41}$$

and

$$\begin{aligned}\|y_k - p\| &= \|J_{M_2,\delta_k}(u_k - \delta_k Bu_k) - J_{M_2,\delta_k}(p - \delta_k Bp)\| \\ &\leq \|(u_k - \delta_k Bu_k) - (p - \delta_k Bp)\| \\ &\leq \|u_k - p\| \\ &\leq \|x_k - p\|. \end{aligned} \tag{4.42}$$

Which yield that

$$\begin{aligned}&\|z_k - p\|^2 \\ = &\left\|\alpha_k(v_k - p) + (1-\alpha_k)\Big(\frac{1}{t_k}\int_0^{t_k} S(s)W_k v_k ds - p\Big)\right\|^2 \\ \leq &\ \alpha_k\|v_k - p\|^2 + (1-\alpha_k)\left\|\frac{1}{t_k}\int_0^{t_k} S(s)W_k v_k ds - p\right\|^2 \\ &-\alpha_k(1-\alpha_k)\left\|v_k - \frac{1}{t_k}\int_0^{t_k} S(s)W_k v_k ds\right\|^2 \\ \leq &\ \alpha_k\|v_k - p\|^2 + (1-\alpha_k)\|v_k - p\|^2 - \alpha_k(1-\alpha_k)\left\|v_k - \frac{1}{t_k}\int_0^{t_k} S(s)W_k v_k ds\right\|^2 \\ \leq &\ \|v_k - p\|^2 - \alpha_k(1-\alpha_k)\left\|v_k - \frac{1}{t_k}\int_0^{t_k} S(s)W_k v_k ds\right\|^2. \end{aligned} \tag{4.43}$$

Applying (4.41) and (4.42), we get

$$\|z_k - p\|^2 \leq \|x_k - p\|^2 - \alpha_k(1-\alpha_k)\left\|v_k - \frac{1}{t_k}\int_0^{t_k} S(s)W_k v_k ds\right\|^2. \tag{4.44}$$

Hence $p \in C_{k+1}$. This implies that $\Theta \subset C_n$ for each $n \geq 1$.

Step 2. Next, we show that $\{x_n\}$ is well defined and C_n is closed and convex for any $n \in \mathbb{N}$.

It is obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \geq 1$. Now, we show that C_{k+1} is closed and convex for some k. For any $p \in C_k$, we obtain

$$\|z_k - p\|^2 \leq \|x_k - p\|^2$$

is equivalent to

$$\|z_k - x_k\|^2 + 2\langle z_k - x_k, x_k - p\rangle \leq 0. \tag{4.45}$$

Thus C_{k+1} is closed and convex. Then, C_n is closed and convex for any $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well-defined.

Step 3. Next, we show that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - x_0\|$ exists. From $x_n = P_{C_n}x_0$, we have

$$\langle x_0 - x_n, x_n - y\rangle \geq 0$$

for each $y \in C_n$. Using $\Theta \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - p\rangle \geq 0, \quad \forall p \in \Theta \quad \text{and} \quad n \in \mathbb{N}.$$

So, for $p \in \Theta$. We observe that

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - p\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - p\rangle \\ &= -\langle x_0 - x_n, x_0 - x_n\rangle + \langle x_0 - x_n, x_0 - p\rangle \\ &\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - p\|. \end{aligned}$$

This implies that

$$\|x_0 - x_n\| \leq \|x_0 - p\|, \quad \forall p \in \Theta \quad \text{and} \quad n \in \mathbb{N}.$$

Hence, we get $\{x_n\}$ is bounded. It follows by (4.41)-(4.43), that $\{v_n\}, \{y_n\}$ and $\{W_n v_n\}$ are also bounded. From $x_n = P_{C_n}x_0$, and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we obtain

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0. \tag{4.46}$$

It follows that, we have for each $n \in \mathbb{N}$

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - x_{n+1}\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\ &= -\langle x_0 - x_n, x_0 - x_n\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\ &\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|. \end{aligned}$$

It follows that

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|.$$

Thus, since the sequence $\{\|x_n - x_0\|\}$ is a bounded and nondecreasing sequence, so $\lim_{n\longrightarrow\infty} \|x_n - x_0\|$ exists, that is

$$m = \lim_{n\longrightarrow\infty} \|x_n - x_0\|. \tag{4.47}$$

Step 4. Next, we show that $\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n\longrightarrow\infty} \|x_n - z_n\| = 0$.

Applying (4.46), we get

$$\begin{aligned} &\|x_n - x_{n+1}\|^2 \\ = &\ \|x_n - x_0 + x_0 - x_{n+1}\|^2 \\ = &\ \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ = &\ \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ = &\ \|x_n - x_0\|^2 - 2\langle x_n - x_0, x_n - x_0\rangle + 2\langle x_n - x_0, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ = &\ -\|x_n - x_0\|^2 + 2\langle x_n - x_0, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ \leq &\ -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2. \end{aligned}$$

Thus, by (4.47), we obtain

$$\lim_{n\longrightarrow\infty} \|x_n - x_{n+1}\| = 0. \tag{4.48}$$

On the other hand, from $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, which implies that

$$\|x_{n+1} - z_n\| \leq \|x_{n+1} - x_n\|. \tag{4.49}$$

It follows by (4.49), we also have

$$\|z_n - x_n\| \leq \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\| \leq 2\|x_n - x_{n+1}\|.$$

By (4.48), we obtain

$$\lim_{n\longrightarrow\infty} \|x_n - z_n\| = 0. \tag{4.50}$$

Step 5. Next, we show that

$$\lim_{n\longrightarrow\infty} \|\Im_n^k x_n - \Im_n^{k-1} x_n\| = 0 \tag{4.51}$$

for every $k \in \{1, 2, 3, \dots, N\}$. Indeed, for $p \in \Theta$, note that $K_{r_{k,n}}^{F_k}$ is the firmly nonexpansie, so we have

$$\begin{aligned}
\|\Im_n^k x_n - \Im_n^k p\|^2 &= \|K_{r_{k,n}}^{F_k} \Im_n^{k-1} x_n - K_{r_{k,n}}^{F_k} p\|^2 \\
&\leq \langle \Im_n^k x_n - p, \Im_n^{k-1} x_n - p\rangle \\
&= \frac{1}{2}\Big\{\|\Im_n^k x_n - p\|^2 + \|\Im_n^{k-1} x_n - p\|^2 - \|\Im_n^k x_n - \Im_n^{k-1} x_n\|^2\Big\}.
\end{aligned}$$

Thus, we get

$$\|\Im_n^k x_n - \Im_n^k p\|^2 \leq \|\Im_n^{k-1} x_n - p\|^2 - \|\Im_n^k x_n - \Im_n^{k-1} x_n\|^2.$$

It follows that

$$\begin{aligned}
\|u_n - p\|^2 &\leq \|\Im_n^k x_n - \Im_n^k p\|^2 \\
&\leq \|\Im_n^{k-1} x_n - p\|^2 - \|\Im_n^k x_n - \Im_n^{k-1} x_n\|^2 \\
&\leq \|x_n - p\|^2 - \|\Im_n^k x_n - \Im_n^{k-1} x_n\|^2. \qquad (4.52)
\end{aligned}$$

By (4.41), (4.42), (4.43) and (4.52), we have for each $k \in \{1, 2, 3, ..., N\}$

$$\begin{aligned} \|z_n - p\|^2 &\leq \|v_n - p\|^2 \\ &\leq \|u_n - p\|^2 \\ &\leq \|x_n - p\|^2 - \|\Im_n^k x_n - \Im_n^{k-1} x_n\|^2. \end{aligned}$$

Consequently, we have

$$\begin{aligned} \|\Im_n^k x_n - \Im_n^{k-1} x_n\|^2 &\leq \|x_n - p\|^2 - \|z_n - p\|^2 \\ &\leq \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|). \end{aligned}$$

Since (4.50) implies that for every $k \in \{1, 2, 3, ..., N\}$

$$\lim_{n \longrightarrow \infty} \|\Im_n^k x_n - \Im_n^{k-1} x_n\| = 0. \tag{4.53}$$

Step 6. Next, we show that $\lim_{n \longrightarrow \infty} \|y_n - v_n\| = 0$ and $\lim_{n \longrightarrow \infty} \|\mathcal{K}_n W_n v_n - v_n\| = 0$, where $\mathcal{K}_n = \frac{1}{t_n} \int_0^{t_n} S(s) ds$

For any given $p \in \Theta$, $\lambda_n \in (0, 2\gamma]$, $\delta_n \in (0, 2\beta]$ and $p = J_{M_1,\lambda_n}(p - \lambda_n Ap) = J_{M_2,\delta_n}(p - \delta_n Bp)$. Since $I - \lambda_n A$ and $I - \delta_n B$ are nonexpansive, we have

$$\begin{aligned} \|v_n - p\|^2 &= \|J_{M_1,\lambda_n}(y_n - \lambda_n Ay_n) - J_{M_1,\lambda_n}(p - \lambda_n Ap)\|^2 \\ &\leq \|(y_n - \lambda_n Ay_n) - (p - \lambda_n Ap)\|^2 \\ &= \|(y_n - p) - \lambda_n(Ay_n - Ap)\|^2 \\ &\leq \|y_n - p\|^2 - 2\lambda_n\langle y_n - p, Ay_n - Ap\rangle + \lambda_n^2\|Ay_n - Ap\|^2 \\ &\leq \|x_n - p\|^2 - 2\lambda_n\gamma\|Ay_n - Ap\|^2 + \lambda_n^2\|Ay_n - Ap\|^2 \\ &\leq \|x_n - p\|^2 + \lambda_n(\lambda_n - 2\gamma)\|Ay_n - Ap\|^2. \end{aligned} \tag{4.54}$$

Similarly, we can show that

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 + \delta_n(\delta_n - 2\beta)\|Bu_n - Bp\|^2. \tag{4.55}$$

Observe that

$$\begin{aligned}
\|z_n - p\|^2 &= \Big\|\alpha_n(v_n - p) + (1-\alpha_n)\Big(\frac{1}{t_n}\int_0^{t_n} S(s)W_n v_n ds - p\Big)\Big\|^2 \\
&\leq \alpha_n\|v_n - p\|^2 + (1-\alpha_n)\Big\|\frac{1}{t_n}\int_0^{t_n} S(s)W_n v_n ds - p\Big\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\Big\|v_n - \frac{1}{t_n}\int_0^{t_n} S(s)W_n v_n ds\Big\|^2 \\
&\leq \alpha_n\|v_n - p\|^2 + (1-\alpha_n)\Big\|\frac{1}{t_n}\int_0^{t_n} S(s)W_n v_n ds - p\Big\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2. \qquad (4.56)
\end{aligned}$$

Substituting (4.54) into (4.56) and using conditions (C4) and (C5), we have

$$\begin{aligned}
\|z_n - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\{\|x_n - p\|^2 + \lambda_n(\lambda_n - 2\gamma)\|Ay_n - Ap\|^2\} \\
&= \|x_n - p\|^2 + (1-\alpha_n)\lambda_n(\lambda_n - 2\gamma)\|Ay_n - Ap\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
(1-d)a_1(2\gamma - b_1)\|Ay_n - Ap\|^2 &\leq (1-\alpha_n)\lambda_n(2\gamma - \lambda_n)\|Ay_n - Ap\|^2 \\
&\leq \|x_n - p\|^2 - \|z_n - p\|^2 \\
&\leq \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|).
\end{aligned}$$

By (4.50), we obtain

$$\lim_{n\longrightarrow\infty}\|Ay_n - Ap\| = 0. \qquad (4.57)$$

Since the resolvent operator J_{M_1,λ_n} is 1-inverse-strongly monotone, we obtain

$$\begin{aligned}
\|v_n - p\|^2 &= \|J_{M_1,\lambda_n}(y_n - \lambda_n Ay_n) - J_{M_1,\lambda_n}(p - \lambda_n Ap)\|^2 \\
&= \|J_{M_1,\lambda_n}(I - \lambda_n A)y_n - J_{M_1,\lambda_n}(I - \lambda_n A)p\|^2 \\
&\leq \Big\langle (I-\lambda_n A)y_n - (I-\lambda_n A)p, v_n - p\Big\rangle \\
&= \frac{1}{2}\Big\{\|(I-\lambda_n A)y_n - (I-\lambda_n A)p\|^2 + \|v_n - p\|^2 \\
&\quad - \|(I-\lambda_n A)y_n - (I-\lambda_n A)p - (v_n - p)\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|y_n - p\|^2 + \|v_n - p\|^2 - \|(y_n - v_n) - \lambda_n(Ay_n - Ap)\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|x_n - p\|^2 + \|v_n - p\|^2 - \|y_n - v_n\|^2 \\
&\quad - \lambda_n^2\|Ay_n - Ap\|^2 + 2\lambda_n\langle y_n - v_n, Ay_n - Ap\rangle\Big\},
\end{aligned}$$

which yields that

$$\|v_n - p\|^2 \leq \|x_n - p\|^2 - \|y_n - v_n\|^2 + 2\lambda_n\|y_n - v_n\|\|Ay_n - Ap\|. \tag{4.58}$$

Similarly, we can obtain

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 - \|u_n - y_n\|^2 + 2\delta_n\|u_n - y_n\|\|Bu_n - Bp\|. \tag{4.59}$$

Substituting (4.58) into (4.56), and using condition (C4) and (C5), we have

$$\begin{aligned}
&\|z_n - p\|^2 \\
\leq\ & \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2 \\
\leq\ & \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\Big\{\|x_n - p\|^2 - \|y_n - v_n\|^2 + 2\lambda_n\|y_n - v_n\|\|Ay_n - Ap\|\Big\} \\
=\ & \|x_n - p\|^2 - (1-\alpha_n)\|y_n - v_n\|^2 + 2(1-\alpha_n)\lambda_n\|y_n - v_n\|\|Ay_n - Ap\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
&(1-\alpha_n)\|y_n - v_n\|^2 \\
\leq\ & \|x_n - p\|^2 - \|z_n - p\|^2 + 2(1-\alpha_n)\lambda_n\|y_n - v_n\|\|Ay_n - Ap\| \\
\leq\ & \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|) + 2(1-\alpha_n)\lambda_n\|y_n - v_n\|\|Ay_n - Ap\|.
\end{aligned}$$

By (4.50) and (4.57), we get

$$\lim_{n\longrightarrow\infty}\|y_n - v_n\| = 0. \tag{4.60}$$

From (4.44) and (C4), we also have

$$\begin{aligned}
\alpha_n(1-\alpha_n)\Big\|v_n - \frac{1}{t_n}\int_0^{t_n} S(s)W_nv_nds\Big\|^2 &\leq \|x_n - p\|^2 - \|z_n - p\|^2 \\
&\leq \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|).
\end{aligned}$$

Since $\mathcal{K}_n = \frac{1}{t_n}\int_0^{t_n} S(s)ds$, we obtain (4.50), we have

$$\lim_{n\longrightarrow\infty}\|\mathcal{K}_nW_nv_n - v_n\| = 0. \tag{4.61}$$

Since $\{W_nv_n\}$ is a bounded sequence in C, from Lemma 2.41 for all $h \geq 0$, we have

$$\begin{aligned}
&\lim_{n\to\infty}\|\mathcal{K}_nW_nv_n - S(h)\mathcal{K}_nW_nv_n\| \\
=\ & \lim_{n\to\infty}\Big\|\frac{1}{t_n}\int_0^{t_n} S(s)W_nv_nds - S(h)\Big(\frac{1}{t_n}\int_0^{t_n} S(s)W_nv_nds\Big)\Big\| = 0. \quad (4.62)
\end{aligned}$$

It follows from (4.61) and (4.62), we get

$$\begin{aligned}
&\|v_n - S(s)v_n\| \\
\leq\ & \|v_n - \mathcal{K}_n W_n v_n\| + \|\mathcal{K}_n W_n v_n - S(s)\mathcal{K}_n W_n v_n\| + \|S(s)\mathcal{K}_n W_n v_n - S(s)v_n\| \\
\leq\ & 2\|v_n - \mathcal{K}_n W_n v_n\| + \|\mathcal{K}_n W_n v_n - S(s)\mathcal{K}_n W_n v_n\|.
\end{aligned}$$

So, we have

$$\lim_{n\to\infty} \|v_n - S(s)v_n\| = 0. \tag{4.63}$$

Step 7. Next, we show that $q \in \Theta := F(\mathcal{S}) \cap F(W) \cap (\cap_{k=1}^{N} SMEP(F_k)) \cap I(A, M_1) \cap I(B, M_2) \neq \emptyset$.

Since $\{v_{n_i}\}$ is bounded, there exists a subsequence $\{v_{n_{i_j}}\}$ of $\{v_{n_i}\}$ which converges weakly to $q \in C$. Without loss of generality, we can assume that $v_{n_i} \rightharpoonup q$.

(1) First, we prove that $q \in F(\mathcal{S})$. Indeed, from Lemma 2.42 and (4.63), we get $q \in F(\mathcal{S})$, i.e., $q = S(s)q, \forall s \geq 0$.

(2) We show that $q \in F(W) = \cap_{n=1}^{\infty} F(W_n)$, where $F(W_n) = \cap_{i=1}^{\infty} F(T_i), \forall n \geq 1$ and $F(W_{n+1}) \subset F(W_n)$. Assume that $q \notin F(W)$, then there exists a positive integer m such that $q \notin F(T_m)$ and so $q \notin \cap_{i=1}^{m} F(T_i)$. Hence for any $n \geq m$, $q \notin \cap_{i=1}^{n} F(T_i) = F(W_n)$, i.e., $q \neq W_n q$. This together with $q = S(s)q$, $\forall s \geq 0$ shows $q = S(s)q \neq S(s)W_n q$, $\forall s \geq 0$, therefore we have $q \neq \mathcal{K}_n W_n q$, $\forall n \geq m$. It follows from the Opial's condition and (4.61) that

$$\begin{aligned}
\liminf_{i\to\infty} \|v_{n_i} - q\| &< \liminf_{i\to\infty} \|v_{n_i} - \mathcal{K}_{n_i} W_{n_i} q\| \\
&\leq \liminf_{i\to\infty} (\|v_{n_i} - \mathcal{K}_{n_i} W_{n_i} v_{n_i}\| + \|\mathcal{K}_{n_i} W_{n_i} v_{n_i} - \mathcal{K}_{n_i} W_{n_i} q\|) \\
&\leq \liminf_{i\to\infty} \|v_{n_i} - q\|,
\end{aligned}$$

which is a contradiction. Thus, we get $q \in F(W)$.

(3) We prove that $q \in \cap_{k=1}^{N} SMEP(F_k, \varphi)$. Since $\Im_n^k = \mathcal{K}_{r_k}^{F_k}, k = 1, 2, \ldots, N$ and $u_n^k = \Im_n^k x_n$, we have

$$F_k(\Im_n^k x_n, x) + \varphi(x) - \varphi(\Im_n^k x_n) + \frac{1}{r_k}\left\langle \mathcal{K}'(\Im_n^k x_n) - \mathcal{K}'(\Im_n^{k-1} x_n), \eta(x, \Im_n^k x_n)\right\rangle \geq 0, \forall x \in C.$$

It follows that

$$\frac{1}{r_k}\left\langle \mathcal{K}'(\Im_{n_i}^k x_{n_i}) - \mathcal{K}'(\Im_{n_i}^{k-1} x_{n_i}), \eta(x, \Im_{n_i}^k x_{n_i})\right\rangle \geq -F_k(\Im_{n_i}^k x_{n_i}, x) - \varphi(x) + \varphi(\Im_{n_i}^k x_{n_i}) \quad (4.64)$$

for all $x \in C$. From (4.53) and by conditions (C1)(c) and (C2), we get

$$\lim_{n_i \to \infty} \frac{1}{r_k}\left\langle \mathcal{K}'(\Im_{n_i}^k x_{n_i}) - \mathcal{K}'(\Im_{n_i}^{k-1} x_{n_i}), \eta(x, \Im_{n_i}^k x_{n_i})\right\rangle = 0.$$

By the assumption and by the condition (H1), we know that the function φ and the mapping $x \longmapsto (-F_k(x,y))$ both are convex and lower semicontinuous, hence they are weakly lower semicontinuous.
These together with $\frac{\mathcal{K}'(\Im_{n_i}^k x_{n_i}) - \mathcal{K}'(\Im_{n_i}^{k-1} x_{n_i})}{r_k} \to 0$ and $\Im_{n_i}^k x_{n_i} \rightharpoonup q$, we have

$$\begin{aligned} 0 &= \liminf_{n_i \to \infty} \left\langle \frac{\mathcal{K}'(\Im_{n_i}^k x_{n_i}) - \mathcal{K}'(\Im_{n_i}^{k-1} x_{n_i})}{r_k}, \eta(x, \Im_{n_i}^k x_{n_i})\right\rangle \\ &\geq \liminf_{n_i \to \infty} \{-F_k(\Im_{n_i}^k x_{n_i}, x) - \varphi(x) + \varphi(\Im_{n_i}^k x_{n_i})\}. \end{aligned}$$

Then, we obtain

$$F_k(q, x) + \varphi(x) - \varphi(q) \geq 0, \quad \forall x \in C, \quad \forall k =, 1, 2, \ldots, N. \tag{4.65}$$

Therefore $q \in \cap_{k=1}^N SMEP(F_k, \varphi)$.

(4) Lastly, we prove that $q \in I(A, M_1) \cap I(B, M_2)$.

We observe that A is an $1/\gamma$-Lipschitz monotone mapping and $D(A) = H$. From Lemma 2.26, we know that $M_1 + A$ is maximal monotone. Let $(v, g) \in G(M_1 + A)$ that is, $g - Av \in M_1(v)$. Since $v_{n_i} = J_{M_1, \lambda_{n_i}}(y_{n_i} - \lambda_{n_i} A y_{n_i})$, we have

$$y_{n_i} - \lambda_{n_i} A y_{n_i} \in (I + \lambda_{n_i} M_1)(v_{n_i}),$$

that is,

$$\frac{1}{\lambda_{n_i}}(y_{n_i} - v_{n_i} - \lambda_{n_i} A y_{n_i}) \in M_1(v_{n_i}). \tag{4.66}$$

By virtue of the maximal monotonicity of $M_1 + A$, we have

$$\left\langle v - v_{n_i}, g - Av - \frac{1}{\lambda_{n_i}}(y_{n_i} - v_{n_i} - \lambda_{n_i} A y_{n_i})\right\rangle \geq 0, \tag{4.67}$$

and so

$$\left\langle v - v_{n_i}, g \right\rangle \geq \left\langle v - v_{n_i}, Av + \frac{1}{\lambda_{n_i}}(y_{n_i} - v_{n_i} - \lambda_{n_i} A y_{n_i}) \right\rangle$$
$$= \left\langle v - v_{n_i}, Av - Av_{n_i} + Av_{n_i} - Ay_{n_i} + \frac{1}{\lambda_{n_i}}(y_{n_i} - v_{n_i}) \right\rangle \quad (4.68)$$
$$\geq 0 + \langle v - v_{n_i}, Av_{n_i} - Ay_{n_i} \rangle + \left\langle v - v_{n_i}, \frac{1}{\lambda_{n_i}}(y_{n_i} - v_{n_i}) \right\rangle.$$

By (4.60), $v_{n_i} \rightharpoonup q$ and A is inverse-strongly monotone, we obtain that $\lim_{n\to\infty} \|Ay_n - Av_n\| = 0$ and it follows that

$$\lim_{n_i \longrightarrow \infty} \langle v - v_{n_i}, g \rangle = \langle v - q, g \rangle \geq 0. \quad (4.69)$$

It follows from the maximal monotonicity of $M_1 + A$ that $\theta \in (M_1 + A)(q)$, that is, $q \in I(A, M_1)$. Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to $q \in C$. Without loss of generality, we can assume that $y_{n_i} \rightharpoonup q$. In similar way, we can obtain $q \in I(B, M_2)$, hence $q \in I(A, M_1) \cap I(B, M_2)$

Step 8. Finally, we show that $x_n \longrightarrow z$ and $u_n \longrightarrow z$, where $z = P_\Theta x_0$.

Since Θ is nonempty closed convex subset of H, there exists a unique $z' \in \Theta$ such that $z' = P_\Theta x_0$. Since $z' \in \Theta \subset C_n$ and $x_n = P_{C_n} x_0$, we have

$$\|x_0 - x_n\| \leq \|x_0 - P_{C_n} x_0\| \leq \|x_0 - z'\| \quad (4.70)$$

for all $n \in \mathbb{N}$. From (4.70) and $\{x_n\}$ is bounded, so $\omega_w(x_n) \neq \emptyset$.
By the weakly lower semicontinuous of the norm, we have

$$\|x_0 - z\| \leq \liminf_{n_i \longrightarrow \infty} \|x_0 - x_{n_i}\| \leq \|x_0 - z'\|. \quad (4.71)$$

However, since $z \in \omega_w(x_n) \subset \Theta$, we have

$$\|x_0 - z'\| \leq \|x_0 - P_{C_n} x_0\| \leq \|x_0 - z\|.$$

Using (4.70) and (4.71), we obtain $z' = z$. Thus $\omega_w(x_n) = \{z\}$ and $x_n \rightharpoonup z$. So, we have

$$\|x_0 - z'\| \leq \|x_0 - z\| \leq \liminf_{n \longrightarrow \infty} \|x_0 - x_n\| \leq \limsup_{n \longrightarrow \infty} \|x_0 - x_n\| \leq \|x_0 - z'\|.$$

Thus, we obtain that

$$\|x_0 - z\| = \lim_{n\longrightarrow\infty} \|x_0 - x_n\| = \|x_0 - z'\|.$$

From $x_n \rightharpoonup z$, we obtain $(x_0 - x_n) \rightharpoonup (x_0 - z)$. Using the Kadec-Klee property, we obtain that

$$\|x_n - z\| = \|(x_n - x_0) - (z - x_0)\| \longrightarrow 0 \quad as \quad n \longrightarrow \infty$$

and hence $x_n \longrightarrow z$ in norm. Finally, noticing $\|u_n - z\| = \|\Im_n^N x_n - \Im_n^N z\| \leq \|x_n - z\|$. We also conclude that $u_n \longrightarrow z$ in norm. This completes the proof. □

Theorem 4.46. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $\{F_k : C \times C \to \mathcal{R},\ \ k = 1, 2, \ldots, N\}$ be a finite family of mixed equilibrium functions satisfying conditions (H1)-(H3). Let $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C and let $\{t_n\}$ be a positive real divergent sequence. Let $\{V_i : C \to C\}_{i=1}^{\infty}$ be a countable family of uniformly ξ-strict pseudo-contractions, $\{T_i : C \to C\}_{i=1}^{\infty}$ be the countable family of nonexpansive mappings defined by $T_i x = tx + (1-t)V_i x, \forall x \in C, \forall i \geq 1, t \in [\xi, 1)$, W_n be the W-mapping defined by* (4.38) *and W be a mapping defined by* (4.39) *with $F(W) \neq \emptyset$. Let $A, B : C \to H$ be γ, β-inverse-strongly monotone mapping. Such that*

$$\Theta := F(\mathcal{S}) \cap F(W) \cap (\cap_{k=1}^{N} SMEP(F_k)) \cap VI(C, A) \cap VI(C, B) \neq \emptyset.$$

Let $r_k > 0, k = 1, 2, \ldots, N$, which are constants. Let $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, $\{z_n\}$ and

$\{u_n\}$ be sequences generated by $x_0 \in C$, $C_1 = C$, $x_1 = P_{C_1}x_0$, $u_n \in C$ and

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrary}, \\ u_n = K^{F_N}_{r_{N,n}} K^{F_{N-1}}_{r_{N-1,n}} K^{F_{N-2}}_{r_{N-2,n}} \cdots K^{F_2}_{r_{2,n}} K^{F_1}_{r_{1,n}} x_n, \\ y_n = P_C(u_n - \delta_n B u_n), \\ v_n = P_C(y_n - \lambda_n A y_n), \\ z_n = \alpha_n v_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} S(s)W_n v_n ds, \\ C_{n+1} = \Big\{ z \in C_n : \|z_n - z\|^2 \le \|x_n - z\|^2 - \alpha_n(1-\alpha_n)\Big\| v_n \\ \quad - \frac{1}{t_n}\int_0^{t_n} S(s)W_n v_n ds \Big\|^2 \Big\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{4.72}$$

where $K^{F_k}_{r_k} : C \to C$, $k = 1, 2, \ldots, N$ is the mapping defined by (2.16) *and $\{\alpha_n\}$ be a sequence in* $(0,1)$ *for all $n \in \mathbb{N}$. Assume the following conditions are satisfied:*

(C1) $\eta_k : C \times C \to H$ is L_k-Lipschitz continuous with constant $k = 1, 2, \ldots, N$ such that

- *(a) $\eta_k(x,y) + \eta_k(y,x) = 0, \quad \forall x, y \in C$,*
- *(b) $x \mapsto \eta_k(x,y)$ is affine,*
- *(c) for each fixed $y \in C$, $y \mapsto \eta_k(x,y)$ is sequentially continuous from the weak topology to the weak topology;*

(C2) $\mathcal{K}_k : C \to \mathcal{R}$ is η_k-strongly convex with constant $\sigma_k > 0$ and its derivative $\mathcal{K}'_k$ is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with a Lipschitz constant $\nu_k > 0$ such that $\sigma_k > L_k\nu_k$;

(C3) For each $k \in \{1, 2, \ldots, N\}$ and for all $x \in C$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that for any $y \in C \backslash D_x$,

$$F_k(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r_k}\Big\langle \mathcal{K}'(y) - \mathcal{K}'(x), \eta(z_x, y)\Big\rangle < 0;$$

(C4) $\{\alpha_n\} \subset [c,d]$ *for some* $c,d \in (\xi,1)$;

(C5) $\{\lambda_n\} \subset [a_1,b_1]$ *for some* $a_1,b_1 \in (0,2\gamma]$;

(C6) $\{\delta_n\} \subset [a_2,b_2]$ *for some* $a_2,b_2 \in (0,2\beta]$;

(C7) $\liminf_{n\to\infty} r_{k,n} > 0$ *for each* $k \in 1,2,3,...,N$.

Then, $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $z = P_\Theta x_0$.

Proof. In Theorem 4.45 take $M_i = \varrho_{iC} : H \longrightarrow 2^H$, where $\varrho_{iC} : 0 \longrightarrow [0,\infty]$ is the indicator function of C, that is,

$$\varrho_{iC}(x) = \begin{cases} 0, & x \in C; \\ +\infty, & x \notin C, \end{cases}$$

for $i = 1,2$. Then (2.8) is equivalent to variational inequality problem , that is, to find $\hat{x} \in C$ such that

$$\langle A\hat{x}, y - \hat{x}\rangle \geq 0, \quad \forall y \in C.$$

Again, since $M_i = \varrho_{iC}$, for $i = 1,2$ then

$$J_{M_1,\lambda_n} = P_C = J_{M_2,\delta_n}.$$

So, we have

$$v_n = P_C(y_n - \lambda_n Ay_n) = J_{M_1,\lambda_n}(y_n - \lambda_n Ay_n),$$

and

$$y_n = P_C(u_n - \delta_n Bu_n) = J_{M_2,\delta_n}(u_n - \delta_n Bu_n).$$

Hence, we can obtain the desired conclusion from Theorem 4.45 immediately. □

Next, we consider another class of important mapping:

Definition 4.47. A mapping $S : C \longrightarrow C$ is called *strictly pseudo-contraction* if there exists a constant $0 \leq \kappa < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y\|^2, \quad \forall x,y \in C.$$

If $\kappa = 0$, then S is nonexpansive. In this case, we say that $S : C \longrightarrow C$ is a κ-strictly pseudo-contraction. Putting $B = I - S$. Then, we have

$$\|(I-B)x - (I-B)y\|^2 \leq \|x-y\|^2 + \kappa\|Bx - By\|^2, \quad \forall x, y \in C.$$

Observe that

$$\|(I-B)x - (I-B)y\|^2 = \|x-y\|^2 + \|Bx - By\|^2 - 2\langle x-y, Bx-By\rangle, \quad \forall x, y \in C.$$

Hence, we obtain

$$\langle x-y, Bx-By\rangle \geq \frac{1-\kappa}{2}\|Bx-By\|^2, \quad \forall x, y \in C.$$

Then, B is $\frac{1-\kappa}{2}$-inverse-strongly monotone mapping.

Now, we obtain the following result.

Theorem 4.48. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $\{F_k : C \times C \to \mathcal{R},\ \ k = 1, 2, \ldots, N\}$ be a finite family of mixed equilibrium functions satisfying conditions (H1)-(H3). Let $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C and let $\{t_n\}$ be a positive real divergent sequence. Let $\{V_i : C \to C\}_{i=1}^{\infty}$ be a countable family of uniformly ξ-strict pseudo-contractions, $\{T_i : C \to C\}_{i=1}^{\infty}$ be the countable family of nonexpansive mappings defined by $T_i x = tx + (1-t)V_i x, \forall x \in C, \forall i \geq 1, t \in [\xi, 1)$, W_n be the W-mapping defined by* (4.38) *and W be a mapping defined by* (4.39) *with $F(W) \neq \emptyset$. Let $A, B : C \to H$ be γ, β-inverse-strongly monotone mapping and S_A, S_B be $\kappa_\gamma, \kappa_\beta$-strictly pseudo-contraction mapping of C into C for some $0 \leq \kappa_\gamma < 1,\ 0 \leq \kappa_\beta < 1$ such that*

$$\Theta := F(\mathcal{S}) \cap F(W) \cap \left(\cap_{k=1}^{N} SMEP(F_k)\right) \cap F(S_A) \cap F(S_B) \neq \emptyset.$$

Let $r_k > 0, k = 1, 2, \ldots, N$, which are constants. Let $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, $\{z_n\}$ and

$\{u_n\}$ be sequences generated by $x_0 \in C$, $C_1 = C$, $x_1 = P_{C_1}x_0$, $u_n \in C$ and

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrary}, \\ u_n = K^{F_N}_{r_{N,n}} K^{F_{N-1}}_{r_{N-1,n}} K^{F_{N-2}}_{r_{N-2,n}} \cdots K^{F_2}_{r_{2,n}} K^{F_1}_{r_{1,n}} x_n, \\ y_n = (1-\delta_n)u_n + \delta_n S_B u_n, \\ v_n = (1-\lambda_n)y_n + \lambda_n S_A y_n, \\ z_n = \alpha_n v_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} S(s)W_n v_n ds, \\ C_{n+1} = \Big\{ z \in C_n : \|z_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n(1-\alpha_n)\Big\| v_n \\ \quad - \frac{1}{t_n}\int_0^{t_n} S(s)W_n v_n ds\Big\|^2 \Big\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{4.73}$$

where $K^{F_k}_{r_k} : C \to C$, $k = 1, 2, \ldots, N$ is the mapping defined by (2.16) and $\{\alpha_n\}$ be a sequence in $(0, 1)$ for all $n \in \mathbb{N}$. Assume the following conditions are satisfied:

(C1) $\eta_k : C \times C \to H$ is L_k-Lipschitz continuous with constant $k = 1, 2, \ldots, N$ such that

(a) $\eta_k(x, y) + \eta_k(y, x) = 0, \quad \forall x, y \in C$,

(b) $x \mapsto \eta_k(x, y)$ is affine,

(c) for each fixed $y \in C$, $y \mapsto \eta_k(x, y)$ is sequentially continuous from the weak topology to the weak topology;

(C2) $\mathcal{K}_k : C \to \mathcal{R}$ is η_k-strongly convex with constant $\sigma_k > 0$ and its derivative $\mathcal{K}'_k$ is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with a Lipschitz constant $\nu_k > 0$ such that $\sigma_k > L_k\nu_k$;

(C3) For each $k \in \{1, 2, \ldots, N\}$ and for all $x \in C$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that for any $y \in C \backslash D_x$,

$$F_k(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r_k}\Big\langle \mathcal{K}'(y) - \mathcal{K}'(x), \eta(z_x, y)\Big\rangle < 0;$$

(C4) $\{\alpha_n\} \subset [c, d]$ *for some* $c, d \in (\xi, 1)$;

(C5) $\{\lambda_n\} \subset [a_1, b_1]$ *for some* $a_1, b_1 \in (0, 2\gamma]$;

(C6) $\{\delta_n\} \subset [a_2, b_2]$ *for some* $a_2, b_2 \in (0, 2\beta]$;

(C7) $\liminf_{n\to\infty} r_{k,n} > 0$ *for each* $k \in 1, 2, 3, ..., N$

Then, $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $z = P_\Theta x_0$.

Proof. Taking $A \equiv I - S_A$ and $B \equiv I - S_B$. Then we see that A, B is $\frac{1-\kappa_\gamma}{2}$, $\frac{1-\kappa_\beta}{2}$-inverse-strongly monotone mapping, respectively. We have $F(S_A) = VI(C, A)$ and $F(S_B) = VI(C, B)$. So, we have

$$y_n = P_C(u_n - \delta_n Bu_n) = P_C((1 - \delta_n)u_n + \delta_n S_B u_n) = (1 - \delta_n)u_n + \delta_n S_B u_n \in C.$$

and

$$v_n = P_C(y_n - \lambda_n Ay_n) = P_C((1 - \lambda_n)y_n + \lambda_n S_A y_n) = (1 - \lambda_n)y_n + \lambda_n S_A y_n \in C.$$

By using Theorem 4.46, it is easy to obtain the desired conclusion. □