CHAPTER III

APPROXIMATION OF FIXED POINTS

In computational mathematics, an iterative method attempts to solve a problem (for example, finding the root of an equation or system of equations) by finding successive approximations to the solution starting from an initial guess. This approach is in contrast to direct methods, which attempt to solve the problem by a finite sequence of operations, and in the absence of rounding errors, would deliver an exact solution. Iterative methods are usually the only choice for nonlinear equations. However, iterative methods are often useful even for linear problems involving a large number of variables (sometimes of the order of millions), where direct methods would be prohibitively expensive (and in some cases impossible) even with the best available computing power.

If an equation can be put into the form $Sx = x$, and a solution x is an attractive fixed point of the function S, then one may begin with a point x_1 in the basin of attraction of x, and let $x_{n+1} = Sx_n$ for $n \geq 1$, and the sequence x_n will converge to the solution x. If the function S is continuously differentiable, a sufficient condition for convergence is that the spectral radius of the derivative is strictly bounded by one in a neighborhood of the fixed point. If this condition holds at the fixed point, then a sufficiently small neighborhood (basin of attraction) must exist.

Iteration Approximation of Fixed Point

In constructing an iterative method, it is frequently advantageous to pick and choose among equivalent formulations of the problem. we illustrate this with several important examples; to begin, consider the equation

$$f(x) = 0, \quad f \text{ is real function.} \tag{3.17}$$

In transforming (3.17) into an equivalent fixed point equation of the form

$$Sx = x,$$

we have at least the following possibilities:

(1) $Sx = x - f(x)$ (simplest version),

(2) $Sx = x - \omega f(x)$ (linear relaxation),

(3) $Sx = x - \omega F(g(x))$ (nonlinear relaxation),

(4) $Sx = x - \frac{f(x)}{f'(x)}$ (Newton's method),

(5) $Sx = h^{-1}(f(x) - k(x))$, where $f(x) = h(x) + k(x)$ (splitting method).

Here ω denote a real, nonzero parameter, while F, k and h are suitable function with h^{-1} denoting the inverse function. The corresponding iterative methods derive from

$$x_{n+1} = Sx_n, \quad \forall n \geq 0.$$

For example, linear relaxation leads to

$$x_{n+1} = x_n - \omega f(x_n), \quad \forall n \geq 0.$$

In order to improve convergence, one frequently alters the parameter in some suitable way at each iteration. Then, we would have

$$x_{n+1} = x_n - \omega_n f(x_n), \quad \forall n \geq 0.$$

The classical iteration processes are often used to approximate a fixed point of a nonlinear mapping $S : C \longrightarrow C$ and studied by many researchers.

Mann iteration

In 1953, Mann [24] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Sx_n \tag{3.18}$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0,1]$. Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results was proved by Reich [30]. In an infinite-dimensional Hilbert space, Mann iteration can yield only weak convergence (see [14] and [3]). Attempts to modify the Mann iteration method (3.18) so that strong convergence is guaranteed have recently been made.

Halpern iteration

In 1967, Halpern [16] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)Sx_n \tag{3.19}$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0,1]$ and prove strong convergence theorem under some certain control condition.

Ishikawa iteration

In 1974, Ishikawa [19] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)Sx_n \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Sy_n. \end{cases} \tag{3.20}$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ and $\{\alpha_n\}$ are real sequence in $[0,1]$ and and prove weak convergence theorem under some certain control condition.

In order to find a common element of $F(S) \cap VI(C,B)$, let $S : H \longrightarrow H$ be a nonexpansive mapping, Yamada [42] introduced the following iterative scheme called the *hybrid steepest descent method*:

$$x_{n+1} = Sx_n - \alpha_n \mu BSx_n, \quad \forall n \geq 1, \tag{3.21}$$

where $x_1 = x \in H$, $\{\alpha_n\} \subset (0,1)$, $B : H \longrightarrow H$ be a strongly monotone and Lipschitz continuous mapping and μ is a positive real number. He proved that the sequence $\{x_n\}$ generated by (3.21) converges strongly to the unique solution of the $F(S) \cap VI(C, B)$.

In 2007, Yao et al. [43] introduced the following so-called **viscosity approximation method**:

$$\begin{cases} x_0 = x \in C, \\ y_n = \beta_n x_n + (1 - \beta_n) S x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) y_n, \quad \forall n \geq 0, \end{cases} \tag{3.22}$$

where S is a nonexpansive mapping of C into itself and f is a contraction on C. They obtained a strong convergence theorem under some mild restrictions on the parameters.

On the other hand, for finding an element of $F(S) \cap VI(A, C) \cap EP(F)$, Su et al. [35], introduced the following iterative scheme by the viscosity approximation method in a Hilbert space: $x_1 \in H$

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S P_C(u_n - \lambda_n A u_n), \end{cases} \tag{3.23}$$

for all $n \in \mathbb{N}$, where $\alpha_n \subset [0, 1)$ and $r_n \subset (0, \infty)$ satisfy some appropriate conditions. Furthermore, they proved $\{x_n\}$ and $\{u_n\}$ converge strongly to the same point $z \in F(S) \cap VI(C, A) \cap EP(F)$ where $z = P_{F(S) \cap VI(C,A) \cap EP(F)} f(z)$.

A typical problem is to minimize a quadratic function over the set of the fixed points a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in F} \frac{1}{2}\langle Ax, x \rangle - \langle x, b \rangle, \tag{3.24}$$

where F is the fixed point set of a nonexpansive mapping T on H and b is a given point in H. Assume that A is *strongly positive* on H if there is a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2, \quad \forall x \in H. \tag{3.25}$$

Moreover, it is shown in [25] that the sequence $\{x_n\}$ defince by the scheme

$$x_{n+1} = \epsilon_n \gamma f(x_n) + (1 - \epsilon_n A) S x_n, \tag{3.26}$$

converges strongly to $z = P_{F(S)}(I - A + \gamma f)(z)$. Recently, Plubtieng and Punpaeng [28] proposed the following iterative algorithm:

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \ \forall y \in H, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + (I - \epsilon_n A) S u_n. \end{cases} \tag{3.27}$$

They prove that if the sequence $\{\epsilon_n\}$ and $\{r_n\}$ of parameters satisfy appropriate condition, then the sequences $\{x_n\}$ and $\{u_n\}$ both converge to the unique solution z of the variational inequality

$$\langle (A - \gamma f)z, x - z\rangle \geq 0, \ \ \forall x \in F(S) \cap EP(F), \tag{3.28}$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(S) \cap EP(F)} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{3.29}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Very recently, Chang et al. [8] introduced an iterative scheme for finding a common element of the set of solutions of the equilibrium problem (2.10) and the set of fixed points of an infinite family of nonexpansive mappings in a Hilbert space. Starting with an arbitrary initial $x_0 \in E$, define a sequence $\{x_n\}$,$\{k_n\}$,$\{y_n\}$ and $\{u_n\}$ recursively by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \ \forall y \in E, \\ y_n = P_E(u_n - \lambda_n B u_n), \\ k_n = P_E(y_n - \lambda_n B y_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n k_n, \ \ \forall n \geq 1, \end{cases} \tag{3.30}$$

where $\{W_n\}$ is the sequence generated by (3.38), $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$, $\{\lambda_n\}$ in a sequence in $[a, b] \subset (0, 2\xi)$. They proved that under certain appropriate conditions imposed on $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{r_n\}$, the sequence $\{x_n\}$ and $\{u_n\}$ generated by (3.30) converge strongly to $z \in \cap_{n=1}^{\infty} F(T_n) \cap VI(E, B) \cap EP(F)$, where $z = P_{\cap_{n=1}^{\infty} F(T_n) \cap VI(E,B) \cap EP(F)} f(z)$.

In 2009, Colao et al. [12] introduced and considered an implicit iterative scheme for finding a common element of the set of solutions of the system equilibrium problems and the set of common fixed points of an infinite family of nonexpansive mappings on C. Starting with an arbitrary initial $x_0 \in C$ and defining a sequence $\{z_n\}$ recursively by

$$x_n = \epsilon_n \gamma f(x_n) + (1 - \epsilon_n A) W_n J_{r_{M,n}}^{F_M} J_{r_{M-1,n}}^{F_{M-1}} J_{r_{M-2,n}}^{F_{M-2}} \cdots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1} x_n, \tag{3.31}$$

where $\{\epsilon_n\}$ be a sequences in $(0, 1)$. It is proved [12] that under certain appropriate conditions imposed on $\{\epsilon_n\}$ and $\{r_n\}$, the sequence $\{x_n\}$ generated by (3.31) converges strongly to $z \in \cap_{n=1}^{\infty} F(T_n) \cap \left(\cap_{k=1}^{M} SEP(F_k)\right)$, where z is the unique solution of the variational inequality and which is the optimality condition for the minimization problem.

In 2010, Colao and Marino [11] introduced the following explicit viscosity scheme with respect to W-mappings for an infinite family of nonexpansive mappings

$$x_{n+1} = \epsilon_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \epsilon_n A) W_n J_{r_n}^F x_n. \tag{3.32}$$

They prove that sequence $\{x_n\}$ and $\{J_{r_n}^F\}$ converge strongly to $z \in \cap_{n=1}^{\infty} F(T_n) \cap EP(F)$, where z is an equilibrium point for F and is the unique solution of the variational inequality:

$$\langle \gamma f z - Az, x - z \rangle \leq 0, \;\; \forall x \in \cap_{n=1}^{\infty} F(T_n) \cap EP(F)$$

or, equivalently, the unique solution of the minimization problem

$$\min_{x \in \cap_{n=1}^{\infty} F(T_n) \cap EP(F)} \left[\frac{1}{2} \langle Ax, x \rangle - h(x) \right],$$

where h is a potential function for γf. Recently, Chantarangsi et al.[9] introduced some iterative processes based on the viscosity hybrid steepest descent method for finding a common solutions of a generalized mixed equilibrium problem, the set of fixed points of a nonexpansive mapping and the set of solutions of variational inequality problem in a real Hilbert space.

In 2007, Takahashi et al. [39] proved the following strong convergence theorem for a nonexpansive mapping by using the **shrinking projection method** in mathematical programming. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, they define a sequence $\{x_n\}$ as follows

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ \ \forall n \geq 0, \end{cases} \tag{3.33}$$

where $0 \leq \alpha_n < a < 1$. They proved that the sequence $\{x_n\}$ generated by (3.33) converges weakly to $z \in F(T)$, where $z = P_{F(T)}x_0$.

In 2008, Takahashi and Takahashi [37] introduced the following iterative scheme for finding a common element of the set of solution of generalized equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. They proved the strong convergence theorems under certain appropriate conditions imposed on parameters. Next, Zhang et al. [44] introduced the following new iterative scheme for finding a common element of the set of solution to the problem (2.8) and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Starting with an arbitrary $x_1 = x \in H$, define a sequence $\{x_n\}$ by

$$\begin{cases} y_n = J_{M,\lambda}(x_n - \lambda B x_n), \\ x_{n+1} = \alpha_n x + (1-\alpha_n)Ty_n, \ \ \forall n \geq 1, \end{cases} \tag{3.34}$$

where $J_{M,\lambda} = (I + \lambda M)^{-1}$ is the resolvent operator associated with M and a positive number λ and $\{\alpha_n\}$ is a sequence in the interval $[0, 1]$. Peng et al. [29] introduced the iterative scheme by the viscosity approximation method for finding a common element of the set of solutions to the problem (2.8), the set of solutions of an equilibrium problem, and the set of fixed points of a nonexpansive mapping in a Hilbert space.

In 2009, Saeidi [31] introduced a more general iterative algorithm for finding a common element of the set of solution for a system of equilibrium problems and the set of common fixed points for a finite family of nonexpansive mappings and a nonexpansive semigroup. In 2010, Katchang and Kumam [20] obtained a strong convergence theorem for finding a common element of the set of fixed points of a family of finitely nonexpansive mappings, the set of solutions of a mixed equilibrium problem and the set of solutions of a variational inclusion problem for an inverse-strongly monotone mapping. Let W_n be W-mapping (defined by (3.38)), f be a contraction mapping and A, B be inverse-strongly monotone mappings. Let $J_{M,\lambda} = (I + \lambda M)^{-1}$ be the resolvent operator associated with M and a positive number λ. Starting with arbitrary initial $x_1 \in H$, defined a sequence $\{x_n\}$ by

$$\begin{cases} F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = J_{M,\lambda}(u_n - \lambda A u_n), \\ v_n = J_{M,\lambda}(y_n - \lambda A y_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)W_n v_n, \quad \forall n \geq 1, \end{cases} \tag{3.35}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$, the sequence $\{x_n\}$ generated by (3.35) converges strongly to $p \in \Omega := \cap_{i=1}^{\infty} F(S_i) \cap I(A, M) \cap MEP(F, \varphi)$, where $p = P_\Omega(I - B + \gamma f)p$. Later, Kumam et al. [21] proved a strongly convergence theorem of the iterative sequence generated by the shrinking projection method for finding a common element of the set of solutions of generalized mixed equilibrium problems, the set of fixed points of a finite family of quasi-nonexpansive mappings, and the set of solutions of variational inclusion problems.