CHAPTER II

PRELIMINARIES

In this chapter, we give some definitions, notations, and some useful results that will be used in the later chapters.

2.1 Basic results.

Definition 2.1. Let X be a linear space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A function $\|\cdot\| : X \longrightarrow \mathbb{R}$ is said to be *a norm on* X if it satisfies the following conditions:

(1) $\|x\| \geq 0, \forall x \in X$;

(2) $\|x\| = 0 \Leftrightarrow x = 0$;

(3) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in X$;

(4) $\|\alpha x\| = |\alpha|\|x\|, \forall x \in X$ and $\forall \alpha \in \mathbb{K}$.

Definition 2.2. Let $(X, \|\cdot\|)$ be a normed space.

(1) A sequence $\{x_n\} \subset X$ is said to *converge strongly* in X if there exists $x \in X$ such that $\lim_{n \longrightarrow \infty} \|x_n - x\| = 0$. That is, if for any $\epsilon > 0$ there exists a positive integer N such that $\|x_n - x\| < \epsilon, \forall n \geq N$. We often write $\lim_{n \longrightarrow \infty} x_n = x$ or $x_n \longrightarrow x$ to mean that x is the limit of the sequence $\{x_n\}$.

(2) A sequence $\{x_n\} \subset X$ is said to be *a Cauchy sequence* if for any $\epsilon > 0$ there exists a positive integer N such that $\|x_m - x_n\| < \epsilon, \forall\, m, n \geq N$. That is, $\{x_n\}$ is *a Cauchy sequence* in X if and only if $\|x_m - x_n\| \longrightarrow 0$ as $m, n \longrightarrow \infty$.

Definition 2.3. A normed space X is called *complete* if every Cauchy sequence in X converges to an element in X.

Definition 2.4. A complete normed linear space over field $\mathbb{K}$ is called *a Banach space over* $\mathbb{K}$

Definition 2.5. An element $x \in C$ is said to be a *fixed point* of a mapping $S : C \longrightarrow C$ proved $Sx = x$. The set of all fixed point of S is denoted by $F(S) = \{x \in C : Sx = x\}$.

Definition 2.6. A family $\mathcal{S} = \{S(s) : 0 \leq s \leq \infty\}$ of mappings of C into itself is called a *nonexpansive semigroup* on C if it satisfies the following conditions:

(1) $S(0)x = x$ for all $x \in C$;

(2) $S(s+t) = S(s)S(t)$ for all $s, t \geq 0$;

(3) $\|S(s)x - S(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;

(4) for all $x \in C$, $s \mapsto S(s)x$ is continuous.

We denoted by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S} = \{S(s) : s \geq 0\}$, i.e., $F(\mathcal{S}) = \cap_{s \geq 0} F(S(s))$. It is know that $F(\mathcal{S})$ is closed and convex.

Definition 2.7. Let F and X be linear spaces over the field $\mathbb{K}$.

(1) A mapping $T : F \longrightarrow X$ is called *a linear operator* if $T(x+y) = Tx+Ty$ and $T(\alpha x) = \alpha Tx, \forall x, y \in F$, and $\forall \alpha \in \mathbb{K}$.

(2) A mapping $T : F \longrightarrow \mathbb{K}$ is called *a linear functional on F* if T is a linear operator.

Definition 2.8. A sequence $\{x_n\}$ in a normed spaces is said to *converge weakly* to some vector x if $\lim_{n \longrightarrow \infty} f(x_n) = f(x)$ holds for every continuous linear functional f. We often write $x_n \rightharpoonup x$ to mean that $\{x_n\}$ converge weakly to x.

Definition 2.9. Let F and X be normed spaces over the field $\mathbb{K}$ and $T : X \longrightarrow F$ a linear operator. T is said to be *bounded* on X, if there exists a real number $M > 0$ such that $\|T(x)\| \leq M\|x\|, \forall x \in X$.

Definition 2.10. Sequence $\{x_n\}_{n=1}^{\infty}$ in a normed linear space X is said to be *a bounded sequence* if there exists $M > 0$; such that $\|x_n\| \leq M, \forall n \in \mathbb{N}$.

Definition 2.11. Let F and X be normed spaces over the field $\mathbb{K}$, $T : F \longrightarrow X$ an operator and $c \in F$. We say that T is *continuous at* c if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\|T(x) - T(c)\| < \epsilon$ whenever $\|x - c\| < \delta$ and $x \in F$. If T is continuous at each $x \in F$, then T is said to be *continuous on* F.

Definition 2.12. Let X and Y be normed spaces. The mapping $T : X \longrightarrow Y$ is said to be *completely continuous* if $T(C)$ is a compact subset of Y for every bounded subset C of X.

Definition 2.13. A subset C of a normed linear space X is said to be *convex subset in* X if $\lambda x + (1 - \lambda)y \in C$ for each $x, y \in C$ and for each scalar $\lambda \in [0, 1]$.

2.2 Hilbert spaces

Definition 2.14. The real-value function of two variables $\langle \cdot, \cdot \rangle : X \times X \longrightarrow \mathbb{R}$ is called *inner product* on a real vector space X if it satisfies the following conditions:

(1) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for all $x, y, z \in X$ and all real number α and β;

(2) $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in X$; and

(3) $\langle x, x \rangle \geq 0$ for each $x \in X$ and $\langle x, x \rangle = 0$ if and only if $x = 0$. A *real inner product space* is a real vector space equipped with an inner product.

Definition 2.15. A **Hilbert spaces** is an inner product space which is complete under the norm induced by its inner product.

Definition 2.16. A sequence of points x_n in a Hilbert space H is said to *converge weakly* to a point x in H if $\lim_{n \longrightarrow \infty} \langle x_n, y \rangle = \langle x, y \rangle$ for all $y \in H$. The notation $x_n \rightharpoonup x$ is sometimes used to denote this kind of convergence.

Definition 2.17. The metric (nearest point) projection P_C from a Hilbert space H to a closed convex subset C of H is defined as follows: Given $x \in H$, $P_C x$ is the only point in C with the property

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

Definition 2.18. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H \tag{2.1}$$

Lemma 2.19. *Let H be a real Hilbert space, C a closed convex subset of H. Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0, \forall z \in C.$$

2.3 Variational inequality problem

Definition 2.20. Let $A : C \longrightarrow H$ be a nonlinear mapping. The classical variational inequality which denoted by $VI(C, A)$, is to find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{2.2}$$

Definition 2.21. $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \tag{2.3}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H, y \in C. \tag{2.4}$$

Lemma 2.22. [36]*Let H be Hilbert space, let C be a nonempty closed convex subset of H and let B be a mapping of C into H. Let $u \in C$. Then for $\lambda > 0$,*

$$u \in VI(C, B) \Longleftrightarrow u = P_C(u - \lambda Bu),$$

where P_C is the metric projection of H onto C.

Definition 2.23. Let $A : C \longrightarrow H$ be nonlinear mappings. Then B is called

(1) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C,$$

(2) v-strongly monotone if there exists a positive real number v such that

$$\langle Ax - Ay, x - y\rangle \geq v\|x - y\|^2, \quad \forall x, y \in C,$$

for constant $v > 0$. This implies that

$$\|Ax - Ay\| \geq v\|x - y\|, \tag{2.5}$$

that is, A is v-expansive and when $v = 1$, it is expansive.

(3) ξ-Lipschitz continuous if there exists a positive real number ξ such that

$$\|Ax - Ay\| \leq \xi\|x - y\|, \quad \forall x, y \in C,$$

(4) u-inverse-strongly monotone, if there exists a positive real number u such that

$$\langle Ax - Ay, x - y\rangle \geq u\|Ax - Ay\|^2, \quad \forall x, y \in C, \tag{2.6}$$

Clearly, every u-inverse-strongly monotone map A is $\frac{1}{u}$-Lipschitz continuous,

(5) Let $f : C \longrightarrow C$ is said to be a α-contraction if there exists a coefficient α $(0 < \alpha < 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|, \quad \forall x, y \in C.$$

(6) An operator A is strongly positive on H if there is a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H. \tag{2.7}$$

(7) A set-valued mapping $T : H \longrightarrow 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$.

A monotone mapping $T : H \longrightarrow 2^H$ is maximal if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping.

It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let B be a monotone map of C into H and let $N_E w_1$ be the *normal cone* to C at $w_1 \in E$, i.e.,

$$N_E w_1 = \{w \in H : \langle w_1 - u, w\rangle \geq 0, \forall u \in C\}.$$

Define

$$Tw_1 = \begin{cases} Aw_1 + N_E w_1, & w_1 \in C; \\ \emptyset, & w_1 \notin C. \end{cases}$$

Then T is the maximal monotone and $0 \in Tw_1$ if and only if $w_1 \in VI(C, A)$.

Definition 2.24. Let $\eta : C \times C \to H$ is called Lipschitz continuous, if there exists a constant $L > 0$ such that

$$\|\eta(x, y)\| \leq L\|x - y\|, \ \forall x, y \in C.$$

Let $\mathcal{K} : C \to \mathcal{R}$ be a differentiable functional on a convex set C, which is called:

(1) *η-convex* [17] if

$$\mathcal{K}(y) - \mathcal{K}(x) \geq \Big\langle \mathcal{K}'(x), \eta(y, x)\Big\rangle, \ \forall x, y \in C,$$

where $\mathcal{K}'(x)$ is the Fréchet derivative of $\mathcal{K}$ at x;

(2) *η-strongly convex* [2] if there exists a constant $\sigma > 0$ such that

$$\mathcal{K}(y) - \mathcal{K}(x) - \Big\langle \mathcal{K}'(x), \eta(y, x)\Big\rangle \geq \frac{\sigma}{2}\|x - y\|^2, \ \forall x, y \in C.$$

In particular, if $\eta(x, y) = x - y$ for all $x, y \in C$, then $\mathcal{K}$ is said to be *strongly convex*.

2.4 Variational Inclusion Problem

Let $B : H \longrightarrow H$ be a single-valued nonlinear mapping and $M : H \longrightarrow 2^H$ be a set-valued mapping. The *variational inclusion problem* is to find $\hat{x} \in H$ such that

$$\theta \in B(\hat{x}) + M(\hat{x}), \tag{2.8}$$

where θ is the zero vecter in H. The set of solutions of problem (2.8) is denoted by $I(B, M)$. A set-valued mapping $M : H \longrightarrow 2^H$ is called *monotone* if for all $x, y \in H$, $f \in M(x)$ and $g \in M(y)$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping M is *maximal* if its graph $G(M) := \{(f, x) \in H \times H : f \in M(x)\}$ of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x, f) \in H \times H$, $\langle x-y, f-g\rangle \geq 0$ for all $(y, g) \in G(M)$ imply $f \in M(x)$.

Definition 2.25. Let $M : H \longrightarrow 2^H$ be a set-valued maximal monotone mapping, then the single-valued mapping $J_{M,\lambda} : H \longrightarrow H$ defined by

$$J_{M,\lambda}(\hat{x}) = (I + \lambda M)^{-1}(\hat{x}), \quad \hat{x} \in H \tag{2.9}$$

is called the *resolvent operator* associated with M, where λ is any positive number and I is the identity mapping. The following characterizes the resolvent operator.

(R1) The resolvent operator $J_{M,\lambda}$ is single-valued and nonexpansive for all $\lambda > 0$, that is,

$$\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\| \leq \|x - y\|, \ \forall x, y \in H \ \text{ and } \ \forall \lambda > 0.$$

(R2) The resolvent operator $J_{M,\lambda}$ is 1-inverse-strongly monotone; see([6]), that is,

$$\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\|^2 \leq \langle x - y, J_{M,\lambda}(x) - J_{M,\lambda}(y)\rangle, \ \forall x, y \in H.$$

(R3) The solution of problem (2.8) is a fixed point of the operator $J_{M,\lambda}(I - \lambda B)$ for all $\lambda > 0$; see also ([22]), that is,

$$I(B, M) = F(J_{M,\lambda}(I - \lambda B)), \ \forall \lambda > 0.$$

(R4) If $0 < \lambda \leq 2\beta$, then the mapping $J_{M,\lambda}(I - \lambda B) : H \longrightarrow H$ is nonexpansive.

(R5) $I(B, M)$ is closed and convex.

Lemma 2.26. [6] *Let $M : H \longrightarrow 2^H$ be a maximal monotone mapping and let $B : H \longrightarrow H$ be a Lipshitz continuous mapping. Then the mapping $L = M + B : H \longrightarrow 2^H$ is a maximal monotone mapping.*

Lemma 2.27. [1] *Let C be a closed convex subset of H. Let $\{x_n\}$ be a bounded sequence in H. Assume that*

(1). The weak ω-limit set $\omega_w(x_n) \subset C$,

(2). For each $z \in C$, $\lim_{n \longrightarrow \infty} \|x_n - z\|$ exists.

Then $\{x_n\}$ is weakly convergent to a point in C.

2.5 Equilibrium Problem

Definition 2.28. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \longrightarrow \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{2.10}$$

The set of solutions of (2.10) is denoted by $EP(F)$. Given a mapping $A : C \longrightarrow H$, let $F(x, y) = \langle Ax, y - x \rangle$ for all $x, y \in C$. Then $z \in EP(F)$ if and only if $\langle Tz, y - z \rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality.

Definition 2.29. Let $\{F_i,\ i = 1, 2, \ldots, N\}$ be a finite family of bifunctions from $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The system of equilibrium problems for $\{F_1, F_2, \ldots, F_N\}$ is to find a common element $x \in C$ such that

$$\begin{cases} F_1(x,y) \geq 0, \ \ \forall y \in C, \\ F_2(x,y) \geq 0, \ \ \forall y \in C, \\ \vdots \\ F_N(x,y) \geq 0, \ \ \forall y \in C. \end{cases} \tag{2.11}$$

We denote the set of solutions of (2.11) by $\cap_{i=1}^{N} SEP(F_i)$, where $SEP(F_i)$ is the set of solutions to the equilibrium problems, that is,

$$F_i(x,y) \geq 0, \quad \forall y \in C. \tag{2.12}$$

If $N = 1$, then the problem (2.11) is reduced to the equilibrium problems.

Definition 2.30. For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions (see [4]):

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for any $x, y \in C$;

(A3) F is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t \longrightarrow 0} F(tz + (1-t)x, y) \leq F(x,y);$$

(A4) $F(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

Lemma 2.31. [4] *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into* **R** *satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \textit{ for all } y \in C.$$

Lemma 2.32. [10] *Assume that $F : C \times C \longrightarrow \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $J_r^F : H \longrightarrow C$ as follows:*

$$J_r^F(x) = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C \right\}$$

for all $z \in H$. Then, the following hold:

(1) J_r^F is single-valued;

(2) J_r^F is firmly nonexpansive, that is, for any $x, y \in H$,

$$\left\|J_r^F x - J_r^F y\right\|^2 \leq \left\langle J_r^F x - J_r^F y, x - y\right\rangle;$$

(3) $F(J_r^F) = EP(F)$;

(4) $EP(F)$ is closed and convex.

2.6 Mixed Equilibrium Problem

Let $\Im = \{F_k\}_{k\in\Gamma}$ be a countable family of bifunctions from $C \times C$ to $\mathcal{R}$ where $\mathcal{R}$ is the set of real numbers and Γ is an arbitrary index set. Let $\varphi : C \to \mathcal{R} \cup \{+\infty\}$ be a proper extended real-valued function. The *system of mixed equilibrium problems* is to find $x \in C$ such that

$$F_k(x,y) + \varphi(y) \geq \varphi(x), \ \forall k \in \Gamma, \ \forall y \in C. \tag{2.13}$$

The set of solutions of (2.13) is denoted by $SMEP(F_k, \varphi)$, that is

$$SMEP(F_k, \varphi) = \{x \in C : F_k(x,y) + \varphi(y) \geq \varphi(x), \ \forall k \in \Gamma, \forall y \in C\}. \tag{2.14}$$

If Γ is a singleton, the the problem (2.13) reduces to fined the following *mixed equilibrium problem* (see also Flores-Bazán [13]). For finding $x \in C$ such that

$$F(x,y) + \varphi(y) \geq \varphi(x), \ \forall y \in C. \tag{2.15}$$

The set of solutions of (2.15) is denoted by $MEP(F,\varphi)$.

For solving the system of mixed equilibrium problem, let us assume that function $F_k : C \times C \longrightarrow \mathcal{R},\ k = 1,2,...,N$ satisfies the following conditions:

(H1) F_k is monotone, i.e., $F_k(x,y) + F_k(y,x) \leq 0, \quad \forall x,y \in C$;

(H2) for each fixed $y \in C$, $x \mapsto F_k(x,y)$ is convex and upper semicontinuous;

(H3) for each fixed $x \in C, y \mapsto F_k(x,y)$ is convex.

Lemma 2.33. [7] *Let C be a nonempty closed convex subset of a real Hilbert space H and let φ be a lower semicontinuous and convex functional from C to $\mathcal{R}$. Let F be a bifunction from $C \times C$ to $\mathcal{R}$ satisfying (H1)-(H3). Assume that*

(i) $\eta : C \times C \to H$ is k Lipschitz continuous with constant $k > 0$ such that;

(a) $\eta(x,y) + \eta(y,x) = 0, \quad \forall x,y \in C$,

(b) $\eta(\cdot,\cdot)$ is affine in the first variable,

(c) for each fixed $x \in C$, $y \mapsto \eta(x,y)$ is sequentially continuous from the weak topology to the weak topology,

(ii) $\mathcal{K} : C \to \mathcal{R}$ is η-strongly convex with constant $\sigma > 0$ and its derivative $\mathcal{K}'$ is sequentially continuous from the weak topology to the strong topology;

(iii) for each $x \in C$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that for any $y \in C \backslash D_x$,

$$F(y,z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\Big\langle \mathcal{K}'(y) - \mathcal{K}'(x), \eta(z_x,y)\Big\rangle < 0.$$

For given $r > 0$, Let $K_r^F : C \to C$ be the mapping defined by:

$$K_r^F(x) = \Big\{ y \in C : F(y,z) + \varphi(z) - \varphi(y) + \frac{1}{r}\Big\langle \mathcal{K}'(y) - \mathcal{K}'(x), \eta(z,y)\Big\rangle \geq 0, \quad \forall z \in C \Big\} \tag{2.16}$$

for all $x \in C$. Then the following hold

(1) K_r^F *is single-valued;*

(2) K_r^F *is nonexpansive if* $\mathcal{K}'$ *is Lipschitz continuous with constant* $\nu > 0$ *such that* $\sigma \geq k\nu$*;*

(3) $F(K_r^F) = MEP(F, \varphi)$*;*

(4) $MEP(F, \varphi)$ *is closed and convex.*

Lemma 2.34. [25]. *Let* C *be a nonempty closed convex subset of* H *and let* f *be a contraction of* H *into itself with* $\alpha \in (0,1)$*, and* A *be a strongly positive linear bounded operator on* H *with coefficient* $\bar{\gamma} > 0$*. Then , for* $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$*,*

$$\Big\langle x - y, (A - \gamma f)x - (A - \gamma f)y \Big\rangle \geq (\bar{\gamma} - \alpha\gamma)\|x - y\|^2, \quad x, y \in H.$$

That is, $A - \gamma f$ *is strongly monotone with coefficient* $\bar{\gamma} - \alpha\gamma$*.*

Lemma 2.35. [25]. *Assume* A *be a strongly positive linear bounded operator on* H *with coefficient* $\bar{\gamma} > 0$ *and* $0 < \rho \leq \|A\|^{-1}$*. Then* $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$*.*

Lemma 2.36. *Let* H *be a real Hilbert space. Then the following inequalities hold:*

(1) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$*;*

(2) $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$*;*

(3) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$, $\forall t \in [0,1]$, $\forall x, y \in H$;

for all $x, y \in H$*.*

Lemma 2.37. [36] *Let* H *be a Hilbert space, let* C *be a nonempty closed convex subset of* H*. Let* $\xi > 0$ *and let* $A : C \longrightarrow H$ *be* ξ*-inverse strongly monotone. If* $0 < \varrho \leq 2\xi$*, then* $I - \varrho B$ *is a nonexpansive mapping of* C *into* H*.*

Lemma 2.38. [27] *The* ***Opial's condition****; for any sequence* $\{x_n\}$ *with* $x_n \rightharpoonup x$*, the inequality* $\liminf_{n \longrightarrow \infty} \|x_n - x\| < \liminf_{n \longrightarrow \infty} \|x_n - y\|$*, holds for every* $y \in H$ *with* $y \neq x$*.*

Lemma 2.39. [34] *Let* $\{x_n\}$ *and* $\{l_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$. *Suppose* $x_{n+1} = (1-\beta_n)l_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\longrightarrow\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then,* $\lim_{n\longrightarrow\infty} \|l_n - x_n\| = 0$.

Lemma 2.40. [40] *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-b_n)a_n + c_n, \; n \geq 0,$$

where $\{b_n\}$ *is a sequence in* $(0,1)$ *and* $\{c_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(1) $\sum_{n=1}^{\infty} b_n = \infty$,

(2) $\limsup_{n\longrightarrow\infty} \frac{c_n}{b_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |c_n| < \infty$,

Then $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 2.41. [32] *Let* C *be a nonempty bounded closed convex subset of a Hilbert space* H *and let* $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ *be a nonexpansive semigroup on* C, *then for any* $h \geq 0$,

$$\lim_{t\longrightarrow\infty} \sup_{x\in C} \left\| \frac{1}{t}\int_0^t T(s)xds - T(h)\Big(\frac{1}{t}\int_0^t T(s)xds\Big) \right\| = 0.$$

Lemma 2.42. [38] *Let* C *be a nonempty bounded closed convex subset of* H, $\{x_n\}$ *be a sequence in* C *and* $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ *be a nonexpansive semigroup on* C. *If the following conditions are satisfied:*

(i) $x_n \rightharpoonup z$;

(ii) $\limsup_{s\longrightarrow\infty} \limsup_{n\longrightarrow\infty} \|S(s)x_n - x_n\| = 0$, *then* $z \in F(\mathcal{S})$.