CHAPTER V

MAIN RESULTS

Strong convergence theorem

Theorem 3.22. *Let H be a Hilbert space, $T, S, K : H \longrightarrow H$ a non-expansive mapping satisfy the condition (A') with $F := F(T) \cap F(S) \cap F(K) \neq \emptyset$. Let $f : H \longrightarrow H$ an η_f-strongly monotone and k_f-Lipschitzian mapping, $g : H \longrightarrow H$ an η_g-strongly monotone and k_g-Lipschitzian mapping, $h : H \longrightarrow H$ an η_h-strongly monotone and k_h-Lipschitzian mapping. For any $x_0 \in H$, $\{x_n\}$ is defined by*

$$\begin{cases} z_n = c_n x_n + (1 - c_n) K_h^{\alpha_n} x_n, \\ y_n = b_n x_n + (1 - b_n) S_g^{\beta_n} z_n, \\ x_{n+1} = a_n x_n + (1 - a_n) T_f^{\lambda_{n+1}} y_n, \quad \forall n \geq 0, \end{cases} \tag{3.13}$$

where

$$\begin{aligned} T_f^{\lambda_{n+1}} x &= Tx - \lambda_{n+1} \mu_f f(Tx), \quad \forall x \in H, \\ S_g^{\beta_n} &= Sx - \beta_n \mu_g g(Sx), \quad \forall x \in H, \\ K_h^{\alpha_n} &= Kx - \alpha_n \mu_h h(Kx), \quad \forall x \in H, \end{aligned} \tag{3.14}$$

and $\{a_n\} \subset (0,1)$, $\{b_n\} \subset (0,1)$, $\{c_n\} \subset (0,1)$ and $\{\lambda_n\} \subset [0,1)$, $\{\beta_n\} \subset [0,1)$, $\{\alpha_n\} \subset [0,1)$ satisfying the following conditions:

(i) $\alpha \leq a_n$, $b_n \leq \beta$, $c_n \leq \gamma$ *for some* $\alpha, \beta, \gamma \in (0,1)$,

(ii) $\sum_{n=1}^{\infty} \lambda_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ *and* $\sum_{n=1}^{\infty} \alpha_n < \infty$,

(iii) $0 < \mu_f < 2\eta_f / k_f^2$, $0 < \mu_g < 2\eta_g / k_g^2$ *and* $0 < \mu_h < 2\alpha_h / k_h^2$.

Then $\{x_n\}$ converges strongly to a common fixed point of T, S and K.

Proof. We shall show that $\{x_n\}$ is bounded.

Take $p \in F$, from Lemma 2.18, we have

$$\begin{aligned} \|S_g^{\beta_n} z_n - p\| &= \|S_g^{\beta_n} z_n - S_g^{\beta_n} p + S_g^{\beta_n} p - p\| \\ &\leq (1 - \beta_n \tau_g)\|z_n - p\| + \beta_n \mu_g \|g(p)\|, \end{aligned} \quad (3.15)$$

$$\begin{aligned} \|K_h^{\alpha_n} x_n - p\| &= \|S_h^{\alpha_n} x_n - S_h^{\alpha_n} p + S_h^{\alpha_n} p - p\| \\ &\leq (1 - \alpha_n \tau_h)\|x_n - p\| + \alpha_n \mu_h \|h(p)\| \end{aligned} \quad (3.16)$$

and

$$\begin{aligned} \|T_f^{\lambda_{n+1}} y_n - p\| &= \|T_f^{\lambda_{n+1}} y_n - T_f^{\lambda_{n+1}} p + T_f^{\lambda_{n+1}} p - p\| \\ &\leq \|T_f^{\lambda_{n+1}} y_n - T_f^{\lambda_{n+1}} p\| + \|T_f^{\lambda_{n+1}} p - p\| \\ &\leq (1 - \lambda_{n+1} \tau_f)\|y_n - p\| + \lambda_{n+1} \mu_f \|f(p)\|, \end{aligned} \quad (3.17)$$

where

$$\tau_g = 1 - \sqrt{1 - \mu_g(2\eta_g - \mu_g k_g^2)}, \ \ \tau_f = 1 - \sqrt{1 - \mu_f(2\eta_f - \mu_f k_f^2)}, \ \ \tau_h = 1 - \sqrt{1 - \mu_h(2\eta_h - \mu_h k_h^2)}.$$

Therefore

$$\begin{aligned} \|y_n - p\| &= \|b_n(x_n - p) + (1 - b_n)(S_g^{\beta_n} z_n - p)\| \\ &\leq [b_n + (1 - b_n)(1 - \beta_n \tau_g)]\|z_n - p\| + (1 - b_n)\beta_n \mu_g \|g(p)\|. \end{aligned} \quad (3.18)$$

Now,by (3.16),we have

$$\begin{aligned} \|z_n - p\| &= \|c_n(x_n - p) + (1 - b_n)(K_g^{\beta_n} z_n - p)\| \\ &\leq [c_n + (1 - c_n)(1 - \alpha_n \tau_h)]\|x_n - p\| + (1 - c_n)\alpha_n \mu_g \|h(p)\|. \end{aligned} \quad (3.19)$$

Substitute (3.19) into (3.15) to get

$$\begin{aligned} \|S_g^{\beta_n} z_n - p\| &\leq (1 - \beta_n \tau_g)\Big[[c_n + (1 - c_n)(1 - \alpha_n \tau_h)]\|x_n - p\| + ((1 - c_n)\alpha_n \mu_g \|h(p)\|\Big] \\ &\quad + \beta_n \mu_g \|g(p)\| \\ &\leq (1 - \beta_n \tau_g)\|x_n - p\| + \alpha_n \mu_g \|h(p)\| + \beta_n \mu_g \|g(p)\|. \end{aligned} \quad (3.20)$$

By (3.17),(3.18) and (3.19), we have

$$
\begin{aligned}
\|x_{n+1}-p\| &= \|a_n(x_n-p)+(1-a_n)(T_f^{\lambda_{n+1}}y_n-p)\| \\
&\leq a_n\|x_n-p\|+(1-a_n)\|T_f^{\lambda_{n+1}}y_n-p\| \\
&\leq a_n\|x_n-p\|+(1-a_n)(1-\lambda_{n+1}\tau_f)\|y_n-p\|+(1-a_n)\lambda_{n+1}\mu_f\|f(p)\| \\
&\leq a_n\|x_n-p\|+(1-a_n)(1-\lambda_{n+1}\tau_f)[b_n+(1-b_n)(1-\beta_n\tau)]\|z_n-p\| \\
&\quad +(1-a_n)(1-\lambda_{n+1}\tau_f)(1-b_n)\beta_n\mu_g\|g(p)\|+(1-a_n)\lambda_{n+1}\mu_f\|f(p)\| \\
&\leq a_n\|x_n-p\|+(1-a_n)(1-\lambda_{n+1}\tau_f)\big[b_n+(1-b_n)(1-\beta_n\tau)\big] \\
&\quad \times[c_n+(1-c_n)(1-\alpha_n\tau_h)]\|x_n-p\| \\
&\quad +(1-a_n)(1-\lambda_{n+1}\tau_f)[b_n+(1-b_n)(1-\beta_n\tau)][(1-c_n)\alpha_n\tau_h)]\|h(p)\| \\
&\quad +(1-a_n)(1-\lambda_{n+1}\tau_f)(1-b_n)\beta_n\mu_g\|g(p)\| \\
&\quad +(1-a_n)\lambda_{n+1}\mu_f\|f(p)\| \\
&\leq \|x_n-p\|+(1-a_n)\alpha_n\mu_h\|h(p)\| \\
&\quad +(1-a_n)\beta_n\mu_g\|g(p)\|+(1-a_n)\lambda_{n+1}\mu_f\|f(p)\|, \qquad (3.21)
\end{aligned}
$$

which implies that

$$
\|x_{n+1}-p\| \leq \|x_n-p\|+\alpha_n\mu_h\|h(p)\|+\beta_n\mu_g\|g(p)\|+\lambda_{n+1}\mu_f\|f(p)\|. \qquad (3.22)
$$

From Lemma 2.19 and the conditions: $\sum_{n=1}^{\infty}\alpha_n<\infty, \sum_{n=1}^{\infty}\beta_n<\infty, \sum_{n=1}^{\infty}\lambda_n<\infty$, it follows that $\lim_{n\longrightarrow\infty}\|x_n-p\|$ exists for each $p\in F$ and $\{x_n\}$ is bounded.

Suppose that

$$
\lim_{n\longrightarrow\infty}\|x_n-p\|=c \quad \text{for some} \quad c\geq 0. \qquad (3.23)
$$

By (3.19), we have

$$
\begin{aligned}
\|y_n-p\| &= \|b_n(x_n-p)+(1-b_n)(S_g^{\beta_n}z_n-p)\| \\
&\leq [b_n+(1-b_n)(1-\beta_n\tau_g)]\|z_n-p\|+(1-b_n)\beta_n\mu_g\|g(p)\| \\
&\leq [b_n+(1-b_n)(1-\beta_n\tau_g)][c_n+(1-c_n)(1-\alpha_n\tau_h)]\|x_n-p\| \\
&\quad +(1-c_n)\alpha_n\mu_h\|h(p)\|+(1-b_n)\beta_n\mu_g\|g(p)\| \\
&\leq \|x_n-p\|+\alpha_n\mu_h\|h(p)\|+\beta_n\mu_g\|g(p)\|. \qquad (3.24)
\end{aligned}
$$

Taking lim sup on both the sides in above inequality, we have

$$\limsup_{n \longrightarrow \infty} \|y_n - p\| \leq c. \tag{3.25}$$

Furthermore, by (3.17), we have

$$\limsup_{n \longrightarrow \infty} \|T_f^{\lambda_{n+1}} y_n - p\| \leq c. \tag{3.26}$$

Since $\lim_{n \longrightarrow \infty} \|x_{n+1} - p\| = c$, it follows that

$$\|x_{n+1} - p\| = \|a_n(x_n - p) + (1 - a_n)(T_f^{\lambda_{n+1}} y_n - p)\| \longrightarrow c$$

as $n \longrightarrow \infty$. Thus by Lemma 2.20, we have

$$\lim_{n \longrightarrow \infty} \|x_n - T_f^{\lambda_{n+1}} y_n\| = 0. \tag{3.27}$$

Next, from (3.17), we consider

$$\begin{aligned} \|x_n - p\| &\leq \|x_n - T_f^{\lambda_{n+1}} y_n\| + \|T_f^{\lambda_{n+1}} y_n - p\| \\ &\leq \|x_n - T_f^{\lambda_{n+1}} y_n\| + \|y_n - p\| + \lambda_{n+1} \mu_f \|f(p)\|, \end{aligned} \tag{3.28}$$

which implies that

$$c \leq \liminf_{n \longrightarrow \infty} \|y_n - p\| \leq \limsup_{n \longrightarrow \infty} \|y_n - p\| \leq c,$$

that is,

$$\lim_{n \longrightarrow \infty} \|y_n - p\| = \lim_{n \longrightarrow \infty} \|b_n(x_n - p) + (1 - b_n)(S_g^{\beta_n} z_n - p)\| = c. \tag{3.29}$$

From (3.16), we know that

$$\|S_g^{\beta_n} z_n - p\| \leq \|x_n - p\| + \alpha_n \mu_h \|h(p)\| + \beta_n \mu_g \|g(p)\|$$

which means

$$\limsup_{n \longrightarrow \infty} \|S_g^{\beta_n} z_n - p\| \leq c. \tag{3.30}$$

By Lemma 2.20, (3.29) and (3.30), we obtain

$$\lim_{n \longrightarrow \infty} \|S_g^{\beta_n} z_n - x_n\| = 0. \tag{3.31}$$

We know that $\{x_n\}$ is bounded and $\{g(S(x_n))\}$ is bounded, thus form (3.31) it follows that

$$\begin{aligned} \|x_n - Sx_n\| &\leq \|x_n - S_g^{\beta_n} z_n\| + \|S_g^{\beta_n} z_n - Sx_n\| \\ &\leq \|x_n - S_g^{\beta_n} z_n\| + \beta_n \mu_g \|g(S(x_n))\| \longrightarrow 0. \end{aligned} \tag{3.32}$$

Now, by (3.19), we have

$$\begin{aligned} \|z_n - p\| &= \|c_n(x_n - p) + (1 - c_n)(K_h^{\alpha_n} x_n - p)\| \\ &\leq \|x_n - p\| + (1 - c_n)\alpha_n \mu_h \|h(p)\|. \end{aligned} \tag{3.33}$$

Taking lim sup on both the sides in above inequality, we have

$$\limsup_{n \longrightarrow \infty} \|z_n - p\| \leq c. \tag{3.34}$$

Next, from (3.31), we consider

$$\begin{aligned} \|x_n - p\| &\leq \|x_n - S_g^{\beta_n} z_n\| + \|S_g^{\beta_n} z_n - p\| \\ &\leq \|x_n - S_g^{\beta_n} z_n\| + \|z_n - p\| + \beta_n \mu_g \|g(p)\|, \end{aligned} \tag{3.35}$$

which implies that

$$c \leq \liminf_{n \longrightarrow \infty} \|z_n - p\| \leq \limsup_{n \longrightarrow \infty} \|z_n - p\| \leq c,$$

i.e.

$$\lim_{n \longrightarrow \infty} \|z_n - p\| = \lim_{n \longrightarrow \infty} \|c_n(x_n - p) + (1 - c_n)(K_h^{\alpha_n} x_n - p)\| = c. \tag{3.36}$$

From (3.16), we know that

$$\|K_h^{\alpha_n} x_n - p\| \leq \|x_n - p\| + \alpha_n \mu_h \|h(p)\|$$

which means

$$\limsup_{n \longrightarrow \infty} \|K_h^{\alpha_n} x_n - p\| \leq c. \tag{3.37}$$

By Lemma 2.20, (3.36) and (3.37), we obtain

$$\lim_{n \longrightarrow \infty} \|K_h^{\alpha_n} x_n - x_n\| = 0. \tag{3.38}$$

We know that $\{x_n\}$ is bounded and $\{h(K(x_n))\}$ is bounded, thus form (3.38) it follows that

$$\begin{aligned} \|x_n - Kx_n\| &\leq \|x_n - K_h^{\alpha_n} x_n\| + \|K_h^{\alpha_n} x_n - Kx_n\| \\ &\leq \|x_n - K_h^{\alpha_n} x_n\| + \alpha_n \mu_h \|h(K(x_n))\| \longrightarrow 0. \end{aligned} \tag{3.39}$$

Moreover, from (3.27) and (3.31), it follows that

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|Tx_n - T_f^{\lambda_{n+1}} y_n\| + \|T_f^{\lambda_{n+1}} y_n - x_n\| \\ &= \|Tx_n - [Ty_n - \lambda_{n+1}\mu_f f(T(y_n))]\| + \|T_f^{\lambda_{n+1}} y_n - x_n\| \\ &\leq \|x_n - y_n\| + \lambda_{n+1}\mu_f \|f(T(y_n))\| + \|T_f^{\lambda_{n+1}} y_n - x_n\| \\ &\leq (1 - b_n)\|x_n - S_g^{\beta_n} z_n\| + \lambda_{n+1}\mu_f \|f(T(y_n))\| + \|T_f^{\lambda_{n+1}} y_n - x_n\| \end{aligned} \tag{3.40}$$

From (3.23) if $c = 0$, there is nothing to prove. Suppose $c > 0$. By (3.40), we know that $\lim_{n\longrightarrow\infty} \|x_n - Tx_n\| = \lim_{n\longrightarrow\infty} \|x_n - Kx_n\| = \lim_{n\longrightarrow\infty} \|x_n - Sx_n\| = 0$. Since T, S, K satisfy the condition (A'), then $f(d(x_n, F)) \leq (1/3)(\|x_n - Tx_n\| + \|x_n - Sx_n\| + \|x_n - Kx_n\|)$. By (3.32), (3.39) and (3.40), we have $\lim_{n\longrightarrow\infty} f(d(x_n, F)) = 0$. Since f is a non-decreasing function and $f(0) = 0$, therefore

$$\liminf_{n\longrightarrow\infty} d(x_n, F) = 0. \tag{3.41}$$

For any $p \in F$, we have

$$\|f(p)\| \leq \|f(p) - f(x_n)\| + \|f(x_n)\| \leq k_f \|x_n - p\| + \|f(x_n)\|, \tag{3.42}$$

$$\|g(p)\| \leq \|g(p) - g(x_n)\| + \|g(x_n)\| \leq k_g \|x_n - p\| + \|g(x_n)\|, \tag{3.43}$$

$$\|h(p)\| \leq \|h(p) - h(x_n)\| + \|h(x_n)\| \leq k_h \|x_n - p\| + \|h(x_n)\|. \tag{3.44}$$

Note the fact that there exist two positive constants M_1, M_2, such that $\|h(x_n)\| \leq M_1$, $\|g(x_n)\| \leq M_2$ and $\|f(x_n)\| \leq M_3$. From (3.22) and the above relations, it follows that

$$\begin{aligned}\|x_{n+1}-p\| &\leq \|x_n-p\|+\alpha_n\mu_h\|h(p)\|+\beta_n\mu_g\|g(p)\|+\lambda_{n+1}\mu_f\|f(p)\| \\ &\leq (1+\alpha_n\mu_h k_h+\beta_n\mu_g k_g+\lambda_{n+1}\mu_f k_f)\|x_n-p\| \\ &\quad +\alpha_n\mu_h\|h(p)\|+\beta_n\mu_g\|g(p)\|+\lambda_{n+1}\mu_f\|f(p)\| \\ &\leq (1+\alpha_n\mu_h k_h+\beta_n\mu_g k_g+\lambda_{n+1}\mu_f k_f)\|x_n-p\| \\ &\quad +\alpha_n\mu_h M_1+\beta_n\mu_g M_2+\lambda_{n+1}\mu_f M_3. \end{aligned} \tag{3.45}$$

Thus

$$d(x_{n+1},F)\leq(1+\alpha_n\mu_h k_h+\beta_n\mu_g k_g+\lambda_{n+1}\mu_f k_f)d(x_n,F)+\alpha_n\mu_h M_1+\beta_n\mu_g M_2+\lambda_{n+1}\mu_f M_3.$$

Since $\sum_{n=0}^{\infty}\alpha_n<\infty$, $\sum_{n=1}^{\infty}\beta_n<\infty$ and $\sum_{n=1}^{\infty}\lambda_n<\infty$, by (3.41), we know that $\lim_{n\longrightarrow\infty}d(x_n,F)=0$. We now prove that $\{x_n\}$ is a Cauchy sequence.

Taking $M=e^{\sum_{i=0}^{\infty}(\alpha_i\mu_h k_h+\beta_i\mu_g k_g+\lambda_{n+1}\mu_f k_f)}$, for any $\varepsilon>0$, there exists positive integer N such that $d(x_n,F)<\varepsilon/(2M)$ and $\sum_{i=N}^{\infty}(\alpha_i\mu_h M_1+\beta_i\mu_g M_2+\lambda_{i+1}\mu_f M_3)<\varepsilon/(2M)$. Let $p\in F$, for any $n,m\geq N$, it follows from (3.45) that

$$\begin{aligned}\|x_{n+1}-x_{m+1}\| &\leq \|x_{n+1}-p\|+\|x_{m+1}-p\| \\ &\leq (1+\alpha_n\mu_h k_h+\beta_n\mu_g k_g+\lambda_{n+1}\mu_f k_f)\|x_n-p\|+\alpha_n\mu_h M_1+\beta_n\mu_g M_2 \\ &\quad +\lambda_{n+1}\mu_f M_3-(1+\alpha_m\mu_h k_h+\beta_m\mu_g k_g+\lambda_{m+1}\mu_f k_f)\|x_m-p\| \\ &\quad +\alpha_m\mu_h M_1+\beta_m\mu_g M_2+\lambda_{m+1}\mu_f M_3\end{aligned}$$

$$\begin{aligned}
\|x_{n+1} - x_{m+1}\| &\leq \prod_{i=1}^{N}(1 + \alpha_i\mu_h k_h + \beta_i\mu_g k_g + \lambda_{i+1}\mu_f k_f)\|x_n - p\| + \alpha_n\mu_h M_1 + \beta_n\mu_g M_2 \\
&+ \lambda_{n+1}\mu_f M_3 + \sum_{i=N}^{n-1}(\alpha_i\mu_h M_1 + \beta_i\mu_g M_2 + \lambda_{i+1}\mu_f M_3) \times \\
&\prod_{j=i+1}^{n}(1 + \alpha_j\mu_h k_h + \beta_j\mu_g k_g + \lambda_{j+1}\mu_f k_f) \\
&+ \prod_{i=N}^{m}(1 + \alpha_i\mu_h k_h + \beta_i\mu_g k_g + \lambda_{i+1}\mu_f k_f)\|x_N - p\| + \alpha_m\mu_h M_1 \\
&+ \beta_m\mu_g M_2 + \lambda_{m+1}\mu_f M_3 + \sum_{i=N}^{m-1}(\alpha_i\mu_h M_1 + \beta_i\mu_g M_2 + \lambda_{i+1}\mu_f M_3) \times \\
&\prod_{j=i+1}^{m}(1 + \alpha_j\mu_h k_h + \beta_j\mu_g k_g + \lambda_{j+1}\mu_f k_f) \\
&\leq 2e^{\sum_{i=N}^{\infty}(\alpha_i\mu_h k_h + \beta_i\mu_g k_g + \lambda_{j+1}\mu_f k_f))}\|x_N - p\| \\
&+ 2e^{\sum_{i=N}^{\infty}(\alpha_i\mu_h k_h + \beta_i\mu_g k_g + \lambda_{j+1}\mu_f k_f))}\|x_N - p\| \\
&\times \sum_{i=N}^{\infty}(\alpha_i\mu_h M_1 + \beta_i\mu_g M_2 + \lambda_{i+1}\mu_f M_3) \qquad (3.46)
\end{aligned}$$

Thus

$$\|x_{n+1} - x_{m+1}\| \leq 2M\|x_N - p\| + 2M\sum_{i=N}^{\infty}(\alpha_i\mu_h M_1 + \beta_i\mu_g M_2 + \lambda_{i+1}\mu_f M_3),$$

which gives

$$\|x_{n+1} - x_{m+1}\| \leq 2Md(x_N - p) + 2M\sum_{i=N}^{\infty}(\alpha_i\mu_h M_1 + \beta_i\mu_g M_2 + \lambda_{i+1}\mu_f M_3) < \varepsilon.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Therefore, there exists $p \in H$ such that $\{x_n\}$ converges strongly to p. It follows from $\|x_n - Tx_n\| \longrightarrow 0$ and $(I - T)$ being continuous that

$$\|(I - T)(x_n - p)\| \longrightarrow 0$$

as $n \longrightarrow \infty$ which implies $p = Tp$. Hence $p \in F$. By the same reasoning, we have $p \in F$. The proof is completed □