CHAPTER IV

SOME METHODS OF ITERATIVE APPROXIMATION

Iteration Approximation of Fixed Point

The interest and importance of construction of fixed points of nonexpansive mappings stem mainly from the fact that it may be applied in many areas, such as imagine recovery and signal processing, solving convex minimization problems. Iterative techniques for approximating fixed points of nonexpansive mappings have been studied by various authors, using famous Mann iteration method, Ishikawa iteration method, and many other iteration methods.

Let D be a subset of a Hilbert space H. Recall that two mappings $S, T : D \longrightarrow D$ are said to satisfy condition (A') which is given in [10] if there exists a nondecreasing function $f : [0, \infty) \longrightarrow [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $(1/2)(\|x - Tx\| + \|x - Sx\|) \geq f(d(x, f))$ for all $x \in D$, where $d(x, \mathcal{F}) = \inf\{\|x - x^*\| : x^* \in \mathcal{F} = F(T) \cap F(S)\}$. We modify this condition for three mappings $S, T, K : C \longrightarrow C$ as follows:

Three mappings $S, T, K : C \longrightarrow C$ where C a subset of H, are said to satisfy condition (A'') if there exists a nondecreasing function $f : [0, \infty) \longrightarrow [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $(1/3)(\|x - Tx\| + \|x - Sx\| + \|x - Kx\|) \geq f(d(x, f))$ for all $x \in C$, where $d(x, \mathcal{F}') = \inf\{\|x - x^*\| : x^* \in \mathcal{F}' = F(T) \cap F(S) \cap F(K)\}$. Note that condition (A'') reduces to condition (A') when $K = S$.

Let $f : H \longrightarrow H$ be a nonlinear mapping and C a nonempty closed convex subset of H. The variational inequality problem with a mapping f on C ($VI(C, f)$

in short) is formulated as finding a point $u^* \in C$ such that

$$\langle f(u^*), v - u^* \rangle \geq 0, \quad \forall v \in C. \tag{2.4}$$

The variational inequalities were initially studied by Kinderlehrer and Stampacchia [11], and ever since have been widely studied. It is well known that the $VI(C, f)$ is equivalent to the fixed point equation

$$u^* = P_C(u^* - \mu f(u^*)), \tag{2.5}$$

where P_C is the projection from H onto C and μ is an arbitrarily fixed constant. In fact, when f is an η-strongly monotone and Lipschitzian mapping on C and $\mu > 0$ small enough, then the mapping defined by the right hand side of (2.5) is a contraction.

Mann iteration

In 1953, Mann [12] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S x_n \tag{2.6}$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0, 1]$. Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results was proved by Reich [18]. In an infinite-dimensional Hilbert space, Mann iteration can yield only weak convergence (see [5] and [1]). Attempts to modify the Mann iteration method (2.6) so that strong convergence is guaranteed have recently been made.

Halpern iteration

In 1967, Halpern [7] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) S x_n \tag{2.7}$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0, 1]$ and prove strong convergence theorem under some certain control condition.

Ishikawa iteration

In 1974, Ishikawa [9] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)S[\beta_n x_n + (1 - \beta_n)Sx_n], \ \forall n \geq 0, \tag{2.8}$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ and $\{\alpha_n\}$ are real sequence in $[0, 1]$ and and prove weak convergence theorem under some certain control condition.

In 2007, Yao et al. [23] introduced the following so-called **viscosity approximation method**:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)[\beta_n x_n + (1 - \beta_n)Sx_n], \ \forall n \geq 0, \tag{2.9}$$

where S is a nonexpansive mapping of C into itself and f is a contraction on C. They obtained a strong convergence theorem under some mild restrictions on the parameters.

Recently, Wang [27] discussed the more general Mann iterative scheme as follows: Let H be a Hilbert space, $T : H \longrightarrow H$ a nonexpansive mapping with $F(T) := \{x \in H, Tx = x\} \neq \emptyset$, and $f : H \longrightarrow H$ an η-strongly monotone and k-Lipschitzian mapping. For any $x_0 \in H$, $\{x_n\}$ is defined by

$$x_{n+1} = a_n x_n + (1 - a_n)T^{\lambda_{n+1}} x_n, \quad \forall n \geq 0, \tag{2.10}$$

where

$$T^{\lambda} x = Tx - \lambda\mu f(Tx), \quad \forall x \in H, \tag{2.11}$$

where $\{a_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset [0, 1)$ with some suitable conditions. Then the sequence $\{x_n\}$ is shown to converge strongly to a fixed point of T, and the necessary and sufficient conditions that $\{x_n\}$ converges strongly to a fixed point of T are obtained.

Theorem I [9]. If C is a compact convex subset of a Hilbert space $H, T : C \longrightarrow C$ is a Lipschitzian pseudo-contractive mapping. For $x_0 \in C$, define the sequence $\{x_n\}$ iteratively by

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \quad \forall n \geq 0, \end{aligned} \tag{2.12}$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences of positive numbers satisfying the conditions

(i) $0 \leq \alpha_n \leq \beta_n < 1$;

(ii) $\lim_{n \longrightarrow \infty} \beta_n = 0$;

(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

Then the sequence $\{x_n\}$ defined by (2.12) converges strongly to a fixed point of T.