CHAPTER II

PRELIMINARIES

In this chapter, we give some definitions, notations, and some useful results that will be used in the later chapters.

2.1 Basic results.

Definition 2.1. Let X be a linear space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A function $\|\cdot\| : X \longrightarrow \mathbb{R}$ is said to be *a norm on* X if it satisfies the following conditions:

(1) $\|x\| \geq 0, \forall x \in X$;

(2) $\|x\| = 0 \Leftrightarrow x = 0$;

(3) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in X$;

(4) $\|\alpha x\| = |\alpha|\|x\|, \forall x \in X$ and $\forall \alpha \in \mathbb{K}$.

Definition 2.2. Let $(X, \|\cdot\|)$ be a normed space.

(1) A sequence $\{x_n\} \subset X$ is said to *converge strongly* in X if there exists $x \in X$ such that $\lim_{n \longrightarrow \infty} \|x_n - x\| = 0$. That is, if for any $\epsilon > 0$ there exists a positive integer N such that $\|x_n - x\| < \epsilon, \forall n \geq N$. We often write $\lim_{n \longrightarrow \infty} x_n = x$ or $x_n \longrightarrow x$ to mean that x is the limit of the sequence $\{x_n\}$.

(2) A sequence $\{x_n\} \subset X$ is said to be *a Cauchy sequence* if for any $\epsilon > 0$ there exists a positive integer N such that $\|x_m - x_n\| < \epsilon, \forall\, m, n \geq N$. That is, $\{x_n\}$ is *a Cauchy sequence* in X if and only if $\|x_m - x_n\| \longrightarrow 0$ as $m, n \longrightarrow \infty$.

Definition 2.3. A normed space X is called *complete* if every Cauchy sequence in X converges to an element in X.

Definition 2.4. An element $x \in C$ is said to be a *fixed point* of a mapping $S : C \longrightarrow C$ proved $Sx = x$. The set of all fixed point of S is denoted by $F(S) = \{x \in$

$C : Sx = x\}$.

Definition 2.5. Let F and X be linear spaces over the field $\mathbb{K}$.

(1) A mapping $T : F \longrightarrow X$ is called *a linear operator* if $T(x+y) = Tx+Ty$ and $T(\alpha x) = \alpha Tx, \forall x, y \in F$, and $\forall \alpha \in \mathbb{K}$.

(2) A mapping $T : F \longrightarrow \mathbb{K}$ is called *a linear functional on F* if T is a linear operator.

Definition 2.6. Let F and X be normed spaces over the field $\mathbb{K}$ and $T : X \longrightarrow F$ a linear operator. T is said to be *bounded* on X, if there exists a real number $M > 0$ such that $\|T(x)\| \leq M\|x\|, \forall x \in X$.

Definition 2.7. Sequence $\{x_n\}_{n=1}^{\infty}$ in a normed linear space X is said to be *a bounded sequence* if there exists $M > 0$; such that $\|x_n\| \leq M, \forall n \in \mathbb{N}$.

Definition 2.8. Let F and X be normed spaces over the field $\mathbb{K}$, $T : F \longrightarrow X$ an operator and $c \in F$. We say that T is *continuous at c* if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\|T(x) - T(c)\| < \epsilon$ whenever $\|x - c\| < \delta$ and $x \in F$. If T is continuous at each $x \in F$, then T is said to be *continuous on F*.

Definition 2.9. A subset C of a normed linear space X is said to be *convex subset in X* if $\lambda x + (1 - \lambda)y \in C$ for each $x, y \in C$ and for each scalar $\lambda \in [0, 1]$.

2.2 Hilbert spaces

Definition 2.10. The real-value function of two variables $\langle \cdot, \cdot \rangle : X \times X \longrightarrow \mathbb{R}$ is called *inner product* on a real vector space X if it satisfies the following conditions:

(1) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for all $x, y, z \in X$ and all real number α and β;

(2) $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in X$; and

(3) $\langle x, x\rangle \geq 0$ for each $x \in X$ and $\langle x, x\rangle = 0$ if and only if $x = 0$. A *real inner product space* is a real vector space equipped with an inner product.

Definition 2.11. A Hilbert spaces is an inner product space which is complete under the norm induced by its inner product.

Definition 2.12. The metric (nearest point) projection P_C from a Hilbert space H to a closed convex subset C of H is defined as follows: Given $x \in H$, $P_C x$ is the only point in C with the property

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

Definition 2.13. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H \tag{2.1}$$

Definition 2.14. Let X be a normed space, $\{x_n\} \subset X$ and $f : X \longrightarrow (-\infty, \infty]$. Then f is said to be

1) *lower semicontinuous* on X if for any $x_0 \in X$, $f(x_0) \leqslant \liminf_{n\longrightarrow\infty} f(x_n)$ whenever $x_n \longrightarrow x_0$.

2) *upper semi (or hemi) continuous* on X if for any $x_0 \in X$, $\limsup_{n\longrightarrow\infty} f(x_n) \leqslant f(x_0)$ whenever $x_n \longrightarrow x_0$.

3) *weakly lower semicontinuous* on X if for any $x_0 \in X$, $f(x_0) \leqslant \liminf_{n\longrightarrow\infty} f(x_n)$ whenever $x_n \rightharpoonup x_0$.

4) *weakly upper semicontinuous* on X if for any $x_0 \in X$, $\limsup_{n\longrightarrow\infty} f(x_n) \leqslant f(x_0)$ whenever $x_n \rightharpoonup x_0$.

Definition 2.15. Let X be a normed space. A mapping $T : X \longrightarrow X$ is said to be *Lipschitzian* if there exists a constant $k \geq 0$ such that for all $x, y \in X$,

$$\|Tx - Ty\| \leq k\|x - y\|. \tag{2.2}$$

The smallest number k for which (2.2) holds is called the *Lipschitz constant* of T and T is called *a contraction (nonexpansive mapping)* if $k \in (0, 1)$ (k=1).

Definition 2.16. An element $x \in X$ is said to be

1) a *fixed point* of a mapping $T : X \longrightarrow X$ provided $Tx = x$.

2) a *common fixed point* of two mappings $S, T : X \longrightarrow X$ provided $Sx = x = Tx$. The set of all fixed points of T is denoted by $F(T)$.

Lemma 2.17. *Let H be a real Hilbert space, C a closed convex subset of H. Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \geq 0, \forall z \in C.$$

Lemma 2.18. [28] *Let $T^\lambda x = Tx - \lambda\mu f(Tx)$, where $T : H \longrightarrow H$ is a nonexpansive mapping from H into itself and f is an η-strongly monotone and k-Lipschitzian mapping from H into itself. If $0 \leq \lambda < 1$ and $0 < \mu < 2\eta/k^2$, then T^λ is a contraction and satisfies*

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in H, \tag{2.3}$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$.

Lemma 2.19. [25] *Let $\{a_n\}$, $\{b_n\}$ and δ_n be sequences of nonnegative real numbers satisfying the inequality,*

$$a_{n+1} \leq (1 - \delta_n)a_n + b_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\longrightarrow\infty} a_n$ exists. If in addition, $\{a_n\}$ has a subsequence which converges strongly to zero, then $\lim_{n\longrightarrow\infty} a_n \longrightarrow 0$.

Lemma 2.20. [26] *Suppose E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers n. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequence of E such that $\limsup_{n \longrightarrow \infty} \|x_n\| \leq r$, $\limsup_{n \longrightarrow \infty} \|y_n\| \leq r$ and $\lim_{n \longrightarrow \infty} \|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n \longrightarrow \infty} \|x_n - y_n\| = 0$.*

Lemma 2.21. [24] *Let C be a nonempty closed convex subset of a real Hilbert space H and T a nonexpansive mapping from C into itself. If T has a fixed point, then $I - T$ is demiclosed at zero, where I is the identity mapping of H, that is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y, it follows that $(I-T)x = y$.*