บทที่ 3
วิธีการดำเนินการวิจัย

1. ระบบสมการเชิงอนุพันธ์ย่อยนาเวียร์ - สโตกส์

ปัญหาการไหลที่ไม่อัดตัวแบบมีความหนืดและขึ้นกับเวลา (Time Dependent) อธิบายได้ด้วยสมการดังต่อไปนี้ [22]

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{3.1}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)-\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial p}{\partial x}=0 \tag{3.2}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)-\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\frac{\partial p}{\partial y}=0 \tag{3.3}$$

โดยที่ u, v แทนความเร็วในแนวแกน x และ y ตามลำดับ มีหน่วยเป็น เมตร/วินาที (m/s)

p แทนความดันมีหน่วยเป็น นิวตัน/ตารางเมตร (N/m^2)

t แทนเวลามีหน่วยเป็น วินาที (s)

ρ แทนความหนาแน่นมีหน่วยเป็น กิโลกรัม/ลูกบาศก์เมตร (Kg/m^3)

μ แทนความหนืดพลศาสตร์มีหน่วยเป็น นิวตัน-วินาที/ตารางเมตร $(N\cdot s/m^2)$

ทำการแปลงตัวแปรต่างๆให้อยู่ในรูปแบบไร้มิติ (Dimensionless Form) [20] โดยกำหนดให้

$$x^*=\frac{x}{L},\ y^*=\frac{y}{L},\ t^*=\frac{tu_a}{L},\ u^*=\frac{u}{u_a},\ v^*=\frac{v}{u_a},\ p^*=\frac{p}{\rho u_a^{\,2}},\ \mathrm{Re}=\frac{\rho u_a L}{\mu} \tag{3.4}$$

โดยที่ L แทนความยาวหรือความกว้างของโดเมน (Characterristic of Dimension)

u_a แทนความเร็วเฉลี่ย (Everage of Velocity)

$*$ แทนตัวแปรที่อยู่ในรูปแบบไร้มิติ (Non - Dimension)

Re แทนเลขเรย์โนลด์

1.1 สมการอนุรักษ์มวลในรูปแบบไร้มิติ

จากข้อกำหนด (3.4) จะได้

$$\begin{aligned}&\frac{\partial u}{\partial x^*}=u_a\frac{\partial u^*}{\partial x^*},\ \frac{\partial x^*}{\partial x}=\frac{1}{L},\ \frac{\partial u}{\partial t^*}=u_a\frac{\partial u^*}{\partial t^*},\ \frac{\partial p}{\partial x^*}=\rho u_a^{\,2}\frac{\partial p^*}{\partial x^*}\\&\frac{\partial v}{\partial y^*}=u_a\frac{\partial v^*}{\partial y^*},\ \frac{\partial y^*}{\partial y}=\frac{1}{L},\ \frac{\partial v}{\partial t^*}=u_a\frac{\partial v^*}{\partial t^*}\end{aligned} \tag{3.5}$$

จากสมการ (3.5) พิจารณา

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial x^*}\cdot\frac{\partial x^*}{\partial x}=\frac{u_a}{L}\frac{\partial u^*}{\partial x^*} \quad \text{และ} \quad \frac{\partial v}{\partial y}=\frac{\partial v}{\partial y^*}\cdot\frac{\partial y^*}{\partial y}=\frac{u_a}{L}\frac{\partial v^*}{\partial y^*} \tag{3.6}$$

ดังนั้นแทนค่าสมการ (3.6) ลงในสมการ (3.1) และนำ $\frac{L}{u_a}$ หารตลอด จะได้สมการอนุรักษ์มวลในรูปแบบไร้มิติ

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}=0 \tag{3.7}$$

1.2 สมการอนุรักษ์โมเมนตัมในรูปแบบไร้มิติ

1.2.1 สมการอนุรักษ์โมเมนตัม (3.2) ในรูปแบบไร้มิติ

พิจารณา

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial y^*}\cdot\frac{\partial y^*}{\partial y}=\frac{u_a}{L}\frac{\partial u^*}{\partial y^*},\quad \frac{\partial v}{\partial x}=\frac{\partial v}{\partial x^*}\cdot\frac{\partial x^*}{\partial x}=\frac{u_a}{L}\frac{\partial v^*}{\partial x^*}$$

$$\frac{\partial p}{\partial x}=\frac{\partial p}{\partial x^*}\cdot\frac{\partial x^*}{\partial x}=\frac{\rho \mathrm{u}_a^{\,2}}{L}\frac{\partial p^*}{\partial x^*},\quad \frac{\partial p}{\partial y}=\frac{\partial p}{\partial y^*}\cdot\frac{\partial y^*}{\partial y}=\frac{\rho \mathrm{u}_a^{\,2}}{L}\frac{\partial p^*}{\partial y^*},$$

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial t^*}\cdot\frac{\partial t^*}{\partial t}=\frac{u_a^{\,2}}{L}\frac{\partial u^*}{\partial t^*},\quad \frac{\partial v}{\partial t}=\frac{\partial v}{\partial t^*}\cdot\frac{\partial t^*}{\partial t}=\frac{u_a^{\,2}}{L}\frac{\partial v^*}{\partial t^*},$$

และ

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial y}{\partial y}\right) \tag{3.8}$$

แทนค่าสมการ (3.4), (3.6) และ (3.8) ลงในสมการ (3.2) จะได้

$$\rho\left(\frac{u_a^{\,2}}{L}\frac{\partial u^*}{\partial t^*}+\left(u^*u_a\right)\frac{u_a}{L}\frac{\partial u^*}{\partial x^*}+\left(v^*u_a\right)\frac{u_a}{L}\frac{\partial u^*}{\partial y^*}\right)-\mu\left(\frac{\partial}{\partial x}\left(\frac{u_a}{L}\frac{\partial u^*}{\partial x^*}\right)+\frac{\partial}{\partial y}\left(\frac{u_a}{L}\frac{\partial v^*}{\partial y^*}\right)\right)+\frac{\rho u_a^{\,2}}{L}\frac{\partial p^*}{\partial x^*}=0 \tag{3.9}$$

นำ $\frac{L}{\rho u_a^{\,2}}$ คูณตลอดสมการ (3.9) จะได้ สมการอนุรักษ์โมเมนตัมในรูปแบบไร้มิติ

$$\left(\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}\right)-\frac{\mu}{\rho u_a L}\left(\frac{\partial}{\partial x}\left(\frac{\partial u^*}{\partial x^*}\right)+\frac{\partial}{\partial y}\left(\frac{\partial v^*}{\partial y^*}\right)\right)+\frac{\partial p^*}{\partial x^*}=0$$

$$\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}-\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 v^*}{\partial y^{*2}}\right)+\frac{\partial p^*}{\partial x^*}=0 \tag{3.10}$$

ในทำนองเดียวกันสมการอนุรักษ์โมเมนตัม (3.3) ในรูปแบบไร้มิติ จะได้

$$\frac{\partial v^*}{\partial t^*}+u^*\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y^*}-\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v^*}{\partial x^{*2}}+\frac{\partial^2 v^*}{\partial y^{*2}}\right)+\frac{\partial p^*}{\partial y^*}=0 \tag{3.11}$$

ดังนั้นระบบสมการนาเวียร์-สโตกส์ที่อยู่ในรูปแบบไร้มิติ ดังสมการ (3.12), (3.13) และ (3.14) ตามลำดับ

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}=0 \tag{3.12}$$

$$\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}+\frac{\partial p^*}{\partial x^*}=\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 v^*}{\partial y^{*2}}\right) \tag{3.13}$$

$$\frac{\partial v^*}{\partial t^*}+u^*\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y^*}+\frac{\partial p^*}{\partial y^*}=\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v^*}{\partial x^{*2}}+\frac{\partial^2 v^*}{\partial y^{*2}}\right) \tag{3.14}$$

โดยที่ u^*, v^* แทนความเร็วที่อยู่ในรูปไร้มิติ

p^* แทนความดันที่อยู่ในรูปไร้มิติ

t^* แทนเวลาที่อยู่ในรูปไร้มิติ

Re แทนเลขเรย์โนลด์

2. การสร้างสมการความดัน (Pressure Equation)

จากระบบสมการ (3.12) - (3.14) สามารถนำมาสร้างเป็นสมการความดันได้ดังนี้ หาอนุพันธ์ย่อยโดยปริยายในสมการ (3.13) เทียบกับตัวแปร x^* และหาอนุพันธ์ย่อยโดยปริยายในสมการ (3.14) เทียบกับตัวแปร y^* จะได้

$$\frac{\partial^2 u^*}{\partial x^*\partial t^*}+u^*\frac{\partial^2 u^*}{\partial x^{*2}}+\left(\frac{\partial u^*}{\partial x^*}\right)^2+v^*\frac{\partial^2 u^*}{\partial x^*\partial y^*}+\frac{\partial u^*}{\partial y^*}\frac{\partial v^*}{\partial x^*}=\frac{1}{\mathrm{Re}}\left(\frac{\partial^3 u^*}{\partial x^{*3}}+\frac{\partial^3 u^*}{\partial x^*\partial y^{*2}}\right)-\frac{\partial^2 p^*}{\partial x^{*2}} \tag{3.15}$$

$$\frac{\partial^2 v^*}{\partial y^*\partial t^*}+u^*\frac{\partial^2 v^*}{\partial y^*\partial x^*}+\frac{\partial v^*}{\partial x^*}\frac{\partial u^*}{\partial y^*}+v^*\frac{\partial^2 v^*}{\partial y^{*2}}+\left(\frac{\partial v^*}{\partial y^*}\right)^2=\frac{1}{\mathrm{Re}}\left(\frac{\partial^3 v^*}{\partial y^*\partial x^{*2}}+\frac{\partial^3 v^*}{\partial y^{*3}}\right)-\frac{\partial^2 p^*}{\partial y^{*2}} \tag{3.16}$$

หาอนุพันธ์ย่อยสมการ (3.12) เทียบกับตัวแปร x^*

$$\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 v^*}{\partial x^*\partial y^*}=0 \text{ และ } \frac{\partial^3 u^*}{\partial x^{*3}}+\frac{\partial^3 v^*}{\partial x^{*2}\partial y^*}=0 \tag{3.17}$$

หาอนุพันธ์ย่อยสมการ (3.12) เทียบกับตัวแปร y^*

$$\frac{\partial^2 u^*}{\partial y^*\partial x^*}+\frac{\partial^2 v^*}{\partial y^{*2}}=0 \text{ และ } \frac{\partial^3 u^*}{\partial y^{*2}\partial x^*}+\frac{\partial^3 v^*}{\partial y^{*3}}=0 \tag{3.18}$$

หาอนุพันธ์ย่อยสมการ (3.12) เทียบกับตัวแปร t^*

$$\frac{\partial^2 u^*}{\partial t^*\partial x^*}+\frac{\partial^2 v^*}{\partial t^*\partial y^*}=0 \tag{3.19}$$

นำสมการ (3.15) บวก สมการ (3.16) พร้อมกับแทนค่าสมการ (3.17), (3.18), (3.19) เพื่อกำจัดตัวแปร

$$\mathrm{Re}\left(\left(\frac{\partial u^*}{\partial x^*}\right)^2 + 2\frac{\partial u^*}{\partial y^*}\frac{\partial v^*}{\partial x^*} + \left(\frac{\partial v^*}{\partial y^*}\right)^2 + \frac{\partial^2 p^*}{\partial x^{*2}} + \frac{\partial^2 p^*}{\partial y^{*2}}\right) = 0 \tag{3.20}$$

พิจารณา

$$\left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*}\right)^2 = \left(\frac{\partial u^*}{\partial x^*}\right)^2 + 2\frac{\partial u^*}{\partial x^*}\frac{\partial v^*}{\partial y^*} + \left(\frac{\partial v^*}{\partial y^*}\right)^2 \tag{3.21}$$

เนื่องจาก $\dfrac{\partial u^*}{\partial x^*} + \dfrac{\partial v^*}{\partial y^*} = 0$

ยกกำลังสองดังนั้นจะได้ว่า

$$\left(\frac{\partial u^*}{\partial x^*}\right)^2 + 2\frac{\partial u^*}{\partial x^*}\frac{\partial v^*}{\partial y^*} + \left(\frac{\partial v^*}{\partial y^*}\right)^2 = 0$$

$$\left(\frac{\partial u^*}{\partial x^*}\right)^2 + \left(\frac{\partial v^*}{\partial y^*}\right)^2 = -2\frac{\partial u^*}{\partial x^*}\frac{\partial v^*}{\partial y^*} \tag{3.22}$$

แทนสมการ (3.22) ลงในสมการ (3.20)ได้สมการความดัน (3.23)

$$2\left(\frac{\partial u^*}{\partial x^*}\frac{\partial v^*}{\partial y^*} - \frac{\partial u^*}{\partial y^*}\frac{\partial v^*}{\partial x^*}\right) = \frac{\partial^2 p^*}{\partial x^{*2}} + \frac{\partial^2 p^*}{\partial y^{*2}} \tag{3.23}$$

3. การสร้างสมการสายธารการไหล (Stream Equation)

ความสัมพันธ์ u^*, v^* กับฟังก์ชันสายธารการไหล φ^*

$$u^* = \frac{\partial \varphi^*}{\partial y^*} \quad \text{และ} \quad v^* = -\frac{\partial \varphi^*}{\partial x^*} \tag{3.24}$$

กำหนดค่าความวน $\omega = \nabla \times U$

$$\omega = \frac{\partial v^*}{\partial x^*} - \frac{\partial u^*}{\partial y^*} \tag{3.25}$$

จาก

$$\nabla^2 \varphi^* = \frac{\partial^2 \varphi^*}{\partial x^{*2}} + \frac{\partial^2 \varphi^*}{\partial y^{*2}} \tag{3.26}$$

แทนค่าสมการ (3.24) ลงในสมการ (3.25) จะได้

$$\omega = -\left(\frac{\partial^2 \varphi^*}{\partial x^{*2}} + \frac{\partial^2 \varphi^*}{\partial y^{*2}}\right)$$

นำ (-1) คูณทั้งสองข้าง

$$-\omega = \left(\frac{\partial^2 \varphi^*}{\partial x^{*2}} + \frac{\partial^2 \varphi^*}{\partial y^{*2}}\right) \tag{3.27}$$

จากสมการ (3.26) และ (3.27) ได้สมการอธิบายสายธารการไหล

$$\nabla^2 \varphi^* = -\omega$$

เขียนอีกรูปแบบได้

$$\frac{\partial\varphi^{*2}}{\partial x^{*2}}+\frac{\partial\varphi^{*2}}{\partial y^{*2}}=\frac{\partial u^*}{\partial y^*}-\frac{\partial v^*}{\partial x^*} \tag{3.28}$$

ดังนั้นระบบสมการที่ใช้ในการคำนวณหาผลเฉลยเชิงตัวเลขคือ

$$\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}+\frac{\partial p^*}{\partial x^*}=\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 u^*}{\partial y^{*2}}\right) \tag{3.29}$$

$$\frac{\partial v^*}{\partial t^*}+u^*\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y^*}+\frac{\partial p^*}{\partial y^*}=\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v^*}{\partial x^{*2}}+\frac{\partial^2 v^*}{\partial y^{*2}}\right) \tag{3.30}$$

$$2\left(\frac{\partial u^*}{\partial x^*}\frac{\partial v^*}{\partial y^*}-\frac{\partial u^*}{\partial y^*}\frac{\partial v^*}{\partial x^*}\right)=\frac{\partial^2 p^*}{\partial x^{*2}}+\frac{\partial^2 p^*}{\partial y^{*2}} \tag{3.31}$$

$$\frac{\partial^2\varphi^*}{\partial x^{*2}}+\frac{\partial^2\varphi^*}{\partial y^{*2}}=\frac{\partial u^*}{\partial y^*}-\frac{\partial v^*}{\partial x^*} \tag{3.32}$$

4. สมการสมาชิกจำกัด

ในการศึกษาครั้งนี้ได้ทำการประดิษฐ์สมการสมาชิกจำกัดโดยใช้ระเบียบวิธีถ่วงน้ำหนักเศษตกค้าง เริ่มจากการคูณเศษตกค้างด้วยฟังก์ชันน้ำหนัก ใช้วิธีบับโนฟ - กาเลอร์คิน (Bubnov-Galerkin) นั่นคือ $W_i = N_i$ จะได้

$$\int_\Omega N_i\left[\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}+\frac{\partial p^*}{\partial x^*}-\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 u^*}{\partial y^{*2}}\right)\right]d\Omega=0 \tag{3.33}$$

$$\int_\Omega N_i\left[\frac{\partial v^*}{\partial t^*}+u^*\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y^*}+\frac{\partial p^*}{\partial y^*}-\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v^*}{\partial x^{*2}}+\frac{\partial^2 v^*}{\partial y^{*2}}\right)\right]d\Omega=0 \tag{3.34}$$

$$\int_\Omega N_i\left[2\left(\frac{\partial u^*}{\partial x^*}\frac{\partial v^*}{\partial y^*}-\frac{\partial u^*}{\partial y^*}\frac{\partial v^*}{\partial x^*}\right)-\left(\frac{\partial^2 p^*}{\partial x^{*2}}+\frac{\partial^2 p^*}{\partial y^{*2}}\right)\right]d\Omega=0 \tag{3.35}$$

$$\int_\Omega N_i\left[\frac{\partial^2\varphi^*}{\partial x^{*2}}+\frac{\partial^2\varphi^*}{\partial y^{*2}}-\frac{\partial u^*}{\partial y^*}+\frac{\partial v^*}{\partial x^*}\right]d\Omega=0 \tag{3.36}$$

ทำการเปลี่ยนรูปแบบการหาปริพันธ์บนพื้นที่ของสมาชิกให้อยู่ในรูปปริพันธ์ตลอดขอบเขตของสมาชิกโดยใช้ทฤษฎีบทของกรีน [13] ดังนั้นสามารถเขียนสมการ (3.33) - (3.36) ใหม่ได้ดังนี้

$$\int_\Omega N_i\frac{\partial u^*}{\partial t^*}d\Omega+\int_\Omega N_i\left(u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}\right)d\Omega+\int_\Omega N_i\left(\frac{\partial p^*}{\partial x^*}\right)d\Omega+\int_\Omega\frac{1}{\mathrm{Re}}\left(\frac{\partial N_i}{\partial x^*}\frac{\partial u^*}{\partial x^*}+\frac{\partial N_i}{\partial y^*}\frac{\partial u^*}{\partial y^*}\right)d\Omega-\int_\Gamma\frac{1}{\mathrm{Re}}N_i\left(\frac{\partial u^*}{\partial \vec{n}}\right)d\Gamma=0 \tag{3.37}$$

$$\int_{\Omega} N_i \frac{\partial v^*}{\partial t^*} d\Omega + \int_{\Omega} N_i \left(u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} \right) d\Omega + \int_{\Omega} N_i \left(\frac{\partial p^*}{\partial y^*} \right) d\Omega + \int_{\Omega} \frac{1}{\mathrm{Re}} \left(\frac{\partial N_i}{\partial x^*} \frac{\partial u^*}{\partial x^*} + \frac{\partial N_i}{\partial y^*} \frac{\partial u^*}{\partial y^*} \right) d\Omega - \int_{\Gamma} \frac{1}{\mathrm{Re}} N_i \left(\frac{\partial u^*}{\partial \vec{n}} \right) d\Gamma = 0 \quad (3.38)$$

$$\int_{\Omega} \mathrm{Re} \left(\frac{\partial N_i}{\partial x^*} \frac{\partial p^*}{\partial x^*} + \frac{\partial N_i}{\partial y^*} \frac{\partial p^*}{\partial y^*} \right) d\Omega - \int_{\Gamma} \mathrm{Re}\, N_i \left(\frac{\partial p^*}{\partial \vec{n}} \right) d\Gamma - \int_{\Omega} 2\,\mathrm{Re}\, N_i \left(\frac{\partial u^*}{\partial x^*} \frac{\partial v^*}{\partial y^*} - \frac{\partial u^*}{\partial y^*} \frac{\partial v^*}{\partial x^*} \right) d\Omega = 0 \quad (3.39)$$

$$\int_{\Omega} \left(\frac{\partial N_i}{\partial x^*} \frac{\partial \varphi^*}{\partial x^*} + \frac{\partial N_i}{\partial y^*} \frac{\partial \varphi^*}{\partial y^*} \right) d\Omega - \int_{\Gamma} N_i \left(\frac{\partial \varphi^*}{\partial \vec{n}} \right) d\Gamma - \int_{\Omega} N_i \frac{\partial u^*}{\partial y^*} d\Omega + \int_{\Omega} N_i \frac{\partial v^*}{\partial x^*} d\Omega = 0 \quad (3.40)$$

พิจารณาพจน์แรกที่มีเวลาเข้ามาเกี่ยวข้อง

$$\int_{\Omega} N_i \left(\frac{\partial u^*}{\partial t^*} \right) d\Omega = \int_{\Omega} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} \{ N_1 \ N_2 \ N_3 \} d\Omega \begin{Bmatrix} \dot{u}_1^* \\ \dot{u}_2^* \\ \dot{u}_3^* \end{Bmatrix} = \int_{\Omega} \{N\} \lfloor N \rfloor d\Omega \left\{ \dot{u}^* \right\}$$

และพจน์ที่ 2

$$\int_{\Omega} N_i \left(u^* \frac{\partial u^*}{\partial x^*} \right) d\Omega = \int_{\Omega} N_i \left(N_1 u^*_1 + N_2 u^*_2 + N_3 u^*_3 \right) \left(\frac{\partial N_1}{\partial x} u^*_1 + \frac{\partial N_2}{\partial x} u^*_2 + \frac{\partial N_3}{\partial x} u^*_3 \right) d\Omega$$

$$= \int_{\Omega} N_i \begin{pmatrix} \left(N_1 \frac{\partial N_1}{\partial x} u^*_1 + N_1 \frac{\partial N_2}{\partial x} u^*_2 + N_1 \frac{\partial N_3}{\partial x} u^*_3 \right) u^*_1 \\ + \left(N_2 \frac{\partial N_1}{\partial x} u^*_1 + N_2 \frac{\partial N_2}{\partial x} u^*_2 + N_2 \frac{\partial N_3}{\partial x} u^*_3 \right) u^*_2 \\ + \left(N_3 \frac{\partial N_1}{\partial x} u^*_1 + N_3 \frac{\partial N_2}{\partial x} u^*_2 + N_3 \frac{\partial N_3}{\partial x} u^*_3 \right) u^*_3 \end{pmatrix} d\Omega$$

$$= \rho \int_{\Omega} N_i \left(k_1 u^*_1 + k_2 u^*_2 + k_3 u^*_3 \right) d\Omega$$

$$= \rho \int_{\Omega} N_i \lfloor K_1 \rfloor \left\{ u^* \right\} d\Omega \quad (3.41)$$

เมื่อ $$k_1 = N_1 \frac{\partial N_1}{\partial x} u^*_1 + N_1 \frac{\partial N_2}{\partial x} u^*_2 + N_1 \frac{\partial N_3}{\partial x} u^*_3 \quad (3.42)$$

$$k_2 = N_2 \frac{\partial N_1}{\partial x} u^*_1 + N_2 \frac{\partial N_2}{\partial x} u^*_2 + N_2 \frac{\partial N_3}{\partial x} u^*_3 \quad (3.43)$$

$$k_3 = N_3 \frac{\partial N_1}{\partial x} u^*_1 + N_3 \frac{\partial N_2}{\partial x} u^*_2 + N_3 \frac{\partial N_3}{\partial x} u^*_3 \quad (3.44)$$

ในทำนองเดียวกัน จะได้ว่า

$$\int_{\Omega} N_i \left(v^* \frac{\partial u^*}{\partial y^*} \right) d\Omega = \int_{\Omega} N_i \lfloor K_2 \rfloor \{v^*\} \, d\Omega \tag{3.45}$$

$$\int_{\Omega} N_i \left(\frac{\partial p^*}{\partial x^*} \right) d\Omega = \int_{\Omega} N_i \left\lfloor \frac{\partial N}{\partial x^*} \right\rfloor \{p^*\} \, d\Omega \tag{3.46}$$

$$\frac{1}{\mathrm{Re}} \int_{\Omega} \frac{\partial N_i}{\partial x^*} \left(\frac{\partial u^*}{\partial x^*} \right) \mathrm{d}\Omega = \frac{1}{\mathrm{Re}} \int_{\Omega} \frac{\partial N_i}{\partial x^*} \left\lfloor \frac{\partial N}{\partial x^*} \right\rfloor \{u^*\} \, \mathrm{d}\Omega \tag{3.47}$$

$$\frac{1}{\mathrm{Re}} \int_{\Omega} \frac{\partial N_i}{\partial y^*} \left(\frac{\partial u^*}{\partial y^*} \right) \mathrm{d}\Omega = \frac{1}{\mathrm{Re}} \int_{\Omega} \frac{\partial N_i}{\partial y^*} \left\lfloor \frac{\partial N}{\partial y^*} \right\rfloor \{u^*\} \, \mathrm{d}\Omega \tag{3.48}$$

ดังนั้นสมการ (3.37)- (3.40) เขียนใหม่ได้ ดังนี้

$$\int_{\Omega} \{N\} \lfloor N \rfloor d\Omega \{\dot{u}^*\} + \int_{\Omega} \left(N_i \lfloor K_1 \rfloor + \frac{1}{\mathrm{Re}} \frac{\partial N_i}{\partial x^*} \left\lfloor \frac{\partial N}{\partial x^*} \right\rfloor + \frac{1}{\mathrm{Re}} \frac{\partial N_i}{\partial y^*} \left\lfloor \frac{\partial N}{\partial y^*} \right\rfloor \right) d\Omega \{u^*\} + \int_{\Omega} N_i \lfloor K_2 \rfloor d\Omega \{v^*\} + \int_{\Omega} N_i \left\lfloor \frac{\partial N}{\partial x^*} \right\rfloor d\Omega \{p^*\}$$

$$= \int_{\Gamma} N_i \frac{\partial u^*}{\partial \vec{n}} d\Gamma \tag{3.49}$$

$$\int_{\Omega} \{N\} \lfloor N \rfloor d\Omega \{\dot{v}^*\} + \int_{\Omega} N_i \lfloor K_3 \rfloor d\Omega \{u^*\} + \int_{\Omega} \left(N_i \lfloor K_4 \rfloor + \frac{1}{\mathrm{Re}} \frac{\partial N_i}{\partial x^*} \left\lfloor \frac{\partial N}{\partial x^*} \right\rfloor + \frac{1}{\mathrm{Re}} \frac{\partial N_i}{\partial y^*} \left\lfloor \frac{\partial N}{\partial y^*} \right\rfloor \right) d\Omega \{v^*\} + \int_{\Omega} N_i \left\lfloor \frac{\partial N}{\partial y^*} \right\rfloor d\Omega \{p^*\}$$

$$= \int_{\Gamma} N_i \frac{\partial v^*}{\partial \vec{n}} d\Gamma \tag{3.50}$$

$$\int_{\Omega} \left(\frac{\partial N_i}{\partial x^*} \left\lfloor \frac{\partial N}{\partial x^*} \right\rfloor + \frac{\partial N_i}{\partial y^*} \left\lfloor \frac{\partial N}{\partial y^*} \right\rfloor \right) d\Omega \{p^*\} - \int_{\Omega} 2 N_i \lfloor K_5 \rfloor d\Omega \{u^*\} + \int_{\Omega} 2 N_i \lfloor k_6 \rfloor d\Omega \{v^*\}$$

$$= \int_{\Gamma} N_i \frac{\partial p^*}{\partial \vec{n}} d\Gamma \tag{3.51}$$

$$\int_{\Omega} \left(\frac{\partial N_i}{\partial x^*} \left\lfloor \frac{\partial N}{\partial x^*} \right\rfloor + \frac{\partial N_i}{\partial y^*} \left\lfloor \frac{\partial N}{\partial y^*} \right\rfloor \right) d\Omega \{\varphi^*\} - \int_{\Omega} N_i \left\lfloor \frac{\partial N}{\partial y^*} \right\rfloor d\Omega \{u^*\} + \int_{\Omega} N_i \left\lfloor \frac{\partial N}{\partial x^*} \right\rfloor d\Omega \{v^*\}$$

$$= \int_{\Gamma} N_i \frac{\partial \varphi^*}{\partial \vec{n}} d\Gamma \tag{3.52}$$

หรือเขียนในรูปเมทริกซ์ได้

$$\lfloor A \rfloor \{\dot{u}^*\}' + \lfloor B \rfloor \{u^*\}' + \lfloor C \rfloor \{v^*\}' + \lfloor D \rfloor \{p^*\}' = \{F_1\}' \tag{3.53}$$

$$\lfloor E \rfloor \{\dot{v}^*\}^t + \lfloor G \rfloor \{u^*\}^t + \lfloor H \rfloor \{v^*\}^t + \lfloor I \rfloor \{p^*\}^t = \{F_2\}^t \tag{3.54}$$

$$\lfloor J \rfloor \{p^*\}^t + \lfloor L \rfloor \{u^*\}^t + \lfloor M \rfloor \{v^*\}^t = \{F_3\}^t \tag{3.55}$$

$$\lfloor N \rfloor \{\varphi^*\}^t - \lfloor O \rfloor \{u^*\}^t + \lfloor Q \rfloor \{v^*\}^t = \{F_4\}^t \tag{3.56}$$

พิจารณาพจน์ที่เกี่ยวข้องกับอนุพันธ์เวลาคำนวณโดยใช้วิธีโดยนัย (Backward Difference Method) จากสมการ (3.53)-(3.54) แทนค่า t ด้วย $t+\Delta t$ ดังนี้

$$\lfloor A \rfloor \{\dot{u}^*\}^{t+\Delta t} + \lfloor B \rfloor \{u^*\}^{t+\Delta t} + \lfloor C \rfloor \{v^*\}^{t+\Delta t} + \lfloor D \rfloor \{p^*\}^{t+\Delta t} = \{F_1\}^{t+\Delta t} \tag{3.57}$$

$$\lfloor E \rfloor \{\dot{v}^*\}^{t+\Delta t} + \lfloor G \rfloor \{u^*\}^{t+\Delta t} + \lfloor H \rfloor \{v^*\}^{t+\Delta t} + \lfloor I \rfloor \{p^*\}^{t+\Delta t} = \{F_2\}^{t+\Delta t} \tag{3.58}$$

เทอมที่มีอนุพันธ์เทียบเวลาคำนวณโดยใช้เทคนิควิธีโดยนัยจะได้

$$\left\{\dot{\mathrm{u}}\right\}^{t+\Delta t} = \frac{\{u\}^{t+\Delta t} - \{u\}^t}{\Delta t} \tag{3.59}$$

$$\left\{\dot{v}\right\}^{t+\Delta t} = \frac{\{v\}^{t+\Delta t} - \{v\}^t}{\Delta t} \tag{3.60}$$

แทนค่าสมการ (3.59)-(3.60) ลงในสมการ (3.57)-(3.58) จะได้

$$\left(\lfloor A \rfloor + \Delta t \lfloor B \rfloor\right) \{u^*\}^{t+\Delta t} - \lfloor A \rfloor \{u\}^t + \Delta t \lfloor C \rfloor \{v^*\}^{t+\Delta t} + \Delta t \lfloor D \rfloor \{p^*\}^{t+\Delta t} = \Delta t \{F_1\}^{t+\Delta t} \tag{3.61}$$

$$\left(\lfloor E \rfloor + \Delta t \lfloor H \rfloor\right) \{v^*\}^{t+\Delta t} - \lfloor E \rfloor \{v\}^t + \Delta t \lfloor G \rfloor \{u^*\}^{t+\Delta t} + \Delta t \lfloor I \rfloor \{p^*\}^{t+\Delta t} = \Delta t \{F_2\}^{t+\Delta t} \tag{3.62}$$

เมื่อกำหนดให้

$$\lfloor A \rfloor = \int_\Omega \{N\} \lfloor N \rfloor d\Omega$$

$$\lfloor B \rfloor = \int_\Omega \left(N_i \lfloor K_1 \rfloor + \frac{1}{\mathrm{Re}} \frac{\partial N_i}{\partial x^*} \left\lfloor \frac{\partial N}{\partial x^*} \right\rfloor + \frac{1}{\mathrm{Re}} \frac{\partial N_i}{\partial y^*} \left\lfloor \frac{\partial N}{\partial y^*} \right\rfloor \right) d\Omega$$

$$\lfloor C \rfloor = \int_\Omega N_i \lfloor K_2 \rfloor d\Omega$$

$$\lfloor D \rfloor = \int_\Omega N_i \left\lfloor \frac{\partial N}{\partial x^*} \right\rfloor d\Omega$$

$$\{F_1\} = \int_\Gamma N_i \frac{\partial u^*}{\partial \vec{n}} d\Gamma$$

$$\lfloor E \rfloor = \int_\Omega \{N\} \lfloor N \rfloor d\Omega$$

$$\lfloor G \rfloor = \int_\Omega N_i \lfloor K_3 \rfloor d\Omega$$

$$\lfloor H \rfloor = \int_{\Omega} \left(N_i \lfloor K_4 \rfloor + \frac{1}{\mathrm{Re}} \frac{\partial N_i}{\partial x^*} \left\lfloor \frac{\partial N}{\partial x^*} \right\rfloor + \frac{1}{\mathrm{Re}} \frac{\partial N_i}{\partial y^*} \left\lfloor \frac{\partial N}{\partial y^*} \right\rfloor \right) d\Omega$$

$$\lfloor I \rfloor = \int_{\Omega} N_i \left\lfloor \frac{\partial N}{\partial y^*} \right\rfloor d\Omega$$

$$\{F_2\} = \int_{\Gamma} N_i \frac{\partial v^*}{\partial \vec{n}} d\Gamma$$

$$\lfloor J \rfloor = \int_{\Omega} \left(\frac{\partial N_i}{\partial x^*} \left\lfloor \frac{\partial N}{\partial x^*} \right\rfloor + \frac{\partial N_i}{\partial y^*} \left\lfloor \frac{\partial N}{\partial y^*} \right\rfloor \right) d\Omega$$

$$\lfloor L \rfloor = -\int_{\Omega} 2N_i \lfloor K_5 \rfloor d\Omega$$

$$\lfloor M \rfloor = \int_{\Omega} 2N_i \lfloor k_6 \rfloor d\Omega$$

$$\{F_3\} = \int_{\Gamma} N_i \frac{\partial p^*}{\partial \vec{n}} d\Gamma$$

$$\lfloor N \rfloor = \int_{\Omega} \left(\frac{\partial N_i}{\partial x^*} \left\lfloor \frac{\partial N}{\partial x^*} \right\rfloor + \frac{\partial N_i}{\partial y^*} \left\lfloor \frac{\partial N}{\partial y^*} \right\rfloor \right) d\Omega$$

$$\lfloor O \rfloor = -\int_{\Omega} N_i \left\lfloor \frac{\partial N}{\partial y^*} \right\rfloor d\Omega$$

$$\lfloor Q \rfloor = \int_{\Omega} N_i \left\lfloor \frac{\partial N}{\partial x^*} \right\rfloor d\Omega$$

$$\{F_4\} = \int_{\Gamma} N_i \frac{\partial \varphi^*}{\partial \vec{n}} d\Gamma$$