วิธีการประมาณค่าแบบใหม่สำหรับหาคำตอบปัญหาจุดตรึงและปัญหาไม่เชิงเส้นในปริภูมิบานาค

New Approximation Methods for Solving Fixed Point Problems and Nonlinear Problems in Banach Spaces

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บทคัดย่อ

จุดประสงค์ของงานวิจัยชิ้นนี้คือ สร้างทฤษฎีบทของกระบวนการทำซ้ำแบบใหม่โดยใช้วิธีการฉาย
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การลู่เข้าอย่างเข้มของกระบวนการทำซ้ำที่สร้างขึ้นภายใต้เงื่อนไขที่เหมาะสมในปริภูมิบานาค ทฤษฎี
บทใหม่ในงานวิจัยชิ้นนี้ได้ขยายและปรับปรุงจากงานวิจัยที่เกี่ยวข้องที่มาก่อน

Abstract

The purpose of this research is to present some new modified hybrid iterative schemes by using the generalized projection operators and the generalized f-projection operators for finding a common element in the set of fixed points of mappings, zeroes of maximal monotone operators and solutions of equilibrium problems in Banach spaces. Under suitable conditions, we prove some strong convergence theorems for the proposed schemes in Banach spaces. The results presented in this research are improvement and generalization of the previously known results.

ประกาศคุณูปการ

งานวิจัยฉบับนี้เสร็จสมบูรณ์ได้เป็นอย่างดี โดยได้รับการสนับสนุนจาก กองทุนวิจัยมหาวิทยาลัย
ทักษิณที่สนับสนุนทุนวิจัย คณะวิทยาศาสตร์ให้ความอนุเคราะห์สถานที่การทำวิจัย และพนักงานเจ้าหน้าที่
สถาบันวิจัยที่อำนวยความสะดวกในการจัดทำผลงานวิจัยเป็นอย่างดี ผู้จัดทำงานวิจัยขอขอบพระคุณมา ณ
โอกาสนี้

สารบัญ

1 Introduction

It is well known that the metric projection operator plays an important role in various fields such as nonlinear functional analysis, optimization theory, fixed point theory, nonlinear programming, game theory, variational inequality and complementarity problems. (see [3]). In 1994, Alber [2] introduced and studied the generalized projections from Hilbert spaces to uniformly convex and uniformly smooth Banach spaces. Moreover Alber [3] presented some applications of the generalized projections to approximately solving variational inequalities and von Neumann intersection problem in Banach spaces. In 2005, Li [50] extended the generalized projection operator from uniformly convex and uniformly smooth Banach spaces to reflexive Banach spaces and studied some properties of the generalized projection operator with applications to solve the variational inequality in Banach spaces. Later, Wu and Huang [85] introduced a new generalized f-projection operator in Banach spaces. They extended the definition of the generalized projection operators introduced by Abler [2] and proved some properties of the generalized f-projection operator. In 2009, Fan et al. [30] presented some basic results for the generalized f-projection operator, and discussed the existence of solutions and approximation of the solutions for generalized variational inequalities in noncompact subsets of Banach spaces.

The study of Ky Fan inequality [31], fixed points of nonlinear mappings, and their approximation algorithms constitutes a topic of intensive research efforts. Many well known problems arising in various branches of science can be studied by using algorithms, which are iterative in their nature. As an example, in computer tomography with limited data, each piece of information implies the existence of a convex set in which the required solution lies.

The problem of finding a point in the intersection of the convex sets is then of crucial interest, and it cannot be usually solved directly. Therefore, an iterative algorithm must be used to approximate such point. The well known convex feasibility problem which captures applications in various disciplines such as image restoration, computer tomography, and radiation therapy treatment planning is to find a point in the intersection of common fixed point sets of a family of nonexpansive mappings. A simple algorithmic solution to the problem of minimizing a quadratic function over the intersection of the convex sets is of extreme value in many applications including set theoretic signal estimation.

For solving convex feasibility problem of a system of generalized Ky Fan inequalities is very general in the sense that it includes, as special cases optimization problems, equilibrium problems, variational inequality problems, Min-Max problems. Moreover, the the generalized Ky Fan inequality was shown in [8] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational inequality problems, vector equilibrium problems, Nash equilibria in noncooperative games. In addition, there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of a generalized Ky Fan inequality. In other words, the generalized Ky Fan inequality and equilibrium problem are an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. In the last two decades, many papers have appeared in the literature on the existence of solutions of a generalized Ky Fan inequality (or equilibrium problem; see, for example [8, 22, 96] and references therein). Some solution methods have been proposed to solve the generalized Ky Fan inequality and equilibrium problems; see, for example, [8, 39, 41, 43, 55, 58, 63, 61] and references therein.

Fixed Point Theory and Applications

Iterative Schemes for Approximating Solution of Nonlinear Operators in Banach Spaces

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Iterative schemes for approximating solution of nonlinear operators in Banach spaces

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Abstract The purpose of this paper is to present a new modified Halpern-Mann type iterative scheme by using the generalized f-projection operator for finding a common element in the set of zeroes of a system of maximal monotone operators, the set of fixed points of a totally quasi-ϕ-asymptotically nonexpansive mapping and the set of solutions of a system of generalized Ky Fan's inequalities in a uniformly smooth and strictly convex Banach space with the Kadec-Klee property. Furthermore, we show that our proposed iterative scheme converges strongly to a common element of the sets mentioned above.

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1 Introduction

In 1972, Ky Fan's inequalities were first introduced by Fan [1]. The study concerning Ky Fan's inequalities, fixed points of nonlinear mappings and their approximation algorithms constitutes a topic of intensive research efforts. Many well-known problems arising in various branches of science can be studied by using algorithms which are iterative in their nature. As an example, in computer tomography with limited data, each piece of information implies the existence of a convex set in which the required solution lies.

Many authors have considered a family of nonexpansive mappings to show the existence of fixed points and related topics. Especially, the well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings and the problem of finding an optimal point that minimizes a given cost function over the set of common fixed points of a family of nonexpansive mappings.

For solving the convex feasibility problem for a system of generalized Ky Fan's inequalities is very general in the sense that it includes, as special cases, optimization problems, equilibrium problems, variational inequality problems, minimax problems. Moreover, the generalized Ky Fan's inequality was shown in [6] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, vector equilibrium problems, Nash equilibria in noncooperative games. In other words, the generalized Ky Fan's inequality and equilibrium problem are a unified model for several problems arising in physics, engineering, science, optimization, economics and related topics.

One of the most interesting and important problems in the theory of maximal monotone operators is to find a zero point of maximal monotone operators. This problem contains the convex minimization problem and the variational inequality problem. A popular method for approximating this problem is called the proximal point algorithm introduced by Martinet [25] in a Hilbert space. In 1976, Rockafellar [31] extended the knowledge of Martinet [25] and proved weak convergence of the proximal point algorithm. The proximal point algorithm of Rockafellar [31] is a successful algorithm for finding a zero point of maximal monotone operators. Thereafter, many papers have shown convergence theorems of the proximal point algorithm in various spaces (see $[12, 13, 16, 21, 23, 26, 30, 37, 43, 44]$).

A point $x \in C$ is a *fixed point* of S provided $Sx = x$. We denote by $F(S)$ the fixed point set of S, that is, $F(S) = \{x \in C : Sx = x\}$. A point p in C is called an *asymptotic fixed point* of S [36] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$. The set of asymptotic fixed points of S is denoted by $\widehat{F}(S)$. Recently, Halpern and Mann iterative algorithms have been considered for approximations of common fixed points by many authors. For example, in 2011, Saewan and Kumam [38] introduced a modified Mann iterative scheme by using the generalized f-projection method for approximating a common fixed point of a countable family for a countable family of relatively quasi-nonexpansive mappings. Chang *et al.* [11] considered a modified Halpern iterative scheme for approximating a common fixed point for a totally quasi-ϕ-asymptotically nonexpansive mapping. Recently, Li *et al.* [24] introduced a hybrid iterative scheme for

approximation of a fixed point of relatively nonexpansive mappings by using the properties of generalized f-projection operators in a uniformly smooth real Banach space, which is also uniformly convex, and proved some strong convergence theorems for the hybrid iterative scheme.

On the other hand, Ofoedu and Shehu [27] extended the algorithm of Li *et al.* [24] to prove strong convergence theorems for a common solution of the set of solutions of a system of Ky Fan's inequalities and the set of common fixed points of a pair of relatively quasi-nonexpansive mappings in a Banach space by using the generalized f-projection operator. Chang *et al.* [10] extended and improved the results of Qin and Su [28] to obtain strong convergence theorems for finding a common element of the set of solutions for a generalized Ky Fan's inequality, the set of solutions for a variational inequality problem and the set of common fixed points for a pair of relatively nonexpansive mappings in a Banach space.

Motivated and inspired by the work mentioned above, in this paper, we introduce a new hybrid iterative scheme of the generalized f-projection operator based on the Halpern-Mann type iterative scheme for finding a common element of the set of zeroes of a system of maximal monotone operators, the set of fixed points of a totally quasi-ϕ-asymptotically nonexpansive mapping and the set of solutions of a system of generalized Ky Fan's inequalities in a uniformly smooth and strictly convex Banach space with the Kadec-Klee property.

2 Preliminaries

A Banach space E with the norm $\|\cdot\|$ is called *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in U$ with $x \neq y$, where $U = \{x \in E : \|x\| = 1\}$ is the unit sphere of E. A Banach space E is called *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also called *uniformly smooth* if the limit exists uniformly for all $x, y \in U$. In this paper, we denote the strong convergence and weak convergence of a sequence $\{x_n\}$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Let E be a real Banach space with the dual space E^* and let C be a nonempty closed and convex subset of E. A mapping $S : C \to C$ is said to be:

(1) *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|$$

for all $x, y \in C$;

(2) *quasi-nonexpansive* if $F(S) \neq \emptyset$ and

$$\|Sx - y\| \leq \|x - y\|$$

for all $x \in C$ and $y \in F(S)$;

(3) *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|S^n x - S^n y\| \leq k_n \|x - y\|$$

for all $x, y \in C$;

(4) *asymptotically quasi-nonexpansive* if $F(S) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|S^n x - y\| \leq k_n \|x - y\|$$

for all $x \in C$ and $y \in F(S)$;

(5) *totally asymptotically nonexpansive* if there exist nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ with $\psi(0) = 0$ such that

$$\|S^n x - S^n y\| \leq \|x - y\| + \mu_n \psi(\|x - y\|) + \nu_n$$

for all $x, y \in C$ and $n \geq 1$.

A mapping $S : C \to C$ is said to be *uniformly L-Lipschitz continuous* if there exists a constant $L > 0$ such that

$$\|S^n x - S^n y\| \leq L \|x - y\| \tag{1}$$

for all $x, y \in C$. A mapping $S : C \to C$ is said to be *closed* if, for any sequence $\{x_n\} \subset C$ such that $\lim_{n \to \infty} x_n = x_0$ and $\lim_{n \to \infty} Sx_n = y_0$, we have $Sx_0 = y_0$.

The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \|x^*\| = \|x\|\}$$

for all $x \in E$. If E is a Hilbert space, then $J = I$, where I is the identity mapping. Consider the functional $\phi : E \times E \to \mathbf{R}$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \tag{2}$$

where J is the normalized duality mapping and $\langle \cdot, \cdot \rangle$ denotes the duality pairing of E and E^*. If E is a Hilbert space, then $\phi(y, x) = \|y - x\|^2$. It is obvious from the definition of ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2 \tag{3}$$

for all $x, y \in E$.

A mapping $S : C \to C$ is said to be:

(1) *relatively nonexpansive* [7,8] if $\hat{F}(S) = F(S)$ and

$$\phi(p, Sx) \leq \phi(p, x)$$

for all $x \in C$ and $p \in F(S)$;

(2) *relatively asymptotically nonexpansive* [2] if $\hat{F}(S) = F(S) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi(p, S^n x) \leq k_n \phi(p, x)$$

for all $x \in C$, $p \in F(S)$ and $n \geq 1$;

(3) *ϕ-nonexpansive* [29, 46] if

$$\phi(Sx, Sy) \leq \phi(x, y)$$

for all $x, y \in C$;

(3) *quasi-ϕ-nonexpansive* [29, 46] if $F(S) \neq \emptyset$ and

$$\phi(p, Sx) \leq \phi(p, x)$$

for all $x \in C$ and $p \in F(S)$;

(4) *asymptotically ϕ-nonexpansive* [46] if there exists a sequence $\{k_n\} \subset [0, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi(S^n x, S^n y) \leq k_n \phi(x, y)$$

for all $x, y \in C$ and $n \geq 1$;

(5) *quasi-ϕ-asymptotically nonexpansive* [46] if $F(S) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [0, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi(p, S^n x) \leq k_n \phi(p, x)$$

for all $x \in C$, $p \in F(S)$ and $n \geq 1$;

(6) *totally quasi-ϕ-asymptotically nonexpansive* if $F(S) \neq \emptyset$ and there exist nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ with $\psi(0) = 0$ such that

$$\phi(p, S^n x) \leq \phi(p, x) + \nu_n \psi(\phi(p, x)) + \mu_n$$

for all $x \in C$, $p \in F(S)$ and $n \geq 1$.

Lemma 1 [9] *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. Let $S : C \to C$ be a closed and totally quasi-ϕ-asymptotically nonexpansive mapping with nonnegative real sequences $\{\nu_n\}$ and $\{\mu_n\}$ with $\nu_n \to 0$ and $\mu_n \to 0$ as $n \to \infty$, respectively, and a strictly increasing continuous function $\zeta : \mathbf{R}^+ \to \mathbf{R}^+$ with $\zeta(0) = 0$. If $\mu_1 = 0$, then the set $F(S)$ of fixed points of S is a closed convex subset of C.*

Alber [5] introduced that the *generalized projection* $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution of the minimization problem

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x). \tag{4}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(y, x)$ and strict monotonicity of the mapping J (see, for example, [3, 5, 14, 20, 40]).

If E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$ and Π_C becomes the metric projection $P_C : H \to C$. If C is a nonempty closed and convex subset of a Hilbert space H, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces, it is not available in more general Banach spaces.

Remark 1 The basic properties of a Banach space E related to the normalized duality mapping J are as follows (see [14]):

(1) If E is an arbitrary Banach space, then J is monotone and bounded;

(2) If E is a strictly convex Banach space, then J is strictly monotone;

(3) If E is a smooth Banach space, then J is single-valued and semicontinuous;

(4) If E is a uniformly smooth Banach space, then J is uniformly norm-to-norm continuous on each bounded subset of E;

(5) If E is a reflexive smooth and strictly convex Banach space, then the normalized duality mapping J is single-valued, one-to-one and onto;

(6) If E is a reflexive strictly convex and smooth Banach space and J is the duality mapping from E into E^*, then J^{-1} is also single-valued, bijective and is also the duality mapping from E^* into E, and thus $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$;

(7) If E is a uniformly smooth Banach space, then E is smooth and reflexive;

(8) E is a uniformly smooth Banach space if and only if E^* is uniformly convex;

(9) If E is a reflexive and strictly convex Banach space, then J^{-1} is norm-weak*-continuous.

Remark 2 If E is a reflexive, strictly convex and smooth Banach space, then $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$, then $x = y$. From (2) we have

$$
\begin{aligned}
0 &= \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \\
&= \|x\|^2 - 2\|x\|\|Jy\| + \|y\|^2 \\
&= \|x\|^2 - 2\|x\|\|Jy\| + \|y\|^2 \\
&= \|x - y\|^2.
\end{aligned}
$$

That is, $\|x\| = \|y\|$. This implies that $\langle x, Jy \rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$ (see [14,40,35] for more details).

In 2006, Wu and Huang [42] introduced a new generalized f-projection operator in a Banach space. They extended the definition of the generalized projection operators introduced by Abler [4] and proved some properties of the generalized f-projection operator. Consider the functional $G : C \times E^* \longrightarrow \mathbf{R} \cup \{+\infty\}$ defined by

$$
G(y, \varpi) = \|y\|^2 - 2\langle y, \varpi \rangle + \|\varpi\|^2 + 2\rho f(y) \tag{5}
$$

for all $(y, \varpi) \in C \times E^*$, where ρ is a positive number and $f : C \to \mathbf{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous. From the definition of G, Wu and Huang [42] proved the following properties:

(1) $G(y, \varpi)$ is convex and continuous with respect to ϖ when y is fixed;

(2) $G(y, \varpi)$ is convex and lower semicontinuous with respect to y when ϖ is fixed.

Definition 1 Let E be a real Banach space with its dual space E^* and let C be a nonempty closed and convex subset of E. We say that $\pi_C^f : E^* \to 2^C$ is a *generalized f-projection operator* if

$$
\pi_C^f \varpi = \{u \in C : G(u, \varpi) = \inf_{y \in C} G(y, \varpi), \forall \varpi \in E^*\}.
$$

Recall that a Banach space E has the Kadec-Klee property [14,18,40] if for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, we have $\|x_n - x\| \to 0$ as $n \to \infty$. It is well known that if E is a uniformly convex Banach space, then E has the Kadec-Klee property.

Lemma 2 [42] *Let E be a real reflexive Banach space with its dual space E^* and let C be a nonempty closed and convex subset of E. The following statements hold:*

(1) *$\pi_C^f \varpi$ is a nonempty, closed and convex subset of C for all $\varpi \in E^*$;*

(2) *If E is smooth, then for all $\varpi \in E^*$, $x \in \pi_C^f \varpi$ if and only if*

$$\langle x - y, \varpi - Jx \rangle + \rho f(y) - \rho f(x) \geq 0$$

for all $y \in C$;

(3) *If E is strictly convex and $f : C \to \mathbf{R} \cup \{+\infty\}$ is positive homogeneous (i.e., $f(tx) = tf(x)$ for all $t > 0$ such that $tx \in C$, where $x \in C$), then $\pi_C^f \varpi$ is a single-valued mapping.*

Recently, Fan *et al.* [17] showed that the condition, f is positive homogeneous, which appears in [17, Lemma 2.1 (iii)], can be removed.

Lemma 3 [17] *Let E be a real reflexive Banach space with its dual space E^* and let C be a nonempty closed and convex subset of E. If E is strictly convex, then $\pi_C^f \varpi$ is single-valued.*

Recall that J is a single-valued mapping when E is a smooth Banach space. There exists a unique element $\varpi \in E^*$ such that $\varpi = Jx$, where $x \in E$. This substitution in (5) gives the following:

$$G(y, Jx) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2 + 2\rho f(y). \tag{6}$$

Now, we consider the second generalized f projection operator in a Banach space (see [24]).

Definition 2 Let E be a real smooth Banach space and let C be a nonempty closed and convex subset of E. We say that $\Pi_C^f : E \to 2^C$ is a *generalized f-projection operator* if

$$\Pi_C^f x = \{u \in C : G(u, Jx) = \inf_{y \in C} G(y, Jx), \forall x \in E\}.$$

Lemma 4 [15] *Let E be a Banach space and let $f : E \to \mathbf{R} \cup \{+\infty\}$ be a lower semicontinuous and convex function. Then there exist $x^* \in E^*$ and $\alpha \in \mathbf{R}$ such that*

$$f(x) \geq \langle x, x^* \rangle + \alpha$$

for all $x \in E$.

Lemma 5 [24] *Let E be a reflexive smooth Banach space and let C be a nonempty closed and convex subset of E. The following statements hold:*

(1) *$\Pi_C^f x$ is a nonempty closed and convex subset of C for all $x \in E$;*

(2) *For all $x \in E$, $\hat{x} \in \Pi_C^f x$ if and only if*

$$\langle \hat{x} - y, Jx - J\hat{x} \rangle + \rho f(y) - \rho f(\hat{x}) \geq 0$$

for all $y \in C$;

(3) *If E is strictly convex, then Π_C^f is a single-valued mapping.*

Lemma 6 [24] *Let E be a real reflexive smooth Banach space and let C be a nonempty closed and convex subset of E. Then, for any $x \in E$ and $\hat{x} \in \Pi_C^f x$,*

$$\phi(y, \hat{x}) + G(\hat{x}, Jx) \leq G(y, Jx)$$

for all $y \in C$.

Lemma 7 [24]. *Let E be a Banach space and let $f : E \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous mapping with convex domain $D(f)$. If $\{x_n\}$ is a sequence in $D(f)$ such that $x_n \rightharpoonup \hat{x} \in D(f)$ and $\lim_{n\to\infty} G(x_n, Jy) = G(\hat{x}, Jy)$, then $\lim_{n\to\infty} \|x_n\| = \|\hat{x}\|$.*

Remark 3 Let E be a uniformly convex and uniformly smooth Banach space and $f(y) = 0$ for all $y \in E$. Then Lemma 6 reduces to the property of the generalized projection operator considered by Alber [5].

If $f(y) \geq 0$ for all $y \in C$ and $f(0) = 0$, then the definition of a totally quasi-ϕ-asymptotically nonexpansive S is equivalent to the following:

If $F(S) \neq \emptyset$ and there exist nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$, respectively, and a strictly increasing continuous function $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ with $\psi(0) = 0$ such that

$$G(p, S^n x) \leq G(p, x) + \nu_n \psi G(p, x) + \mu_n$$

for all $x \in C$, $p \in F(S)$ and $n \geq 1$.

Let θ be a bifunction from $C \times C$ to $\mathbf{R}$, where $\mathbf{R}$ denotes the set of real numbers. *The equilibrium problem* (for short, (EP)) is to find $\hat{x} \in C$ such that

$$\theta(\hat{x}, y) \geq 0 \tag{7}$$

for all $y \in C$. The set of solutions of (EP) (7) is denoted by $EP(\theta)$.

For solving the equilibrium problem for a bifunction $\theta : C \times C \to \mathbf{R}$, let us assume that θ satisfies the following conditions:

(A1) $\theta(x, x) = 0$ for all $x \in C$;

(A2) θ is monotone, *i.e.*, $\theta(x, y) + \theta(y, x) \leq 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$,

$$\lim_{t\downarrow 0} \theta(tz + (1 - t)x, y) \leq \theta(x, y);$$

(A4) for all $x \in C$, $y \mapsto \theta(x, y)$ is convex and lower semicontinuous.

For example, let B be a continuous and monotone operator of C into E^* and define

$$\theta(x, y) = \langle Bx, y - x \rangle$$

for all $x, y \in C$. Then θ satisfies (A1)-(A4).

Lemma 8 [6] *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E and let θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying the conditions (A1)-(A4). Then, for any $r > 0$ and $x \in E$, there exists $z \in C$ such that*

$$\theta(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0$$

for all $y \in C$.

Lemma 9 [41] *Let C be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E and let θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying the conditions (A1)-(A4). For all $r > 0$ and $x \in E$, define a mapping $T_r^\theta : E \to C$ as follows:*

$$T_r^\theta x = \{z \in C : \theta(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C\}.$$

Then the following hold:
(1) *T_r^θ is single-valued;*
(2) *T_r^θ is a firmly nonexpansive-type mapping [22], that is, for all $x, y \in E$,*

$$\langle T_r^\theta x - T_r^\theta y, JT_r^\theta x - JT_r^\theta y \rangle \leq \langle T_r^\theta x - T_r^\theta y, Jx - Jy \rangle;$$

(3) *$F(T_r^\theta) = EP(\theta)$;*
(4) *$EP(\theta)$ is closed and convex.*

Lemma 10 [41] *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E and let θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying the conditions (A1)-(A4). Then, for any $r > 0$, $x \in E$ and $q \in F(T_r^\theta)$,*

$$\phi(q, T_r^\theta x) + \phi(T_r^\theta x, x) \leq \phi(q, x).$$

An operator $A \subset E \times E^*$ is said to be *monotone* if

$$\langle x - y, x^* - y^* \rangle \geq 0$$

for all $(x, x^*), (y, y^*) \in A$. A point $z \in E$ is called a *zero point* of A if

$$0 \in Az. \tag{8}$$

We denote the set of zeroes of the operator A by $A^{-1}0$, that is,

$$A^{-1}0 = \{z \in E : 0 \in Az\}.$$

A monotone $A \subset E \times E^*$ is said to be *maximal* if its graph $G(A) = \{(x, y^*) : y^* \in Ax\}$ is not property contained in the graph of any other monotone operator. If A is maximal monotone, then the solution set $A^{-1}0$ is closed and convex.

Let E be a smooth strictly convex and reflexive Banach space, let C be a nonempty closed convex subset of E and let $A \subset E \times E^*$ be a monotone operator satisfying $D(A) \subset C \subset J^{-1}(\cap_{\lambda > 0} R(J + \lambda A))$. Then the *resolvent* $J_\lambda : C \to D(A)$ of A is defined by

$$J_\lambda x = \{ z \in D(A) : Jx \in Jz + \lambda Az, \forall x \in C \}.$$

J_λ is a single-valued mapping from E to $D(A)$. On the other words, $J_\lambda = (J + \lambda A)^{-1} J$ for all $\lambda > 0$.

For any $\lambda > 0$, the *Yosida approximation* $A_\lambda : C \to E^*$ of A is defined by $A_\lambda x = \frac{Jx - JJ_\lambda x}{\lambda}$ for all $x \in C$. We know that $A_\lambda x \in A(J_\lambda x)$ for all $\lambda > 0$ and $x \in E$. Since relatively quasi-nonexpansive mappings and quasi-ϕ-nonexpansive mappings are the same, we can see that J_λ is a quasi-ϕ-nonexpansive mapping (see [39, Theorem 4.7]).

Lemma 11 [19] *Let E be a smooth strictly convex and reflexive Banach space, let C be a nonempty closed convex subset of E and let $A \subset E \times E^*$ be a monotone operator satisfying $D(A) \subset C \subset J^{-1}(\cap_{\lambda > 0} R(J + \lambda A))$. For any $\lambda > 0$, let J_λ and A_λ be the resolvent and the Yosida approximation of A, respectively. Then the following hold:*

(1) $\phi(p, J_\lambda x) + \phi(J_\lambda x, x) \le \phi(p, x)$ for all $x \in C$ and $p \in A^{-1}0$;
(2) (b) $(J_\lambda x, A_\lambda x) \in A$ for all $x \in C$;
(3) (c) $F(J_\lambda) = A^{-1}0$.

Lemma 12 [32] *Let E be a reflexive strictly convex and smooth Banach space. Then an operator $A \subset E \times E^*$ is maximal monotone if and only if $R(J + \lambda A) = E^*$ for all $\lambda > 0$.*

3 Main result

Now, we give the main results in this paper.

Theorem 1 *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. For each $i = 1, 2, \cdots, m$, let θ_i be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying the conditions (A1)-(A4). Let $A_j \subset E \times E^*$ be a maximal monotone operator satisfying $D(A_j) \subset C$ and $J^{A_j}_{\lambda_{j,n}} = (J + \lambda_{j,n} A_j)^{-1} J$ for all $\lambda_{j,n} > 0$ and $j = 1, 2, \cdots, l$. Let $S : C \to C$ be a closed and totally quasi-ϕ-asymptotically nonexpansive mapping with nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$, respectively, and a strictly increasing continuous function $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ with $\psi(0) = 0$. Let $f : E \to \mathbf{R}^+$ be a convex and lower semicontinuous function with $C \subset int(D(f))$ and $f(0) = 0$. Assume that S is uniformly L-Lipschitz continuous and $\mathcal{F} = F(S) \cap (\cap_{i=1}^m EP(\theta_i)) \cap (\cap_{j=1}^l A_j^{-1}0) \neq \emptyset$. For any initial point $x_1 \in E$, define $C_1 = C$ and the sequence $\{x_n\}$ in C by*

$$\begin{cases} z_n = J^{A_l}_{\lambda_{l,n}} \circ J^{A_{l-1}}_{\lambda_{l-1,n}} \circ \cdots \circ J^{A_1}_{\lambda_{1,n}} x_n, \\ u_n = T^{\theta_m}_{r_{m,n}} \circ T^{\theta_{m-1}}_{r_{m-1,n}} \circ \cdots \circ T^{\theta_1}_{r_{1,n}} z_n, \\ y_n = J^{-1}(\alpha_n Jx_1 + \beta_n JS^n x_n + \gamma_n Ju_n), \\ C_{n+1} = \{ v \in C_n : G(v, Jy_n) \le \alpha_n G(v, Jx_1) + (1 - \alpha_n) G(v, Jx_n) + \zeta_n \}, \\ x_{n+1} = \Pi^f_{C_{n+1}} x_1 \end{cases} \qquad (9)$$

for each $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\zeta_n = \nu_n \sup_{q \in \mathcal{F}} \psi(G(q, Jx_n)) + \mu_n$ and for each $i = 1, 2, \cdots, m$, $\{r_{i,n}\} \subset [d, \infty)$ for some $d > 0$. If, for each $j = 1, 2, \cdots, l$, $\liminf_{n \to \infty} \lambda_{j,n} > 0$, $\lim_{n \to \infty} \alpha_n = 0$ and $\liminf_{n \to \infty} \beta_n < 1$, then the sequence $\{x_n\}$ converges strongly to a point $\Pi_{\mathcal{F}}^f x_1$.

Proof. We split the proof into five steps.

Step 1. We first show that C_n is closed and convex for all $n \geq 1$. From the definitions of $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for all $n \geq 1$. For any $b \in C_n$, we know that $G(b, Jy_n) \leq G(b, Jx_n) + \zeta_n$ is equivalent to the following:

$$2\alpha_n \langle b, Jx_1 \rangle + 2(1 - \alpha_n)\langle b, Jx_n \rangle - 2\langle b, Jy_n \rangle$$
$$\leq \alpha_n \|x_1\|^2 + (1 - \alpha_n)\|x_n\|^2 - \|y_n\|^2 + \zeta_n.$$

Therefore, C_{n+1} is closed and convex for all $n \geq 1$.

Step 2. We show that $\mathcal{F} \subset C_n$ for all $n \geq 1$. Now, we show by induction that $\mathcal{F} \subset C_n$ for all $n \geq 1$. It is obvious that $\mathcal{F} \subset C_1 = C$. Suppose that $\mathcal{F} \subset C_n$ for some $n \geq 1$. Define $u_n = K_n^m z_n$, when $K_n^i = T_{r_{i,n}}^{\theta_i} T_{r_{i-1,n}}^{\theta_{i-1}} \cdots T_{r_{1,n}}^{\theta_1}$ for all $j = 1, 2, \cdots, m$ with $K_n^0 = I$ and define $z_n = \Delta_n^l x_n$ when $\Delta_n^j := J_{\lambda_{j,n}}^{A_j} \circ J_{\lambda_{j-1,n}}^{A_{j-1}} \circ \cdots \circ J_{\lambda_{1,n}}^{A_1}$ for all $j = 1, 2, \cdots, l$ with $\Delta_n^0 = I$. Let $q \in \mathcal{F}$. Then we have

$$\begin{aligned}
G(q, Ju_n) &= G(q, JK_n^m z_n) \\
&\leq G(q, Jz_n) \\
&= G(q, J(\Delta_n^l x_n)) \\
&\leq G(q, Jx_n).
\end{aligned} \tag{10}$$

Since S is a totally quasi-ϕ-asymptotically nonexpansive mapping, from (10) we have

$$\begin{aligned}
&G(q, Jy_n) \\
&= G(q, (\alpha_n Jx_1 + \beta_n JS^n x_n + \gamma_n Ju_n)) \\
&= \|q\|^2 - 2\alpha_n \langle q, Jx_1 \rangle - 2\beta_n \langle q, JS^n x_n \rangle - 2\gamma_n \langle q, Ju_n \rangle \\
&\quad + \|\alpha_n Jx_1 + \beta_n JS^n x_n + \gamma_n Ju_n\|^2 + 2\rho f(q) \\
&\leq \|q\|^2 - 2\alpha_n \langle q, Jx_1 \rangle - 2\beta_n \langle q, JS^n x_n \rangle - 2\gamma_n \langle q, Ju_n \rangle \\
&\quad + \alpha_n \|Jx_n\|^2 + \beta_n \|JS^n x_n\|^2 + \gamma_n \|Ju_n\|^2 + 2\rho f(q) \\
&= \alpha_n G(q, Jx_1) + \beta_n G(q, JS^n x_n) + \gamma_n G(q, Ju_n) \\
&\leq \alpha_n G(q, Jx_1) + \beta_n(G(q, Jx_n) + \nu_n \psi(G(q, Jx_n)) + \mu_n) + \gamma_n G(q, Ju_n) \\
&\leq \alpha_n G(q, Jx_1) + \beta_n G(q, Jx_n) + \gamma_n G(q, Ju_n) + \beta_n(\nu_n \psi(G(q, Jx_n)) + \mu_n) \\
&\leq \alpha_n G(q, Jx_1) + \beta_n G(q, Jx_n) + \gamma_n G(q, Ju_n) + \nu_n \sup_{q \in \mathcal{F}} \psi(G(q, Jx_n)) + \mu_n \\
&\leq \alpha_n G(q, Jx_1) + \beta_n G(q, Jx_n) + \gamma_n G(q, Ju_n) + \zeta_n \\
&\leq \alpha_n G(q, Jx_1) + \beta_n G(q, Jx_n) + \gamma_n G(q, Jz_n) + \zeta_n \\
&\leq \alpha_n G(q, Jx_1) + \beta_n G(q, Jx_n) + \gamma_n G(q, Jx_n) + \zeta_n \\
&\leq \alpha_n G(q, Jx_1) + (\beta_n + \gamma_n) G(q, Jx_n) + \zeta_n \\
&= \alpha_n G(q, Jx_1) + (1 - \alpha_n) G(q, Jx_n) + \zeta_n.
\end{aligned}$$
$$\tag{11}$$

This shows that $q \in C_{n+1}$, which implies that $\mathcal{F} \subset C_{n+1}$ and so $\mathcal{F} \subset C_n$ for all $n \geq 1$ and the sequence $\{x_n\}$ is well defined.

Step 3. We show that $x_n \to p$, $y_n \to p$, $z_n \to p$ and $u_n \to p$ as $n \to \infty$. Since $f : E \to \mathbf{R}$ is a convex and lower semi-continuous function, from Lemma 4, we known that there exist $x^* \in E^*$ and $\alpha \in \mathbf{R}$ such that

$$f(x) \geq \langle x, x^* \rangle + \alpha$$

for all $x \in E$. Since $x_n \in C_n \subset E$, it follows that

$$
\begin{aligned}
G(x_n, Jx_1) &= \|x_n\|^2 - 2\langle x_n, Jx_1 \rangle + \|x_1\|^2 + 2\rho f(x_n) \\
&\geq \|x_n\|^2 - 2\langle x_n, Jx_1 \rangle + \|x_1\|^2 + 2\rho\langle x_n, x^* \rangle + 2\rho\alpha \\
&= \|x_n\|^2 - 2\langle x_n, Jx_1 - \rho x^* \rangle + \|x_1\|^2 + 2\rho\alpha \\
&\geq \|x_n\|^2 - 2\|x_n\|\|Jx_1 - \rho x^*\| + \|x_1\|^2 + 2\rho\alpha \\
&= (\|x_n\| - \|Jx_1 - \rho x^*\|)^2 + \|x_1\|^2 - \|Jx_1 - \rho x^*\|^2 + 2\rho\alpha.
\end{aligned}
\tag{12}
$$

For all $q \in \mathcal{F}$ and $x_n = \Pi_{C_n}^f x_1$, we have

$$G(q, Jx_1) \geq G(x_n, Jx_1) \geq (\|x_n\| - \|Jx_1 - \rho x^*\|)^2 + \|x_1\|^2 - \|Jx_1 - \rho x^*\|^2 + 2\rho\alpha.$$

That is, $\{x_n\}$ is bounded and so are $\{G(x_n, Jx_1)\}$ and $\{y_n\}$. By using the fact that $x_{n+1} = \Pi_{C_{n+1}}^f x_1 \in C_{n+1} \subset C_n$ and $x_n = \Pi_{C_n}^f x_1$, it follows from Lemma 6 and (3) that

$$0 \leq (\|x_{n+1}\| - \|x_n\|)^2 \leq \phi(x_{n+1}, x_n) \leq G(x_{n+1}, Jx_1) - G(x_n, Jx_1). \tag{13}$$

This implies that $\{G(x_n, Jx_1)\}$ is nondecreasing and so $\lim_{n\to\infty} G(x_n, Jx_1)$ exists. Taking $n \to \infty$, we obtain

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{14}$$

Since $\{x_n\}$ is bounded, E is reflexive and C_n is closed and convex for all $n \geq 1$. We can assume that $x_n \rightharpoonup p \in C_n$ as $n \to \infty$. From the fact that $x_n = \Pi_{C_n}^f x_1$, we get

$$G(x_n, Jx_1) \leq G(p, Jx_1) \tag{15}$$

for all $n \geq 1$. Since f is convex and lower semi-continuous, we have

$$
\begin{aligned}
\liminf_{n\to\infty} G(x_n, Jx_1) &= \liminf_{n\to\infty} \{\|x_n\|^2 - 2\langle x_n, Jx_1 \rangle + \|x_1\|^2 + 2\rho f(x_n)\} \\
&\geq \|p\|^2 - 2\langle p, Jx_1 \rangle + \|x_1\|^2 + 2\rho f(p) \\
&= G(x_n, Jx_1).
\end{aligned}
\tag{16}
$$

By (15) and (16), we get

$$G(p, Jx_1) \leq \liminf_{n\to\infty} G(x_n, Jx_1) \leq \limsup_{n\to\infty} G(x_n, Jx_1) \leq G(p, Jx_1).$$

That is, $\lim_{n\to\infty} G(x_n, Jx_1) = G(p, Jx_1)$, which implies that $\|x_n\| \to \|p\|$ as $n \to \infty$. Since E has the Kadec-Klee property, we obtain

$$\lim_{n\to\infty} x_n = p. \tag{17}$$

We also have

$$\lim_{n\to\infty} x_{n+1} = p. \tag{18}$$

From (17), we get

$$\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} \left(\nu_n \sup_{q\in\mathcal{F}} \psi(G(q, Jx_n)) + \mu_n\right) = 0. \tag{19}$$

From (17) and (18), we have $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. Since J is uniformly norm-to-norm continuous, it follows that

$$\lim_{n\to\infty} \|Jx_n - Jx_{n+1}\| = 0. \tag{20}$$

Moreover, since $x_{n+1} = \Pi_{C_{n+1}}^f x_1 \in C_{n+1} \subset C_n$ and (9), we have

$$G(x_{n+1}, Jy_n) \le \alpha_n G(x_{n+1}, Jx_1) + (1 - \alpha_n)G(x_{n+1}, Jx_n) + \zeta_n$$

is equivalent to the following:

$$\phi(x_{n+1}, y_n) \le \alpha_n \phi(x_{n+1}, x_1) + (1 - \alpha_n)\phi(x_{n+1}, x_n) + \zeta_n. \tag{21}$$

Since $\lim_{n\to\infty} \alpha_n = 0$, (14) and (19), we have

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0. \tag{22}$$

By (3), it follows that

$$\|y_n\| \to \|p\| \tag{23}$$

as $n \to \infty$. Since J is uniformly norm-to-norm continuous, we obtain

$$\|Jy_n\| \to \|Jp\| \tag{24}$$

as $n \to \infty$. This implies that $\{\|Jy_n\|\}$ is bounded in E^*, Since E^* is reflexive, we assume that $Jy_n \rightharpoonup y^* \in E^*$ as $n \to \infty$. In view of $J(E) = E^*$, there exists $y \in E$ such that $Jy = y^*$. It follows that

$$\begin{aligned}\phi(x_{n+1}, y_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n\rangle + \|y_n\|^2 \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n\rangle + \|Jy_n\|^2.\end{aligned} \tag{25}$$

Taking $\liminf_{n\to\infty}$ on both sides of the equality above, since $\|\cdot\|$ is weak lower semi-continuous, this yields that

$$\begin{aligned}0 &\ge \|p\|^2 - 2\langle p, y^*\rangle + \|y^*\|^2 \\ &= \|p\|^2 - 2\langle p, Jy\rangle + \|Jy\|^2 \\ &= \|p\|^2 - 2\langle p, Jy\rangle + \|y\|^2 \\ &= \phi(p, y).\end{aligned} \tag{26}$$

From Remark 2, $p = y$, which implies that $y^* = Jp$. It follows that $Jy_n \rightharpoonup Jp \in E^*$ as $n \to \infty$. From (24) and the Kadec-Klee property of E^*, we have $Jy_n \to Jp$ as $n \to \infty$. Note that $J^{-1} : E^* \to E$ is norm-weak*-continuous, that is, $y_{n_i} \rightharpoonup p$ as $n \to \infty$. From (23) and the Kadec-Klee property of E, we have

$$\lim_{n\to\infty} y_n = p. \tag{27}$$

From (11), we have

$$\begin{aligned}G(q, Jy_n) &\le \alpha_n G(q, Jx_1) + \beta_n G(q, Jx_n) + \gamma_n G(q, Jz_n) + \zeta_n \\ &\le \alpha_n G(q, Jx_1) + (1 - \alpha_n)G(q, Jx_n) + \zeta_n.\end{aligned}$$

From (17), (19), (27) and the conditions $\lim_{n\to\infty} \alpha_n = 0$, $\liminf_{n\to\infty} \beta_n < 1$, it follows that for any $q \in F$, $\lim_{n\to\infty} \phi(q, z_n) = \phi(q, p)$. Let $z_n = \Delta_n^l x_n$ for all $n \geq 1$. From Lemma 11(1), it follows that for any $q \in F$,

$$
\begin{aligned}
\phi(z_n, x_n) &= \phi(\Delta_n^l x_n, x_n) \\
&\leq \phi(q, x_n) - \phi(q, \Delta_n^l x_n) \\
&= \phi(q, x_n) - \phi(q, z_n).
\end{aligned}
$$

Taking $n \to \infty$ on both sides of the inequality above, we have

$$
\lim_{n\to\infty} \phi(z_n, x_n) = 0.
$$

From (3), it follows that $(\|x_n\| - \|z_n\|)^2 \to 0$ as $n \to \infty$. Since $\|x_n\| \to \|p\|$ as $n \to \infty$, we have

$$
\|z_n\| \to \|p\| \tag{28}
$$

as $n \to \infty$. Since J is uniformly norm-to-norm continuous on bounded subsets of E, it follows that

$$
\|Jz_n\| \to \|Jp\| \tag{29}
$$

as $n \to \infty$. This implies that $\{\|Jz_n\|\}$ is bounded in E^*. Since E^* is reflexive, we can assume that $Jz_n \rightharpoonup z^* \in E^*$ as $n \to \infty$. In view of $J(E) = E^*$, there exists $z \in E$ such that $Jz = z^*$, and so

$$
\begin{aligned}
\phi(x_n, z_n) &= \|x_n\|^2 - 2\langle x_n, Jz_n \rangle + \|z_n\|^2 \\
&= \|x_n\|^2 - 2\langle x_n, Jz_n \rangle + \|Jz_n\|^2.
\end{aligned} \tag{30}
$$

Taking $\liminf_{n\to\infty}$ on both sides of the equality above, from the weak lower semi-continuity of the norm $\|\cdot\|$, it follows that

$$
\begin{aligned}
0 &\geq \|p\|^2 - 2\langle p, z^* \rangle + \|z^*\|^2 \\
&= \|p\|^2 - 2\langle p, Jz \rangle + \|Jz\|^2 \\
&= \|p\|^2 - 2\langle p, Jz \rangle + \|z\|^2 \\
&= \phi(p, z).
\end{aligned} \tag{31}
$$

From Remark 2, we have $p = z$, which implies that $z^* = Jp$ and so $Jz_n \rightharpoonup Jp \in E^*$ as $n \to \infty$. From (29) and the Kadec-Klee property of E^*, we have $Jz_n \to Jp$ as $n \to \infty$. Since J^{-1} is norm-weak*-continuous, that is, $z_n \rightharpoonup p$, from (28) and the Kadec-Klee property of E, it follows that

$$
\lim_{n\to\infty} z_n = p. \tag{32}
$$

From (11), we have

$$
\begin{aligned}
G(q, Jy_n) &\leq \alpha_n G(q, Jx_1) + \beta_n G(q, Jx_n) + \gamma_n G(q, Ju_n) + \zeta_n \\
&\leq \alpha_n G(q, Jx_1) + (1 - \alpha_n) G(q, Jx_n) + \zeta_n.
\end{aligned}
$$

From (17), (19), (27) and the conditions $\lim_{n\to\infty} \alpha_n = 0$, $\liminf_{n\to\infty} \beta_n < 1$, it follows that $\lim_{n\to\infty} \phi(q, u_n) = \phi(q, p)$. From Lemma 9, it follows that for any $q \in F$ and $u_n = K_n^m z_n$,

$$
\begin{aligned}
\phi(u_n, x_n) &= \phi(K_n^m z_n, x_n) \\
&\leq \phi(q, z_n) - \phi(q, K_n^m x_n) \\
&= \phi(q, z_n) - \phi(q, u_n).
\end{aligned}
$$

Taking $n \to \infty$ on both sides of the inequality above, we have

$$\lim_{n \to \infty} \phi(x_n, u_n) = 0. \tag{33}$$

From (3), we have

$$(\|x_n\| - \|u_n\|)^2 \to 0 \tag{34}$$

as $n \to \infty$. Since $\|x_n\| \to \|p\|$, we have

$$\|u_n\| \to \|p\| \tag{35}$$

as $n \to \infty$, and so

$$\|Ju_n\| \to \|Jp\| \tag{36}$$

as $n \to \infty$. That is, $\{\|Ju_n\|\}$ is bounded in E^*. Since E^* is reflexive, we can assume that $Ju_n \rightharpoonup u^* \in E^*$ as $n \to \infty$. In view of $J(E) = E^*$, there exists $u \in E$ such that $Ju = u^*$. It follows that

$$\begin{aligned} \phi(x_{n+1}, u_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n \rangle + \|u_n\|^2 \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n \rangle + \|Ju_n\|^2. \end{aligned} \tag{37}$$

Taking $\liminf_{n \to \infty}$ on both sides of the equality above, since $\|\cdot\|$ is weak lower semi-continuous, it follows that

$$\begin{aligned} 0 &\geq \|p\|^2 - 2\langle p, u^* \rangle + \|u^*\|^2 \\ &= \|p\|^2 - 2\langle p, Ju \rangle + \|Ju\|^2 \\ &= \|p\|^2 - 2\langle p, Ju \rangle + \|u\|^2 \\ &= \phi(p, u). \end{aligned} \tag{38}$$

From Remark 2, $p = u$, that is, $u^* = Jp$. It follows that $Ju_n \rightharpoonup Jp \in E^*$. From (36) and the Kadec-Klee property of E^*, we have $Ju_n \to Jp$ as $n \to \infty$. Since J^{-1} is norm-weak*-continuous, that is, $u_n \rightharpoonup p$ as $n \to \infty$. From (35) and the Kadec-Klee property of E, we have

$$\lim_{n \to \infty} u_n = p. \tag{39}$$

Step 4. We show that $p \in \mathcal{F} = F(S) \cap (\cap_{i=1}^m EP(\theta_i)) \cap (\cap_{j=1}^l A_j^{-1} 0)$. First, we show that $p \in \cap_{j=1}^l A_j^{-1} 0$. Let $z_n = \Delta_n^l x_n$ for each $n \geq 1$. Then, for any $q \in F$, it follows that for each $j = 1, 2, \cdots, l$,

$$\begin{aligned} \phi(q, z_n) &= \phi(q, \Delta_n^l x_n) \\ &\leq \phi(q, \Delta_n^{l-1} x_n) \\ &\leq \phi(q, \Delta_n^{l-2} x_n) \\ &\quad \cdots \\ &\leq \phi(q, \Delta_n^j x_n). \end{aligned} \tag{40}$$

By Lemma 11, for each $j = 1, 2, \cdots, m$, we have

$$\begin{aligned} \phi(\Delta_n^j x_n, x_n) &\leq \phi(q, x_n) - \phi(q, \Delta_n^j x_n) \\ &\leq \phi(q, x_n) - \phi(q, z_n). \end{aligned} \tag{41}$$

Since $x_n \to p$ and $z_n \to p$ as $n \to \infty$, we get $\phi(\Delta_n^j x_n, x_n) \to 0$ as $n \to \infty$ for all $j = 1, 2, \cdots, m$. From (3), it follows that

$$(\|\Delta_n^j x_n\| - \|x_n\|)^2 \to 0$$

Computational of generalized projection method for maximal monotone operators and a countable family of relatively quasi-nonexpansive mappings

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Computational of generalized projection method for maximal monotone operators and a countable family of relatively quasi-nonexpansive mappings

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We introduce a new hybrid projection method in mathematical programming for finding a common element of the set of common fixed points of a countable family of relatively quasi-nonexpansive mappings, the set of the variational inequality for an α-inverse-strongly monotone operator, the set of solutions of the mixed equilibrium problem and a zero of a maximal monotone operator in the framework of a real Banach space. We obtain a strong convergence theorem for the sequences generated by this process in a 2-uniformly convex and uniformly smooth Banach space. Furthermore, we give some numerical examples which support our main theorem in the last part.

1. Introduction and preliminaries

Let E be a real Banach space with $\|\cdot\|$ and E^* the dual space of E. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. A Banach space E is said to be *strictly convex* if $\left\|\frac{x+y}{2}\right\| < 1$ for all $x, y \in U$. A Banach space E is said to be *smooth* if the limit $\lim_{t \to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in U$. Let E be a Banach space. The *modulus of convexity* of E is the function $\delta : [0, 2] \to [0, 1]$ defined by $\delta(\varepsilon) = \inf \left\{ 1 - \left\|\frac{x+y}{2}\right\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon \right\}$. A Banach space E is *uniformly convex* if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0, 2]$; see [1,2] for more details. Observe that every p-uniform convex is uniformly convex. One should note that a Banach space is p-uniform convex for $1 < p < 2$. It is well known that a Hilbert space is *2-uniformly convex*, uniformly smooth. For each $p > 1$, the *generalized duality mapping*

$J_p : E \to 2^{E^*}$ is defined by $J_p(x) = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1} \right\}$ for all $x \in E$. In particular, $J = J_2$ is called the *normalized duality mapping*. If E is a Hilbert space, then $J = I$, where I is the identity mapping. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

Let E be a real Banach space and let C be a nonempty closed convex subset of E and $A : C \to E^*$ be an operator. *The classical variational inequality problem* for an operator A is to find $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solution of (1.1) is denoted by $VI(A, C)$. Recall that let $A : C \to E^*$ be a mapping. Then, A is called

(i) *monotone* if
$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C,$$

(ii) $\alpha-$*inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that
$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

The class of inverse-strongly monotone has been studied by many researchers to approximating a common fixed point; see [3–6] for more details.

Consider the problem of finding:

$$v \in E \text{ such that } 0 \in T(v), \tag{1.2}$$

where T is an operator from E into E^*. Such $v \in E$ is called a *zero point* of T. When T is a maximal monotone operator, a well-know methods for solving (1.2) in a Hilbert space H is the *proximal point algorithm*: $x_1 = x \in H$ and,

$$x_{n+1} = J_{r_n} x_n, \quad n = 1, 2, 3, \ldots, \tag{1.3}$$

where $J_{r_n} = (I + r_n T)^{-1}$ and $\{r_n\} \subset (0, \infty)$, then Rockafellar [7] proved that the sequence $\{x_n\}$ converges weakly to an element of $T^{-1}(0)$.

Let θ be a bifunction of $C \times C$ into $\mathbb{R}$ and $\varphi : C \to \mathbb{R}$ be a real-valued function. The *mixed equilibrium problem*, denoted by $MEP(\theta, \varphi)$ is to find $x \in C$ such that

$$\theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.4}$$

If $\varphi \equiv 0$, the problem (1.4) reduce into the *equilibrium problem for θ*, denoted by $EP(\theta)$, is to find $x \in C$ such that
$$\theta(x, y) \geq 0, \quad \forall y \in C. \tag{1.5}$$

If $\theta \equiv 0$, the problem (1.4) reduce into the *minimize problem*, denoted by $Argmin(\varphi)$ is to find $x \in C$ such that
$$\varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.6}$$

The above formulation (1.5) was shown in [8] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational inequality problems, vector equilibrium problems and Nash equilibria in noncooperative games. In addition, there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem,

which can also be written in the form of an $EP(\theta)$. In other words, the $EP(\theta)$ is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. In the last two decades, many papers have appeared in the literature on the existence of solutions of $EP(\theta)$; see, for example,[8,9] and references therein. Some solution methods have been proposed to solve the $EP(\theta)$; see, for example,[3,4,8,10–14] and references therein.

Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \text{for } x, y \in E. \tag{1.7}$$

It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(y, x) \le (\|y\| + \|x\|)^2, \quad \forall x, y \in E. \tag{1.8}$$

Remark 1.1 If E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. From (1.7), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy \rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$; see [15,16] for more details.

As well know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [17] recently introduced the *generalized projection* $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x), \tag{1.9}$$

existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, for example, [15–19]).

If E is a Hilbert space, then $\phi(y, x) = \|y - x\|^2$ and Π_C becomes the metric projection of E onto C. Let Π_C be the generalized projection from a smooth, strictly convex and reflexive Banach space E onto a nonempty closed convex subset C of E. Then, Π_C is a closed relatively quasi-nonexpansive mapping from E onto C with $F(\Pi_C) = C$ (see also [13]).

Let C be a closed convex subset of E, a mapping $S : C \to C$ is said to be *nonexpansive* if $\|Sx - Sy\| \le \|x - y\|$, $\forall x, y \in C$. A point $x \in C$ is a *fixed point* of S provided $Sx = x$. Denote by $F(S)$ the set of fixed points of S; that is, $F(S) = \{x \in C : Sx = x\}$. Recall that a point p in C is said to be an *asymptotic fixed point* of S [20] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$. The set of asymptotic fixed points of S will be denoted by $\widehat{F(S)}$. A mapping S from C into itself is said to be *relatively nonexpansive* [21–23] if $\widehat{F(S)} = F(S)$ and $\phi(p, Sx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(S)$. The asymptotic behaviour of a relatively nonexpansive mapping was studied in [24–26]. S is said to be *ϕ-nonexpansive*, if $\phi(Sx, Sy) \le \phi(x, y)$ for $x, y \in C$. S is said to be *relatively quasi-nonexpansive* if $F(S) \ne \emptyset$ and $\phi(p, Sx) \le \phi(p, x)$ for $x \in C$ and $p \in F(S)$. We note that the class of relatively quasi-nonexpansive mappings is more general than the class of relatively nonexpansive mappings [24–28] which requires the strong restriction: $F(S) = \widehat{F(S)}$.

Definition 1.2   Let C be a nonempty subset of E and let $\{T_n\}_{n=1}^{\infty}$ be a countable family of mappings from C into E. A point p in C is called an *asymptotic fixed point* of $\{T_n\}_{n=1}^{\infty}$ [20] if C contains a sequence $\{x_n\}_{n=1}^{\infty}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$. The asymptotic fixed point set of $\{T_n\}_{n=1}^{\infty}$ will be denoted by $\widehat{F}\left(\{T_n\}_{n=1}^{\infty}\right)$. A mapping T_n from C into itself is called *countable family of relatively nonexpansive mappings* if

(R1) $F\left(\{T_n\}_{n=1}^{\infty}\right)$ is nonempty;
(R2) $\phi(p, T_n x) \leq \phi(p, x)$ for all $x \in C$ and $p \in F\left(\{T_n\}_{n=1}^{\infty}\right)$;
(R3) $\widehat{F}\left(\{T_n\}_{n=1}^{\infty}\right) = F\left(\{T_n\}_{n=1}^{\infty}\right)$.

A sequence $\{T_n\}_{n=1}^{\infty}$ is called *countable family of relatively quasi-nonexpansive mappings* (or *countable family of quasi-ϕ-nonexpansive mappings*) if conditions $(R1)$ and $(R2)$ hold. It is obvious that a countable family of relatively nonexpansive mappings is a countable family of relatively quasi-nonexpansive mappings but the converse is not true. In order to explain this better, we give the following example.

Example 1.3   Let $E = \mathbb{R}$ with the usual norm and define a mapping $T_n : E \longrightarrow E$ by

$$T_n(x) = \begin{cases} 0, & \text{if } x \leq \frac{1}{n}; \\ \frac{1}{n}, & \text{if } x > \frac{1}{n}, \end{cases}$$

for all $n \geq 0$ and for each $x \in \mathbb{R}$.

It is easy to see that $\bigcap_{n=1}^{\infty} F(T_n) = F(T_n) = \{0\}$ and

$$\phi(0, T_n x) = \|0 - T_n x\| \leq \|0 - x\| = \phi(0, x), \qquad \forall x \in \mathbb{R}.$$

Therefore, *T* is a relatively quasi-nonexpansive mapping but not a relatively nonexpansive mapping.

In 2004, Matsushita and Takahashi [29] introduced the following iteration: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J S x_n), \tag{1.10}$$

where the initial guess element $x_0 \in C$ is arbitrary, $\{\alpha_n\}$ is a real sequence in $[0, 1]$, S is a relatively nonexpansive mapping and Π_C denotes the generalized projection from E onto a closed convex subset C of E. They proved that the sequence $\{x_n\}$ converges weakly to a fixed point of S. In the sane year, Kamimura et al. [30] considered the algorithm (1.11) in a uniformly smooth and uniformly convex Banach space E, namely

$$x_{n+1} = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J(J_{r_n} x_n)), \quad n = 1, 2, 3, \ldots. \tag{1.11}$$

They proved that the algorithm (1.11) converges weakly to some element of $T^{-1} 0$.

In 2005, Matsushita and Takahashi [27] proposed the following hybrid iteration method (it is also called the CQ method) with generalized projection for relatively nonexpansive mapping S in a Banach space E:

$$\begin{cases} x_0 \in C \ \text{chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J S x_n), \\ C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, J x_0 - J x_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0. \end{cases} \quad (1.12)$$

They proved that $\{x_n\}$ converges strongly to $\Pi_{F(S)} x_0$, where $\Pi_{F(S)}$ is the generalized projection from C onto $F(S)$. In 2008, Iiduka and Takahashi [31] introduced the following iterative scheme for finding a solution of the variational inequality problem for an inverse-strongly monotone operator A in a 2-uniformly convex and uniformly smooth Banach space $E : x_1 = x \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(J x_n - \lambda_n A x_n), \quad (1.13)$$

for every $n = 1, 2, 3, \ldots$, where Π_C is the generalized metric projection from E onto C, J is the duality mapping from E into E^* and $\{\lambda_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (1.13) converges weakly to some element of $VI(A, C)$.

Let $\{S_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$ satisfy the following condition (*the AKTT-condition*): if for each bounded subset B of C

$$\sum_{n=1}^{\infty} \sup\{\|S_{n+1} z - S_n z\| : z \in B\} < \infty. \quad (1.14)$$

Assume that if the mapping $S : C \to C$ defined by $Sx = \lim_{n \to \infty} S_n x$ for all $x \in C$, then $\lim_{n \to \infty} \sup\{\|Sz - S_n z\| : z \in C\} = 0$. Aoyama et al. [32, Lemma 3.1] proved that the sequence $\{S_n\}$ converges strongly to a point in C for all $x \in C$. Takahashi et al. [33] studied the strong convergence theorem by the new hybrid method for a countable family of nonexpansive mappings in Hilbert spaces: $x_0 \in H$, $C_1 = C$ and $x_1 = P_{C_1} x_0$ and let

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n) S_n x_n, \\ C_{n+1} = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases} \quad (1.15)$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$ and $\{S_n\}$ is a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$. They prove that if $\{S_n\}$ satisfies some appropriate conditions, then $\{x_n\}$ converge strongly to $P_{\bigcap_{n=1}^{\infty} F(S_n)} x_0$.

In 2009, Takahashi and Zembayashi [34] studied the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem in the framework of Banach spaces. Also, Takahashi and Zembayashi [35] proved the following iteration for a relatively nonexpansive mapping:

$$\begin{cases} y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J S x_n), \\ u_n \in C \ \text{such that} \ \theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x, \end{cases} \quad (1.16)$$

where J is the duality mapping on E. Then, $\{x_n\}$ converges strongly to $\Pi_{F(S) \cap EP(\theta)} x$, where $\Pi_{F(S) \cap EP(\theta)}$ is the generalized projection of E onto $F(S) \cap EP(\theta)$.

In 2009, Inoue et al. [36] proved strong convergence theorem for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a

relatively nonexpansive mapping by using the hybrid method for defined a sequence $\{x_n\}$ as follows:

$$
\begin{cases}
u_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J S J_{r_n} x_n), \\
C_n = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n), \\
Q_n = \{z \in C_n : \langle x_n - z, J x_0 - J x_n \rangle \geq 0\}, \\
x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 1,
\end{cases}
\tag{1.17}
$$

and, under some control conditions, they proved that the sequence $\{x_n\}$ converge strongly to a point $\Pi_{F(S) \cap T^{-1}0}$. After that, Klin-eam et al. [9], extended Inoue et al. [36] to obtain the strong convergence theorem for finding a common element of the zero point set of a maximal monotone operator and the set fixed point set of two relatively nonexpansive mappings in a Banach space.

On the other hand, we also consider the following condition for the countable family mapping was introduced by Nakajo et al. [37]. Let C be a nonempty closed convex subset of a Hilbert space H, let $\{S_n\}$ be a family of mappings of C into itself with $F := \cap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and $\omega_w(z_n)$ denotes the set of all weak subsequential limits of a bounded sequence $\{z_n\}$ in C. $\{S_n\}$ is said to satisfy *the NST-condition* if for every bounded sequence $\{z_n\}$ in C,

$$
\lim_{n \to \infty} \|z_n - S_n z_n\| = 0 \quad \text{implies} \quad \omega_w(z_n) \subset F.
$$

In 2010, Boonchari and Saejung [38] used the more general condition so-called *the (∗)-condition* (see (3.1) and more also example see [39]) for a countable family of relatively quasi-nonexpansive mappings to obtained the strong convergence theorems in a real Banach space.

Remark 1.4   The following questions naturally arise in connection with the above results.

- Is it possible to construct an approximate fixed point sequence for finding common fixed points of a countable family of more general relatively quasi nonexpansive mappings the set of the variational inequality for an α-inverse-strongly monotone operator, the set of solutions of the mixed equilibrium problem and a zero of a maximal monotone operator in more general Banach spaces?
- Can the algorithms (1.15), (1.16) and (1.17) still be valid for relatively quasi-nonexpansive mappings which more general than relatively nonexpansive mappings?
- Can the algorithms (1.16) and (1.17) still valid for a countable family of relatively quasi-nonexpansive mappings under some weaker conditions?

Employing the ideas of Iiduka and Takahashi [31], Takahashi and Zembayashi [34, 35], Cholamjiak [40], Inoue et al. [36], Klin-eam et al. [9] and Plubtieng and Sriprad [41], we introduce a new hybrid projection method in mathematical programming which modifying and combining the algorithms (1.11), (1.13), (1.15), (1.16) and (1.17) by using the new condition. Consequently, we obtain strong convergence theorems for finding a common element of the set of fixed points of a countable family of more general relatively quasi nonexpansive mappings the set of the variational inequality for an α-inverse-strongly monotone operator, the set of solutions of the mixed equilibrium problem and a zero of a maximal monotone operator in a real uniformly smooth and uniformly convex Banach

space. Our main theorems in this paper improve and unify most of the results that have been proved for this important class of non-linear operators.

2. Basic results

We also need the following lemmas for the proof of our main results.

LEMMA 2.1 (Beauzamy [42] and Xu [43]) *If E be a 2-uniformly convex Banach space. Then for all $x, y \in E$ we have*

$$\|x - y\| \leq \frac{2}{c^2} \|Jx - Jy\|,$$

where J is the normalized duality mapping of E and $0 < c \leq 1$.

The best constant $\frac{1}{c}$ in Lemma is called the *p-uniformly convex constant* of E.

LEMMA 2.2 (Beauzamy [42] and Zalinescu [44]) *If E be a p-uniformly convex Banach space and let p be a given real number with $p \geq 2$. Then for all $x, y \in E$, $j_x \in J_p x$ and $j_y \in J_p y$*

$$\langle x - y, j_x - j_y \rangle \geq \frac{c^p}{2^{p-2} p} \|x - y\|^p,$$

where J_p is the generalized duality mapping of E and $\frac{1}{c}$ is the p-uniformly convexity constant of E.

LEMMA 2.3 (Kamimura and Takahashi [19]) *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

LEMMA 2.4 (Alber [17]) *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

LEMMA 2.5 (Alber [17]) *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

LEMMA 2.6 (Qin et al. [13]) *Let E be a uniformly convex and smooth Banach space, let C be a closed convex subset of E, and let T be a closed relatively quasi-nonexpansive mapping from C into itself. Then $F(T)$ is a closed convex subset of C.*

For solving the equilibrium problem for a bifunction $\theta : C \times C \to \mathbb{R}$, let us assume that θ satisfies the following conditions:

(A1) $\theta(x, x) = 0$ for all $x \in C$;
(A2) θ is monotone, i.e., $\theta(x, y) + \theta(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} \theta(tz + (1-t)x, y) \leq \theta(x, y);$$

(A4) for each $x \in C$, $y \mapsto \theta(x, y)$ is convex and lower semi-continuous.

LEMMA 2.7 (Blum and Oettli [8]) *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, let θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), and let $r > 0$ and $x \in E$. Then there exists $z \in C$ such that*

$$\theta(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C.$$

The following lemma is a spacial case of Zhang [45].

LEMMA 2.8 (Zhang [45]) *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let $\varphi : C \to \mathbb{R}$ is convex and lower semi-continuous and θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). For $r > 0$ and $x \in E$, then there exists $u \in C$ such that*

$$\theta(u, y) + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx \rangle \geq 0, \quad \forall y \in C.$$

Define a mapping $K_r : C \longrightarrow C$ as follows:

$$K_r(x) = \left\{ u \in C : \theta(u, y) + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx \rangle \geq 0, \; \forall y \in C \right\}$$

(2.1)

for all $x \in C$. Then the followings hold:

(1) K_r *is single-valued;*
(2) K_r *is firmly nonexpansive, i.e., for all $x, y \in E$, $\langle K_r x - K_r y, J K_r x - J K_r y \rangle \leq \langle K_r x - K_r y, Jx - Jy \rangle$;*
(3) $F(K_r) = MEP(\theta, \varphi)$;
(4) $MEP(\theta, \varphi)$ *is closed and convex;*
(5) $\phi(p, K_r z) + \phi(K_r z, z) \leq \phi(p, z), \forall p \in F(K_r)$ *and $z \in E$.*

Let E be a reflexive, strictly convex, smooth Banach space and J be the duality mapping from E into E^*. Then, J^{-1} is also single value, one-to-one and surjective, and it is the duality mapping from E^* into E. We make use of the following mapping V studied in Alber [17]

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2,$$

(2.2)

for all $x \in E$ and $x^* \in E^*$, that is, $V(x, x^*) = \phi(x, J^{-1}(x^*))$.

LEMMA 2.9 (Alber [17]) *Let E be a reflexive, strictly convex smooth Banach space and let V be as in (2.2). Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^* \rangle \leq V(x, x^* + y^*),$$

for all $x \in E$ and $x^*, y^* \in E^*$.

An operator $T \subset E \times E^*$ is said to be *monotone* if $\langle x - y, x^* - y^* \rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in T$. We denote the set $\{x \in E : 0 \in Tx\}$ by $T^{-1}0$. A monotone T is said to be *maximal* if its graph $G(T) = \{(x, y) : y \in Tx\}$ is not properly contained in the graph of any other monotone operator. If T is maximal monotone, then the solution set $T^{-1}0$ is closed and convex. Let E be a reflexive, strictly convex and smooth Banach space, it is known that T is a maximal monotone if and only if $R(J + rT) = E^*$ for all $r > 0$. Define the *resolvent* of T by $J_r = (J + rT)^{-1}J$ for all $r > 0$. J_r is a single-valued mapping from E to $D(T)$. Also, $T^{-1}(0) = F(J_r)$ for all $r > 0$, where $F(J_r)$ is the set of all fixed points of J_r. Define, for $r > 0$, the *Yosida approximation* of T by $T_r = (J - JJ_r)/r$. We know that $T_r x \in T(J_r x)$ for all $r > 0$ and $x \in E$.

LEMMA 2.10 (Kohsaka and Takahashi [46]) *Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty closed convex subset of E and let $T \subset E \times E^*$ be a monotone operator satisfying $D(T) \subset C \subset J^{-1}(\cap_{r>0} R(J + rT))$. Let $r > 0$, let J_r and T_r be the resolvent and the Yosida approximation of T, respectively. Then the following hold:*

   (i) $\phi(u, J_r x) + \phi(J_r x, x) \leq \phi(u, x), \quad \forall x \in C, u \in T^{-1}0;$
   (ii) $(J_r x, T_r x) \in T, \forall x \in C;$
   (iii) $F(J_r) = T^{-1}0.$

Let A be an inverse-strongly monotone mapping of C into E^* which is said to be *hemicontinuous* if for all $x, y \in C$, the mapping F of $[0, 1]$ into E^*, defined by $F(t) = A(tx + (1 - t)y)$, is continuous with respect to the weak* topology of E^*. We define by $N_C(v)$ *the normal cone* for C at a point $v \in C$, that is,

$$N_C(v) = \left\{ x^* \in E^* : \langle v - y, x^* \rangle \geq 0, \ \forall y \in C \right\}. \tag{2.3}$$

LEMMA 2.11 (Rockafellar [47]) *Let C be a nonempty, closed convex subset of a Banach space E and A a monotone, hemicontinuous operator of C into E^*. Let $U \subset E \times E^*$ be an operator defined as follows:*

$$Uv = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & otherwise. \end{cases} \tag{2.4}$$

Then U is maximal monotone and $U^{-1}0 = VI(A, C)$.

3. Main results

Let C be a closed subset of a Banach space E. Recall that a mapping $S : C \longrightarrow C$ is closed if for each $\{x_n\}$ in C, if $x_n \longrightarrow x$ and $Sx_n \longrightarrow y$, then $Sx = y$. Let $\{S_n\}$ be a family of mappings of C into itself with $F := \cap_{n=1}^{\infty} F(S_n) \neq \emptyset$, and $\{S_n\}$ is said to satisfy the $(*)$-condition if for each bounded sequence $\{z_n\}$ in C,

$$\lim_{n \longrightarrow \infty} \|z_n - S_n z_n\| = 0, \quad z_n \longrightarrow z \text{ imply } z \in F. \tag{3.1}$$

Remark 3.1 It follows directly from the definitions above that if $\{S_n\}$ satisfies NST-condition, then $\{S_n\}$ satisfies $(*)$-condition (see example [39, Example 1.6]). If $S_n \equiv S$ and S is closed, then $\{S_n\}$ satisfies $(*)$-condition.

In this section, by using the $(*)$-condition, we prove the new convergence theorem for finding a common element of the set of solutions of the mixed equilibrium problem, the common fixed point set of a countable family of relatively quasi-nonexpansive mappings, a zero of maximal monotone operators and the solution set of variational inequalities for an α-inverse strongly monotone mapping in a *2*-uniformly convex and uniformly smooth Banach space.

THEOREM 3.2 *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let $T \subset E \times E^*$ be a maximal monotone operator satisfying $D(T) \subset C$ and let $J_{r_n} = (J + r_n T)^{-1} J$ for all $r_n > 0$. Let θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), and let $\varphi : C \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $\|Ay\| \leq \|Ay - Au\|$, $\forall y \in C$ and $u \in VI(A, C) \neq \emptyset$. Let $S_n : C \to C$ be a countable family of relatively quasi-nonexpansive mappings such that satisfies the $(*)$-condition and $\Theta := \left(\cap_{n=1}^\infty F(S_n)\right) \cap T^{-1}0 \cap MEP(\theta, \varphi) \cap VI(A, C) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define the sequence $\{x_n\}$ as follows:*

$$
\begin{cases}
z_n = \Pi_C J^{-1}(J x_n - \lambda_n A x_n), \\
y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J S_n J_{r_n} z_n), \\
u_n \in C \text{ such that } \theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\
C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \alpha_n \phi(z, x_n) + (1 - \alpha_n)\phi(z, z_n) \leq \phi(z, x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1,
\end{cases}
$$

(3.2)

where J is the duality mapping on E, $\{\alpha_n\}$ is sequence in $[0, 1]$ and $\{r_n\} \subset [d, \infty)$ for some $d > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2 \alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E. If $\liminf_{n \to \infty}(1 - \alpha_n) > 0$, then $\{x_n\}$ converges strongly to $p \in \Theta$, where $p = \Pi_\Theta x_0$.

Proof We first show that C_{n+1} is closed and convex for each $n \geq 1$. Clearly, $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for each $n \in \mathbb{N}$. Since for any $z \in C_n$, we known $\phi(z, u_n) \leq \phi(z, x_n)$ is equivalent to $2\langle z, J x_n - J u_n \rangle \leq \|x_n\|^2 - \|u_n\|^2$. So, C_{n+1} is closed and convex. Next, we show that $\Theta \subset C_n$ for all $n \geq 1$. It is obvious that $\Theta \subset C_1 = C$, suppose that $\Theta \subset C_n$ for $n \in \mathbb{N}$. Indeed, put $u_n = K_{r_n} y_n$ and $v_n = J_{r_n} z_n$ for all $n \geq 1$. On the other hand, from Lemma 2.8, Lemma 2.10, by the convexity of $\| \cdot \|^2$, and S_n is a family of relatively quasi-nonexpansive mappings, for each $q \in \Theta \subset C_n$, we have

$$
\begin{aligned}
\phi(q, u_n) &= \phi(q, K_{r_n} y_n) \\
&\leq \phi(q, y_n) \\
&= \phi(q, J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J S_n v_n) \\
&= \|q\|^2 - 2\langle q, \alpha_n J x_n + (1 - \alpha_n) J S_n v_n \rangle + \|\alpha_n J x_n + (1 - \alpha_n) J S_n v_n\|^2 \\
&\leq \|q\|^2 - 2\alpha_n \langle q, J x_n \rangle - 2(1 - \alpha_n)\langle q, J S_n v \rangle + \alpha_n \|x_n\|^2 + (1 - \alpha_n)\|S_n v_n\|^2 \\
&= \alpha_n \phi(q, x_n) + (1 - \alpha_n)\phi(q, S_n v_n) \\
&\leq \alpha_n \phi(q, x_n) + (1 - \alpha_n)\phi(q, v_n)
\end{aligned}
$$

$$= \alpha_n \phi(q, x_n) + (1 - \alpha_n) \phi(q, J_{r_n} z_n)$$
$$\leq \alpha_n \phi(q, x_n) + (1 - \alpha_n) \phi(q, z_n). \tag{3.3}$$

It follows from Lemma 2.9, that

$$
\begin{aligned}
\phi(q, z_n) &= \phi(q, \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)) \\
&\leq \phi(q, J^{-1}(Jx_n - \lambda_n Ax_n)) \\
&= V(q, Jx_n - \lambda_n Ax_n) \\
&\leq V(q, (Jx_n - \lambda_n Ax_n) + \lambda_n Ax_n) - 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - q, \lambda_n Ax_n \rangle \\
&= V(q, Jx_n) - 2\lambda_n \langle J^{-1}(Jx_n - \lambda_n Ax_n) - q, Ax_n \rangle \\
&= \phi(q, x_n) - 2\lambda_n \langle x_n - q, Ax_n \rangle + 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n \rangle.
\end{aligned}
\tag{3.4}
$$

Since $q \in VI(A, C)$ and from A be an α-inverse-strongly monotone mapping, we have

$$
\begin{aligned}
-2\lambda_n \langle x_n - q, Ax_n \rangle &= -2\lambda_n \langle x_n - q, Ax_n - Aq \rangle - 2\lambda_n \langle x_n - q, Aq \rangle \\
&\leq -2\lambda_n \langle x_n - q, Ax_n - Aq \rangle \\
&= -2\alpha\lambda_n \|Ax_n - Aq\|^2.
\end{aligned}
\tag{3.5}
$$

From Lemma 2.1 and A be an α-inverse-strongly monotone mapping, we also have

$$
\begin{aligned}
2\langle J^{-1}&(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n \rangle \\
&= 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}(Jx_n), -\lambda_n Ax_n \rangle \\
&\leq 2\|J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}(Jx_n)\| \|\lambda_n Ax_n\| \\
&\leq \frac{4}{c^2} \|JJ^{-1}(Jx_n - \lambda_n Ax_n) - JJ^{-1}(Jx_n)\| \|\lambda_n Ax_n\| \\
&= \frac{4}{c^2} \|Jx_n - \lambda_n Ax_k - Jx_n\| \|\lambda_n Ax_n\| \\
&= \frac{4}{c^2} \|\lambda_n Ax_n\|^2 \\
&= \frac{4}{c^2} \lambda_n^2 \|Ax_n\|^2 \\
&\leq \frac{4}{c^2} \lambda_n^2 \|Ax_n - Aq\|^2.
\end{aligned}
\tag{3.6}
$$

Substituting (3.5) and (3.6) into (3.4), we obtain

$$
\begin{aligned}
\phi(q, z_n) &\leq \phi(q, x_n) - 2\alpha\lambda_n \|Ax_n - Aq\|^2 + \frac{4}{c^2} \lambda_n^2 \|Ax_n - Aq\|^2 \\
&= \phi(q, x_n) + 2\lambda_n \left(\frac{2}{c^2} \lambda_n - \alpha \right) \|Ax_n - Aq\|^2 \\
&\leq \phi(q, x_n).
\end{aligned}
\tag{3.7}
$$

Substituting (3.7) into (3.3), we also have

$$
\begin{aligned}
\phi(q, u_n) &\leq \alpha_n \phi(q, x_n) + (1 - \alpha_n) \phi(q, x_n), \\
&= \phi(q, x_n).
\end{aligned}
\tag{3.8}
$$

This show that $q \in C_{n+1}$ implies that $\Theta \subset C_{n+1}$ and hence, $\Theta \subset C_n$ for all $n \geq 1$. This implies that the sequence $\{x_n\}$ is well defined.

From definition of C_{n+1} that $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 1. \tag{3.9}$$

By Lemma 2.5, we get

$$
\begin{aligned}
\phi(x_n, x_0) &= \phi(\Pi_{C_n}(x_0), x_0) \\
&\leq \phi(p, x_0) - \phi(p, x_n) \\
&\leq \phi(p, x_0), \quad \forall p \in \Theta.
\end{aligned}
\tag{3.10}
$$

From (3.9) and (3.10), then $\{\phi(x_n, x_0)\}$ are nondecreasing and bounded. So, we obtain that $\lim_{n\to\infty} \phi(x_n, x_0)$ exists. In particular, by (1.8), the sequence $\{(\|x_n\| - \|x_0\|)^2\}$ is bounded. This implies $\{x_n\}$ is also bounded. So, we have $\{u_n\}$ and $\{z_n\}$ are bounded.

Next, we show that $\{x_n\}$ is a Cauchy sequence in C. Since $x_m = \Pi_{C_m} x_0 \in C_m \subset C_n$, for $m > n$, by Lemma 2.5, we have

$$
\begin{aligned}
\phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0) \\
&\leq \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\
&= \phi(x_m, x_0) - \phi(x_n, x_0).
\end{aligned}
$$

Taking $m, n \to \infty$, we have $\phi(x_m, x_n) \to 0$. From Lemma 2.3, we get $\|x_n - x_m\| \longrightarrow 0$. Thus $\{x_n\}$ is a Cauchy sequence and by the completeness of E and the closedness of C, we can assume that there exist $p \in C$ such that $x_n \to p \in C$ as $n \to \infty$. Now, we claim that $\|Ju_n - Jx_n\| \to 0$, as $n \to \infty$. By definition of $\Pi_{C_n} x_0$, we have

$$
\begin{aligned}
\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_0) \\
&\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\
&= \phi(x_{n+1}, x_0) - \phi(x_n, x_0).
\end{aligned}
$$

Since $\lim_{n\to\infty} \phi(x_n, x_0)$ exists, we also have

$$
\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0.
\tag{3.11}
$$

Form Lemma 2.3, that

$$
\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.
\tag{3.12}
$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we have

$$
\lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = 0.
\tag{3.13}
$$

Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$ and the definition of C_{n+1}, we have

$$
\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n), \quad n \longrightarrow \infty.
$$

Again applying Lemma 2.3, we get

$$
\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0.
\tag{3.14}
$$

Hence

$$
\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0.
\tag{3.15}
$$

From

$$
\begin{aligned}
\|u_n - x_n\| &= \|u_n - x_{n+1} + x_{n+1} - x_n\| \\
&\leq \|u_n - x_{n+1}\| + \|x_{n+1} - x_n\|.
\end{aligned}
$$

It follows that

$$
\lim_{n\to\infty} \|u_n - x_n\| = 0.
\tag{3.16}
$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we also have

$$\lim_{n \to \infty} \|Ju_n - Jx_n\| = 0. \tag{3.17}$$

Next, we will show that $x_n \to p \in \Theta := MEP(\theta, \varphi) \cap (\cap_{n=1}^{\infty} F(S_n)) \cap VI(A, C) \cap T^{-1}0$

(a) First, we show that $x_n \to p \in MEP(\theta, \varphi)$. From (3.3) and (3.7), we get $\phi(p, y_n) \leq \phi(p, x_n)$. By Lemma 2.8 (5) and $u_n = K_{r_n} y_n$, we observe that

$$
\begin{aligned}
\phi(u_n, y_n) &= \phi(K_{r_n} y_n, y_n) \\
&\leq \phi(p, y_n) - \phi(p, K_{r_n} y_n) \\
&\leq \phi(p, x_n) - \phi(p, K_{r_n} y_n) \\
&= \phi(p, x_n) - \phi(p, u_n) \\
&= \|p\|^2 - 2\langle p, Jx_n \rangle + \|x_n\|^2 - \left(\|p\|^2 - 2\langle p, Ju_n \rangle + \|u_n\|^2 \right) \\
&= \|x_n\|^2 - \|u_n\|^2 - 2\langle p, Jx_n - Ju_n \rangle \\
&\leq \|x_n - u_n\|(\|x_n + u_n\|) + 2\|p\|\|Jx_n - Ju_n\|.
\end{aligned}
\tag{3.18}
$$

From (3.16), (3.17) and Lemma 2.3, we also have

$$\lim_{n \to \infty} \|u_n - y_n\| = 0. \tag{3.19}$$

Since J is uniformly norm-to-norm continuous, we have

$$\lim_{n \to \infty} \|Ju_n - Jy_n\| = 0. \tag{3.20}$$

By using the triangle inequality, we obtain

$$
\begin{aligned}
\|x_{n+1} - y_n\| &= \|x_{n+1} - u_n + u_n - y_n\| \\
&\leq \|x_{n+1} - u_n\| + \|u_n - y_n\|.
\end{aligned}
\tag{3.21}
$$

By (3.15) and (3.19), we get

$$\lim_{n \to \infty} \|x_{n+1} - y_n\| = 0. \tag{3.22}$$

Since J is uniformly norm-to-norm continuous, we obtain

$$\lim_{n \to \infty} \|Jx_{n+1} - Jy_n\| = 0. \tag{3.23}$$

From (A2), that

$$\varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq \theta(y, u_n), \quad \forall y \in C,$$

$$\varphi(y) - \varphi(u_n) + \left\langle y - u_n, \frac{(Ju_n - Jy_n)}{r_n} \right\rangle \geq \theta(y, u_n), \quad \forall y \in C.$$

From $r_n > 0$ then $\frac{\|Ju_n - Jy_n\|}{r_n} \to 0$ as $n \to \infty$, we have

$$\varphi(y) - \varphi(u_n) \geq \theta(y, u_n),$$

since $u_n \to p$ as $n \to \infty$, we obtain

$$\theta(y, p) + \varphi(p) - \varphi(y) \leq 0.$$

For t with $0 < t < 1$ and $y \in C$, let $y_t = ty + (1-t)p$. Then, $y_t \in C$ and hence $\theta(y_t, p) + \varphi(p) - \varphi(y_t) \leq 0$. By the conditions (A1), (A4) and convexity of φ, we have

$$
\begin{aligned}
0 &= \theta(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\
&\leq t\theta(y_t, y) + (1-t)\theta(y_t, p) + t\varphi(y) + (1-t)\varphi(p) - \varphi(y_t) \\
&\leq t(\theta(y_t, y) + \varphi(y) - \varphi(y_t)) + (1-t)(\theta(y_t, p) + \varphi(p) - \varphi(y_t)) \\
&\leq \theta(y_t, y) + \varphi(y) - \varphi(y_t).
\end{aligned}
$$

From (A3) and the weakly lower semicontinuity of φ, we also have $\theta(p, y) + \varphi(y) - \varphi(p) \geq 0$, $\forall y \in C$. This implies $p \in MEP(\theta, \varphi)$.

(b)   We show that $x_n \to p \in \cap_{n=1}^{\infty} F(S_n)$. From definition of C_{n+1}, we have

$$
\alpha_n \phi(z, x_n) + (1 - \alpha_n)\phi(z, z_n) \leq \phi(z, x_n) \Leftrightarrow \phi(z, z_n) \leq \phi(z, x_n).
$$

Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1}$, we get $\phi(x_{n+1}, z_n) \leq \phi(x_{n+1}, x_n)$. It follows from (3.11), that

$$
\lim_{n \to \infty} \phi(x_{n+1}, z_n) = 0 \tag{3.24}
$$

again form Lemma 2.3, that

$$
\lim_{n \to \infty} \|x_{n+1} - z_n\| = 0. \tag{3.25}
$$

Since

$$
\|z_n - x_n\| \leq \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\|
$$

from (3.12) and (3.25), we also have

$$
\lim_{n \to \infty} \|z_n - x_n\| = 0. \tag{3.26}
$$

Since J is uniformly norm-to-norm continuous, we obtain

$$
\lim_{n \to \infty} \|Jz_n - Jx_n\| = 0. \tag{3.27}
$$

Since $v_n = J_{r_n} z_n$, we compute

$$
\begin{aligned}
\|Jx_{n+1} - Jy_n\| &= \|Jx_{n+1} - (\alpha_n Jx_n + (1 - \alpha_n)JS_n v_n\| \\
&= \|\alpha_n Jx_{n+1} - \alpha_n Jx_n + (1 - \alpha_n)Jx_{n+1} - (1 - \alpha_n)JS_n v_n\| \\
&= \|\alpha_n(Jx_{n+1} - Jx_n) + (1 - \alpha_n)(Jx_{n+1} - JS_n v_n)\| \\
&= \|(1 - \alpha_n)(Jx_{n+1} - JS_n v_n) - \alpha_n(Jx_n - Jx_{n+1})\| \\
&\geq (1 - \alpha_n)\|Jx_{n+1} - JS_n v_n\| - \alpha_n\|Jx_n - Jx_{n+1}\|,
\end{aligned}
$$

and hence

$$
\|Jx_{n+1} - JS_n v_n\| \leq \frac{1}{1 - \alpha_n}(\|Jx_{n+1} - Jy_n\| + \alpha_n\|Jx_n - Jx_{n+1}\|). \tag{3.28}
$$

From (3.23), (3.13) and $\liminf_{n \to \infty}(1 - \alpha_n) > 0$, we obtain that

$$
\lim_{n \to \infty} \|Jx_{n+1} - JS_n v_n\| = 0. \tag{3.29}
$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$
\lim_{n \to \infty} \|x_{n+1} - S_n v_n\| = 0. \tag{3.30}
$$

Using the triangle inequality, we get

$$\|z_n - S_n v_n\| = \|z_n - x_{n+1} + x_{n+1} - S_n v_n\|$$
$$\leq \|z_n - x_{n+1}\| + \|x_{n+1} - S_n v_n\|.$$

From (3.25) and (3.30), we have

$$\lim_{n \to \infty} \|z_n - S_n v_n\| = 0. \tag{3.31}$$

Since $\phi(q, v_n) \geq \frac{1}{1-\alpha_n}(\phi(q, u_n) - \alpha_n \phi(q, x_n))$ and Lemma 2.10, we observe that

$$\begin{aligned}
\phi(v_n, z_n) &= \phi(J_{r_n} z_n, z_n) \\
&\leq \phi(q, z_n) - \phi(q, J_{r_n} z_n) \\
&= \phi(q, z_n) - \phi(q, v_n) \\
&\leq \phi(q, z_n) - \frac{1}{1-\alpha_n}(\phi(q, u_n) - \alpha_n \phi(q, x_n) \\
&\leq \phi(q, x_n) - \frac{1}{1-\alpha_n}(\phi(q, u_n) - \alpha_n \phi(q, x_n) \\
&= \frac{1}{1-\alpha_n}(\phi(q, x_n) - \phi(q, u_n)) \\
&= \frac{1}{1-\alpha_n}\left(\|x_n\|^2 - \|u_n\|^2 - 2\langle q, Jx_n - Ju_n\rangle\right) \\
&\leq \frac{1}{1-\alpha_n}\left((\|x_n\|^2 - \|u_n\|^2) + 2|\langle q, Jx_n - Ju_n\rangle|\right) \\
&\leq \frac{1}{1-\alpha_n}((\|x_n\| - \|u_n\|)(\|x_n\| + \|u_n\|) + 2\|q\|\|Jx_n - Ju_n\|) \\
&\leq \frac{1}{1-\alpha_n}((\|x_n\| - \|u_n\|)(\|x_n\| + \|u_n\|) + 2\|q\|\|Jx_n - Ju_n\|).
\end{aligned}$$

It follows from (3.16) and (3.17), we get

$$\lim_{n \to \infty} \phi(v_n, z_n) = 0.$$

From Lemma 2.3, that

$$\lim_{n \to \infty} \|v_n - z_n\| = 0. \tag{3.32}$$

By using the triangle inequality, we get

$$\|v_n - S_n v_n\| = \|v_n - z_n + z_n - S_n v_n\|$$
$$\leq \|v_n - z_n\| + \|z_n - S_n v_n\|.$$

From (3.31) and (3.32), we have

$$\lim_{n \to \infty} \|v_n - S_n v_n\| = 0. \tag{3.33}$$

Since $\|x_n - v_n\| \leq \|x_n - z_n\| + \|z_n - v_n\|$, (3.26) and (3.32), then

$$\lim_{n \to \infty} \|x_n - v_n\| = 0. \tag{3.34}$$

Since $x_n \longrightarrow p$ as $n \to \infty$, we have $v_n \longrightarrow p$ as $n \to \infty$. It follows from $(*)$-condition, that $p \in \cap_{n=1}^{\infty} F(S_n)$.

(c) We show that $x_n \to p \in VI(A, C)$. Indeed, define $U \subset E \times E^*$ by Theorem 2.11, U is maximal monotone and $U^{-1}0 = VI(A, C)$. Let $(v, w) \in G(U)$. Since $w \in Uv = Av + N_C(v)$, we get $w - Av \in N_C(v)$.

From $z_n \in C$, we have

$$\langle v - z_n, w - Av \rangle \geq 0. \tag{3.35}$$

On the other hand, since $z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$. Then by Lemma 2.4, we have

$$\langle v - z_n, Jz_n - (Jx_n - \lambda_n Ax_n) \rangle \geq 0,$$

and thus

$$\left\langle v - z_n, \frac{Jx_n - Jz_n}{\lambda_n} - Ax_n \right\rangle \leq 0. \tag{3.36}$$

It follows from (3.35) and (3.36), that

$$
\begin{aligned}
\langle v - z_n, w \rangle &\geq \langle v - z_n, Av \rangle \\
&\geq \langle v - z_n, Av \rangle + \left\langle v - z_n, \tfrac{Jx_n - Jz_n}{\lambda_n} - Ax_n \right\rangle \\
&= \langle v - z_n, Av - Ax_n \rangle + \left\langle v - z_n, \tfrac{Jx_n - Jz_n}{\lambda_n} \right\rangle \\
&= \langle v - z_n, Av - Az_n \rangle + \langle v - z_n, Az_n - Ax_n \rangle + \langle v - z_n, \tfrac{Jx_n - Jz_n}{\lambda_n} \rangle \\
&\geq -\|v - z_n\| \tfrac{\|z_n - x_n\|}{\alpha} - \|v - z_n\| \tfrac{\|Jx_n - Jz_n\|}{a} \\
&\geq -M \left(\tfrac{\|z_n - x_n\|}{\alpha} + \tfrac{\|Jx_n - Jz_n\|}{a} \right),
\end{aligned}
$$

where $M = \sup_{n \geq 1} \|v - z_n\|$. Take the limit as $n \to \infty$ and (3.27), we obtain $\langle v - p, w \rangle \geq 0$. By the maximality of U, we have $p \in U^{-1}0$, that is $p \in VI(A, C)$.

(d) We show that $x_n \to p \in T^{-1}0$. Since J is uniformly norm-to-norm continuous on bounded sets, from (3.32) and J is uniformly norm-to-norm continuous

$$\lim_{n \to \infty} \|Jz_n - Jv_n\| = 0.$$

From $r_n > 0$, we have

$$\lim_{n \to \infty} \frac{1}{r_n} \|Jz_n - Jv_n\| = 0.$$

Therefore,

$$\lim_{n \to \infty} \|T_{r_n} z_n\| = \lim_{n \to \infty} \frac{1}{r_n} \|Jz_n - Jv_n\| = 0.$$

For $(w, w^*) \in T$, from the monotonicity of T, we have $\langle w - v_n, w^* - T_{r_n} z_n \rangle \geq 0$ for all $n \geq 0$. Letting $n \to \infty$, we get $\langle w - p, w^* \rangle \geq 0$. From the maximality of T, we have $p \in T^{-1}0$. Hence, from (a)–(d), we obtain $p \in \Theta$.

Finally, we show that $p = \Pi_\Theta x_0$. From $x_n = \Pi_{C_n} x_0$, we have $\langle Jx_0 - Jx_n, x_n - z \rangle \geq 0$, $\forall z \in C_n$. Since $\Theta \subset C_n$, we also have

$$\langle Jx_0 - Jx_n, x_n - y \rangle \geq 0, \quad \forall y \in \Theta.$$

Taking limit $n \to \infty$, we obtain

$$\langle Jx_0 - Jp, p - y \rangle \geq 0, \quad \forall y \in \Theta.$$

By Lemma 2.4, we can conclude that $p = \Pi_\Theta x_0$ and $x_n \to p$ as $n \to \infty$. This completes the proof. $\qquad\square$

Setting $S_n \equiv S$ in Theorem 3.2 and Remark 3.1, then we obtain the following result.

COROLLARY 3.3 *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let $T \subset E \times E^*$ be a maximal monotone operator*

satisfying $D(T) \subset C$ and let $J_{r_n} = (J + r_n T)^{-1} J$ for all $r_n > 0$. Let θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), and let $\varphi : C \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $\|Ay\| \leq \|Ay - Au\|$, $\forall y \in C$ and $u \in VI(A, C) \neq \emptyset$. Let $S : C \to C$ be a closed relatively quasi-nonexpansive mappings such that $\Theta := F(S) \cap T^{-1}0 \cap MEP(\theta, \varphi) \cap VI(A, C) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define the sequence $\{x_n\}$ as follows:

$$\begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JSJ_{r_n} z_n), \\ u_n \in C \text{ such that } \theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \alpha_n \phi(z, x_n) + (1 - \alpha_n)\phi(z, z_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases}$$

(3.37)

where J is the duality mapping on E, $\{\alpha_n\}$ is sequence in $[0, 1]$ and $\{r_n\} \subset [d, \infty)$ for some $d > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2 \alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E. If $\liminf_{n \to \infty}(1 - \alpha_n) > 0$, then $\{x_n\}$ converges strongly to $p \in \Theta$, where $p = \Pi_\Theta x_0$.

4. Application to complementarity problem

Let K be a nonempty, closed convex cone E and A be an operator of K into E^*. We define its *polar* in E^* to be the set

$$K^* = \left\{ y^* \in E^* : \langle x, y^* \rangle \geq 0, \forall x \in K \right\}. \tag{4.1}$$

Then, the element $u \in K$ is called a solution of the *complementarity problem* if

$$Au \in K^*, \langle u, Au \rangle = 0. \tag{4.2}$$

The set of solutions of the complementarity problem is denoted by $C(K, A)$.

THEOREM 4.1 *Let K be a nonempty and closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let $T \subset E \times E^*$ be a maximal monotone operator satisfying $D(T) \subset K$ and let $J_{r_n} = (J + r_n T)^{-1} J$ for all $r_n > 0$. Let θ be a bifunction from $K \times K$ to $\mathbb{R}$ satisfying (A1)–(A4), and let $\varphi : C \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of E into E^* satisfying $\|Ay\| \leq \|Ay - Au\|$, $\forall y \in K$ and $u \in C(K, A) \neq \emptyset$. Let $S_n : K \to K$ be a countable family of relatively quasi-nonexpansive mappings such that satisfies the (∗)-condition and $\Theta := (\cap_{n=1}^{\infty} F(S_n)) \cap T^{-1}0 \cap MEP(\theta, \varphi) \cap C(K, A) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{K_1}$ and $K_1 = K$, define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} z_n = \Pi_K J^{-1}(Jx_n - \lambda_n Ax_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JS_n J_{r_n} z_n), \\ u_n \in C \text{ such that } \theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \alpha_n \phi(z, x_n) + (1 - \alpha_n)\phi(z, z_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases}$$

(4.3)

where J is the duality mapping on E, $\{\alpha_n\}$ is sequences in $[0, 1]$ and $\{r_n\} \subset [d, \infty)$ for some $d > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E. If $\liminf_{n \to \infty}(1 - \alpha_n) > 0$, then $\{x_n\}$ converges strongly to $p \in \Theta$, where $p = \Pi_\Theta x_0$.

Proof   As in the proof of Takahashi in [16, Lemma 7.11], we get that $VI(K, A) = C(K, A)$. So, we obtain the result.   □

5. Numerical examples

In this section, in order to demonstrate the effectiveness, realization and convergence of algorithm of Theorem (3.1), we consider the following simple example.

Example 5.1   For simplicity, we assume $E = \mathbb{R}$ and $C = [-1, 1]$. Let $\theta(z, y) = -5z^2 + zy + 4y^2$ and $\varphi(x) = 4x^2$. Find $z \in [-1, 1]$ such that

$$\theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in [-1, 1].$$

Solution   It can easily be seen that θ, φ and F are satisfied the conditions in Theorem 3.1. For any $r > 0$ and $x \in [-1, 1]$, by Lemma 2.8, we can see that there exists $z \in [-1, 1]$ such that, for any $y \in [-1, 1]$,

$$\theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0,$$

$$-5z^2 + 4y^2 + zy + 4y^2 - 4z^2 + \frac{1}{r}\langle y - z, z - x \rangle) \geq 0,$$

$$8ry^2 + (rz + z - x)y + (-9rz^2 - z^2 + xz) \geq 0.$$

Let $G(y) = 8ry^2 + (rz + z - x)y + (-9rz^2 - z^2 + xz)$. Then G is a quadratic function of y with coefficient $a = 8r$, $b = rz + z - x$ and $c = -9rz^2 - z^2 + xz$. Therefore, we can compute the discriminant Δ of E as follows:

$$\Delta = b^2 - 4ac$$
$$= [rz + z - x]^2 - 4(8r)\left(-9rz^2 - z^2 + xz\right)$$
$$= 289r^2z^2 + 34rz^2 - 34rxz + z^2 - 2zx + x^2$$
$$= \left(289r^2 + 34r + 1\right)z^2 + (34r^2 - 2)xz + x^2$$
$$= (17r + 1)^2z^2 + 2(17r + 1)xz + x^2$$
$$= [(17r + 1)z + x]^2.$$

We know that $G(y) \geq 0$ for all $y \in [-1, 1]$ if it has at most one solution in $[-1, 1]$. Thus $\Delta \leq 0$ and hence $z = \left(\dfrac{1}{17r + 1}\right)x$.

By Example 1.3, we can define a countable family of a relatively quasi-nonexpansive mapping $S_n : C \longrightarrow C$ as follow:

$$S_n(x) = \begin{cases} 0, & \text{if } x \leq \frac{1}{n}; \\ \frac{1}{n}, & \text{if } x > \frac{1}{n}, \end{cases}$$
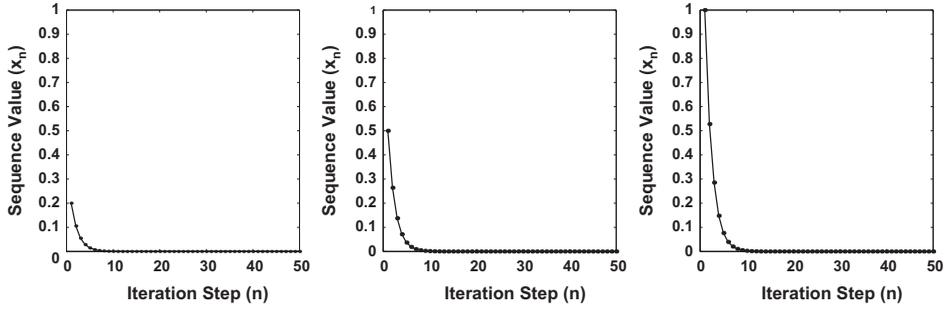
Figure 1. The convergence comparison of different initial values $x_1 = 0.2, 0.5, 1$.

Table 1. The numerical results for initial guess $x_1 = 0.2, 0.5, 1$.

n (iterative number)	initial guess		
	$x_1 = 0.2$	$x_1 = 0.5$	$x_1 = 1$
5	0.0147	0.0368	0.0765
10	0.0005	0.0013	0.0027
15	1.9548×10^{-5}	4.8872×10^{-5}	1.0145×10^{-4}
20	7.0621×10^{-7}	1.7655×10^{-6}	3.6648×10^{-6}
25	2.5466×10^{-8}	6.3665×10^{-8}	1.3215×10^{-7}

In Theorem 3.1, we set $\theta \equiv 0$, $A \equiv 0$, $J_{r_n} \equiv I$, $\alpha_n = \frac{n+1}{2n}$ and $r_n = 1$. We apply it to find the fixed point of S_n of Example 5.1.

Under the above assumption, then Theorem 3.1 is simplified as follows:

$$
\begin{cases}
x_0 \in E \text{ chosen arbitrary,} \\
C_1 = E, \, x_1 = P_E x_0 \text{ and } z_n = x_n, \\
y_n = \left(\frac{n+1}{2n}\right) x_n + \left(\frac{n+1}{2n}\right) S_n x_n, \\
u_n = K_{r_n}(x_n) = \dfrac{x_n}{18}, \\
C_{n+1} = \{z \in C_n : \|z - u_n\| \leq \|z - x_n\|\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1,
\end{cases}
\tag{5.1}
$$

In fact, in one-dimensional case, the C_{n+1} is closed interval. If we set $[a_{n+1}, b_{n+1}] := C_{n+1}$, then the projection point x_{n+1} of $x_1 \in C$ onto C_{n+1} can be expressed as:

$$
x_{n+1} = P_{C_{n+1}}(x_1) \begin{cases}
x_1, & \text{if } x_1 \in [a_{n+1}, b_{n+1}]; \\
b_{n+1}, & \text{if } x_1 > b_{n+1}; \\
a_{n+1}, & \text{if } x_1 < a_{n+1}.
\end{cases}
$$

The numerical results for initial guess $x_1 = 0.2, 0.5, 1$ were shown in Table 1. From the table, we see that the iterations converge to 0 which is the unique fixed point of S. The convergence of each iteration was also shown in Figure 1 for comparison.

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RESEARCH　　　　　　　　　　　　　　　　　　　　　**Open Access**

Strong convergence theorem for total quasi-ϕ-asymptotically nonexpansive mappings in a Banach space

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Abstract

In this paper, we prove strong convergence theorems to a point which is a fixed point of multi-valued mappings, a zero of an α-inverse-strongly monotone operator and a solution of the equilibrium problem. Next, we obtain strong convergence theorems to a solution of the variational inequality problem, a fixed point of multi-valued mappings and a solution of the equilibrium problem. The results presented in this paper are improvement and generalization of the previously known results.

Keywords: total quasi-ϕ-asymptotically nonexpansive multi-valued mappings; hybrid scheme; equilibrium problem; variational inequality problems; inverse-strongly monotone operator

1 Introduction

Let E be a real Banach space with dual E^*, and let C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be an operator. A is called *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C;$$

α-*inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C;$$

L-*Lipschitz continuous* if there exists a constant $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

If A is α-inverse strongly monotone, then it is $\frac{1}{\alpha}$-Lipschitz continuous, *i.e.*,

$$\|Ax - Ay\| \leq \frac{1}{\alpha}\|x - y\|, \quad \forall x, y \in C.$$

A monotone operator A is said to be *maximal* if its graph $G(A) = \{(x, x^*) : x^* \in Ax\}$ is not properly contained in the graph of any other monotone operator.

Let A be a monotone operator. We consider the problem of finding $x \in E$ such that

$$0 \in Ax, \tag{1.1}$$

a point $x \in E$ is called a *zero point* of A. Denote by $A^{-1}0$ the set of all points $x \in E$ such that $0 \in Ax$. This problem is very important in optimization theory and related fields.

Let A be a monotone operator. *The classical variational inequality problem* for an operator A is to find $\hat{z} \in C$ such that

$$\langle A\hat{z}, y - \hat{z} \rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $VI(A, C)$. This problem is connected with the convex minimization problem, the complementary problem, the problem of finding a point $x \in E$ satisfying $Ax = 0$.

The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^* \rangle$ or $x^*(x)$. For each $p > 1$, the *generalized duality mapping* $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1} \right\}$$

for all $x \in E$. In particular, $J = J_2$ is called the *normalized duality mapping*. If E is a Hilbert space, then $J = I$, where I is the identity mapping.

Consider the functional defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2 \quad \text{for } x, y \in E, \tag{1.3}$$

where J is the normalized duality mapping. It is obvious from the definition of ϕ that

$$\left(\|y\| - \|x\| \right)^2 \leq \phi(y, x) \leq \left(\|y\| + \|x\| \right)^2, \quad \forall x, y \in E. \tag{1.4}$$

Alber [1] introduced that the *generalized projection* $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution of the minimization problem

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x), \tag{1.5}$$

existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J.

Iiduka and Takahashi [2] introduced the following iterative scheme for finding a solution of the variational inequality problem for an inverse-strongly monotone operator A in a 2-uniformly convex and uniformly smooth Banach space E: $x_1 = x \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \quad \forall n \geq 1, \tag{1.6}$$

where Π_C is the generalized projection from E onto C, J is the duality mapping from E into E^* and $\{\lambda_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (1.6) converges weakly to some element of $VI(A, C)$. In connection, Iiduka and Takahashi [3] studied the following iterative scheme for finding a zero point of a monotone

operator A in a 2-uniformly convex and uniformly smooth Banach space E:

$$\begin{cases} x_1 = x \in E \quad \text{chosen arbitrarily,} \\ y_n = J^{-1}(Jx_n - \lambda_n Ax_n), \\ X_n = \{z \in E : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Y_{n+1} = \{z \in E : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{X_n \cap Y_n}(x), \end{cases} \tag{1.7}$$

where $\Pi_{X_n \cap Y_n}$ is the generalized projection from E onto $X_n \cap Y_n$, J is the duality mapping from E into E^* and $\{\lambda_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ converges strongly to an element of $A^{-1}0$. Moreover, under the additional suitable assumption they proved that the sequence $\{x_n\}$ converges strongly to some element of $VI(A, C)$. Some solution methods have been proposed to solve the variational inequality problem; see, for instance, [4–6].

A mapping $T : C \to C$ is said to be *ϕ-nonexpansive* [7, 8] if

$$\phi(Tx, Ty) \leq \phi(x, y), \quad \forall x, y \in C.$$

T is said to be *quasi-ϕ-nonexpansive* [7, 8] if $F(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, p \in F(T).$$

T is said to be *total quasi-ϕ-asymptotically nonexpansive*, if $F(T) \neq \emptyset$ and there exist nonnegative real sequences v_n, μ_n with $v_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$ such that

$$\phi(p, T^n x) \leq \phi(p, x) + v_n \varphi(\phi(p, x)) + \mu_n, \quad \forall n \geq 1, \forall x \in C, p \in F(T).$$

Let 2^C be the family of all nonempty subsets of C, and let $S : C \to 2^C$ be a multi-valued mapping. For a point $q \in C$, $n \geq 1$ define an iterative sequence as follows:

$$Sq := \{q_1 : q_1 \in Sq\},$$

$$S^2 q = SSq := \bigcup_{q_1 \in Sq} Sq_1,$$

$$S^3 q = SS^2 q := \bigcup_{q_2 \in T^2 q} Sq_2,$$

$$\vdots$$

$$S^n q = SS^{n-1} q := \bigcup_{q_{n-1} \in S^{n-1} q} Sq_{n-1}.$$

A point $p \in C$ is said to be an *asymptotic fixed point* of S if there exists a sequence $\{x_n\}$ in C such that $\{x_n\}$ converges weakly to p and

$$\lim_{n \to \infty} d(x_n, Sx_n) := \lim_{n \to \infty} \inf_{x \in Sx_n} \|x_n - x\| = 0.$$

The asymptotic fixed point set of S is denoted by $\widehat{F}(S)$.

A multi-valued mapping S is said to be *total quasi-ϕ-asymptotically nonexpansive* if $F(S) \neq \emptyset$ and there exist nonnegative real sequences ν_n, μ_n with $\nu_n \to 0, \mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$ such that for all $x \in C, p \in F(S)$,

$$\phi(p, w_n) \leq \phi(p, x) + \nu_n \varphi\big(\phi(p, x)\big) + \mu_n, \quad \forall n \geq 1, w_n \in S^n x.$$

S is said to be *closed* if for any sequence $\{x_n\}$ and $\{w_n\}$ in C with $w_n \in S x_n$ if $x_n \to x$ and $w_n \to w$, then $w \in Sx$.

A multi-valued mapping S is said to be *uniformly asymptotically regular* on C if

$$\lim_{n \to \infty} \Big(\sup_{x \in C} \|s_{n+1} - s_n\| \Big) = 0, \quad s_n \in S^n x.$$

Every quasi-ϕ-asymptotically nonexpansive multi-valued mapping implies a quasi-ϕ-asymptotically nonexpansive mapping but the converse is not true.

In 2012, Chang *et al.* [9] introduced the concept of total quasi-ϕ-asymptotically non-expansive multi-valued mapping and then proved some strong convergence theorem by using the hybrid shrinking projection method.

Let $f : C \times C \to \mathbb{R}$ be a bifunction, the *equilibrium problem* is to find $x \in C$ such that

$$f(x, y) \geq 0, \quad \forall y \in C. \tag{1.8}$$

The set of solutions of (1.8) is denoted by $EP(f)$. The equilibrium problem is very general in the sense that it includes, as special cases, optimization problems, variational inequality problems, min-max problems, saddle point problem, fixed point problem, Nash EP. In 2008, Takahashi and Zembayashi [10, 11] introduced iterative sequences for finding a common solution of an equilibrium problem and a fixed point problem. Some solution methods have been proposed to solve the equilibrium problem; see, for instance, [12–21].

For a mapping $A : C \to E^*$, let $f(x, y) = \langle Ax, y - x \rangle$ for all $x, y \in C$. Then $x \in EP(f)$ if and only if $\langle Tx, y - x \rangle \geq 0$ for all $y \in C$; *i.e.*, x is a solution of the variational inequality.

Motivated and inspired by the work mentioned above, in this paper, we introduce and prove strong convergence of a new hybrid projection algorithm for a fixed point of total quasi-ϕ-asymptotically nonexpansive multi-valued mappings, the solution of the equilibrium problem, a zero point of monotone operators. Moreover, we prove strong convergence to the solution of the variation inequality in a uniformly smooth and 2-uniformly convex Banach space.

2 Preliminaries

A Banach space E with the norm $\|\cdot\|$ is called *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. A Banach space E is called *smooth* if the limit $\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$ exists for each $x, y \in U$. It is also called *uniformly smooth* if the limit exists uniformly for all $x, y \in U$. The *modulus of convexity* of E is the function $\delta : [0, 2] \to [0, 1]$ defined by

$$\delta(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon \right\}.$$

A Banach space E is *uniformly convex* if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0, 2]$. Observe that every p-uniform convex is uniformly convex. Every uniformly convex Banach space E has the *Kadec-Klee property*, that is, for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.

Let E be a real Banach space with dual E^*, E is uniformly smooth if and only if E^* is a uniformly convex Banach space. If E is a uniformly smooth Banach space, then E is a smooth and reflexive Banach space.

Remark 2.1
- If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.
- If E is reflexive smooth and strictly convex, then the normalized duality mapping J is single-valued, one-to-one and onto.
- If E is a reflexive strictly convex and smooth Banach space and J is the duality mapping from E into E^*, then J^{-1} is also single-valued, bijective and is also the duality mapping from E^* into E and thus $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$.

See [22] for more details.

Remark 2.2 If E is a reflexive, strictly convex and smooth Banach space, then $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$, then $x = y$. From (1.3) we have $\|x\| = \|y\|$. This implies that $\langle x, Jy \rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$ (see [22, 23] for more details).

Lemma 2.3 (Beauzamy [24] and Xu [25]) *If E is a 2-uniformly convex Banach space, then, for all $x, y \in E$, we have*

$$\|x - y\| \leq \frac{2}{c^2}\|Jx - Jy\|,$$

where J is the normalized duality mapping of E and $0 < c \leq 1$.

The best constant $\frac{1}{c}$ in the lemma is called the *p-uniformly convex constant* of E.

Lemma 2.4 (Beauzamy [24] and Zalinescu [26]) *If E is a p-uniformly convex Banach space, and let p be a given real number with $p \geq 2$, then, for all $x, y \in E$, $J_x \in J_p(x)$ and $J_y \in J_p(y)$,*

$$\langle x - y, J_x - J_y \rangle \geq \frac{c^p}{2^{p-2}p}\|x - y\|^p,$$

where J_p is the generalized duality mapping of E and $\frac{1}{c}$ is the p-uniformly convex constant of E.

Lemma 2.5 (Kamimura and Takahashi [27]) *Let E be a uniformly convex and smooth Banach space, and let $\{x_n\}$, $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 2.6 (Alber [1]) *Let C be a nonempty closed convex subset of a smooth Banach space E, and let $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.7 (Alber [1]) *Let E be a reflexive strictly convex and smooth Banach space, C be a nonempty closed convex subset of E, and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 2.8 (Chang *et al.* [9]) *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. Let $S : C \to 2^C$ be a closed and total quasi-ϕ-asymptotically nonexpansive multi-valued mapping with non-negative real sequence ν_n and μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$. If $\mu_1 = 0$, then the fixed point set $F(S)$ is a closed convex subset of C.*

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, let us assume that f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, *i.e.*, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y);$$

(A4) for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semi-continuous.

Lemma 2.9 (Blum and Oettli [28]) *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4), *and let $r > 0$ and $x \in E$. Then there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.10 (Takahashi and Zembayashi [11]) *Let C be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying conditions* (A1)-(A4). *For all $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}.$$

Then the following hold:

(1) T_r *is single-valued*;

(2) T_r *is a firmly nonexpansive-type mapping* [29], *that is, for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle;$$

(3) $F(T_r) = EP(f)$;

(4) $EP(f)$ is closed and convex.

Lemma 2.11 (Takahashi and Zembayashi [11]) *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

Let A be an inverse-strongly monotone mapping of C into E^* which is said to be *hemicontinuous* if for all $x, y \in C$, the mapping h of $[0, 1]$ into E^*, defined by $h(t) = A(tx + (1-t)y)$, is continuous with respect to the weak* topology of E^*. We define by $N_C(v)$ *the normal cone* for C at a point $v \in C$, that is,

$$N_C(v) = \left\{ x^* \in E^* : \langle v - y, x^* \rangle \geq 0, \forall y \in C \right\}. \tag{2.1}$$

Theorem 2.12 (Rockafellar [30]) *Let C be a nonempty, closed convex subset of a Banach space E, and let A be a monotone, hemicontinuous operator of C into E^*. Let $B \subset E \times E^*$ be an operator defined as follows*:

$$Bv = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & \text{otherwise.} \end{cases} \tag{2.2}$$

Then B is maximal monotone and $B^{-1}0 = VI(A, C)$.

Theorem 2.13 (Takahashi [31]) *Let C be a nonempty subset of a Banach space E, and let A be a monotone, hemicontinuous operator of C into E^* with $C = D(A)$. Then*

$$VI(A, C) = \left\{ u \in C : \langle v - u, Av \rangle \geq 0, \forall v \in C \right\}. \tag{2.3}$$

It is obvious that the set $VI(A, C)$ is a closed and convex subset of C and the set $A^{-1}0 = VI(A, E)$ is a closed and convex subset of E.

Theorem 2.14 (Takahashi [31]) *Let C be a nonempty compact convex subset of a Banach space E, and let A be a monotone, hemicontinuous operator of C into E^* with $C = D(A)$. Then $VI(A, C)$ is nonempty.*

We make use of the following mapping V studied in Alber [1]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2, \quad \forall x \in E, x^* \in E^*, \tag{2.4}$$

that is, $V(x, x^*) = \phi(x, J^{-1}(x^*))$.

Lemma 2.15 (Alber [1]) *Let E be a reflexive strictly convex smooth Banach space, and let V be as in (2.4). Then we have*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^* \rangle \leq V(x, x^* + y^*), \quad \forall x \in E, x^*, y^* \in E^*.$$

Lemma 2.16 (Beauzamy [24] and Xu [25]) *If E is a 2-uniformly convex Banach space, then, for all $x, y \in E$, we have*

$$\|x - y\| \leq \frac{2}{c^2} \|Jx - Jy\|,$$

where J is the normalized duality mapping of E and $0 < c \leq 1$.

Lemma 2.17 (Cho *et al.* [32]) *Let E be a uniformly convex Banach space, and let $B_r(0) = \{x \in E : \|x\| \leq r\}$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \|\lambda x\|^2 + \|\mu y\|^2 + \|\gamma z\|^2 - \lambda \mu g(\|x - y\|)$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

Lemma 2.18 (Pascali and Sburlan [33]) *Let E be a real smooth Banach space, and let $A : E \to 2^{E^*}$ be a maximal monotone mapping. Then $A^{-1}0$ is a closed and convex subset of E and the graph $G(A)$ of A is demiclosed in the following sense: if $\{x_n\} \subset D(A)$ with $x_n \rightharpoonup x \in E$ and $y_n \in Ax_n$ with $y_n \to y \in E^*$, then $x \in D(A)$ and $y \in Ax$.*

3 Main results

Theorem 3.1 *Let C be a nonempty closed and convex subset of a uniformly smooth and 2-uniformly convex Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying conditions (A1)-(A4), and let A be an α-inverse-strongly monotone mapping of E into E^*. Let $S : C \to 2^C$ be a closed and total quasi-ϕ-asymptotically nonexpansive multi-valued mapping with nonnegative real sequences v_n, μ_n with $v_n \to 0, \mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$. Assume that S is uniformly asymptotically regular on C with $\mu_1 = 0$ and $F := F(S) \cap EP(f) \cap A^{-1}0 \neq \emptyset$. For arbitrary $x_1 \in C, C_1 = C$, generate a sequence $\{x_n\}$ by*

$$\begin{cases} z_n = J^{-1}(Jx_n - \lambda_n Ax_n), \\ u_n = T_{r_n} z_n, \\ y_n = J^{-1}(\alpha_n Jx_n + \beta_n Jw_n + \gamma_n Ju_n), \quad w_n \in S^n x_n, \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, z_n) \leq \phi(v, x_n) + K_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $K_n = v_n \sup_{q \in F} \psi(\phi(q, x_n)) + \mu_n$. Assume that the control sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\lambda_n\}$ and $\{r_n\}$ satisfy the following conditions:

1. *$\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\liminf_{n \to \infty} \alpha_n \beta_n > 0$,*
2. *$\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < \frac{c^2 \alpha}{2}$ and $\frac{1}{c}$ is the 2-uniformly convex constant of E,*
3. *$\{r_n\} \subset [d, \infty)$ for some $d > 0$,*

then $\{x_n\}$ converges strongly to $\Pi_F x_1$.

Proof We will show that C_n is closed and convex for all $n \in \mathbb{N}$. Since $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for all $n \in \mathbb{N}$. For any $v \in C_n$, we know that $\phi(v, y_n) \le \phi(v, x_n) + K_n$ is equivalent to

$$2\langle v, Jx_n - Jy_n \rangle \le \|x_n\|^2 - \|y_n\|^2 + K_n.$$

That is, C_{n+1} is closed and convex, hence C_n is closed and convex for all $n \in \mathbb{N}$.

We show by induction that $F \subset C_n$ for all $n \in \mathbb{N}$. It is obvious that $F \subset C = C_1$. Suppose that $F \subset C_n$ where $n \in \mathbb{N}$. Let $q \in F$, we have

$$
\begin{aligned}
\phi(q, z_n) &= \phi\big(q, J^{-1}(Jx_n - \lambda_n Ax_n)\big) \\
&= V(q, Jx_n - \lambda_n Ax_n) \\
&\le V\big(q, (Jx_n - \lambda_n Ax_n) + \lambda_n Ax_n\big) - 2\big\langle J^{-1}(Jx_n - \lambda_n Ax_n) - q, \lambda_n Ax_n \big\rangle \\
&= V(q, Jx_n) - 2\lambda_n \big\langle J^{-1}(Jx_n - \lambda_n Ax_n) - q, Ax_n \big\rangle \\
&= \phi(q, x_n) - 2\lambda_n \langle x_n - q, Ax_n \rangle + 2\big\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n \big\rangle.
\end{aligned}
$$
(3.2)

Since A is an α-inverse-strongly monotone mapping, we get

$$
\begin{aligned}
-2\lambda_n \langle x_n - q, Ax_n \rangle &= -2\lambda_n \langle x_n - q, Ax_n - Aq \rangle - 2\lambda_n \langle x_n - q, Aq \rangle \\
&\le -2\lambda_n \langle x_n - q, Ax_n - Aq \rangle \\
&= -2\alpha\lambda_n \|Ax_n - Aq\|^2.
\end{aligned}
$$
(3.3)

It follows from Lemma 2.17 that

$$
\begin{aligned}
2\big\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n \big\rangle &= 2\big\langle J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}(Jx_n), -\lambda_n Ax_n \big\rangle \\
&\le 2\big\| J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}(Jx_n) \big\| \|\lambda_n Ax_n\| \\
&\le \frac{4}{c^2} \big\| JJ^{-1}(Jx_n - \lambda_n Ax_n) - JJ^{-1}(Jx_n) \big\| \|\lambda_n Ax_n\| \\
&= \frac{4}{c^2} \|Jx_n - \lambda_n Ax_n - Jx_n\| \|\lambda_n Ax_n\| \\
&= \frac{4}{c^2} \|\lambda_n Ax_n\|^2 \\
&= \frac{4}{c^2} \lambda_n^2 \|Ax_n\|^2 \\
&\le \frac{4}{c^2} \lambda_n^2 \|Ax_n - Aq\|^2.
\end{aligned}
$$
(3.4)

Replacing (3.2) by (3.3) and (3.4), we get

$$
\begin{aligned}
\phi(q, z_n) &\le \phi(q, x_n) - 2\alpha\lambda_n \|Ax_n - Aq\|^2 + \frac{4}{c^2} \lambda_n^2 \|Ax_n - Aq\|^2 \\
&= \phi(q, x_n) + 2\lambda_n \left(\frac{2}{c^2} \lambda_n - \alpha \right) \|Ax_n - Aq\|^2 \\
&\le \phi(q, x_n).
\end{aligned}
$$
(3.5)

From Lemma 2.11, we know that

$$\phi(q, u_n) = \phi(q, T_{r_n} z_n) \le \phi(q, z_n) \le \phi(q, x_n). \tag{3.6}$$

Since S is a total quasi-ϕ-asymptotically nonexpansive multi-valued mapping and $w_n \in S^n x_n$, it follows that

$$
\begin{aligned}
\phi(q, y_n) &= \phi\big(q, J^{-1}(\alpha_n J x_n + \beta_n J w_n + \gamma_n J u_n)\big) \\
&= \|q\|^2 - 2\langle q, \alpha_n J x_n + \beta_n J w_n + \gamma_n J u_n \rangle + \|\alpha_n J x_n + \beta_n J w_n + \gamma_n J u_n\|^2 \\
&\le \alpha_n \phi(q, x_n) + \beta_n \phi(q, w_n) + \gamma_n \phi(q, u_n) \\
&\le \alpha_n \phi(q, x_n) + \beta_n \phi(q, x_n) + \beta_n \nu_n \psi\big(\phi(q, x_n)\big) + \beta_n \mu_n + \gamma_n \phi(q, u_n) \\
&\le \alpha_n \phi(q, x_n) + \beta_n \phi(q, x_n) + \nu_n \sup_{q \in F} \psi\big(\phi(q, x_n)\big) + \mu_n + \gamma_n \phi(q, u_n) \\
&= \alpha_n \phi(q, x_n) + \beta_n \phi(q, x_n) + \gamma_n \phi(q, u_n) + K_n \\
&\le \alpha_n \phi(q, x_n) + \beta_n \phi(q, x_n) + \gamma_n \phi(q, T_{r_n} z_n) + K_n \\
&\le \alpha_n \phi(q, x_n) + \beta_n \phi(q, x_n) + \gamma_n \phi(q, z_n) + K_n \\
&\le \alpha_n \phi(q, x_n) + \beta_n \phi(q, x_n) + \gamma_n \phi(q, x_n) + K_n \\
&\le \phi(q, x_n) + K_n, \tag{3.7}
\end{aligned}
$$

where $K_n = \nu_n \sup_{q \in F} \psi(\phi(q, x_n)) + \mu_n$.

This shows that $q \in C_{n+1}$, which implies that $F \subset C_{n+1}$. Hence $F \subset C_n$ for all $n \in \mathbb{N}$ and the sequence $\{x_n\}$ is well defined.

From the definition of C_{n+1} with $x_n = \Pi_{C_n} x_1$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, it follows that

$$\phi(x_n, x_1) \le \phi(x_{n+1}, x_1), \quad \forall n \ge 1, \tag{3.8}$$

that is, $\{\phi(x_n, x_1)\}$ is nondecreasing. By Lemma 2.7, we get

$$
\begin{aligned}
\phi(x_n, x_1) &= \phi(\Pi_{C_n} x_1, x_1) \\
&\le \phi(q, x_1) - \phi(q, x_n) \\
&\le \phi(q, x_1), \quad \forall q \in F. \tag{3.9}
\end{aligned}
$$

This implies that $\{\phi(x_n, x_1)\}$ is bounded and so $\lim_{n \to \infty} \phi(x_n, x_1)$ exists. In particular, by (1.4), the sequence $\{(\|x_n\| - \|x_1\|)^2\}$ is bounded. This implies $\{x_n\}$ is also bounded. So, we have $\{u_n\}$, $\{z_n\}$ and $\{y_n\}$ are also bounded.

Since $x_m = \Pi_{C_m} x_1 \in C_m \subset C_n$ for all $m, n \ge 1$ with $m > n$, by Lemma 2.7, we have

$$
\begin{aligned}
\phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_1) \\
&\le \phi(x_m, x_1) - \phi(\Pi_{C_n} x_1, x_1) \\
&= \phi(x_m, x_1) - \phi(x_n, x_1),
\end{aligned}
$$

taking $m, n \to \infty$, we have $\phi(x_m, x_n) \to 0$. This implies that $\{x_n\}$ is a Cauchy sequence. From Lemma 2.5, it follows that $\|x_n - x_m\| \to 0$ and $\{x_n\}$ is a Cauchy sequence. By the completeness of E and the closedness of C, we can assume that there exists $p \in C$ such that

$$\lim_{n \to \infty} x_n = p, \tag{3.10}$$

we also get that

$$\lim_{n \to \infty} K_n = \lim_{n \to \infty} \nu_n \sup_{q \in F} \psi\big(\phi(q, x_n)\big) + \mu_n = 0. \tag{3.11}$$

Next, we show that $p \in F := F(S) \cap A^{-1}0 \cap EP(f)$.

(a) We show that $p \in F(S)$. By the definition of $\Pi_{C_n} x_1$, we have

$$\begin{aligned}
\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_1) \\
&\leq \phi(x_{n+1}, x_1) - \phi(\Pi_{C_n} x_1, x_1) \\
&= \phi(x_{n+1}, x_1) - \phi(x_n, x_1).
\end{aligned}$$

Since $\lim_{n \to \infty} \phi(x_n, x_1)$ exists, we get

$$\lim_{n \to \infty} \phi(x_{n+1}, x_n) = 0. \tag{3.12}$$

It follows from Lemma 2.5 that

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{3.13}$$

From the definition of C_{n+1} and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we have $\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n) + K_n \to 0$ as $n \to \infty$. By Lemma 2.5, it follows that

$$\lim_{n \to \infty} \|x_{n+1} - y_n\| = 0. \tag{3.14}$$

From $\lim_{n \to \infty} x_n = p$, we also have

$$\lim_{n \to \infty} y_n = p. \tag{3.15}$$

By using the triangle inequality, we get $\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \to 0$ as $n \to \infty$. Since J is uniformly norm-to-norm continuous, we obtain $\|Jx_n - Jy_n\| \to 0$ as $n \to \infty$. On the other hand, we note that

$$\begin{aligned}
\phi(q, x_n) - \phi(q, y_n) &= \|x_n\|^2 - \|y_n\|^2 - 2\langle q, Jx_n - Jy_n \rangle \\
&\leq \|x_n - y_n\|\big(\|x_n + y_n\|\big) + 2\|q\|\|Jx_n - Jy_n\|.
\end{aligned}$$

In view of $\|x_n - y_n\| \to 0$ and $\|Jx_n - Jy_n\| \to 0$ as $n \to \infty$, we obtain that

$$\phi(q, x_n) - \phi(q, y_n) \to 0 \quad \text{as } n \to \infty. \tag{3.16}$$

From Lemma 2.17, we have

$$
\begin{aligned}
\phi(q, y_n) &= \phi\big(q, J^{-1}[\alpha_n J x_n + \beta_n J w_n + \gamma_n J u_n]\big) \\
&\leq \|q\|^2 - 2\langle q, \alpha_n J x_n + \beta_n J w_n + \gamma_n J u_n \rangle + \|\alpha_n J x_n + \beta_n J w_n + \gamma_n J u_n\|^2 \\
&\quad - \alpha_n \beta_n g\big(\|J x_n - J w_n\|\big) \\
&= \alpha_n \phi(q, x_n) + \beta_n \phi(q, w_n) + \gamma_n \phi(q, u_n) - \alpha_n \beta_n g\big(\|J x_n - J w_n\|\big) \\
&\leq \phi(q, x_n) + K_n - \alpha_n \beta_n g\big(\|J x_n - J w_n\|\big).
\end{aligned}
\tag{3.17}
$$

It follows from $\liminf_{n\to\infty} \alpha_n \beta_n > 0$, (3.16), (3.11) and the property of g that

$$
\lim_{n\to\infty} \|J x_n - J w_n\| = 0.
$$

Since J^{-1} is uniformly norm-to-norm continuous, we obtain

$$
\lim_{n\to\infty} \|x_n - w_n\| = 0.
\tag{3.18}
$$

From (3.10) it follows that

$$
\lim_{n\to\infty} \|w_n - p\| = 0.
\tag{3.19}
$$

For $w_n \in S^n x_n$, generate a sequence $\{s_n\}$ by

$$
s_2 \in S w_1 \subset S^2 x_1,
$$

$$
s_3 \in S w_2 \subset S^3 x_2,
$$

$$
s_4 \in S w_3 \subset S^4 x_3,
$$

$$
\vdots
$$

$$
s_{n+1} \in S w_n \subset S^{n+1} x_n.
$$

On the other hand, we have $\|s_{n+1} - p\| \leq \|s_{n+1} - w_n\| + \|w_n - p\|$. Since S is uniformly asymptotically regular, it follows that

$$
\lim_{n\to\infty} \|s_{n+1} - p\| = 0,
\tag{3.20}
$$

we have

$$
\lim_{n\to\infty} \big\|S^{n+1} x_n - p\big\| = 0,
\tag{3.21}
$$

that is, $S S^n x_n \to p$ as $n \to \infty$. From the closedness of S, we have $p \in F(S)$.

(b) We show that $p \in A^{-1}0$.

From the definition of C_{n+1} and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we have $\phi(x_{n+1}, z_n) \leq \phi(x_{n+1}, x_n) + K_n \to 0$ as $n \to \infty$. By Lemma 2.5, it follows that $\lim_{n\to\infty} \|x_{n+1} - z_n\| = 0$. By the

triangle inequality, we get $\|x_n - z_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \to 0$ as $n \to \infty$. From $\lim_{n \to \infty} \|z_n - x_n\| = 0$ and from (3.10), it follows that

$$\lim_{n \to \infty} z_n = p. \tag{3.22}$$

Since J is uniformly norm-to-norm continuous, we also have

$$\lim_{n \to \infty} \|Jz_n - Jx_n\| = 0. \tag{3.23}$$

Hence, from the definition of the sequence $\{z_n\}$, it follows that

$$\|Ax_n\| = \frac{\|Jz_n - Jx_n\|}{\lambda_n}. \tag{3.24}$$

From (3.23) and the definition of the sequence $\{\lambda_n\}$, we have

$$\lim_{n \to \infty} \|Ax_n\| = 0, \tag{3.25}$$

that is,

$$\lim_{n \to \infty} Ax_n = 0. \tag{3.26}$$

Since A is Lipschitz continuous, it follows from (3.10) that

$$Ap = 0. \tag{3.27}$$

Again, since A is Lipschitz continuous and monotone so it is maximal monotone. It follows from Lemma 2.18 that $p \in A^{-1}0$.

(c) We show that $p \in EP(f)$.

From $x_n, y_n \to 0$ and $K_n \to 0$ as $n \to \infty$ and applying (3.7) for any $q \in F$, we get $\lim_{n \to \infty} \phi(q, u_n) \to \phi(q, p)$, it follows that

$$\begin{aligned}
\phi(u_n, x_n) &= \phi(T_{r_n}, x_n) \\
&\leq \phi(q, x_n) - \phi(q, T_{r_n} x_n) \\
&= \phi(q, x_n) - \phi(q, u_n).
\end{aligned}$$

Taking limit as $n \to \infty$ on the both sides of the inequality, we have $\lim_{n \to \infty} \phi(u_n, x_n) = 0$. From Lemma 2.5, it follows that

$$\lim_{n \to \infty} \|u_n - x_n\| = 0 \tag{3.28}$$

and

$$\lim_{n \to \infty} u_n = p. \tag{3.29}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we obtain

$$\lim_{n \to \infty} \|Ju_n - Jz_n\| = 0.$$

Since $r_n > 0$ for all $n \geq 1$, we have $\frac{\|Ju_n - Jz_n\|}{r_n} \to 0$ as $n \to \infty$ and

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jz_n\rangle \geq 0, \quad \forall y \in C.$$

From (A2), the fact that

$$\|y - u_n\|\frac{\|Ju_n - Jz_n\|}{r_n} \geq \frac{1}{r_n}\langle y - u_n, Ju_n - Jz_n\rangle$$
$$\geq -f(u_n, y)$$
$$\geq f(y, u_n), \quad \forall y \in C,$$

taking the limit as $n \to \infty$ in the above inequality and from the fact that $u_n \to p$ as $n \to \infty$, it follows that $f(y, p) \leq 0$ for all $y \in C$. For any $0 < t < 1$, define $y_t = ty + (1 - t)p$. Then $y_t \in C$, which implies that $f(y_t, p) \leq 0$. Thus it follows from (A1) that

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1 - t)\theta(y_t, p) \leq tf(y_t, y),$$

and so $f(y_t, y) \geq 0$. From (A3) we have $f(p, y) \geq 0$ for all $y \in C$ and so $p \in EP(f)$. Hence, by (a), (b) and (c), that is, $p \in F(S) \cap A^{-1}0 \cap EP(f)$.

Finally, we show that $p = \Pi_F x_1$. From $x_n = \Pi_{C_n} x_1$, we have $\langle Jx_1 - Jx_n, x_n - z\rangle \geq 0$ for all $z \in C_n$. Since $F \subset C_n$, we also have

$$\langle Jx_1 - Jx_n, x_n - \hat{p}\rangle \geq 0, \quad \forall \hat{p} \in F.$$

Taking limit $n \to \infty$, we obtain

$$\langle Jx_1 - Jp, p - \hat{p}\rangle \geq 0, \quad \forall \hat{p} \in F.$$

By Lemma 2.6, we can conclude that $p = \Pi_F x_1$ and $x_n \to p$ as $n \to \infty$. The proof is completed. $\square$

Next, we define $z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$ and assume that $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in VI(A, C) \neq \emptyset$. We can prove the strong convergence theorem for finding the set of solutions of the variational inequality problem in a real uniformly smooth and 2-uniformly convex Banach space.

Remark 3.2 (Qin *et al.* [7]) Let Π_C be the generalized projection from a smooth strictly convex and reflexive Banach space E onto a nonempty closed convex subset C of E. Then Π_C is a closed quasi-ϕ-nonexpansive mapping from E onto C with $F(\Pi_C) = C$.

Corollary 3.3 *Let C be a nonempty closed and convex subset of a uniformly smooth and 2-uniformly convex Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying conditions* (A1)-(A4), *and let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in VI(A, C) \neq \emptyset$. Let $S : C \to 2^C$ be a closed and total quasi-ϕ-asymptotically nonexpansive multi-valued mapping with nonnegative real sequences v_n, μ_n with $v_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous*

function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\psi(0) = 0$. *Assume that S is uniformly asymptotically regular on C with* $\mu_1 = 0$ *and* $F := F(S) \cap EP(f) \cap VI(A, C) \neq \emptyset$. *For arbitrary* $x_1 \in C$, $C_1 = C$, *generate a sequence* $\{x_n\}$ *by*

$$
\begin{cases}
z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\
u_n = T_{r_n} x_n, \\
y_n = J^{-1}(\alpha_n Jx_n + \beta_n Jw_n + \gamma_n Ju_n), \quad w_n \in S^n x_n, \\
C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, z_n) \leq \phi(v, x_n) + K_n\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \in \mathbb{N},
\end{cases}
\tag{3.30}
$$

where $K_n = \nu_n \sup_{q \in F} \psi(\phi(q, x_n)) + \mu_n$. *Assume that the control sequences* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ *and* $\{r_n\}$ *satisfy the following conditions:*

1. $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0, 1)$ *such that* $\alpha_n + \beta_n + \gamma_n = 1$, $\liminf_{n \to \infty} \alpha_n \beta_n > 0$,
2. $\{\lambda_n\} \subset [a, b]$ *for some* a, b *with* $0 < a < b < \frac{c^2 \alpha}{2}$ *and* $\frac{1}{c}$ *is the 2-uniformly convex constant of* E,
3. $\{r_n\} \subset [d, \infty)$ *for some* $d > 0$,

then $\{x_n\}$ *converges strongly to* $\Pi_F x_1$.

Proof For $q \in F$ and Π_C is quasi-ϕ-nonexpansive mapping, we have

$$
\phi(q, z_n) = \phi\big(q, \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)\big) \leq \phi\big(q, J^{-1}(Jx_n - \lambda_n Ax_n)\big).
$$

So, we can show that $p \in VI(A, C)$.

Define $B \subset E \times E^*$ by Theorem 2.14, B is maximal monotone and $B^{-1} 0 = VI(A, C)$. Let $(z, w) \in G(B)$. Since $w \in Bz = Az + N_C(z)$, we get $w - Az \in N_C(z)$.

From $z_n \in C$, we have

$$
\langle z - z_n, w - Az \rangle \geq 0. \tag{3.31}
$$

On the other hand, since $z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$. Then, by Lemma 2.6, we have

$$
\big\langle z - z_n, Jz_n - (Jx_n - \lambda_n Ax_n) \big\rangle \geq 0,
$$

and thus

$$
\left\langle z - z_n, \frac{Jx_n - Jz_n}{\lambda_n} - Ax_n \right\rangle \leq 0. \tag{3.32}
$$

It follows from (3.31) and (3.32) that

$$
\begin{aligned}
\langle z - z_n, w \rangle &\geq \langle z - z_n, Az \rangle \\
&\geq \langle z - z_n, Az \rangle + \left\langle z - z_n, \frac{Jx_n - Jz_n}{\lambda_n} - Ax_n \right\rangle \\
&= \langle z - z_n, Az - Ax_n \rangle + \left\langle z - z_n, \frac{Jx_n - Jz_n}{\lambda_n} \right\rangle
\end{aligned}
$$

$$= \langle z - z_n, Az - Az_n \rangle + \langle z - z_n, Az_n - Ax_n \rangle + \left\langle z - z_n, \frac{Jx_n - Jz_n}{\lambda_n} \right\rangle$$

$$\geq -\|z - z_n\| \frac{\|z_n - x_n\|}{\alpha} - \|z - z_n\| \frac{\|Jx_n - Jz_n\|}{a}$$

$$\geq -M \left(\frac{\|z_n - x_n\|}{\alpha} + \frac{\|Jx_n - Jz_n\|}{a} \right),$$

where $M = \sup_{n \geq 1} \|z - z_n\|$. From $\|x_n - z_n\| \to 0$ as $n \to \infty$ and (3.23), taking $\lim_{n \to \infty}$ on the both sides of the equality above, we have $\langle z - p, w \rangle \geq 0$. By the maximality of B, we have $p \in B^{-1}0$, that is, $p \in VI(A, C)$. From Theorem 3.1, we have $p \in F(S) \cap EP(f) \cap VI(A, C)$. The proof is completed. $\square$

Let A be a strongly monotone mapping with constant k, Lipschitz with constant $L > 0$, that is,

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in D(A),$$

which implies that

$$\frac{1}{L}\|Ax - Ay\| \leq \|x - y\|, \quad \forall x, y \in D(A).$$

It follows that

$$\langle Ax - Ay, x - y \rangle \geq k\|x - y\|^2 \geq \frac{k}{L}\|Ax - Ay\|^2$$

hence A is α-inverse-strongly monotone with $\alpha = \frac{k}{L}$. Therefore, we have the following corollaries.

Corollary 3.4 *Let C be a nonempty closed and convex subset of a uniformly smooth and 2-uniformly convex Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying conditions (A1)-(A4), and let $A : E \to E^*$ be a strongly monotone mapping with constant k, Lipschitz with constant $L > 0$. Let $S : C \to 2^C$ be a closed and total quasi-ϕ-asymptotically nonexpansive multi-valued mapping with nonnegative real sequences v_n, μ_n with $v_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$. Assume that S is uniformly asymptotically regular on C with $\mu_1 = 0$ and $F := F(S) \cap EP(f) \cap A^{-1}0 \neq \emptyset$. For arbitrary $x_1 \in C$, $C_1 = C$, a sequence $\{x_n\}$ is generated by*

$$\begin{cases} z_n = J^{-1}(Jx_n - \lambda_n Ax_n), \\ u_n = T_{r_n}z_n, \\ y_n = J^{-1}(\alpha_n Jx_n + \beta_n Jw_n + \gamma_n Ju_n), \quad w_n \in S^n x_n, \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, z_n) \leq \phi(v, x_n) + K_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \in \mathbb{N}, \end{cases} \tag{3.33}$$

where $K_n = v_n \sup_{q \in F} \psi(\phi(q, x_n)) + \mu_n$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ and $\{r_n\}$ satisfy the following conditions:

1. $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\liminf_{n\to\infty} \alpha_n\beta_n > 0$,

2. $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < \frac{c^2 k}{2L}$ and $\frac{1}{c}$ is the 2-uniformly convex constant of E,

3. $\{r_n\} \subset [d,\infty)$ for some $d > 0$,

then $\{x_n\}$ converges strongly to $\Pi_F x_1$.

Corollary 3.5 *Let C be a nonempty closed and convex subset of a uniformly smooth and 2-uniformly convex Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying conditions (A1)-(A4), and let $A : C \to E^*$ be a strongly monotone mapping with constant k, Lipschitz with constant $L > 0$ satisfying $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in VI(A,C) \neq \emptyset$. Let $S : C \to 2^C$ be a closed and total quasi-ϕ-asymptotically nonexpansive multi-valued mapping with nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$. Assume that S is uniformly asymptotically regular on C with $\mu_1 = 0$ and $F := F(S) \cap EP(f) \cap VI(A,C) \neq \emptyset$. For arbitrary $x_1 \in C$, $C_1 = C$, generate a sequence $\{x_n\}$ by*

$$\begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ u_n = T_{r_n} x_n, \\ y_n = J^{-1}(\alpha_n Jx_n + \beta_n Jw_n + \gamma_n Ju_n), \quad w_n \in S^n x_n, \\ C_{n+1} = \{v \in C_n : \phi(v,y_n) \leq \phi(v,z_n) \leq \phi(v,x_n) + K_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \in \mathbb{N}, \end{cases} \quad (3.34)$$

where $K_n = \nu_n \sup_{q \in F} \psi(\phi(q,x_n)) + \mu_n$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ and $\{r_n\}$ satisfy the following conditions:

1. $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\liminf_{n\to\infty} \alpha_n\beta_n > 0$,

2. $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < \frac{c^2 k}{2L}$ and $\frac{1}{c}$ is the 2-uniformly convex constant of E,

3. $\{r_n\} \subset [d,\infty)$ for some $d > 0$,

then $\{x_n\}$ converges strongly to $\Pi_F x_1$.

Let F be a Fréchet differentiable functional in a Banach space E and ∇F be the gradient of F, denote $(\nabla F)^{-1}0 = \{x \in E : F(x) = \min_{y \in E} F(y)\}$. Baillon and Haddad [34] proved the following lemma.

Lemma 3.6 (Baillon and Haddad [34]) *Let E be a Banach space. Let F be a continuously Fréchet differentiable convex functional on E and ∇F be the gradient of F. If ∇F is $\frac{1}{\alpha}$-Lipschitz continuous, then ∇F is an α-inverse strongly monotone mapping.*

We replace A in Theorem 3.1 by ∇F, then we can obtain the following corollary.

Corollary 3.7 *Let C be a nonempty closed and convex subset of a uniformly smooth and 2-uniformly convex Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying conditions (A1)-(A4). Let F be a continuously Fréchet differentiable convex functional*

on E and ∇F be $\frac{1}{\alpha}$-*Lipschitz continuous. Let* $S : C \to 2^C$ *be a closed and total quasi-ϕ-asymptotically nonexpansive multi-valued mapping with nonnegative real sequences* v_n, μ_n *with* $v_n \to 0$, $\mu_n \to 0$ *as* $n \to \infty$ *and a strictly increasing continuous function* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\psi(0) = 0$. *Assume that* S *is uniformly asymptotically regular on* C *with* $\mu_1 = 0$ *and* $F := F(S) \cap F(T) \cap EP(f) \cap A^{-1}0 \neq \emptyset$. *For an initial point* $x_1 \in E$, $C_1 = C$, *define the sequence* $\{x_n\}$ *by*

$$
\begin{cases}
z_n = J^{-1}(Jx_n - \lambda_n \nabla Fx_n), \\
u_n = T_{r_n} x_n, \\
y_n = J^{-1}(\alpha_n Jx_n + \beta_n Jw_n + \gamma_n Ju_n), \quad w_n \in S^n x_n, \\
C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, z_n) \leq \phi(v, x_n) + K_n\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \in \mathbb{N},
\end{cases}
\tag{3.35}
$$

where $\mu_n = \sup\{\mu_n^S, \mu_n^T\}$, $v_n = \sup\{v_n^S, v_n^T\}$, $\psi = \sup\{\psi^S, \psi^T\}$, $k_n = v_n \sup_{q \in \mathcal{F}} \psi(\phi(q, x_n)) + \mu_n$.

Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ *and* $\{r_n\}$ *satisfy the following conditions*:

1. $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0,1)$ *such that* $\alpha_n + \beta_n + \gamma_n = 1$, $\liminf_{n\to\infty} \alpha_n \beta_n > 0$ *and* $\liminf_{n\to\infty} \alpha_n \gamma_n > 0$,

2. $\{\lambda_n\} \subset [a, b]$ *for some* a, b *with* $0 < a < b < \frac{c^2 \alpha}{2}$ *and the 2-uniformly convex constant* $\frac{1}{c}$ *of* E,

3. $\{r_n\} \subset [d, \infty)$ *for some* $d > 0$,

then $\{x_n\}$ *converges strongly to* $\Pi_F x_1$.

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as $n \to \infty$ for all $j = 1, 2, \cdots, m$. Since $\|x_n\| \to \|p\|$ as $n \to \infty$, we also have

$$\|\Delta_n^j x_n\| \to \|p\| \tag{42}$$

as $n \to \infty$ for all $j = 1, 2, \cdots, m$. This implies that for each $j = 1, 2, \cdots, m$, $\{\Delta_n^j x_n\}$ is bounded. Since E is reflexive, without loss of generality, we can assume that $\Delta_n^j x_n \rightharpoonup k$ as $n \to \infty$. Since C_n is closed and convex for each $n \geq 1$, it is obvious that $k \in C_n$. Again, since

$$\phi(\Delta_n^j x_n, x_n) = \|\Delta_n^j x_n\|^2 - 2\langle \Delta_n^j x_n, J x_n \rangle + \|x_n\|^2,$$

taking $\liminf_{n \to \infty}$ on both sides of the equality above, we have

$$0 \geq \|k\|^2 - 2\langle k, Jp \rangle + \|p\|^2 = \phi(k, p). \tag{43}$$

That is, $k = p$ and it follows that for all $j = 1, 2, \cdots, l$,

$$\Delta_n^j x_n \rightharpoonup p \tag{44}$$

as $n \to \infty$. Thus, from (42), (44) and the Kadec-Klee property, it follows that

$$\lim_{n \to \infty} \Delta_n^j x_n = p \tag{45}$$

for all $j = 1, 2, \cdots, m$. We also have

$$\lim_{n \to \infty} \Delta_n^{j-1} x_n = p \tag{46}$$

for all $j = 1, 2, \cdots, m$, and so

$$\lim_{n \to \infty} \|\Delta_n^j x_n - \Delta_n^{j-1} x_n\| = 0 \tag{47}$$

for all $j = 1, 2, \cdots, m$. Since J is uniformly norm-to-norm continuous on bounded subsets of E and $\liminf_{n \to \infty} \lambda_{j,n} > 0$ for each $j = 1, 2, \cdots, l$, we have

$$\lim_{n \to \infty} \tfrac{1}{\lambda_{j,n}} \|J \Delta_n^j x_n - J \Delta_n^{j-1} x_n\| = 0. \tag{48}$$

Let $\Delta_n^j x_n = J_{\lambda_{j,n}}^j \Delta_n^{j-1} x_n$ for each $j = 1, 2, \cdots, l$. Then we have

$$\lim_{n \to \infty} \|A_{\lambda_{j,n}} \Delta_n^{j-1} x_n\| = \lim_{n \to \infty} \tfrac{1}{\lambda_{j,n}} \|J \Delta_n^j x_n - J \Delta_n^{j-1} x_n\| = 0. \tag{49}$$

For any $(w, w^*) \in G(A_j)$ and $(\Delta_n^j x_n, A_{\lambda_{j,n}} \Delta_n^{j-1} x_n) \in G(A_j)$ for each $j = 1, 2, \cdots, l$, it follows from the monotonicity of A_j that for all $n \geq 1$,

$$\langle w - \Delta_n^j x_n, w^* - A_{\lambda_{j,n}} \Delta_n^{j-1} x_n \rangle \geq 0$$

for all $j = 1, 2, \cdots, l$. Letting $n \to \infty$ in the inequality above, we get $\langle w - p, w^* \rangle \geq 0$ for all $j = 1, 2, \cdots, l$. Since A_j is maximal monotone for all $j = 1, 2, \cdots, l$, we obtain $p \in \cap_{j=1}^l A_j^{-1} 0$.

Next, we show that $p \in \cap_{i=1}^{m} EP(\theta_i)$. For any $q \in F$ and $u_n = K_n^m z_n$, we observe that

$$
\begin{aligned}
\phi(q, u_n) &= \phi(q, K_n^m z_n) \\
&\leq \phi(q, K_n^{m-1} z_n) \\
&\leq \phi(q, K_n^{m-2} z_n) \\
&\cdots \\
&\leq \phi(q, K_n^i z_n).
\end{aligned}
\tag{50}
$$

By Lemma 10, for $i = 1, 2, \cdots, m$, we have

$$
\begin{aligned}
\phi(K_n^i z_n, x_n) &\leq \phi(q, x_n) - \phi(q, K_n^i z_n) \\
&\leq \phi(q, x_n) - \phi(q, u_n).
\end{aligned}
\tag{51}
$$

Since $x_n \to p$ and $u_n \to p$ as $n \to \infty$, we get $\phi(K_n^i z_n, x_n) \to 0$ as $n \to \infty$ for all $i = 1, 2, \cdots, m$. From (3), it follows that

$$
(\|K_n^i z_n\| - \|x_n\|)^2 \to 0
$$

as $n \to \infty$. Since $\|x_n\| \to \|p\|$ as $n \to \infty$, we also have

$$
\|K_n^i z_n\| \to \|p\|
\tag{52}
$$

as $n \to \infty$. Since $\{K_n^i z_n\}$ is bounded and E is reflexive, without loss of generality, we assume that $K_n^i z_n \rightharpoonup h$ as $n \to \infty$. Since C_n is closed and convex for each $n \geq 1$, it is obvious that $h \in C_n$. Again, since

$$
\phi(K_n^i z_n, x_n) = \|K_n^i z_n\|^2 - 2\langle K_n^i z_n, J x_n \rangle + \|x_n\|^2,
$$

taking $\liminf_{n\to\infty}$ on both sides of the equality above, we have

$$
0 \geq \|h\|^2 - 2\langle h, Jp \rangle + \|p\|^2 = \phi(h, p).
\tag{53}
$$

That is, $h = p$ and it follows that for all $i = 1, 2, \cdots, m$, it follows that

$$
K_n^i z_n \rightharpoonup p
\tag{54}
$$

as $n \to \infty$. Thus, from (520, (54) and the Kadec-Klee property, it follows that

$$
\lim_{n\to\infty} K_n^i z_n = p
\tag{55}
$$

for all $i = 1, 2, \cdots, m$. We also have

$$
\lim_{n\to\infty} K_n^{i-1} z_n = p
\tag{56}
$$

for all $i = 1, 2, \cdots, m$, and so

$$
\|K_n^i z_n - K_n^{i-1} z_n\| = 0
$$

for all $i = 1, 2, \cdots, m$. Since J is uniformly norm-to-norm continuous, we obtain

$$
\lim_{n\to\infty} \|J K_n^i z_n - J K_n^{i-1} z_n\| = 0
$$

for all $i = 1, 2, \cdots, m$. From $r_{i,n} > 0$ for all $i = 1, 2, \cdots, m$, we have

$$
\frac{\|J K_n^i z_n - J K_n^{i-1} z_n\|}{r_{i,n}} \to 0
$$

as $n \to \infty$ and

$$\theta_i(K_n^i z_n, y) + \frac{1}{r_{i,n}} \langle y - K_n^i z_n, JK_n^i z_n - JK_n^{i-1} z_n \rangle \geq 0 \qquad (57)$$

for all $y \in C$. Thus, by (A2), we have

$$
\begin{aligned}
\|y - K_n^i z_n\| \frac{\|JK_n^i z_n - JK_n^{i-1} z_n\|}{r_{i,n}} &\geq \frac{1}{r_{i,n}} \langle y - K_n^i z_n, JK_n^i z_n - JK_n^{i-1} z_n \rangle \\
&\geq -\theta_i(K_n^i z_n, y) \\
&\geq \theta_i(y, K_n^i z_n)
\end{aligned}
\qquad (58)
$$

for all $y \in C$ and $K_n^i z_n \to p$ as $n \to \infty$, and so $\theta_i(y, p) \leq 0$ for all $y \in C$. For any t with $0 < t < 1$, define $y_t = ty + (1-t)p$. Then $y_t \in C$, which implies that $\theta_i(y_t, p) \leq 0$ for all $i = 1, 2, \cdots, m$. Thus, from (A1), it follows that

$$0 = \theta_i(y_t, y_t) \leq t\theta_i(y_t, y) + (1-t)\theta_i(y_t, p) \leq t\theta_i(y_t, y),$$

and so $\theta_i(y_t, y) \geq 0$ for all $i = 1, 2, \cdots, m$. From (A3), we have $\theta_i(p, y) \geq 0$ for all $y \in C$ and $i = 1, 2, \cdots, m$, that is, $p \in EP(\theta_i)$ for all $i = 1, 2, \cdots, m$. This implies that $p \in \cap_{i=1}^m EP(\theta_i)$.

Finally, we show that $p \in F(S)$. Since $\{x_n\}$ is bounded, the mapping S is also bounded. From $y_n \to p$ as $n \to \infty$ and (9), we have

$$\|JS^n x_n\| \to \|Jp\| \qquad (59)$$

as $n \to \infty$. Since $J^{-1} : E^* \to E$ is norm-weak*-continuous,

$$S^n x_n \rightharpoonup p \qquad (60)$$

as $n \to \infty$.

On the other hand, in view of (59), it follows that

$$\| \|S^n x_n\| - \|p\| \| = \| \|J(S^n x_n)\| - \|Jp\| \| \leq \|J(S^n x_n) - Jp\| = 0$$

and so $\|S^n x_n\| \to \|p\|$. Since E has the Kadee-Klee property, we get

$$S^n x_n \to p \qquad (61)$$

for all $n \geq 1$. By using the triangle inequality, since S is uniformly L-Lipschitz continuous, we get

$$
\begin{aligned}
&\|S^{n+1} x_n - S^n x_n\| \\
&\leq \|S^{n+1} x_n - S^{n+1} x_{n+1}\| + \|S^{n+1} x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - S^n x_n\| \\
&\leq (L+1)\|x_{n+1} - x_n\| + \|S^{n+1} x_{n+1} - x_{n+1}\| + \|x_n - S^n x_n\|.
\end{aligned}
\qquad (62)
$$

Since $S^n x_n \to p$ as $n \to \infty$, we get $S^{n+1} x_n \to p$ as $n \to \infty$, and so $SS^n x_n \to p$ as $n \to \infty$. In view of the closeness of S, we have $Sp = p$, which implies that $p \in F(S)$. Hence $p \in \mathcal{F}$.

Step 5. We show that $p = \Pi_{\mathcal{F}}^f x_1$. Since $\mathcal{F}$ is a closed and convex set, it follows from Lemma 5 that $\Pi_{\mathcal{F}}^f x_1$ is single-valued, which is denoted by $\hat{p}$. By the definitions of $x_n = \Pi_{C_n}^f x_1$ and $\hat{p} \in \mathcal{F} \subset C_n$, we also have

$$G(x_n, Jx_1) \leq G(\hat{p}, Jx_1)$$

for all $n \geq 1$. By the definitions of G and f, we know that for any $x \in E$, $G(\xi, Jx)$ is convex and lower semicontinuous with respect to ξ, and so

$$G(p, Jx_1) \leq \liminf_{n \to \infty} G(x_n, Jx_1) \leq \limsup_{n \to \infty} G(x_n, Jx_1) \leq G(\hat{p}, Jx_1).$$

From the definition of $\Pi_{\mathcal{F}}^f x_1$, since $p \in \mathcal{F}$, we conclude that $\hat{p} = p = \Pi_{\mathcal{F}}^f x_1$ and $x_n \to p$ as $n \to \infty$. This completes the proof.

Setting $\nu_n = (k_n - 1)$, $\mu_n = 0$ and $\psi : \mathbf{R}^+ \to 1$ in Theorem 1, we have the following result.

Corollary 1 *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. For each $i = 1, 2, \cdots, m$, let θ_i be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying the conditions (A1)-(A4). Let $A_j \subset E \times E^*$ be a maximal monotone operator satisfying $D(A) \subset C$ and $J_{\lambda_{j,n}}^{A_j} = (J + \lambda_{j,n} A_j)^{-1} J$ for all $\lambda_{j,n} > 0$ and $j = 1, 2, \cdots, l$. Let $S : C \to C$ be a closed and quasi-ϕ-asymptotically nonexpansive mapping. Let $f : E \to \mathbf{R}^+$ be a convex and lower semicontinuous function with $C \subset int(D(f))$ and $f(0) = 0$. Assume that S is uniformly L-Lipschitz continuous and $\mathcal{F} = F(S) \cap (\cap_{i=1}^m EP(\theta_i)) \cap (\cap_{j=1}^l A_j^{-1} 0) \neq \emptyset$. For an initial point $x_1 \in E$, define $C_1 = C$ and the sequence $\{x_n\}$ in C by*

$$\begin{cases} z_n = J_{\lambda_{l,n}}^{A_l} \circ J_{\lambda_{l-1,n}}^{A_{l-1}} \circ \cdots \circ J_{\lambda_{1,n}}^{A_1} x_n, \\ u_n = T_{r_{m,n}}^{\theta_m} \circ T_{r_{m-1,n}}^{\theta_{m-1}} \circ \cdots \circ T_{r_{1,n}}^{\theta_1} z_n, \\ y_n = J^{-1}(\alpha_n Jx_1 + \beta_n JS^n x_n + \gamma_n Ju_n), \\ C_{n+1} = \{v \in C_n : G(v, Jy_n) \leq \alpha_n G(q, Jx_1) + (1 - \alpha_n) G(q, Jx_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}^f x_1 \end{cases} \tag{63}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\zeta_n = \sup_{q \in \mathcal{F}}(k_n - 1) G(q, Jx_n)$ and, for each $i = 1, 2, 3, \cdots, m$, $\{r_{i,n}\} \subset [d, \infty)$ for some $d > 0$. If $\lim_{n \to \infty} \alpha_n = 0$, $\liminf_{n \to \infty} \beta_n < 1$ and $\liminf_{n \to \infty} \lambda_{j,n} > 0$ for all $j = 1, 2, \cdots, l$, then the sequence $\{x_n\}$ converges strongly to a point $\Pi_{\mathcal{F}}^f x_1$.

If $f(x) = 0$ for all $x \in E$ in Theorem 1, then $G(x, Jy) = \phi(x, y)$ and $\Pi_{\mathcal{F}}^f = \Pi_{\mathcal{F}}$ and so we have the following corollary.

Corollary 2 *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. For each $i = 1, 2, \cdots, m$, let θ_i be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying the conditions (A1)-(A4). Let $A_j \subset E \times E^*$ be a maximal monotone operator satisfying $D(A_j) \subset C$ and $J_{\lambda_{j,n}}^{A_j} = (J + \lambda_{j,n} A_j)^{-1} J$ for all $\lambda_{j,n} > 0$ and $j = 1, 2, \cdots, l$. Let $S : C \to C$ be a closed and totally quasi-ϕ-asymptotically nonexpansive mapping with nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$, respectively, and a strictly increasing continuous function $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ with $\psi(0) = 0$. Assume that S is uniformly L-Lipschitz continuous and $\mathcal{F} = F(S) \cap (\cap_{i=1}^m EP(\theta_i)) \cap (\cap_{j=1}^l A_j^{-1} 0) \neq \emptyset$. For an initial point $x_1 \in E$, define $C_1 = C$ and the sequence $\{x_n\}$ in C by*

$$\begin{cases} z_n = J_{\lambda_{l,n}}^{A_l} \circ J_{\lambda_{l-1,n}}^{A_{l-1}} \circ \cdots \circ J_{\lambda_{1,n}}^{A_1} x_n, \\ u_n = T_{r_{m,n}}^{\theta_m} \circ T_{r_{m-1,n}}^{\theta_{m-1}} \circ \cdots \circ T_{r_{1,n}}^{\theta_1} z_n, \\ y_n = J^{-1}(\alpha_n Jx_1 + \beta_n JS^n x_n + \gamma_n Ju_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \alpha_n \phi(v, x_1) + (1 - \alpha_n) \phi(v, x_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1 \end{cases} \tag{64}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $(0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\zeta_n = \nu_n \sup_{q \in \mathcal{F}} \psi(\phi(q, x_n)) + \mu_n$ and, for each $i = 1, 2, 3, \cdots, m$, $\{r_{i,n}\} \subset [d, \infty)$ for some $d > 0$. If $\lim_{n \to \infty} \alpha_n = 0$, $\liminf_{n \to \infty} \beta_n < 1$ and $\liminf_{n \to \infty} \lambda_{j,n} > 0$ for each $j = 1, 2, \cdots, l$, then the sequence $\{x_n\}$ converges strongly to a point $\Pi_{\mathcal{F}} x_1$.

Setting $\nu_n = (k_n - 1)$, $\mu_n = 0$ and $\psi(x) = x$ in Theorem 1, we have the following corollary.

Corollary 3 *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. For each $i = 1, 2, \cdots, m$, let θ_i be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying the conditions (A1)-(A4). Let $A_j \subset E \times E^*$ be a maximal monotone operator satisfying $D(A) \subset C$ and $J_{\lambda_{j,n}}^{A_j} = (J + \lambda_{j,n} A_j)^{-1} J$ for all $\lambda_{j,n} > 0$ and $j = 1, 2, \cdots, l$. Let $S : C \to C$ be a closed and quasi-ϕ-asymptotically nonexpansive mapping. Assume that S uniformly L-Lipschitz continuous and $\mathcal{F} = F(S) \cap (\cap_{i=1}^m EP(\theta_i)) \cap (\cap_{j=1}^l A_j^{-1} 0) \neq \emptyset$. For an initial point $x_1 \in E$, define $C_1 = C$ and the sequence $\{x_n\}$ in C by*

$$
\begin{cases}
z_n = J_{\lambda_{l,n}}^{A_l} \circ J_{\lambda_{l-1,n}}^{A_{l-1}} \circ \cdots \circ J_{\lambda_{1,n}}^{A_1} x_n, \\
u_n = T_{r_{m,n}}^{\theta_m} \circ T_{r_{m-1,n}}^{\theta_{m-1}} \circ \cdots \circ T_{r_{1,n}}^{\theta_1} z_n, \\
y_n = J^{-1}(\alpha_n J x_1 + \beta_n J S^n x_n + \gamma_n J u_n), \\
C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \alpha_n \phi(q, x_1) + (1 - \alpha_n) \phi(q, x_n) + \zeta_n\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_1
\end{cases}
\tag{65}
$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $(0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\zeta_n = \sup_{q \in \mathcal{F}} (k_n - 1) \phi(q, x_n)$ and, for each $i = 1, 2, 3, \cdots, m$, $\{r_{i,n}\} \subset [d, \infty)$ for some $d > 0$. If $\lim_{n \to \infty} \alpha_n = 0$, $\liminf_{n \to \infty} \beta_n < 1$ and $\liminf_{n \to \infty} \lambda_{j,n} > 0$ for each $j = 1, 2, \cdots, l$, then the sequence $\{x_n\}$ converges strongly to a point $\Pi_{\mathcal{F}} x_1$.

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สรุปผลการดำเนินงาน

นักวิจัยต่างประเทศมาทำกิจกรรมในประเทศไทย

เมื่อ Prof. Yeol Je Cho เดินทางมาถึงมหาวิทยาลัยทักษิณ ได้ร่วมพูดคุยถึงความคืบหน้าของงานวิจัยที่ได้ทำร่วมกัน และได้ให้คำแนะนำ เสนอแนวคิดใหม่ ๆ ในการแก้ปัญหางานวิจัย นอกจากนั้นแล้ว Prof. Yeol Je Cho ยังมีหัวข้องานวิจัยที่น่าสนใจมาแนะนำเพิ่มเติม ในวันถัดมา Prof. Yeol Je Cho ได้ร่วมพูดคุยกับนักศึกษาและผู้สนใจในเรื่องการเรียนการสอนและการวิจัยทางคณิตศาสตร์ในประเทศเกาหลี ก่อนเดินทางกลับ Prof. Yeol Je Cho ได้กรุณาช่วยอ่านงานวิจัยเพื่อปรับปรุงภาษาและเพื่อทำให้งานวิจัยน่าสนใจมากขึ้น

Let $f : C \times C \to \mathbb{R}$ be a bifunction, the *equilibrium problem*, is to find $x \in C$ such that

$$f(x, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.2) is denoted by $EP(f)$. The equilibrium problem is very general in the sense that it includes, as special cases optimization problems, variational inequality problems, Min-Max problems, saddle point problem, fixed point problem, Nash EP. In 2008, Takahashi and Zembayashi [82, 83], introduced iterative sequences for finding a common solution of an equilibrium problem and fixed point problem. Some solution methods have been proposed to solve the equilibrium problem; see, for instance, [60, 18, 36, 73, 26, 92, 27, 28, 93, 94].

Let θ be a bifunction of $C \times C$ into $\mathbb{R}$ and $\varphi : C \to \mathbb{R}$ be a real-valued function. The *mixed equilibrium problem*, denoted by $MEP(\theta, \varphi)$, is to find $x \in C$ such that

$$\theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.2}$$

If $\varphi \equiv 0$, the problem (1.2) reduce into the *equilibrium problem for θ*, denoted by $EP(\theta)$, is to find $x \in C$ such that

$$\theta(x, y) \geq 0, \quad \forall y \in C. \tag{1.3}$$

If $\theta \equiv 0$, the problem (1.2) reduce into the *minimize problem*, denoted by $Argmin(\varphi)$, is to find $x \in C$ such that

$$\varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.4}$$

The above formulation (1.3) was shown in [8] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational inequality problems, vector equilibrium problems, Nash equilibria in noncooperative games. In addition, there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an $EP(\theta)$. In other words, the $EP(\theta)$ is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. In the last two decades, many papers have appeared in the literature on the existence of solutions of $EP(\theta)$; see, for example [8, 42] and references therein. Some solution methods have been proposed to solve the $EP(\theta)$; see, for example, [8, 39, 41, 43, 55, 63, 61] and references therein.

Let E be a real Banach space with dual E^* and let C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be an operator. A is called *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C;$$

$\alpha-$*inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C;$$

L-Lipschitz continuous if there exists a constant $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

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If A is α-inverse strongly monotone, then it is $\frac{1}{\alpha}$-Lipschitz continuous, i.e.,

$$\|Ax - Ay\| \le \frac{1}{\alpha}\|x - y\|, \quad \forall x, y \in C.$$

A monotone operator A is said to be *maximal* if its graph $G(A) = \{(x, x^*) : x^* \in Ax\}$ is not properly contained in the graph of any other monotone operator.

Let A be a monotone operator. We consider the problem for finding $x \in E$ such that

$$0 \in Ax, \tag{1.5}$$

a point $x \in E$ is called a *zero point* of A. Denote by $A^{-1}0$ the set of all point $x \in E$ such that $0 \in Ax$. This problem is very important in optimization theory and related fields.

Let A be a monotone operator. *The classical variational inequality problem* for an operator A is to find $\hat{z} \in C$ such that

$$\langle A\hat{z}, y - \hat{z} \rangle \ge 0, \quad \forall y \in C. \tag{1.6}$$

The set of solution of (1.6) is denoted by $VI(A, C)$. This problem is connected with the convex minimization problem, the complementary problem, the problem of finding a point $x \in E$ satisfying $Ax = 0$.

The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^* \rangle$ or $x^*(x)$. For each $p > 1$, the *generalized duality mapping* $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\}$$

for all $x \in E$. In particular, $J = J_2$ is called the *normalized duality mapping*. If E is a Hilbert space, then $J = I$, where I is the identity mapping.

Consider the functional defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2, \quad \text{for } x, y \in E, \tag{1.7}$$

where J is the normalized duality mapping. It is obvious from the definition of ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(y, x) \le (\|y\| + \|x\|)^2, \quad \forall x, y \in E. \tag{1.8}$$

Alber [3] introduced The *generalized projection* $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution of the minimization problem

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x), \tag{1.9}$$

existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J.

Iiduka and Takahashi [37] introduced the following iterative scheme for finding a solution of the variational inequality problem for an inverse-strongly monotone operator A in a 2-uniformly convex and uniformly smooth Banach space E: $x_1 = x \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \quad \forall n \ge 1, \tag{1.10}$$

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where Π_C is the generalized projection from E onto C, J is the duality mapping from E into E^* and $\{\lambda_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (1.10) converges weakly to some element of $VI(A, C)$. In connection, Iiduka and Takahashi [34] studied the following iterative scheme for finding a zero point of a monotone operator A in a 2-uniformly convex and uniformly smooth Banach space E:

$$\begin{cases} x_1 = x \in E \text{ chosen arbitrarily,} \\ y_n = J^{-1}(Jx_n - \lambda_n Ax_n), \\ X_n = \{z \in E : \phi(z, y_n) \le \phi(z, x_n)\}, \\ Y_{n+1} = \{z \in E : \langle x_n - z, Jx - Jx_n \rangle \ge 0\}, \\ x_{n+1} = \Pi_{X_n \cap Y_n}(x). \end{cases} \tag{1.11}$$

where $\Pi_{X_n \cap Y_n}$ is the generalized projection from E onto $X_n \cap Y_n$, J is the duality mapping from E into E^* and $\{\lambda_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ converges strongly to an element of $A^{-1}0$. Moreover, under the additional suitable assumption they proved that the sequence $\{x_n\}$ converges strongly to some element of $VI(A, C)$. Some solution methods have been proposed to solve the variational inequality problem; see, for instance, [19, 47, 48].

Let E be a real Banach space with the dual space E^* and C be a nonempty closed and convex subset of E. A mapping $S : C \to C$ is said to be:

(1) *nonexpansive* if

$$\|Sx - Sy\| \le \|x - y\|$$

for all $x, y \in C$;

(2) *quasi-nonexpansive* if $F(S) \ne \emptyset$ and

$$\|Sx - y\| \le \|x - y\|$$

for all $x \in C$ and $y \in F(S)$;

(3) *quasi-ϕ-nonexpansive* if $F(S) \ne \emptyset$ and

$$\phi(p, Sx) \le \phi(p, x), \forall x \in C, \ p \in F(S).$$

;

(4) *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|S^n x - S^n y\| \le k_n \|x - y\|$$

for all $x, y \in C$;

(5) *asymptotically quasi-nonexpansive* if $F(S) \ne \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|S^n x - y\| \le k_n \|x - y\|$$

for all $x \in C$ and $y \in F(S)$;

(6) *total asymptotically nonexpansive* if there exist the nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ with $\psi(0) = 0$ such that

$$\|S^n x - S^n y\| \le \|x - y\| + \mu_n \psi(\|x - y\|) + \nu_n$$

for all $x, y \in C$ and $n \ge 1$;

(7) *total quasi-ϕ-asymptotically nonexpansive*, if $F(S) \ne \emptyset$ and there exist nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$ such that

$$\phi(p, S^n x) \le \phi(p, x) + \nu_n \varphi(\phi(p, x)) + \mu_n, \ \forall n \ge 1, \ \forall x \in C, \ p \in F(S).$$

A mapping $S : C \to C$ is said to be *uniformly L-Lipschitz continuous* if there exists a constant $L > 0$ such that

$$\|S^n x - S^n y\| \le L\|x - y\| \tag{1.12}$$

for all $x, y \in C$. A mapping $S : C \to C$ is said to be *closed* if, for any sequence $\{x_n\} \subset C$ such that $\lim_{n \to \infty} x_n = x_0$ and $\lim_{n \to \infty} Sx_n = y_0$, we have $Sx_0 = y_0$.

Let 2^C be the family of all nonempty subsets of C and let $S : C \to 2^C$ be a multi-valued mapping. For a point $q \in C$, $n \ge 1$ define an iterative sequence as follows:

$$Sq := \{q_1 : q_1 \in Sq\}$$

$$S^2 q = SSq := \bigcup_{q_1 \in Sq} Sq_1$$

$$S^3 q = SS^2 q := \bigcup_{q_2 \in T^2 q} Sq_2$$

$$\vdots$$

$$S^n q = SS^{n-1} q := \bigcup_{q_{n-1} \in S^{n-1} q} Sq_{n-1}.$$

A point $p \in C$ is said to be an *asymptotic fixed point* of S, if there exists a sequence $\{x_n\}$ in C such that $\{x_n\}$ converges weakly to p and

$$\lim_{n \to \infty} d(x_n, Sx_n) := \lim_{n \to \infty} \inf_{x \in Sx_n} \|x_n - x\| = 0.$$

The asymptotic fixed point set of S denoted by $\widehat{F}(S)$.

A multi-valued mapping S is said to be *total quasi-ϕ-asymptotically nonexpansive*, if $F(S) \ne \emptyset$ and there exist nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$ such that for all $x \in C$, $p \in F(S)$

$$\phi(p, w_n) \le \phi(p, x) + \nu_n \varphi(\phi(p, x)) + \mu_n, \ \forall n \ge 1, \ w_n \in S^n x.$$

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S is said to be *closed* if for any sequence $\{x_n\}$ and $\{w_n\}$ in C with $w_n \in Sx_n$ if $x_n \to x$ and $w_n \to w$, then $w \in Sx$.

A multi-valued mapping S is said to be *uniformly asymptotically regular* on C if

$$\lim_{n \to \infty} \left(\sup_{x \in C} \|s_{n+1} - s_n\| \right) = 0, \ s_n \in S^n x.$$

Every quasi-ϕ-asymptotically nonexpansive multi-valued mappings implies quasi-ϕ-asymptotically nonexpansive mappings but the converse is not true.

In 2012, Chang et al. [13] introduced the concept of total quasi-ϕ-asymptotically nonexpansive multi-valued mapping and then proved some strong convergence theorem by using the hybrid shrinking projection method.

For a mapping $A : C \to E^*$, let $f(x,y) = \langle Ax, y - x \rangle$ for all $x, y \in C$. Then $x \in EP(f)$ if and only if $\langle Tx, y - x \rangle \geq 0$ for all $y \in C$; i.e., x is a solution of the variational inequality.

Motivated and inspired by the work mentioned above, in this work, we introduce and prove strong convergence theorems of hybrid projection algorithm for a fixed point set. In the first theorem, we prove strong convergence theorem of a family of relatively quasi-nonexpansive mapping. Next, we extend a mapping to a total quasi-ϕ-asymopotically nonexpansive mapping and then we can prove strong convergence theorem. Finally, we extend from a single-value mapping to a multi-valued mapping. In the final theorem we show the fixed set of a total quasi-ϕ-asymopotically nonexpansive multi-valued mappings. Moreover, we prove strong convergence to solution of the equilibrium problem, zero points of monotone operators, the solution of the variation inequality in Banach space.

2 Preliminaries

A Banach space E with norm $\| \cdot \|$, is called *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. A Banach space E is called *smooth* if the limit $\lim_{t \to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U$. It is also called *uniformly smooth* if the limit exists uniformly for all $x, y \in U$. The *modulus of convexity* of E is the function $\delta : [0,2] \to [0,1]$ defined by

$$\delta(\varepsilon) = \inf\{1 - \|\tfrac{x+y}{2}\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\}.$$

A Banach space E is a *uniformly convex* if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be a *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0, 2]$. Observe that every p-uniform convex is uniformly convex. Every uniformly convex Banach space E has the *Kadec-Klee property*, that is, for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.

Let E be a real Banach space with dual E^*, E is a uniformly smooth if and only if E^* is a uniformly convex Banach space. If E is a uniformly smooth Banach space, then E is a smooth and reflexive Banach space.

Remark 2.1.   • If E is smooth, then J is single-valued;

- If E is strictly convex, then J is one-to-one and $\langle x - y, x^* - y^* \rangle > 0$ holds for all $(x, x^*), (y, y^*) \in J$ with $x \neq y$;

- If E is a strictly convex, then J is strictly monotone;

- If E is a smooth, then J is single valued and semi-continuous;

- If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E;

- If E is reflexive smooth and strictly convex, then the normalized duality mapping J is single valued, one-to-one and onto;

- If E be a reflexive strictly convex and smooth Banach space and J is the duality mapping from E into E^*, then J^{-1} is also single-value, bijective and is also the duality mapping from E^* into E and thus $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$;

- If E is uniformly smooth, then E is smooth and reflexive;

- E is uniformly smooth if and only if E^* is uniformly convex.

See [17] for more details.

Remark 2.2. If E is a reflexive, strictly convex and smooth Banach space, then $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. From (1.7), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy \rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$ (see [17, 68] for more details).

Lemma 2.3. (*Beauzamy* [10] *and Xu*[86]) *If E be a 2-uniformly convex Banach space. Then, for all $x, y \in E$ we have*

$$\|x - y\| \leq \frac{2}{c^2} \|Jx - Jy\|,$$

where J is the normalized duality mapping of E and $0 < c \leq 1$.

The best constant $\frac{1}{c}$ in Lemma is called the *p-uniformly convex constant* of E.

Lemma 2.4. (*Beauzamy* [10] *and Zalinescu*[89]) *If E be a p-uniformly convex Banach space and let p be a given real number with $p \geq 2$. Then for all $x, y \in E$, $J_x \in J_p(x)$ and $J_y \in J_p(y)$*

$$\langle x - y, J_x - J_y \rangle \geq \frac{c^p}{2^{p-2}p} \|x - y\|^p,$$

where J_p is the generalized duality mapping of E and $\frac{1}{c}$ is the p-uniformly convexity constant of E.

Lemma 2.5. (Kamimura and Takahashi [45]) *Let E be a uniformly convex and smooth Banach space and $\{x_n\}, \{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 2.6. (Alber [3]) *Let C be a nonempty closed convex subset of a smooth Banach space E and let $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

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Lemma 2.7. (Alber [3]) *Let E be a reflexive strictly convex and smooth Banach space, C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 2.8. (*Qin et al.* [63]). *Let E be a uniformly convex and smooth Banach space, let C be a closed convex subset of E, and let T be a closed relatively quasi-nonexpansive mapping from C into itself. Then $F(T)$ is a closed convex subset of C.*

Lemma 2.9. ([21]) *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. Let $S : C \to C$ be a closed and total quasi-ϕ-asymptotically nonexpansive mapping with the nonnegative real sequences $\{\nu_n\}$ and $\{\mu_n\}$ with $\nu_n \to 0$ and $\mu_n \to 0$ as $n \to \infty$, respectively, and a strictly increasing continuous function $\zeta : \mathbf{R}^+ \to \mathbf{R}^+$ with $\zeta(0) = 0$. If $\mu_1 = 0$, then the set $F(S)$ of fixed points of S is a closed convex subset of C.*

Lemma 2.10. (Change et al.[13]) *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. Let $S : C \to 2^C$ be a closed and total quasi-ϕ-asymptotically nonexpansive multi-valued mapping with nonnegtive real sequence ν_n and μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$. If $\mu_1 = 0$, then the fixed point set $F(S)$ is a closed convex subset of C.*

Let A be an inverse-strongly monotone mapping of C into E^* which is said to be *hemicontinuous* if for all $x, y \in C$, the mapping h of $[0,1]$ into E^*, defined by $h(t) = A(tx + (1-t)y)$, is continuous with respect to the weak* topology of E^*. We define by $N_C(v)$ *the normal cone* for C at a point $v \in C$, that is,

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^* \rangle \geq 0, \ \forall y \in C\}. \tag{2.1}$$

Theorem 2.11. (*Rockafellar* [72]) *Let C be a nonempty, closed convex subset of a Banach space E and A a monotone, hemicontinuous operator of C into E^*. Let $B \subset E \times E^*$ be an operator defined as follows:*

$$Bv = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & otherwise. \end{cases} \tag{2.2}$$

Then B is maximal monotone and $B^{-1}0 = VI(A, C)$.

Theorem 2.12. (*Takahashi* [81]) *Let C be a nonempty subset of a Banach space E and A a monotone, hemicontinuous operator of C into E^* with $C = D(A)$. Then*

$$VI(A, C) = \{u \in C : \langle v - u, Av \rangle \geq 0, \ \forall v \in C\}. \tag{2.3}$$

It is obvious that the set $VI(A, C)$ is a closed and convex subset of C and the set $A^{-1}0 = VI(A, E)$ is closed and convex subset of E.

Theorem 2.13. (*Takahashi* [81]) *Let C be a nonempty compact convex subset of a Banach space E and A a monotone, hemicontinuous operator of C into E^* with $C = D(A)$. Then $VI(A, C)$ is nonempty.*

We make use of the following mapping V studied in Alber [3]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2, \quad \forall x \in E, \ x^* \in E^*, \tag{2.4}$$

that is, $V(x, x^*) = \phi(x, J^{-1}(x^*))$.

Lemma 2.14. (Alber [3]) *Let E be a reflexive strictly convex smooth Banach space and V be as in (2.4). Then we have*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^* \rangle \leq V(x, x^* + y^*), \quad \forall x \in E, \ x^*, y^* \in E^*.$$

Lemma 2.15. (Cho et al.[16]) *Let E be a uniformly convex Banach space and $B_r(0) = \{x \in E : \|x\| \leq r\}$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \|\lambda x\|^2 + |\mu y\|^2 + |\gamma z\|^2 - \lambda \mu g(\|x - y\|),$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

Lemma 2.16. (Pascali and Sburlan [59]) *Let E be a real smooth Banach spaces and $A : E \to 2^{E^*}$ be a maximal monotone mapping. Then $A^{-1}0$ is closed and convex subset of E and the graph $G(A)$ of A is demiclosed in the following sense: if $\{x_n\} \subset D(A)$ with $x_n \rightharpoonup x \in E$ and $y_n \in Ax_n$ with $y_n \to y \in E^*$, then $x \in D(A)$ and $y \in Ax$.*

In 2006, Wu and Huang [85] introduced a new generalized f-projection operator in Banach space. They extended the definition of the generalized projection operators introduced by Abler [2] and proved some properties of the generalized f-projection operator. Consider the functional $G : C \times E^* \longrightarrow \mathbf{R} \cup \{+\infty\}$ defined by

$$G(y, \varpi) = \|y\|^2 - 2\langle y, \varpi \rangle + \|\varpi\|^2 + 2\rho f(y) \tag{2.5}$$

for all $(y, \varpi) \in C \times E^*$, where ρ is a positive number and $f : C \to \mathbf{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous. From the definition of G, Wu and Huang [85] proved the following properties:

(1) $G(y, \varpi)$ is convex and continuous with respect to ϖ when y is fixed;

(2) $G(y, \varpi)$ is convex and lower semicontinuous with respect to y when ϖ is fixed.

Let E be a real Banach space with its dual space E^* and C be a nonempty closed and convex subset of E. We say that $\pi_C^f : E^* \to 2^C$ is a *generalized f-projection operator* if

$$\pi_C^f \varpi = \{u \in C : G(u, \varpi) = \inf_{y \in C} G(y, \varpi), \forall \varpi \in E^*\}.$$

Recall that a Banach space E has the Kadec-Klee property ([17, 33, 81]) if, for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, we have $\|x_n - x\| \to 0$ as $n \to \infty$. It is well-known that, if E is a uniformly convex Banach space, then E has the Kadec-Klee property.

Lemma 2.17. ([85]) *Let E be a real reflexive Banach space with its dual space E^* and C be a nonempty closed and convex subset of E. The following statement hold:*

    (1) $\pi_C^f \varpi$ *is a nonempty, closed and convex subset of C for all $\varpi \in E^*$;*

    (2) *If E is smooth, then for all $\varpi \in E^*$, $x \in \pi_C^f \varpi$ if and only if*

$$\langle x - y, \varpi - Jx \rangle + \rho f(y) - \rho f(x) \geq 0$$

for all $y \in C$;

    (3) *If E is strictly convex and $f : C \to \mathbf{R} \cup \{+\infty\}$ is positive homogeneous (i.e., $f(tx) = tf(x)$ for all $t > 0$ such that $tx \in C$, where $x \in C$), then $\pi_C^f \varpi$ is single valued mapping.*

Recently, Fan et al. [30] showed that the condition, f is positive homogeneous, which appears in [30, Lemma 2.1 (iii)], can be removed.

Lemma 2.18. ([30]) *Let E be a real reflexive Banach space with its dual space E^* and let C be a nonempty closed and convex subset of E. If E is strictly convex, then $\pi_C^f \varpi$ is single-valued.*

Recall that J is single-value mapping when E is a smooth Banach space. There exists a unique element $\varpi \in E^*$ such that $\varpi = Jx$, where $x \in E$. This substitution in (2.5) gives the following:

$$G(y, Jx) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2 + 2\rho f(y). \tag{2.6}$$

Now, we consider the second generalized f projection operator in a Banach space (see [50]).

Let E be a real smooth Banach space and C be a nonempty closed and convex subset of E. We say that $\Pi_C^f : E \to 2^C$ is a *generalized f-projection operator* if

$$\Pi_C^f x = \{u \in C : G(u, Jx) = \inf_{y \in C} G(y, Jx), \forall x \in E\}.$$

Lemma 2.19. ([29]) *Let E be a Banach space and $f : E \to \mathbf{R} \cup \{+\infty\}$ be a lower semicontinuous and convex function. Then there exist $x^* \in E^*$ and $\alpha \in \mathbf{R}$ such that*

$$f(x) \geq \langle x, x^* \rangle + \alpha$$

for all $x \in E$.

Lemma 2.20. ([50]) *Let E be a reflexive smooth Banach space and C be a nonempty closed and convex subset of E. The following statements hold:*

    (1) $\Pi_C^f x$ *is nonempty closed and convex subset of C for all $x \in E$;*

    (2) *For all $x \in E$, $\hat{x} \in \Pi_C^f x$ if and only if*

$$\langle \hat{x} - y, Jx - J\hat{x} \rangle + \rho f(y) - \rho f(\hat{x}) \geq 0$$

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for all $y \in C$;

(3) *If E is strictly convex, then Π_C^f is single-valued mapping.*

Lemma 2.21. ([50]) *Let E be a real reflexive smooth Banach space and C be a nonempty closed and convex subset of E. Then, for any $x \in E$ and $\hat{x} \in \Pi_C^f x$,*

$$\phi(y, \hat{x}) + G(\hat{x}, Jx) \leq G(y, Jx)$$

for all $y \in C$.

Lemma 2.22. ([50]). *Let E be a Banach space and $f : E \to \mathbf{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous mapping with convex domain $D(f)$. If $\{x_n\}$ is a sequence in $D(f)$ such that $x_n \rightharpoonup \hat{x} \in D(f)$ and $\lim_{n \to \infty} G(x_n, Jy) = G(\hat{x}, Jy)$, then $\lim_{n \to \infty} \|x_n\| = \|\hat{x}\|$.*

Remark 2.23. Let E be a uniformly convex and uniformly smooth Banach space and $f(y) = 0$ for all $y \in E$. Then Lemma 2.21 reduces to the property of the generalized projection operator considered by Alber [3].

Let $f(y) \geq 0$ for all $y \in C$ and $f(0) = 0$, then the definition of a totally quasi-ϕ-asymptotically nonexpansive S is equivalent to the following:

If $F(S) \neq \emptyset$ and there exist the nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$, respectively, and a strictly increasing continuous function $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ with $\psi(0) = 0$ such that

$$G(p, S^n x) \leq G(p, x) + \nu_n \psi G(p, x) + \mu_n$$

for all $x \in C$, $p \in F(S)$ and $n \geq 1$.

Let θ be a bifunction from $C \times C$ to $\mathbf{R}$, where $\mathbf{R}$ denotes the set of real numbers. *The equilibrium problem* (for short, (EP)) is to find $\hat{x} \in C$ such that

$$\theta(\hat{x}, y) \geq 0 \tag{2.7}$$

for all $y \in C$. The set of solutions of the (EP) 2.7 is denoted by $EP(\theta)$.

An operator $A \subset E \times E^*$ is said to be *monotone* if

$$\langle x - y, x^* - y^* \rangle \geq 0$$

for all $(x, x^*), (y, y^*) \in A$. A point $z \in E$ is called a *zero point* of A if

$$0 \in Az. \tag{2.8}$$

We denote the set of zeroes of the operator A by $A^{-1}0$, that is,

$$A^{-1}0 = \{z \in E : 0 \in Az\}.$$

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A monotone $A \subset E \times E^*$ is said to be *maximal* if its graph $G(A) = \{(x, y^*) : y^* \in Ax\}$ is not property contained in the graph of any other monotone operator. If A is maximal monotone, then the solution set $A^{-1}0$ is closed and convex.

Let E be a smooth strictly convex and reflexive Banach space, C be a nonempty closed convex subset of E and $A \subset E \times E^*$ be a monotone operator satisfying $D(A) \subset C \subset J^{-1}(\cap_{\lambda>0} R(J + \lambda A))$. Then the *resolvent* $J_\lambda : C \to D(A)$ of A is defined by

$$J_\lambda x = \{z \in D(A) : Jx \in Jz + \lambda Az, \forall x \in C\}.$$

J_λ is a single-valued mapping from E to $D(A)$. On the other words, $J_\lambda = (J + \lambda A)^{-1}J$ for all $\lambda > 0$.

For any $\lambda > 0$, the *Yosida approximation* $A_\lambda : C \to E^*$ of A is defined by $A_\lambda x = \frac{Jx - JJ_\lambda x}{\lambda}$ for all $x \in C$. We know that $A_\lambda x \in A(J_\lambda x)$ for all $\lambda > 0$ and $x \in E$. Since relatively quasi nonexpansive mappings and quasi-ϕ-nonexpansive mappings are same, we can see that J_λ is a quasi-ϕ-nonexpansive mapping (see [76, Theorem 4.7]).

Lemma 2.24. ([44]) *Let E be a smooth strictly convex and reflexive Banach space, C be a nonempty closed convex subset of E and $A \subset E \times E^*$ be a monotone operator satisfying $D(A) \subset C \subset J^{-1}(\cap_{\lambda>0} R(J + \lambda A))$. For any $\lambda > 0$, let J_λ and A_λ be the resolvent and the Yosida approximation of A, respectively. Then the following hold:*

(1) $\phi(p, J_\lambda x) + \phi(J_\lambda x, x) \leq \phi(p, x)$ *for all $x \in C$ and $p \in A^{-1}0$;*

(2) (b) $(J_\lambda x, A_\lambda x) \in A$ *for all $x \in C$;*

(3) (c) $F(J_\lambda) = A^{-1}0.$

Lemma 2.25. ([72]) *Let E be a reflexive strictly convex and smooth Banach space. Then an operator $A \subset E \times E^*$ is maximal monotone if and only if $R(J + \lambda A) = E^*$ for all $\lambda > 0$.*

For solving the equilibrium problem for a bifunction $\theta : C \times C \to \mathbb{R}$, let us assume that θ satisfies the following conditions:

(A1) $\theta(x, x) = 0$ for all $x \in C$;

(A2) θ is monotone, i.e., $\theta(x, y) + \theta(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,
$$\lim_{t\downarrow 0} \theta(tz + (1 - t)x, y) \leq \theta(x, y);$$

(A4) for each $x \in C$, $y \mapsto \theta(x, y)$ is convex and lower semi-continuous.

Lemma 2.26. (*Blum and Oettli* [8]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, let θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), and let $r > 0$ and $x \in E$. Then there exists $z \in C$ such that*
$$\theta(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C.$$

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The following lemma is a spacial case of Zhang [96].

Lemma 2.27. (*Zhang* [96]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let $\varphi : C \to \mathbb{R}$ is convex and lower semi-continuous and θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). For $r > 0$ and $x \in E$, then there exists $u \in C$ such that*

$$\theta(u, y) + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx \rangle \geq 0, \quad \forall y \in C.$$

Define a mapping $K_r : C \to C$ as follows:

$$K_r(x) = \{u \in C : \theta(u, y) + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx \rangle \geq 0, \quad \forall y \in C\} \tag{2.9}$$

for all $x \in C$. Then the followings hold:

1. *K_r is single-valued;*

2. *K_r is firmly nonexpansive, i.e., for all $x, y \in E$, $\langle K_r x - K_r y, JK_r x - JK_r y \rangle \leq \langle K_r x - K_r y, Jx - Jy \rangle$;*

3. *$F(K_r) = MEP(\theta, \varphi)$;*

4. *$MEP(\theta, \varphi)$ is closed and convex;*

5. *$\phi(p, K_r z) + \phi(K_r z, z) \leq \phi(p, z), \forall p \in F(K_r)$ and $z \in E$.*

3   Methodology

1. Studying and investigating on fixed point problems.

2. Studying and investigating on nonlinear problems.

3. Establishing new theorems for fixed point problems and nonlinear problems in Banach spaces.

4. Finding necessary and sufficient conditions which are established for mappings to have fixed points and common solutions of nonlinear mappings.

4   Main results

Let C be a closed subset of a Banach space E. Recall that a mapping $S : C \to C$ is closed if for each $\{x_n\}$ in C, if $x_n \to x$ and $Sx_n \to y$, then $Sx = y$. Let $\{S_n\}$ be a family of mappings of C in to itself with $F := \cap_{n=1}^{\infty} F(S_n) \neq \emptyset$, $\{S_n\}$ is said to satisfy the $(*)$-condition if for each bounded sequence $\{z_n\}$ in C,

$$\lim_{n \to \infty} \|z_n - S_n z_n\| = 0, \ z_n \to z \text{ imply } z \in F. \tag{4.1}$$

Remark 4.1. It follows directly from the definitions above that if $\{S_n\}$ satisfies NST-condition, then $\{S_n\}$ satisfies $(*)$-condition. If $S_n \equiv S$ and S is closed, then $\{S_n\}$ satisfies $(*)$-condition.

Using the $(*)$-condition, we prove the new convergence theorems for finding a common element of the set of solutions of an equilibrium problems, the common fixed point set of a family of relatively quasi-nonexpansive mappings, a zero of maximal monotone operators and the solution set of variational inequalities for an α-inverse strongly monotone mapping in a 2-uniformly convex and uniformly smooth Banach space.

Theorem 4.2. *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let $T \subset E \times E^*$ be a maximal monotone operator satisfying $D(T) \subset C$ and let $J_{r_n} = (J + r_n T)^{-1} J$ for all $r_n > 0$. Let θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $\|Ay\| \leq \|Ay - Au\|, \quad \forall y \in C$ and $u \in VI(A,C) \neq \emptyset$. Let $S_n : C \to C$ be a family of relatively quasi-nonexpansive mappings such that satisfies the $(*)$-condition and $\Theta := (\cap_{n=1}^{\infty} F(S_n)) \cap T^{-1}0 \cap MEP(\theta,\varphi) \cap VI(A,C) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JS_n J_{r_n} z_n), \\ u_n \in C \text{ such that } \theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \alpha_n \phi(z, x_n) + (1-\alpha_n)\phi(z, z_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \qquad (4.2)$$

where J is the duality mapping on E, $\{\alpha_n\}$ is sequence in $[0,1]$ and $\{r_n\} \subset [d, \infty)$ for some $d > 0$ and $\{\lambda_n\} \subset [a,b]$ for some a,b with $0 < a < b < c^2\alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E. If $\liminf_{n\to\infty}(1-\alpha_n) > 0$, then $\{x_n\}$ converges strongly to $p \in \Theta$, where $p = \Pi_\Theta x_0$.

When we extend mapping S to totally quasi-ϕ-asymptotically nonexpansive, we have the following Theorem.

Theorem 4.3. *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. For each $i = 1, 2, \cdots, m$, let θ_i be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying the conditions (A1)-(A4). Let $A_j \subset E \times E^*$ be a maximal monotone operator satisfying $D(A_j) \subset C$ and $J_{\lambda_{j,n}}^{A_j} = (J + \lambda_{j,n} A_j)^{-1} J$ for all $\lambda_{j,n} > 0$ and $j = 1, 2, \cdots, l$. Let $S : C \to C$ be a closed and totally quasi-ϕ-asymptotically nonexpansive mapping with the nonnegative real sequences $\{\nu_n\}, \{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$, respectively, and a strictly increasing continuous function $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ with $\psi(0) = 0$. Let $f : E \to \mathbf{R}^+$ be a convex and lower semicontinuous function with $C \subset int(D(f))$ and $f(0) = 0$. Assume that S is uniformly L-Lipschitz continuous and $\mathcal{F} = F(S) \cap (\cap_{i=1}^m EP(\theta_i)) \cap (\cap_{j=1}^l A_j^{-1} 0) \neq \emptyset$. For any initial point $x_1 \in E$, define $C_1 = C$ and the sequence $\{x_n\}$ in C by*

$$\begin{cases} z_n = J_{\lambda_{l,n}}^{A_l} \circ J_{\lambda_{l-1,n}}^{A_{l-1}} \circ \cdots \circ J_{\lambda_{1,n}}^{A_1} x_n, \\ u_n = T_{r_{m,n}}^{\theta_m} \circ T_{r_{m-1,n}}^{\theta_{m-1}} \circ \cdots \circ T_{r_{1,n}}^{\theta_1} z_n, \\ y_n = J^{-1}(\alpha_n Jx_1 + \beta_n JS^n x_n + \gamma_n Ju_n), \\ C_{n+1} = \{v \in C_n : G(v, Jy_n) \leq \alpha_n G(v, Jx_1) + (1-\alpha_n)G(v, Jx_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}^f x_1 \end{cases} \qquad (4.3)$$

for each $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\zeta_n = \nu_n \sup_{q \in \mathcal{F}} \psi(G(q, Jx_n)) + \mu_n$ and for each $i = 1, 2, \cdots, m$, $\{r_{i,n}\} \subset [d, \infty)$ for some $d > 0$. If, for each $j = 1, 2, \cdots, l$, $\liminf_{n\to\infty} \lambda_{j,n} > 0$, $\lim_{n\to\infty} \alpha_n = 0$ and $\liminf_{n\to\infty} \beta_n < 1$, then the sequence $\{x_n\}$ converges strongly to a point $\Pi_{\mathcal{F}}^f x_1$.

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Theorem 4.4. *Let C be a nonempty closed and convex subset of a uniformly smooth and 2-uniformly convex Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions $(A1)$-$(A4)$ and let A be an α-inverse-strongly monotone mapping of E into E^*. Let $S : C \to 2^C$ be a closed and total quasi-ϕ-asymptotically nonexpansive multi-valued mapping with nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$. Assume that S is uniformly asymptotically regular on C with $\mu_1 = 0$ and $F := F(S) \cap EF(f) \cap A^{-1}0 \neq \emptyset$. For arbitrary $x_1 \in C$, $C_1 = C$, generate a sequences $\{x_n\}$ by*

$$
\begin{cases}
z_n = J^{-1}(Jx_n - \lambda_n Ax_n), \\
u_n = T_{r_n}z_n, \\
y_n = J^{-1}(\alpha_n Jx_n + \beta_n Jw_n + \gamma_n Ju_n), \ w_n \in S^n x_n, \\
C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, z_n) \leq \phi(v, x_n) + K_n\}, \\
x_{n+1} = \Pi_{C_{n+1}}x_1, \quad n \in \mathbb{N},
\end{cases}
\tag{4.4}
$$

where $K_n = \nu_n \sup_{q \in F} \psi(\phi(q, x_n)) + \mu_n$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ and $\{r_n\}$ satisfy the following conditions:

1. *$\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\liminf_{n \to \infty} \alpha_n\beta_n > 0$,*

2. *$\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < \frac{c^2\alpha}{2}$ and $\frac{1}{c}$ is the 2-uniformly convexity constant of E,*

3. *$\{r_n\} \subset [d, \infty)$ for some $d > 0$,*

then $\{x_n\}$ converges strongly to $\Pi_F x_1$.

5 Conclusion and Suggestion

In this work, we introduce new hybrid iterations of the generalized projection operators and the generalized f-projection operators for finding a common element of the fixed point set and solution of nonlinear problems. This research has three theorems for fixed point problems and nonlinear problems. Each theorem compound by difference mapping and differences conditions for get the strong convergence. For other researcher should study other mappings or iterations and study the necessary conditions to get some new convergence.

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