ITERATIVE METHODS FOR APPROXIMATING COMMON SOLUTIONS OF SYSTEMS OF EQUILIBRIUM PROBLEMS AND SYSTEMS OF VARIATIONAL INEQUALITIES

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Abstract

The purposes of this dissertation are to construct new iterative algorithms for the fixed point approximation of nonexpansive mappings, and to solve many mathematical problems in a real Hilbert space. In this dissertation, we propose the proof of convergence theorems of iterative approximation method for finding the common element of (1) the set of common fixed points of nonexpansive, strict pseudo-contractions and Lipschitz continuous mappings; (2) the set of variational inequality problems and variational inclusion for nonlinear mappings; and (3) the set of solutions of an equilibrium problem in a real Hilbert space. Moreover, the supplementary applications involving optimization problems are also obtained by using the results from our convergence theorems.

Keywords : Equilibrium Problem / Fixed Point / Hilbert Space / Nonexpansive Mapping / Strong Convergence / Variational Inclusion / Variational Inequality

หัวข้อวิทยานิพนธ์	ระเบียบวิธีทำซ้ำสำหรับการประมาณคำตอบร่วมของระบบของปัญหาดุลยภาพและระบบอสมการเชิงแปรผัน
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บทคัดย่อ

จุดประสงค์ของวิทยานิพนธ์นี้ คือการสร้างระเบียบวิธีการประมาณค่าทำซ้ำชนิดใหม่ สำหรับการประมาณค่าจุดตรึงของการส่งแบบไม่ขยาย และเพื่อแก้ไขปัญหาต่างๆ ทางคณิตศาสตร์ในปริภูมิค่าจริงฮิลเบิร์ต ในวิทยานิพนธ์นี้ได้เสนอวิธีพิสูจน์ทฤษฎีบทการลู่เข้าของระเบียบวิธีการประมาณค่าทำซ้ำสำหรับการหาสมาชิกร่วมของ (1) เซตของคำตอบร่วมจุดตรึงของการส่งแบบไม่ขยาย การส่งแบบหดตัวเทียมโดยแท้ และการส่งแบบต่อเนื่องลิพชิทซ์ (2) เซตคำตอบของระบบอสมการเชิงแปรผัน และการแปรผันของเซตย่อยสำหรับการส่งแบบไม่เชิงเส้น และ (3) เซตคำตอบคำตอบของระบบปัญหาเชิงดุลยภาพในปริภูมิค่าจริงฮิลเบิร์ต นอกจากนี้ยังได้บทประยุกต์ที่เกี่ยวกับการหาค่าเหมาะที่สุดโดยใช้ผลจากทฤษฎีการลู่เข้าดังกล่าว

คำสำคัญ : ปัญหาเชิงดุลยภาพ / จุดตรึง / ปริภูมิฮิลเบิร์ต / การส่งแบบไม่ขยาย / การลู่เข้าอย่างเข้ม การเป็นเซตย่อยการแปรผัน /อสมการการแปรผัน /

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CHAPTER 1 INTRODUCTION

1.1 Background

The fixed point theory is the most important tool to solve problems in many branches of sciences and applied sciences. Moreover, the construction of fixed point iteration processes for nonlinear mappings takes an important role to find solutions of many problems in applied areas as it application. In case, the problem in science had been transformed into a mathematical model such as equality, inequality, equality system and inequality system.There are two questions following.

(1). Does the answer of model exist ?

(2). How can find the answer ?

Fixed-point iteration processes for nonlinear mappings in Hilbert spaces including Mann and Ishikawa iterations process have been studied extensively by many authors to approximate fixed points of various classes of operators such as nonexpansive mappings, contraction mappings, strict pseudo-contraction mappings, resolvent operators and to solve variational inequalities in Hilbert spaces for the detail please see [1],[2] and the references therein.

The Equilibrium theory represents an important area of mathematical sciences such as optimization, operation research, game theory, financial mathematics and mechanics. Equilibrium problems include variational inequalities, optimization problems, Nash equilibria problems, saddle point problems, fixed point problems, and complementarity problems as special cases., see [3]. Related to the equilibrium problems, we also have the problem of finding the fixed points of the nonexpansive mappings which is the subject of current interest in functional analysis. It is natural to construct a unified approach for these problems. In this direction, several authors have introduced some iterative schemes for finding a common element of a set of the solutions of the equilibrium problems, the set solutions of the variational inequality problems for nonlinear mappings and a set of the fixed points of an infinite (a finite) family of nonexpansive mappings.

The theory of variational inequality represents, in fact, a very natural generalization of the theory of boundary value problems and allows us to consider new problems arising from many fields of applied sciences such as applied mathematics, mechanics, physics, engineering, the theory of convex programming and the theory of control.

Variational inequalities was introduced by Stampacchia [4] in the early sixties and have had a great impact and influence in the development of almost all branches of pure and applied sciences. It is well-known that the variational inequalities are equivalent to the fixed point problems. This alternative equivalent formulation has been used to suggest and analyze in variational inequalities. In particular, the solution of the variational inequalities can be computed to use for the iterative projection methods. It is well-known that the convergence of a projection method requires the operator to be strongly monotone and Lipschitz continuous. These conditions are very stict and rule out its application in several important problems.

Let C be a nonempty set and $T : C \to C$.

In 1953, Mann [5] introduced the Mann's iteration which is defined by the initial guess x_0 and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.1.1}$$

where $\{\alpha_n\} \in [0, 1]$. In an infinite-dimensional Hilbert space, Mann iteration can yield only weak convergence.

In 1967, Halpern [6] defined the iteration with starting from x_0, for fixed u and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.1.2}$$

which is satisfied the conditions $\lim_{n\to\infty} \alpha_n = 0$ and $\Sigma_{n=1}^{\infty}\alpha_n = \infty$. Then $\{x_n\}$ converges strongly to a fixed point of T.

In 1974, The Ishikawa's iteration process is defined by Ishikawa [7] as the following:

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Ty_n, \quad n \geq 0, \end{cases} \tag{1.1.3}$$

where the initial guess x_0 is taken in C arbitrarily and the sequences $\{\alpha_n\}, \{\beta_n\} \in [0, 1]$.

In 2003, Nakajo and Takahashi [8] proposed the modification of the following Mann iteration method (1.1.1).

$$\begin{cases} x_0 \in C \quad \text{is arbitrary}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2 \ldots, \end{cases} \tag{1.1.4}$$

where P_C is metric projection on the set C.

In 2004, Takahashi and Toyoda [9] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)TP_C(x_n - \lambda_n Ax_n), \quad n \geq 0. \tag{1.1.5}$$

where $x_0 \in C, \{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$, and obtained weak convergence theorem in a Hilbert space H.

In 2004, Xu [10] studied the iteration process $\{x_n\}$ called viscosity approximation method as shown in the following:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \text{ for } n \geq 1,$$

where $\{\alpha_n\} \subset (0, 1)$ and $f : C \to C$ is a contraction mapping.

Let C be a nonempty closed and convex subset of a real Hilbert space H. The classical variational inequality problem is to find a $u \in C$ such that $\langle v - u, Bu \rangle \geq 0$ for all $v \in C$. The set of solutions of the variational inequality is denoted by $VI(C, B)$. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ $x, y \in C$. we denote by $F(T)$ the set of fixed points of T; that is $F(T) = \{x \in C : x = Tx\}$

In 2005, Iiduka and Takahashi [11] proposed a new iterative scheme as following:

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_n x + (1-\alpha_n)TP_C(x_n - \lambda_n Bx_n), \ \forall n \geq 0, \end{cases} \tag{1.1.6}$$

where B is β-inverse-strongly monotone, $\{\alpha_n\}$ is a sequence in (0, 1), and $\{\lambda_n\}$ is a sequence in $(0, 2\beta)$. They showed that if $F(T) \cap VI(C, B)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.1.6) converges strongly to some $z \in F(T) \cap VI(C, B)$.

In 2007, Moudafi [12] intoduced the following Krasnoselski-Mann algorithm:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(\beta_n S x_n + (1-\beta_n)T x_n), \tag{1.1.7}$$

where $S, T : C \to C$ are two nonexpansive mappings, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$.

In 2007, Tada and Takahashi [13] introduced the following iterative scheme by the hybrid method in a Hilbert space H. Let $x_0 = x \in H$ and

$$\begin{cases} u_n \in C \text{ such that} \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ w_n = (1-\alpha_n)x_n + \alpha_n S u_n, \\ C_n = \{z \in H : \|w_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.1.8}$$

for every $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$.

In 2009, Cianciaruso et al. [14] introduced a two step algorithm to solve the following problem:

$$\langle x^* - Sx^*, x - x^*\rangle \geq 0, \quad \forall x \in F(T), \tag{1.1.9}$$

and defined the following algorithm:

$$\begin{cases} y_n = \beta_n S x_n + (1-\beta_n)x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T y_n, \end{cases} \tag{1.1.10}$$

where $f : C \to C$ is a contraction mapping, S and $T : C \to C$ are two nonexpansive mappings, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$.

In 2010, Yao et al. [15] modified the two step algorithm (1.1.10) to extend range of f from C to H by using the metric projection of H onto C. They introduced the following iterative scheme:

$$\begin{cases} y_n = \beta_n S x_n + (1-\beta_n)x_n, \\ x_{n+1} = P_C[\alpha_n f(x_n) + (1-\alpha_n)T y_n], \end{cases} \tag{1.1.11}$$

where $f : C \to H$ is a contraction mapping, S and $T : C \to C$ are two nonexpansive mappings, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$.

In 2011, Yao and Xu [16] independently introduced two iterative methods for finding the minimum-norm fixed point of nonexpansive mapping which is defined on closed convex subset C of H. The proposed algorithms are based on the well-known Browder's iterative method [17] and Halpern's iterative method [18].

Recently, In 2011 Gu et al.[19] introduced the following iterative algorithm:

$$\begin{cases} y_n = P_C[\beta_n S x_n + (1-\beta_n)x_n], \\ x_{n+1} = P_C[\alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)T_i y_n], \quad \forall n \geq 1, \end{cases} \tag{1.1.12}$$

where $f : C \to H$ is a contraction mapping, $S : C \to H$ is a nonexpansive mapping, $\{T_i\}_{i=1}^{\infty} : C \to C$ is a countable family of nonexpansive mappings, $\alpha_0 = 1$, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$.

Very recently, Phan Tu Vuong et. al. [20], considered the sequences $\{x_n\}, \{y_n\}, \{z_n\}$, and $\{t_n\}$ generated by $x_0 \in C$ and

$$\begin{cases} y_n = \arg\min_{y \in C}\{\lambda_n F(x_n, y) + \frac{1}{2}\|y - x_n\|^2\}, \\ z_n = \arg\min_{y \in C}\{\lambda_n F(y_n, y) + \frac{1}{2}\|y - x_n\|^2\}, \\ t_n = \alpha_n x_n + (1-\alpha_n)[\beta_n z_n + (1-\beta_n)S z_n], \\ x_{n+1} = t_n, \end{cases} \tag{1.1.13}$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0, 1[$, $\{\beta_n\} \subset]0, 1[$, and $\{\lambda_n\} \subset]0, 1]$.

Motivated by above, we introduce and modify new iterative schemes for approximate the common solution of fixed point problem, the variational inequalities, the system of variational inclusions and the system of equilibrium problems.

1.2 Objectives

There are three main objectives:

1. To study and extend the classical fixed point theory, the variational inequalities and the system of equilibrium problems of nonlinear mappings.

2. To prove new convergence theorems for finding the common solution of fixed point problems, systems of variational inequalities and systems of equilibrium problems for nonlinear mappings.

3. To apply our results to solve some problems of classical variational inequality problems and optimization problems.

1.3 The Summary of the Study

The summary of the dissertation is concluded as follows:

In Chapter 1, we give a brief introduction and review the background of this dissertation.

In Chapter 2, we give the necessary tools and concern some well-known definitions and preliminaries which will be useful in the later chapters.

In Chapter 3, we introduce a new iterative scheme for solving variational inequality problems, fixed points for nonexpensive and pseudocontraction mappings.

In Chapter 4, we introduce a new iterative scheme for finding a common solution of the set of fixed point, the system of mixed equilibrium problems and the variational inclusion.

In Chapter 5, we introduce a new iterative method for finding a common fixed point of a countable family of strictly pseudocontractive mappings, and solutions of the equilibrium problems, the variational inequality problems and the fixed point problems for a strict pseudocontraction mapping.

Finally, we conclude this research in Chapter 6.

CHAPTER 2 BASIC CONCEPTS AND PRELIMINARIES

In this chapter, we give some definitions, notations, lemmas and some useful results that will be used in the later chapters. Throughout this dissertation, we let $\mathbb{R}$ be the set of all real numbers, $\mathbb{N}$ be the set of all natural numbers, H be a Hilbert space.

2.1 Basic Concepts

Definition 2.1.1. [21] Let X denote any nonempty set that contains each of its elements x and each real number α, a unique element $\alpha \cdot x$, written as αx, which is called a scalar multiple of x. (One could also include complex numbers α as well, but we restrict ourselves here to the real case.) Also, assume that for each two elements $x, y \in X$ there exists a unique element $x + y \in X$ called the sum of x and y. The system $(X, \cdot, +)$ is called a linear space (over $\mathbb{R}$) if the following conditions are satisfied: [Here $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$.]

(1) $x + y = y + x$;

(2) $x + (y + z) = (x + y) + z$;

(3) $\alpha(x + y) = \alpha x + \alpha y$;

(4) $x + y = x + z$ implies $y = z$;

(5) $(\alpha + \beta)x = \alpha x + \beta x$;

(6) $(\alpha\beta)x = \alpha(\beta x)$;

(7) $1x = x$.

Definition 2.1.2. [22] Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}$, satisfying the following condition for all x, y and z in X:

(M1) $d(x, y) = 0 \Longleftrightarrow x = y$;

(M2) $d(x, y) = d(y, x)$;

(M3) $d(x, y) \leq d(x, z) + d(z, y)$.

The function d assigns to each pair (x, y) of element of X a nonnegative real number $d(x, y)$, which does not on the order of the elements; $d(x, y)$ is called the *distance* between x and y. The set X together with a metric, denoted by (X, d), is called a *metric space*. The conditions (M1)-(M3) are usually called the *metric axioms*.

Definition 2.1.3. [22] Let X be a linear space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A function $\|\cdot\| : X \to \mathbb{R}$ is said to be *a norm on* X if it satisfies the following conditions:

(N1) $\|x\| \geq 0, \forall x \in X$;

(N2) $\|x\| = 0 \Leftrightarrow x = 0$;

(N3) $\|x + y\| \leq \|x\| + \|y\|, \quad \forall x, y \in X$;

(N4) $\|\alpha x\| = |\alpha|\|x\|, \quad \forall x \in X$ and $\forall \alpha \in \mathbb{K}$.

From this norm we can define a metric, induced by the norm $\|\cdot\|$, by

$$d(x, y) = \|x - y\|, \quad (x, y \in X).$$

A linear space X equipped with the norm $\|\cdot\|$ is called a *normed linear space*.

Definition 2.1.4. [23] Let $(X, \|\cdot\|)$ be a normed space.

(1) A sequence $\{x_n\} \subset X$ is said to *converge strongly* in X if there exists $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. That is, if for any $\epsilon > 0$ there exists a positive integer N such that $\|x_n - x\| < \epsilon, \forall n \geq N$. We often write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ to mean that x is the limit of the sequence $\{x_n\}$.

(2) A sequence $\{x_n\} \subset X$ is said to be *a Cauchy sequence* if for any $\epsilon > 0$ there exists a positive integer N such that $\|x_m - x_n\| < \epsilon, \forall\ m, n \geq N$. That is, $\{x_n\}$ is *a Cauchy sequence* in X if and only if $\|x_m - x_n\| \to 0$ as $m, n \to \infty$.

Definition 2.1.5. [21] Sequence $\{x_n\}_{n=1}^{\infty}$ in a normed linear space X is said to be *a bounded sequence* if there exists $M > 0$ such that $\|x_n\| \leq M, \forall n \in \mathbb{N}$.

Definition 2.1.6. [21] A subset C of a normed linear space X is said to be *convex subset in* X if $\lambda x + (1 - \lambda)y \in C$ for each $x, y \in C$ and for each scalar $\lambda \in [0, 1]$.

Definition 2.1.7. [21] The real-value function of two variables $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ is called *inner product* on a real vector space X if for any $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$ the following conditions are satisfied:

(I1) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$;

(I2) $\langle x, y\rangle = \langle y, x\rangle$;

(I3) $\langle x, x\rangle \geq 0$ for each $x \in X$ and $\langle x, x\rangle = 0$ if and only if $x = 0$. A *real inner product space* is a real vector space equipped with an inner product.

Definition 2.1.8. [21] A **Hilbert spaces** is an inner product space which is complete under the norm induced by its inner product.

An inner product on X defines a norm on X given by $\|x\| = \sqrt{\langle x, x\rangle}$.

Lemma 2.1.9. [23]**(The Schwarz inequality)**
If x and y are any two vector in an inner product space X, then

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Lemma 2.1.10. [23] Let H be a real Hilbert space. Then the following inequalities hold:

(i) $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$,

(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$,

(iii)$\|x + y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$,

Definition 2.1.11. [21] The metric projection (or *nearest point*) from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in C$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

Lemma 2.1.12. [23] Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in C$. Then

(i) $z = P_C x \Longleftrightarrow \langle z - x, y - x\rangle \geq 0, \quad \forall y \in C$,

(ii) $\|P_C x - P_C y\| \leq \|x - y\|, \quad \forall x, y \in H$,

(iii) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H$,

(iv) $\langle x - P_C x, y - P_C x\rangle \leq 0, \quad \forall x \in H, y \in C$,

(v) $\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H, y \in C$.

Definition 2.1.13. [21] A normed space $(X, \|\cdot\|)$ is called strictly convex if for all $x, y \in X$, $x \neq y$, $\|x\| = \|y\| = 1$, we have $\|\lambda x + (1 - \lambda)y\| < 1$, $\forall \lambda \in (0, 1)$.

Definition 2.1.14. [21] A sequence $\{x_n\}$ in a normed spaces is said to *converge weakly* to some vector x if $\lim_{n\to\infty} f(x_n) = f(x)$ holds for every continuous linear functional f. We often write $x_n \rightharpoonup x$ to mean that $\{x_n\}$ converges weakly to x.

Lemma 2.1.15. [23] Let $\{x_n\}$ be a sequence of a normed space $(X, \|\cdot\|)$, $x \in X$ and let $x_n \to x$ if and only if, for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exist a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converging to x.

Definition 2.1.16. [23] A normed space X is called *complete* if every Cauchy sequence in X converges to an element in X.

Lemma 2.1.17. [24] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Definition 2.1.18. [21] Let F and X be linear spaces over the field $\mathbb{K}$.

(1) A mapping $T : F \to X$ is called *a linear operator* if $T(x+y) = Tx + Ty$ and $T(\alpha x) = \alpha Tx, \forall x, y \in F$, and $\forall \alpha \in \mathbb{K}$.

(2) A mapping $T : F \to \mathbb{K}$ is called *a linear functional on F* if T is a linear operator.

Definition 2.1.19. [23] Let F and X be normed spaces over the field $\mathbb{K}$ and $T : X \to F$ a linear operator. T is said to be *bounded* on X if there exists a real number $M > 0$ such that $\|T(x)\| \leq M\|x\|, \forall x \in X$.

Remark 2.1.20. [23] In a Hilbert space H, weak convergence is defined by $\lim_{n\to\infty}\langle x_n, y\rangle = \langle x, y\rangle$ for all $y \in H$. The notation $x_n \rightharpoonup x$ is sometimes used to denote this kind of convergence.

Remark 2.1.21. If $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$, then $x = y$.

Lemma 2.1.22. [23] Let X be an inner product space and $\{x_n\}$ be a bounded sequence of H such that $x_n \rightharpoonup x$. Then following inequality holds:

$$\|x\| \leq \lim_{n\to\infty} \inf \|x_n\|.$$

Lemma 2.1.23. [10] Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n,\ n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

2.2 Basic Concepts in Hilbert Spaces

Let C be a closed convex subset of a real Hilbert space H with inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively. We have the following are hold:

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle, \tag{2.2.1}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.2.2}$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$.

Lemma 2.2.1. [25] Let $(E, \langle .,. \rangle)$ be an inner product space. Then for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2.$$

Lemma 2.2.2. [26] A Hilbert space H satisfies the **Opial condition** that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$, holds for every $y \in H$ with $y \neq x$.

Definition 2.2.3. [27] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let f be a function of C into $(-\infty, \infty]$, where $(-\infty, \infty] = \mathbb{R} \cup \{\infty\}$. Then, f is called *lower semicontinuous* if for any $a \in \mathbb{R}$, the set $\{x \in C : f(x) \leq a\}$ is closed.

Lemma 2.2.4. [21]**(Demi-closedness Principle)** Assume that S is a nonexpansive self-mapping of a nonempty closed convex subset C of a real Hilbert space H. If S has a fixed point, the $I - S$ is demi-closed: that is, whenever $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$ (for short, $x_n \rightharpoonup x \in C$), and the sequence $\{(I - S)x_n\}$ converges strongly to some y (for short, $(I - S)x_n \to y$), it follows that $(I - S)x = y$. Here I is the identity operator of H.

Lemma 2.2.5. [29] A Hilbert space H satisfies the **Kadec-Klee property** that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $\|x_n - x\| \to 0$.

2.3 The Classical of Fixed Point Theory

Definition 2.3.1. [21] Let H be a Hilbert space and let C a nonempty bounded convex subset of H. A mapping $T : C \to C$ is called *nonexpansive* on C if

$$\|Tx - Ty\| \leqslant \|x - y\|, \quad \forall x, y \in C.$$

Definition 2.3.2. [22] An element $x \in C$ is said to be a *fixed point* of a mapping $T : C \to C$. The set of all fixed points of T is denoted by $F(T) = \{x \in C : Tx = x\}$.

Lemma 2.3.3. [23] Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let S be a nonexpansive mapping of C into itself. Then, $F(T) \neq \emptyset$.

Definition 2.3.4. [23] Let H be a Hilbert space and let C a nonempty bounded convex subset of H. A mapping $f : C \to C$ is called *a contraction* on C if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leqslant \alpha\|x - y\|, \quad \forall x, y \in C.$$

2.4 Some Nonlinear Mappings in Hilbert Spaces

Let C be a closed convex subset of a real Hilbert space H with inner product and norm are denoted by $\langle ., . \rangle$ and $\|.\|$, respectively. Let $T : C \to C$ a nonlinear mapping.

Definition 2.4.1. [23] The *metric projection (nearest point)* P_C from a Hilbert space H to a closed convex subset C of H is defined as follows: given $x \in H$, $P_C x$ is the only point in C with the property

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - P_C x, P_C x - z\rangle \geq 0, \quad \forall z \in C; \tag{2.4.1}$$

$$\|(x - y) - (P_C x - P_C y)\|^2 \geq \|x - y\|^2 - \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.4.2}$$

Theorem 2.4.2. [22] **(Banach Contraction Mapping Principle)** Let (X, d) be a complete metric space and $f : X \to X$ be a contraction. Then f has a unique fixed point, i.e. there exists a unique $x^* \in X$ such that $Tx^* = x^*$.

Definition 2.4.3. [23] A mapping $S : C \to C$ is called *strictly pseudo-contractive* if there exists a constant $0 \leq k < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C.$$

Remark 2.4.4. If $k = 0$, then S is nonexpansive.

In this case, we say that $S : C \to C$ is a k-strictly pseudo-contraction.

Putting $B = I - S$. Then, we have

$$\|(I - B)x - (I - B)y\|^2 \leq \|x - y\|^2 + k\|Bx - By\|^2, \quad \forall x, y \in C.$$

Observe that

$$\|(I - B)x - (I - B)y\|^2 = \|x - y\|^2 + \|Bx - By\|^2 - 2\langle x - y, Bx - By\rangle, \quad \forall x, y \in C.$$

Hence, we obtain

$$\langle x - y, Bx - By\rangle \geq \frac{1 - k}{2}\|Bx - By\|^2, \quad \forall x, y \in C.$$

Then, B is $\frac{1-k}{2}$-inverse-strongly monotone mapping.

Lemma 2.4.5. [30] Assume that C is a closed convex subset of Hilbert space H, and let $S : C \to C$ be a self-mapping of C

(i) If S is a k-strict pseudo-contraction, then S satisfies the Lipscchitz condition

$$\|Sx - Sy\| \leq \frac{1+k}{1-k}\|x-y\| \qquad \forall x, y \in C.$$

(ii) If S is a k-strict pseudo-contraction, then the mapping $I - S$ is demiclosed (at 0). That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I-S)x_n \to 0$, $then (I-S)\tilde{x} = 0$.

(iii) If S is a k-strict pseudo-contraction, then the fixed point set $F(S)$ of S is closed and convex so that the projection $P_{F(S)}$ is well defined.

Lemma 2.4.6. [31] Let C be a nonempty closed convex subset of a real Hilbert space H and let $S : C \to C$ be a k-strict pseudo-contraction mapping with a fixed point. Then $F(S)$ is closed and convex. Define $S_k : C \to C$ by $S_k = kx + (1-k)Sx$ for each $x \in C$. Then S_k is nonexpansive such that $F(S_k) = F(S)$.

Definition 2.4.7. [21] Let A be a strongly positive on H if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H. \tag{2.4.3}$$

A typical problem is that of minimizing a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in F(S)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{2.4.4}$$

where A is a nonexpansive mapping and b is a given point in H.

Optimization problem (for short, OP) as the following

$$\min_{x\in F} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x-u\|^2 - h(x), \tag{2.4.5}$$

where $F = \cap_{n=1}^{\infty} C_n$, $C_1, C_2, \ldots$ are infinitely closed convex subsets of H such that $\cap_{n=1}^{\infty} C_n \neq \emptyset$, $u \in H$, $\mu \geq 0$ is a real number, A is a strongly positive linear bounded operator on H and h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Lemma 2.4.8. [32] Let C be a nonempty closed convex subset of a real Hilbert space H, and $g : C \to \mathbb{R} \cup \{\infty\}$ be a proper lower-semicontinuous differentiable convex function. If z is a solution to the minimization problem

$$g(z) = \inf_{x \in C} g(x),$$

then

$$\Big\langle g'(x), x - z \Big\rangle \geq 0, \quad x \in C.$$

In particular, if z solves problem OP, then

$$\Big\langle u + \big[\gamma f - (I + \mu A)\big] z, x - z \Big\rangle \leq 0.$$

Lemma 2.4.9. [33] Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.

Definition 2.4.10. [21] A mapping A of C into H is called *monotone* if

$$\langle Au - Av, u - v \rangle \geq 0, \quad \forall u, v \in C. \tag{2.4.6}$$

Definition 2.4.11. [21] A is called *α-inverse-strongly monotone* if there exists a positive real number α such that

$$\langle Au - Av, u - v \rangle \geq \alpha \|Au - Av\|^2, \quad \forall u, v \in C. \tag{2.4.7}$$

Lemma 2.4.12. [21] Let $A : H \to H$ be a α-inverse-strongly monotone mapping. If $\lambda \leq 2\alpha$, for any $\lambda > 0$ and $\alpha > 0$ then $I - \lambda A$ is a nonexpansive mapping from H into itself.

Proof. Let $u, v \in H$ and $\lambda > 0$,

$$\begin{aligned} \|(I - \lambda A)u - (I - \lambda A)v\|^2 &= \|(u - v) - \lambda(Au - Av)\|^2 \\ &= \|u - v\|^2 - 2\lambda\langle u - v, Au - Av \rangle + \lambda^2 \|Au - Av\|^2 \\ &\leq \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|Au - Av\|^2. \end{aligned}$$

□

Definition 2.4.13. [21] A mapping $A : C \to C$ is called *L-Lipschitz-continuous* if there exists a positive real number L such that

$$\|Au - Av\| \leq L\|u - v\|, \quad \forall u, v \in C. \tag{2.4.8}$$

Remark 2.4.14. It is easy to see that if A is an α-inverse-strongly monotone mapping of C into H, then A is $\frac{1}{\alpha}$–Lipschitz continuous.

Definition 2.4.15. [21] Let $\eta : C \times C \to H$ and $B : C \to H$ be two mappings. B is said to be:

(1) *monotone* if

$$\langle Bx - By, \eta(x, y)\rangle \geq 0, \quad \forall x, y \in C;$$

(2) σ*-strongly monotone* if there exists a positive real number σ such that

$$\langle Bx - By, \eta(x, y)\rangle \geq \sigma\|x - y\|^2, \quad \forall x, y \in C;$$

(3) L*-Lipschitz continuous* if there exists a constant $L > 0$ such that

$$\|\eta(x, y)\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

Lemma 2.4.16. [31] Let $V : C \to H$ be a k-strict pseudo-contraction, then
(1) the fixed point set $F(V)$ of V is closed convex so that the projection $P_{F(V)}$ is well defined;
(2) define a mapping $T : C \to H$ by

$$Tx = tx + (1 - t)Vx, \ \forall x \in C. \tag{2.4.9}$$

If $t \in [k, 1)$, then T is a nonexpansive mapping such that $F(V) = F(T)$.

Definition 2.4.17. [34] For the infinite family of nonexpansive mapping of $T_1, T_2, ...,$ we define the mapping W_n of C into itself as follows:

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \lambda_n T_n U_{n,n+1} + (1 - \lambda_n)I, \\ U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1 - \lambda_{n-1})I, \\ \vdots \\ U_{n,k} = \lambda_k T_k U_{n,k+1} + (1 - \lambda_k)I, \\ U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1 - \lambda_{k-1})I, \\ \vdots \\ U_{n,2} = \lambda_2 T_2 U_{n,3} + (1 - \lambda_2)I, \\ W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1 - \lambda_1)I, \end{cases} \tag{2.4.10}$$

where $T_1, T_2, ...$ be an infinite family of nonexpansive mappings of C into itself and $\lambda_1, \lambda_2, ...$ be real numbers such that $0 \leq \lambda_n \leq 1$ for every $n \in \mathbb{N}$.

Lemma 2.4.18. [34] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, ...$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty, let $\mu_1, \mu_2, ...$ be real numbers such that $0 \leq \mu_n \leq b < 1$ for every $n \geq 1$. Then,

(1) W_n is nonexpansive and $F(W_n) = \cap_{i=1}^{n} F(T_i)$, $\forall n \geq 1$;

(2) for every $x \in C$ and $k \in \mathbb{N}$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists;

(3) a mapping $W : C \to C$ defined by

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \forall x \in C \tag{2.4.11}$$

is a nonexpansive mapping satisfying $F(W) = \cap_{i=1}^{\infty} F(T_i)$ and it is called the W-mapping generated by $T_1, T_2, ...$ and $\mu_1, \mu_2, ...$

Lemma 2.4.19. [35] Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_i : C \to C\}$ be a countable family of nonexpansive mappings with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$, $\{\mu_i\}$ be a real sequence such that $0 < \mu_i \leq b < 1, \forall i \geq 1$. If D is any bounded subset of C, then

$$\lim_{n\to\infty} \sup_{x\in D} \|Wx - W_n x\| = 0.$$

2.5 Variational Inequalities in Hilbert Spaces

Let C be a closed convex subset of a real Hilbert space H with inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively.

Definition 2.5.1. [21] Let $B : C \to H$ be a nonlinear mapping. The *variational inequality problem* is to find $x \in C$ such that

$$\langle Bx, y - x \rangle \geq 0, \quad \forall y \in C. \tag{2.5.1}$$

We denote by $VI(C, B)$ the set of solutions of the variational inequality problem, that is,

$$VI(C, B) = \{x \in C : \langle Bx, y - x \rangle \geq 0, \quad \forall y \in C\}. \tag{2.5.2}$$

Lemma 2.5.2. [23] Let H be Hilbert space, let C be a nonempty closed convex subset of H and let B be a mapping of C into H. Let $u \in C$. Then, for $\lambda > 0$,

$$u \in VI(C, B) \Longleftrightarrow u = P_C(u - \lambda Bu),$$

where P_C is the metric projection of H onto C.

Lemma 2.5.3. [23] Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let $\xi > 0$ and let $B : C \to H$ be ξ-inverse strongly monotone. Then, $VI(C, B) \neq \emptyset$.

Definition 2.5.4. [37] Let $A, B : C \to H$ be two mappings. We consider the following problem for finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \ \forall x \in C, \\ &\langle \mu Bx^* + y^* - x^*, x - y^* \rangle \geq 0, \ \forall x \in C, \end{aligned} \tag{2.5.3}$$

which is called a *general system of variational inequalities* where $\lambda \geq 0$ and $\mu \geq 0$ are two constants. The set of solution of (2.5.3) is denoted by $GVI(C, A, B)$.

Remark 2.5.5. If $A = B$, then the problem (2.5.3) is reduced into the *new system of variational inequalities* for finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \ \forall x \in C, \\ &\langle \mu Ax^* + y^* - x^*, x - y^* \rangle \geq 0, \ \forall x \in C, \end{aligned} \tag{2.5.4}$$

which is defined by Verma [37] (see also Verma [38]).

Remark 2.5.6. If $x^* = y^*$, then the problem (2.5.4) is reduced into the *classical variational inequalities* for finding $x^* \in C$ such that

$$\langle Ax^*, v - x^* \rangle \geq 0 \tag{2.5.5}$$

for all $v \in C$, which was originally introduced and studied by Stampacchia [4]. The set of solution of (2.5.5) is denoted by $VI(C, A)$.

Definition 2.5.7. [21] Let $A : H \to H$ be a single-valued nonlinear mapping and $M : H \to 2^H$ be a set-valued mapping. We consider the following *variational inclusion problem*, which is to find a point $u \in H$ such that

$$\theta \in A(u) + M(u), \tag{2.5.6}$$

where θ is the zero vector in H. The set of solutions of problem (2.5.6) is denoted by $VI(A, M)$.

Remark 2.5.8. If $M = \partial\delta_C$, where C is a nonempty closed convex subset of H and $\delta_C : H \to [0, \infty]$ is the indicator function of C, i.e.,

$$\delta_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases}$$

Then the variational inclusion problem (2.5.6) is equivalent to the classical variational inequality (2.5.5).

Definition 2.5.9. [21] A set-valued mapping $M : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in Mx$ and $g \in My$ imply $\langle x - y, f - g\rangle \geq 0$.

Definition 2.5.10. [31] A monotone mapping $M : H \to 2^H$ is *maximal* if the graph of $G(M)$ of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(M)$ implies $f \in Mx$.

Definition 2.5.11. [39] Let the set-valued mapping $M : H \to 2^H$ be a maximal monotone. We define the *resolvent operator* $J_{M,\lambda}$ associate with M and λ as follows:

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \quad u \in H, \tag{2.5.7}$$

where λ is a positive number. It is worth mentioning that the resolvent operator $J_{M,\lambda}$ is single-valued, nonexpansive and 1-inverse strongly monotone.

Remark 2.5.12. ([41],[42]) Let A be an inverse-strongly monotone mapping of C into H and let N_Cv be the *normal cone* to C at $v \in C$, i.e.,

$$N_Cv = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$$

and define

$$Tv = \begin{cases} Av + N_Cv, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$.

2.6 Equilibrium Problems

Definition 2.6.1. [40] Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The *equilibrium problem* for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{2.6.1}$$

The set of solutions of (2.6.1) is denoted by $EP(F)$, that is,

$$EP(F) = \{x \in C : F(x, y) \geq 0, \quad \forall y \in C\}.$$

Given a mapping $B : C \to H$, let $F(x, y) = \langle Bx, y - x\rangle$ for all $x, y \in C$. Then, $z \in EP(F)$ if and only if $\langle Bz, y - z\rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality.

Let $\Im = \{F_k\}_{k\in\Lambda}$ be a family of bifunctions from $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The system of equilibrium problems for $\Im = \{F_k\}_{k\in\Lambda}$ is to determine common equilibrium points for $\Im = \{F_k\}_{k\in\Lambda}$ such that

$$F_k(x, y) \geq 0, \quad \forall k \in \Lambda, \quad \forall y \in C, \tag{2.6.2}$$

where Λ is an arbitrary index set. The set of solutions of (2.6.2) is denoted by $SEP(\Im)$, that is,

$$SEP(\Im) = \left\{ x \in C : F_k(x, y) \geq 0, \quad \forall k \in \Lambda, \quad \forall y \in C \right\}. \tag{2.6.3}$$

If Λ is a singleton, then the problem (2.6.2) is reduced to the problem (2.6.1).

Definition 2.6.2. [3] For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;

(A3) F is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t\to 0} F(tz + (1 - t)x, y) \leq F(x, y);$$

(A4) $F(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$, (A5) for each $y \in C, x \mapsto F(x,y)$ is weakly upper semicontinuous;

(B1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z,y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0; \tag{2.6.4}$$

(B2) C is a bounded set.

Definition 2.6.3. [43] Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, $\Psi : C \to H$ be a nonlinear mapping and $\varphi : C \to \mathbb{R}$ be a real-valued function. The *generalized mixed equilibrium problem* is to find $x \in C$ such that

$$F(x,y) + \langle \Psi x, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{2.6.5}$$

The set of solutions of (2.6.5) is denoted by $GMEP(F,\varphi,\Psi)$, that is

$$GMEP(F,\varphi,\Psi) = \{x \in C : F(x,y) + \langle \Psi x, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C\}.$$

Remark 2.6.4. [48] If $\Psi \equiv 0$, then the problem (2.6.5) is reduced into the *mixed equilibrium problem* for finding $x \in C$ such that

$$F(x,y) + \varphi(y) \geq \varphi(x), \forall y \in C. \tag{2.6.6}$$

The set of solutions of (2.6.6) is denoted by $MEP(F,\varphi)$. We see that x is a solution of problem (2.6.6) implies that $x \in \text{dom } \varphi = \{x \in C | \varphi(x) < +\infty\}$.

Remark 2.6.5. If $\varphi \equiv 0$, the problem (2.6.5) is reduced into the *generalized equilibrium problem* is to find $x \in C$ such that

$$F(x,y) + \langle \varphi x, y - x\rangle \geq 0, \quad \forall y \in C. \tag{2.6.7}$$

The set of solution of (2.6.7) is denoted by $GEP(F,B)$, that is,

$$GEP(F,\varphi) = \Big\{x \in C : F(x,y) + \langle \varphi x, y - x\rangle \geq 0, \quad \forall y \in C\Big\}.$$

In the case of $\varphi \equiv 0$, then the problem (2.6.7) is reduced to the problem (2.6.1). In the case of $F \equiv 0$, the problem (2.6.7) is reduced to the classical variational inequality problem (2.5.1).

Remark 2.6.6. [48] If $\varphi \equiv 0$, then the mixed equilibrium problem (2.6.6) is reduced into the *equilibrium problem* (2.6.1)

Remark 2.6.7. If $F \equiv 0$ and $\Psi \equiv 0$, then the problem (2.6.5) is reduced into the *minimize problem* for finding $x \in C$ such that

$$\varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{2.6.8}$$

The set of solutions of (2.6.8) is denoted by $Argmin(\varphi)$.

Lemma 2.6.8. [30] Let C be a nonempty closed convex subset of H and let f be a contraction of H into itself with $\alpha \in (0,1)$, and A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$. Then , for $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$,

$$\Big\langle x - y, (A - \gamma f)x - (A - \gamma f)y \Big\rangle \geq (\bar{\gamma} - \alpha\gamma)\|x - y\|^2, \quad x, y \in H.$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\bar{\gamma} - \alpha\gamma$.

Lemma 2.6.9. [30] Assume A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.

Lemma 2.6.10. [45] Let C be a closed convex subset of H. Let $\{x_n\}$ be a bounded sequence in H. Assume that

(i) The weak ω-limit set $\omega_w(x_n) \subset C$,

(ii) For each $z \in C$, $\lim_{n\to\infty} \|x_n - z\|$ exists. Then, $\{x_n\}$ is weakly convergent to a point in C.

Lemma 2.6.11. [49] Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C.$$

Lemma 2.6.12. [50] Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$$

for all $x \in H$. Then, the following hold:

(1) T_r is single-valued;

(2) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,
$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;

(3) $F(T_r) = EP(F)$;

(4) $EP(F)$ is closed and convex.

Lemma 2.6.13. [55] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction mapping satisfies (A1)-(A4) and let $\varphi : C \to \mathbb{R}$ is convex and lower semicontinuous such that $C \cap dom\varphi \neq \emptyset$. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, then there exists $u \in C$ such that

$$F(u, y) + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, u - x\rangle \geq 0.$$

Define a mapping $K_r : H \to C$ as follows:

$$K_r(x) = \Big\{u \in C : F(u, y) + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, u - x\rangle \geq 0, \forall y \in C\Big\}$$

for all $x \in H$. Then, the following hold:

(1) K_r is single-valued;

(2) K_r is firmly nonexpansive, i.e., for any $x, y \in H$, $\|K_r x - K_r y\|^2 \leq \langle K_r x - K_r y, x - y\rangle$;

(3) $F(K_r) = MEP(F)$;

(4) $MEP(F)$ is closed and convex.

Let $\varphi : C \to \mathbb{R}$ be a real-valued function and let $\{F_k : C \times C \to \mathbb{R}, \quad k = 1, 2, \ldots, N\}$ be a finite family of equilibrium functions, i.e., $F_k(u, u) = 0$ for each $u \in C$. The *system of mixed equilibrium problems* (for short, SMEP) for function $(F_1, F_2, \ldots, F_N, \varphi)$ which is to find $z \in C$ such that

$$\begin{cases} F_1(z, y) + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C, \\ F_2(z, y) + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C, \\ \quad \vdots \\ F_N(z, y) + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C. \end{cases} \tag{2.6.9}$$

The set of solutions of (2.6.9) is denoted by $\cap_{k=1}^{N} MEP(F_k, \varphi)$, where $MEP(F_k, \varphi)$ is the set of solutions of the *mixed equilibrium problem.* If $\varphi \equiv 0$, and $N = 1$, then the problem (2.6.9) reduces to the *equilibrium problem.*

Lemma 2.6.14. ([40],[53]) For solving the system of mixed equilibrium problems (2.6.9), let us assume that function $F_k : C \times C \to \mathbb{R}, k = 1, 2, \ldots, N$ satisfies the following conditions:

(H1) F_k is monotone, i.e., $F_k(x, y) + F_k(y, x) \leq 0, \quad \forall x, y \in C$;

(H2) for each fixed $y \in C$, $x \mapsto F_k(x, y)$ is convex and upper semicontinuous;

(H3) for each $x \in C, y \mapsto F_k(x, y)$ is convex.

Lemma 2.6.15. [16] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a *Banach space E* and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf \beta_n \leq \limsup \beta_n < 1$. Suppose $x_{n+1} = \beta_n y_n + (1 - \beta_n)x_n$ for all $n \geq 0$ and

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{2.6.10}$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.6.16. Let C be a convex subset of a Hilbert space H. Let $x \in H$ and $x_0 \in C$. Then $x_0 = P_C x$ if and only if

$$\langle z - x_0, x_0 - x\rangle \leq 0, \quad \forall z \in C.$$

Lemma 2.6.17. [53] Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $A : C \to H$ be a continuous accretive mapping. Then, for $r > 0$ and $x \in H$, there exist $z \in C$ such that

$$\langle y - z, Az\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Moreover, by a similar argument of the proof of lemma 2.8 and 2.9 of [64].

Lemma 2.6.18. [53] Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $A : C \to H$ be a continuous accretive mapping. For $r > 0$ and $x \in H$, define a mapping $F_r : H \to C$ as follows :

$$F_r x := \{z \in C : \langle y - z, Az\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\}$$

for all $x \in H$. Then the following hold:

(1) F_r is single - valued;

(2) F_r is a firmly nonexpansive type mapping, i.e., for all $x, y \in H$,

$$\|F_rx - F_ry\|^2 \leq \langle F_rx - F_ry, x - y\rangle;$$

(3) $F(F_r) = VI(C, A)$;

(4) $VI(C, A)$ is closed and convex.

Lemma 2.6.19. [53] Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $T : C \to H$ be a continuous pseudo-contractive mappings. Then, for $r > 0$ and $x \in H$, there exist $z \in C$ such that

$$\langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0, \quad \forall y \in C.$$

Lemma 2.6.20. [53] Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a continuous pseudo-contractive mappings. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows :

$$T_rx := \{z \in C : \langle y - z, Tz\rangle + \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0, \quad \forall y \in C\}$$

for all $x \in H$. Then the following hold:

(1) T_r is single - valued;

(2) T_r is a firmly nonexpansive type mapping, i.e., for all $x, y \in H$,

$$\|T_rx - T_ry\|^2 \leq \langle T_rx - T_ry, x - y\rangle;$$

(3) $F(T_r) = F(T)$;

(4) F(T) is closed and convex.

Lemma 2.6.21. [11] The function $u \in C$ is a solution of the variational inequality if and only if $u \in C$ satisfies the relation $u = P_C(u - \lambda Bu)$ for all $\lambda > 0$.

Lemma 2.6.22. [62] Let $M : H \to 2^H$ be a maximal monotone mapping and let $B : H \to H$ be a monotone and Lipshitz continuous mapping. Then the mapping $L = M + B : H \to 2^H$ is a maximal monotone mapping.

Lemma 2.6.23. [26] Each Hilbert space H satisfies Opial's condition, that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$, hold for each $y \in H$ with $y \neq x$.

Lemma 2.6.24. [57] Let C be a closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at zero, that is,

$$x_n \rightharpoonup x \textit{ and } \|x_n - Tx_n\| \to 0$$

implies $x = Tx$.

Lemma 2.6.25. [56] Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i\in N}$ be an infinite family of nonexpansive mappings of C into itself such that $\cap_{i\in N}F(T_i) \neq \emptyset$ and let $\{\lambda_i\}$ be an real sequence such that $0 \leq \lambda_i \leq b < 1$ for every $i \in N$. Then $F(W) = \cap_{i\in N}F(T_i) \neq \emptyset$.

Lemma 2.6.26. [56] Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}$ be an infinite family of nonexpansive mappings of C into itself and let $\{\lambda_i\}$ be a real sequence such that $0 \leq \lambda_i \leq b < 1$ for every $i \in N$. Then, for every $x \in C$ and $k \in N$, the $\lim_{n\longrightarrow\infty} U_{n,k}$ exist.

In view of the previous lemma, we define

$$Wx := \lim_{n\longrightarrow\infty} U_{n,1}x = \lim_{n\longrightarrow\infty} W_n x.$$

Lemma 2.6.27. [57] Let H be a Hilbert space, C is a closed convex subset of H and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$; in particular, if $y = 0$ then $x \in F(T)$.

Lemma 2.6.28. [45] Let C be a nonempty closed convex subset of a real Hilbert space H. If $T : C \to C$ is a k-strict pseudo-contraction, then the mapping $I - T$ is demiclosed at 0. That is, if $\{x_n\}$ is a sequence in C weakly converging to x and $\{(I - T)x_n\}$ converges strongly to 0, then $(I - T)x = 0$.

Lemma 2.6.29. [58] Let H be a Hilbert space, C be a closed convex subset of H, $f : C \to H$ be a contraction with coefficient $0 < \rho < 1$ and $T : C \to C$ be a nonexpansive mapping. Then, for $0 < \gamma < \bar{\gamma}/\rho$, for $x, y \in C$,

1. the mapping $(I - f)$ is strongly monotone with coefficient $(1 - \rho)$ that is

$$\langle x - y, (I - f)x - (I - f)y\rangle \geq (1 - \rho)\|x - y\|^2,$$

2. the mapping $(I - T)$ is monotone, that is

$$\langle x - y, (I - T)x - (I - T)y\rangle \geq 0.$$

Lemma 2.6.30. [45] Let C be a closed convex subset of H. Let $\{x_n\}$ be a bounded sequence in H. Assume that

(1) The weak ω-limit set $\omega_w(x_n) \subset C$.

(2) For each $z \in C$, $\lim_{n\to\infty} \|x_n - z\|$ exists.

Then $\{x_n\}$ is weakly convergent to a point in C.

Lemma 2.6.31. [31] Let H be a real Hilbert space, C be a closed and convex subset of H, and T be a k-strict pseudo-contraction mapping on C, then $F(T)$ is closed convex, so that the projection $P_{F(T)}$ is well defined.

Lemma 2.6.32. [31] Let H be a Hilbert space, C be a closed and convex subset of H, and $T : C \to H$ be a k-strict pseudo-contraction mapping. Define a mapping $V : C \to H$ by $Vx = \lambda x + (1 - \lambda)Tx$ for all $x \in C$. Then, as $\lambda \in [k, 1)$, V is a nonexpansive mapping such that $F(V) = F(T)$.

Lemma 2.6.33. [19] Let H be a Hilbert space and C be a nonempty closed and convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Then

$$\|Tx - x\|^2 \leq 2\langle x - Tx, x - x'\rangle, \quad \forall x' \in F(T), \forall x \in C.$$

Lemma 2.6.34. [20] For solving the Ky Fan inequality or equilibrium problem, let us assume that the following conditions are satisfied on the bifunction $F : C \times C \to \mathbb{R}$.

(A1) $F(x, x) = 0$ for every $x \in C$;

(A2) F is pseudomonotone on C, i.e., $F(x, y) \geq 0 \Rightarrow F(y, x) \leq 0, \quad \forall x, y \in C$;

(A3) F is jointly weakly continuous on $C \times C$ in the sense that, if $x, y \in C$ and $\{x_n\}$ and $\{y_n\}$ are two sequences in C converging weakly to x and y, respectively, then $F(x_n, y_n) \to F(x, y)$;

(A4) $F(x, \cdot)$ is convex, lower semicontinuous, and subdifferentiable on C for very $x \in C$;

(A5) F satisfies the Lipschitz-type condition, there exist positive integer c_1 and c_2, such that for every $x, y, z \in C$,

$$F(x, y) + F(y, z) \geq F(x, z) - c_1\|y - x\|^2 - c_2\|z - y\|^2.$$

If F satisfies the properties (A1)-(A4), then the set $EP(F)$ of solutions to the Ky Fan inequality is closed and convex.

Proposition 2.6.35. ([60], Lemma 3.1) For every $x^* \in EP(F)$, and every $n \in \mathbb{N}$, one has

1. $\langle x_n - y_n, y - y_n\rangle \leq \lambda_n F(x_n, y) - \lambda_n F(x_n, y_n), \quad \forall y \in C;$

2. $\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (1 - 2\lambda_n c_1)\|y_n - x_n\|^2 - (1 - 2\lambda_n c_2)\|z_n - y_n\|^2.$

Proposition 2.6.36. [61] Let K be a nonempty closed and convex subset of H. Let $u \in H$ and let $\{x_n\}$ be a sequence in H. If any weak limit point of $\{x_n\}$ belongs to K, and $\|x_n - u\| \leq \|u - P_K u\|$ for all $n \in \mathbb{N}$, then $x_n \to P_K u$.

CHAPTER 3 ITERATIVE ALGORITHMS FOR FIXED POINT PROBLEMS AND VARIATIONAL INEQUALITY PROBLEMS

In this section, we show a strong convergence theorem for finding the least norm of fixed points for strict pseudo mappings, a common element of the set of fixed points and the solution set of variational inequality in Hilbert spaces.

3.1 [An Algorithm for Minimum-Norm of Fixed Point for Nonexpansive Mappings]

Theorem 3.1.1. *Let C be a closed convex of a real Hilbert space H. Let $S : C \to C$ be a nonexpansive mapping. Let A be an α-inverse strongly monotone and $\Omega := F(S) \cap VI(C, A) \neq \emptyset$. Assume that a sequence $\{\alpha_n\} \subset (0, 1)$ satisfies the conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\Sigma_{n=0}^{\infty} \alpha_n = +\infty$.

Then the sequence $\{x_n\}$ generated by the following algorithm

$$x_{n+1} = (1 - \alpha_n)[\lambda SP_C(I - \lambda A)x_n + (1 - \lambda)x_n] \tag{3.1.1}$$

converges strongly to a fixed point of S which is a minimal norm and the unique solution of the variational inequality:

$$x^* \in \Omega, \langle x^*, x - x^*\rangle \geq 0, \forall x \in \Omega.$$

Proof First, we prove that the sequence $\{x_n\}$ is bounded. Let $q \in \Omega$. By (3.1.1), we have

$$\begin{aligned}
\|x_{n+1} - q\| &= \|(1 - \alpha_n)[\lambda SP_C(I - \lambda A)x_n + (1 - \lambda)x_n] - q\| \\
&\leq \|(1 - \alpha_n)[(1 - \lambda)(x_n - q) + \lambda(SP_C(I - \lambda A)x_n - q)] - \alpha_n q\| \\
&\leq \|(1 - \alpha_n)[(1 - \lambda)\|(x_n - q)\| + \lambda\|(x_n - q)\|] - \|\alpha_n q\| \\
&\leq (1 - \alpha_n)\|x_n - q\| + \alpha_n\|q\| \\
&\leq \max\{\|x_n - q\|, \|q\|\}.
\end{aligned}$$

By induction, it follows that

$$\|x_n - q\| \leq \max\{\|x_0 - q\|, \|q\|\},$$

for all $n \geq 0$. Then $\{x_n\}$ is bounded. Therefore, $\{SP_C(I - \lambda A)x_n\}$ is also bounded.

Let $y_n = \frac{(1-\alpha_n)\lambda SP_C(I-\lambda A)x_n}{\alpha_n+(1-\alpha_n)\lambda}$, then the iterative sequence (3.1.1) is equivalent to

$$x_{n+1} = (\alpha_n + (1 - \alpha_n)\lambda)y_n + (1 - \alpha_n - (1 - \alpha_n)\lambda)x_n. \tag{3.1.2}$$

Since $\lim_{n\to\infty}(\alpha_n + (1 - \alpha_n)\lambda) = \lambda$, then

$$\begin{aligned}
\|y_n - q\| &= \left\|\frac{(1 - \alpha_n)\lambda SP_C(I - \lambda A)x_n}{\alpha_n + (1 - \alpha_n)\lambda} - q\right\| \\
&= \left\|\frac{(1 - \alpha_n)\lambda SP_C(I - \lambda A)x_n - (\alpha_n + (1 - \alpha_n)\lambda)q}{\alpha_n + (1 - \alpha_n)\lambda}\right\| \\
&= \left\|\frac{(1 - \alpha_n)\lambda SP_C(I - \lambda A)x_n - \alpha_n q - (1 - \alpha_n)\lambda q}{\alpha_n + (1 - \alpha_n)\lambda}\right\| \\
&\leq \frac{(1 - \alpha_n)\lambda\|x_n - q\| - \alpha_n\|q\|}{\alpha_n + (1 - \alpha_n)\lambda} \\
&= \frac{\alpha_n}{\alpha_n + (1 - \alpha_n)\lambda}\|q\| + (1 - \frac{\alpha_n}{\alpha_n + (1 - \alpha_n)\lambda})\|x_n - q\| \\
&\leq \max\{\|x_n - q\|, \|q\|\}.
\end{aligned}$$

Thus, $\{y_n\}$ is bounded. Hence by nonexpansiveness of S and P_C, we have

$$\begin{aligned}
&\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \\
= &\left\|\frac{(1 - \alpha_{n+1})\lambda SP_C(I - \lambda A)x_{n+1}}{\alpha_{n+1} + (1 - \alpha_{n+1})\lambda} - \frac{(1 - \alpha_n)\lambda SP_C(I - \lambda A)x_n}{\alpha_n + (1 - \alpha_n)\lambda}\right\| - \|x_{n+1} - x_n\| \\
\leq &\frac{(1 - \alpha_{n+1})\lambda}{\alpha_{n+1} + (1 - \alpha_{n+1})\lambda}\|SP_C(I - \lambda A)x_{n+1} - SP_C(I - \lambda A)x_n\| \\
&+ \left|\frac{(1 - \alpha_{n+1})\lambda}{\alpha_{n+1} + (1 - \alpha_{n+1})\lambda} - \frac{(1 - \alpha_n)\lambda}{\alpha_n + (1 - \alpha_n)\lambda}\right|\|SP_C(I - \lambda A)x_n\| \\
\leq &\frac{(1 - \alpha_{n+1})\lambda}{\alpha_{n+1} + (1 - \alpha_{n+1})\lambda}\|Sx_{n+1} - Sx_n\| \\
&+ \left|\frac{(1 - \alpha_{n+1})\lambda}{\alpha_{n+1} + (1 - \alpha_{n+1})\lambda} - \frac{(1 - \alpha_n)\lambda}{\alpha_n + (1 - \alpha_n)\lambda}\right|\|SP_C(I - \lambda A)x_n\| \\
&-\|x_{n+1} - x_n\| \\
\leq &\left(\frac{(1 - \alpha_{n+1})\lambda}{\alpha_{n+1} + (1 - \alpha_{n+1})\lambda} - 1\right)\|x_{n+1} - x_n\| \\
&+\left|\frac{(1 - \alpha_{n+1})\lambda}{\alpha_{n+1} + (1 - \alpha_{n+1})\lambda} - \frac{(1 - \alpha_n)\lambda}{\alpha_n + (1 - \alpha_n)\lambda}\right|\|SP_C(I - \lambda A)x_n\|.
\end{aligned}$$

From $\{x_n\}$ and $\{SP_C(I-\lambda A)x_n\}$ are bounded sequences and $\lim_{n\to\infty}\alpha_n = 0$, then

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \leq 0.$$

By Lemma 2.6.15, we obtain that $\lim_{n\to\infty}\|y_n - x_n\| = 0$. Therefore,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(\alpha_n + (1-\alpha_n)\lambda)\|y_n - x_n\| = 0. \tag{3.1.3}$$

On the other hand, we consider

$$\begin{aligned}\|x_n - SP_C(I-\lambda A)x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - SP_C(I-\lambda A)x_n\| \\ &= \|x_n - x_{n+1}\| + \|(1-\alpha_n)(\lambda SP_C(I-\lambda A)x_n + (1-\lambda)x_n) \\ &\quad -SP_C(I-\lambda A)x_n\| \\ &\leq \|x_n - x_{n+1}\| + (1-\alpha_n)(1-\lambda)\|x_n - SP_C(I-\lambda A)x_n\| \\ &\quad +\alpha_n\|SP_C(I-\lambda A)x_n\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_n - SP_C(I-\lambda A)x_n\| &\leq \frac{1}{1-(1-\alpha_n)(1-\lambda)}\|x_n - x_{n+1}\| \\ &\quad +\frac{1}{1-(1-\alpha_n)(1-\lambda)}\alpha_n\|SP_C(I-\lambda A)x_n\| \to 0 \text{ as } n\to\infty.\end{aligned}$$

Next, we prove that $\limsup_{n\to\infty}\langle x^* - x_n, x^*\rangle \leq 0$.

Since $\{x_n\}$ is bounded. Then, we can take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x^* - x_n, x^*\rangle = \lim_{i\to\infty}\langle x^* - x_{n_i}, x^*\rangle.$$

Again, since $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_{n_i} \rightharpoonup x^{'}$. Consequently,

$$\limsup_{n\to\infty}\langle x^* - x_n, x^*\rangle = \langle x^* - x^{'}, x^*\rangle \leq 0.$$

Notice that $\lim_{n\to\infty}\|x_n - SP_C(I-\lambda A)x_n\| = 0$. By the demiclosedness principle of nonexpansive mapping S, we have $x' \in Fix(S)$. Since $x^* = P_{Fix(S)}(0)$. It follows from the properties of projection operator that

$$\limsup_{n\to\infty}\langle x^* - x_n, x^*\rangle = \langle x^* - x', x^*\rangle \leq 0. \tag{3.1.4}$$

By (3.1.1), we have

$$\begin{aligned}
\|x_{n+1}-(1-\alpha_n)x^*\|^2 &= \|[(1-\alpha_n)\lambda SP_C(I-\lambda A)x_{n+1}+(1-\lambda)x_n]-(1-\alpha_n)x^*\|^2 \\
&= \|(1-\alpha_n)[\lambda SP_C(I-\lambda A)x_n+(1-\lambda)x_n]-x^*\|^2 \\
&= \|(1-\alpha_n)[\lambda SP_C x_n-(1-\lambda)x_n]-(1-\lambda+\lambda)x^*\|^2 \\
&\leq (1-\alpha_n)\|\lambda(Sx_n-x^*)+(1-\lambda)(x_n-x^*)\|^2 \\
&\leq (1-\alpha_n)\|\lambda(x_n-x^*)+(1-\lambda)(x_n-x^*)\|^2 \\
&\leq (1-\alpha_n)\|x_n-x^*\|^2. \qquad (3.1.5)
\end{aligned}$$

Observe that

$$\begin{aligned}
\|x_{n+1}-(1-\alpha_n)x^*\|^2 &= \|x_{n+1}-x^*\|^2-2\alpha_n\langle -x^*, x_{n+1}-x^*\rangle+\alpha_n^2\|x^*\|^2 \\
&\geq \|x_{n+1}-x^*\|^2-2\alpha_n\langle x_{n+1}-x^*, x^*\rangle. \qquad (3.1.6)
\end{aligned}$$

Therefore by (3.1.5) and (3.1.6), we get

$$\|x_{n+1}-x^*\|^2 \leq (1-\alpha_n)\|x_n-x^*\|^2+2\alpha_n\langle x_{n+1}-x^*, x^*\rangle. \qquad (3.1.7)$$

By the condition of (ii) and the inequality (3.1.4), we can apply Lemma (2.1.23) to (3.1.7) and conclude that $\{x_n\}$ converges strongly to x^* as $n \to \infty$ that is, the minimum - norm fixed point of S. This completes the proof. □

3.2 An Algorithm for Variational Inequalities and Fixed Point for Pseudo-Contractive Mappings

Theorem 3.2.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a continuous pseudo-contractive mapping and $A : C \to H$ be a continuous monotone mapping such that $\mathfrak{F} := F(T) \cap VI(C, A) \neq \emptyset$. For $x \in H$, define T_{r_n} and F_{r_n} as follows:*

$$T_r x := \{z \in C : \langle y-z, Tz\rangle + \frac{1}{r}\langle y-z, (1+r)z-x\rangle \leq 0, \quad \forall y \in C\}$$

and

$$F_r x := \{z \in C : \langle y-z, Az\rangle + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad \forall y \in C\}.$$

Let f be a contraction of H into itself with a contraction constant β and let $B : H \to H$ be a strongly positive linear bounded self-adjoint operator with coefficients $\overline{\beta} > 0$ and let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$\begin{cases} y_n = F_{r_n} x_n \\ x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + [(1-\delta_n)I - \alpha_n B]T_{r_n} y_n, \end{cases} \tag{3.2.1}$$

where $\{\alpha_n\}, \{\delta_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ such that

(C1) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) $\lim_{n\to\infty} \delta_n = 0, \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$*;*

(C3) $\liminf_{n\to\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$

Then, the sequence $\{x_n\}$ converges strongly to $z \in \mathfrak{F}$, which is the unique solution of the variational inequality:

$$\langle (B - \gamma f)z, x - z \rangle \geq 0, \quad \forall x \in \mathfrak{F}. \tag{3.2.2}$$

Equivalently, $z = P_{\mathfrak{F}}(I - B + \gamma f)z$, which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2} \langle Az, z \rangle - h(z),$$

where h is a potential function for γf (i.e.,$h'(z) = \gamma f(z)$ for $z \in H$).

Remark (1) The variational inequality (3.2.2) has the unique solution; (see [62]).
(2) It follows from condition (C1) that $(1-\delta_n)I - \alpha_n B$ is positive and $\|(1-\delta_n)I - \alpha_n B\| \leq I - \delta_n - \alpha_n \overline{\beta}$ for all $n \geq 1$; (see [66]).

Proof. We processed the proof with following four steps:

Step 1 First, we will prove that the sequence $\{x_n\}$ is bounded.
Let $v \in \mathfrak{F}$ and let $u_n = T_{r_n} y_n$ and $y_n = F_{r_n} x_n$. Then, from Lemmas 2.6.18 and 2.6.20 that

$$\|u_n - v\| = \|T_{r_n} y_n - T_{r_n} v\| \leq \|y_n - v\| = \|F_{r_n} x_n - F_{r_n} v\| \leq \|x_n - v\|. \tag{3.2.3}$$

Moreover, from (3.2.1) and (3.2.2), we compute

$$\begin{aligned}
\|x_{n+1}-v\| &= \|\alpha_n(\gamma(f(x_n)-Bv))+\delta_n(x_n-v)+[(1-\delta_n)I-\alpha_nB]T_{r_n}-v\| \\
&\le \alpha_n\|\gamma f(x_n)-Bv)\|+\delta_n\|x_n-v\|+\|(1-\delta_n)I-\alpha_nB\|\|T_{r_n}-v\| \\
&\le \alpha_n\beta\gamma\|x_n-v\|+\alpha_n\|\gamma f(v)-Bv\|+\delta_n\|x_n-v\| \\
&\quad +(1+\delta_n-\alpha_n\overline{\beta})\|T_{r_n}y_n-v\| \\
&\le \alpha_n\beta\gamma\|x_n-v\|+\alpha_n\|\gamma f(v)-Bv\|+\delta_n\|x_n-v\| \\
&\quad +(1+\delta_n-\alpha_n\overline{\beta})\|u_n-v\| \\
&\le \alpha_n\beta\gamma\|x_n-v\|+\alpha_n\|\gamma f(v)-Bv\|+\delta_n\|x_n-v\| \\
&\quad +(1+\delta_n-\alpha_n\overline{\beta})\|x_n-v\| \\
&= \alpha_n\beta\gamma\|x_n-v\|+\alpha_n\|\gamma f(v)-Bv\|+\delta_n\|x_n-v\|+\|x_n-v\| \\
&\quad -\delta_n\|x_n-v\|-\alpha_n\overline{\beta}\|x_n-v\| \\
&= \alpha_n\beta\gamma\|x_n-v\|+\alpha_n\|\gamma f(v)-Bv\|+\|x_n-v\|-\alpha_n\overline{\beta}\|x_n-v\| \\
&\le (\alpha_n\beta\gamma+1-\alpha_n\overline{\beta})\|x_n-v\|+\alpha_n\|\gamma f(v)-Bv\| \\
&= (1-\alpha_n(\overline{\beta}-\beta\gamma))\|x_n-v\|+\alpha_n\|\gamma f(v)-Bv\| \\
&\le \max\left\{\|x_n-v\|,\frac{\|\gamma f(v)-Bv\|}{\overline{\beta}-\beta\gamma}\right\}, \quad \forall n\ge 1.
\end{aligned}$$

Therefore, by the simple introduction, we have

$$\|x_n-v\|=\max\left\{\|x_1-v\|,\frac{\|\gamma f(v)-Bv\|}{\overline{\beta}-\beta\gamma}\right\}, \quad \forall n\ge 1$$

which show that $\{x_n\}$ is bounded, so $\{y_n\},\{u_n\}$ and $\{f(x_n)\}$ are bounded.

Step 2 We will show that $\|x_{n+1}-x_n\|\to 0$ and $\|u_n-y_n\|\to 0$ as $n\to\infty$. Notice that each T_{r_n} and F_{r_n} are firmly nonexpansive. Hence, we have

$$\begin{aligned}
\|u_{n+1}-u_n\| &= \|T_{r_n}y_{n+1}-T_{r_n}y_n\|\le\|y_{n+1}-y_n\| \\
&= \|F_{r_n}x_{n+1}-F_{r_n}x_n\|\le\|x_{n+1}-x_n\|.
\end{aligned}$$

From (3.2.1), we note that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n \gamma f(x_n) + \delta_n x_n + [(1-\delta_n)I - \alpha_n B]T_{r_n} y_n \\
&\quad -\alpha_{n-1}\gamma f(x_{n-1}) - \delta_{n-1}x_{n-1} - [(1-\delta_{n-1})I - \alpha_{n-1}B]T_{r_n} y_{n-1}\| \\
&= \|\alpha_n \gamma f(x_n) + \delta_n x_n + (I - \delta_n - \alpha_n B)u_n \\
&\quad -\alpha_{n-1}\gamma f(x_{n-1}) - \delta_{n-1}x_{n-1} - (I - \delta_{n-1} - \alpha_{n-1}B)u_{n-1}\| \\
&\leq \|\alpha_n \gamma f(x_n) - \alpha_n \gamma f(x_{n-1}) + \alpha_n \gamma f(x_{n-1}) \\
&\quad +\delta_n x_n - \delta_{n-1}x_{n-1} - \alpha_{n-1}\gamma f(x_{n-1}) + (I - \delta_n - \alpha_n B)u_n \\
&\quad -(I - \delta_n - \alpha_n B)u_{n-1} + (I - \delta_n - \alpha_n B)u_{n-1} \\
&\quad -(I - \delta_{n-1} - \alpha_{n-1}B)u_{n-1}\| \\
&\leq \|\alpha_n \gamma f(x_n) - \alpha_n \gamma f(x_{n-1})\| + \|\alpha_n \gamma f(x_{n-1}) \\
&\quad -\alpha_{n-1}\gamma f(x_{n-1})\| + \|\delta_n x_n - \delta_{n-1}x_{n-1}\| \\
&\quad +\|(I - \delta_n - \alpha_n B)u_n - (I - \delta_n - \alpha_n B)u_{n-1}\| \\
&\quad +\|(I - \delta_n - \alpha_n B)u_{n-1} - (I - \delta_{n-1} - \alpha_{n-1}B)u_{n-1}\| \\
&= \alpha_n \gamma \|f(x_n) - f(x_{n-1})\| + |\alpha_n - \alpha_{n-1}|\|\gamma f(x_{n-1})\| \\
&\quad +\delta_n x_n - \delta_{n-1}x_{n-1} + \delta_n x_{n-1} - \delta_n x_{n-1} \\
&\quad +(I - \delta_n - \alpha_n B)\|u_n - u_{n-1}\| \\
&\quad +\|(I - \delta_n - \alpha_n B - I + \delta_{n-1} + \alpha_{n-1}B)u_{n-1}\|
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\gamma\|f(x_{n-1})\| \qquad (3.2.4)\\
&\quad +\delta_n\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|\|x_{n-1}\|\\
&\quad +|I - \delta_n - \alpha_n B|\|u_n - u_{n-1}\|\\
&\quad +|\delta_{n-1} - \delta_n + \alpha_{n-1}B + \alpha_n B|\|u_{n-1}\|\\
&\leq \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\gamma\|f(x_{n-1})\|\\
&\quad +\delta_n\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|\|x_{n-1}\|\\
&\quad +|I - \delta_n - \alpha_n B|\|x_n - x_{n-1}\|\\
&\quad +|\delta_{n-1} - \delta_n|\|u_{n-1}\| + |\alpha_{n-1}B + \alpha_n B|\|u_{n-1}\|\\
&\leq \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\gamma\|f(x_{n-1})\|\\
&\quad +\delta_n\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|\|x_{n-1}\|\\
&\quad +|I - \delta_n - \alpha_n B|\|x_n - x_{n-1}\| + |\delta_{n-1} - \delta_n|\|x_{n-1}\|\\
&\quad -|\alpha_{n-1} - \alpha_n|B\|x_{n-1}\|\\
&\leq \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\gamma\|f(x_{n-1})\|\\
&\quad +\delta_n\|x_n - x_{n-1}\| + |I - \delta_n - \alpha_n B|\|x_n - x_{n-1}\| - |\alpha_{n-1} - \alpha_n|B\|x_{n-1}\| \qquad (3.2.5)\\
&\leq \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\gamma\|f(x_{n-1})\| + |\delta_n + I - \delta_n\\
&\quad -\alpha_n B|\|x_n - x_{n-1}\| - |\alpha_{n-1} - \alpha_n|B\|x_{n-1}\|\\
&\leq \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\gamma\|f(x_{n-1})\|\\
&\quad +|I - \alpha_n B|\|x_n - x_{n-1}\| - |\alpha_n - \alpha_{n-1}|B\|x_{n-1}\|\\
&\leq \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|\gamma f(x_{n-1}) - Bx_{n-1}\| + |I - \alpha_n B|\|x_n - x_{n-1}\|\\
&\leq \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|\gamma f(x_{n-1}) - Bx_{n-1}\| + |I - \alpha_n B|\|y_n - y_{n-1}\|\\
&\leq \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|K + |I - \alpha_n B|\|y_n - y_{n-1}\|,
\end{aligned}
$$

where $K = \|\gamma f(x_{n-1}) - Bx_{n-1}\| = 2\sup\{\|f(x_n)\| + \|u_n\| : n \in N\}$.
Moreover, since $y_n = F_{r_n}x_n$ and $y_{n+1} = F_{r_{n+1}}x_{n+1}$, we get

$$\langle y - y_n, Ay_n\rangle + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq 0, \quad \forall y \in C \qquad (3.2.6)$$

and

$$\langle y - y_{n+1}, Ay_{n+1}\rangle + \frac{1}{r_{n+1}}\langle y - y_{n+1}, y_{n+1} - x_{n+1}\rangle \geq 0, \quad \forall y \in C. \qquad (3.2.7)$$

Putting $y = y_{n+1}$ in (3.2.6) and $y = y_n$ in (3.2.7), we obtain

$$\langle y_{n+1} - y_n, Ay_n\rangle + \frac{1}{r_n}\langle y_{n+1} - y_n, y_n - x_n\rangle \geq 0 \tag{3.2.8}$$

and

$$\langle y_n - y_{n+1}, Ay_{n+1}\rangle + \frac{1}{r_{n+1}}\langle y_n - y_{n+1}, y_{n+1} - x_{n+1}\rangle \geq 0. \tag{3.2.9}$$

Adding (3.2.8) and (3.2.9), we have

$$\langle y_{n+1} - y_n, Ay_n - Ay_{n+1}\rangle + \left\langle y_{n+1} - y_n, \frac{y_n - x_n}{r_n} - \frac{y_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle \geq 0$$

which implies that

$$-\langle y_{n+1} - y_n, Ay_{n+1} - Ay_n\rangle + \left\langle y_{n+1} - y_n, \frac{y_n - x_n}{r_n} - \frac{y_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle \geq 0.$$

Using the fact that A is monotone, we get

$$\left\langle y_{n+1} - y_n, \frac{y_n - x_n}{r_n} - \frac{y_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle \geq 0,$$

and hence

$$\left\langle y_{n+1} - y_n, y_n - y_{n+1} + y_{n+1} - y_n - \frac{r_n}{r_{n+1}}(y_{n+1} - x_{n+1})\right\rangle \geq 0.$$

We observe that

$$\begin{aligned} & \|y_{n+1} - y_n\|^2 \\ \leq\ & \left\langle y_{n+1} - y_n, x_{n+1} - x_n(1 - \frac{r_n}{r_{n+1}})(y_{n+1} - x_{n+1})\right\rangle \\ \leq\ & \|y_{n+1} - y_n\| \left\{\|x_{n+1} - x_n\| + |(1 - \frac{r_n}{r_{n+1}})|\|(y_{n+1} - x_{n+1})\|\right\}. \end{aligned} \tag{3.2.10}$$

Without loss of generality, let k be a real number such that $r_n > k > 0$ for all $n \in N$. Then, we have

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|y_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{k}|r_{n+1} - r_n|M, \end{aligned} \tag{3.2.11}$$

where $M = \sup\{\|y_n - x_n\| : n \in N\}$. Furthermore, from (3.2.4) and (3.2.11), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \alpha_n\gamma\beta\|x_n - x_{n-1}\| + \|\alpha_n - \alpha_{n-1}\|K + (1 - \alpha_n)(\|x_n - x_{n-1}\| \\ &\quad + \frac{1}{k}|r_n - r_{i-1}|M) \\ &= (1 - \alpha_n + \alpha_n\gamma\beta)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|K + \frac{1}{k}|r_n - r_{n-1}|M \\ &= (1 - \alpha_n(1 - \gamma\beta))\|x_n - x_{n-1}\| + K|\alpha_n - \alpha_{n-1}| + \frac{M}{k}|r_n - r_{n-1}|. \end{aligned}$$

Using Lemma 2.1.23, and by the conditions (C1) and (C3), we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Consequently, from (3.2.11), we obtain

$$\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0. \tag{3.2.12}$$

Since $u_n = T_{r_n} y_n$ and $u_{n+1} = T_{r_{n+1}} y_{n+1}$, we have

$$\langle y - u_n, Tu_n\rangle - \frac{1}{r_n}\langle y - u_n, (1 - r_n)u_n - y_n\rangle \leq 0, \quad \forall y \in C \tag{3.2.13}$$

and

$$\langle y - u_{n+1}, Tu_{n+1}\rangle - \frac{1}{r_{n+1}}\langle y - u_{n+1}, (1 - r_{n+1})u_{n+1} - y_{n+1}\rangle \leq 0, \quad \forall y \in C. \tag{3.2.14}$$

Putting $y := u_{n+1}$ in (3.2.13) and $y := u_n$ in (3.2.14), we get

$$\langle u_{n+1} - u_n, Tu_n\rangle - \frac{1}{r_n}\langle u_{n+1} - u_n, (1 - r_n)u_n - y_n\rangle \leq 0, \tag{3.2.15}$$

and

$$\langle u_n - u_{n+1}, Tu_{n+1}\rangle - \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, (1 - r_{n+1})u_{n+1} - y_{n+1}\rangle \leq 0. \tag{3.2.16}$$

Adding (3.2.15) and (3.2.16), we have

$$\langle u_{n+1} - u_n, Tu_n - Tu_{n+1}\rangle - \left\langle u_{n+1} - u_n, \frac{(1 - r_n)u_n - y_n}{r_n} - \frac{(1 - r_{n+1})u_{n+1} - y_{n+1}}{r_{n+1}}\right\rangle \leq 0.$$

Using the fact that T is pseudo-contractive, we get

$$\left\langle u_{n+1} - u_n, \frac{u_n - y_n}{r_n} - \frac{u_{n+1} - y_{n+1}}{r_{n+1}}\right\rangle \geq 0$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - y_n - \frac{r_n}{r_{n+1}}(u_{n+1} - y_{n+1})\right\rangle \geq 0.$$

Thus, using the methods in (3.2.10) and (3.2.11), we can obtain

$$\|u_{n+1} - u_n\| \leq \|y_{n+1} - y_n\| + \frac{1}{r_{n+1}}|r_{n+1} + r_n|M_1, \tag{3.2.17}$$

where $M_1 = \sup\{\|u_n - y_n\| : n \in N\}$. Therefore, from (3.2.12) and property of $\{r_n\}$, we get

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0.$$

Furthermore, since $x_n = \alpha_{n-1}\gamma f(x_{n+1}) + \delta_{n-1}x_{n-1} + [(1-\delta_{n-1})I - \alpha_{n-1}B]T_{r_n}y_{n-1}$, we have

$$\begin{aligned}
\|x_n - u_n\| &\le \|x_n - u_{n-1}\| + \|u_{n-1} - u_n\| \\
&= \|\alpha_{n-1}\gamma f(x_{n-1}) + \delta_{n-1}x_{n-1} + [(1-\delta_{n-1})I - \alpha_{n-1}B]T_{r_n}y_{n-1} - u_{n-1}\| \\
&\quad + \|u_{n-1} - u_n\| \\
&= \alpha_{n-1}\gamma f(x_{n-1}) + \delta_{n-1}x_{n-1} + (I - \delta_{n-1} - \alpha_{n-1}B)u_{n-1} - u_{n-1}\| \\
&\quad + \|u_{n-1} - u_n\| \\
&= \alpha_{n-1}\gamma f(x_{n-1}) + \delta_{n-1}x_{n-1} + u_{n-1} - \delta_{n-1}u_{n-1} - \alpha_{n-1}Bu_{n-1} - u_{n-1}\| \\
&\quad + \|u_{n-1} - u_n\| \\
&\le \alpha_{n-1}\gamma f(x_{n-1}) - \alpha_{n-1}Bu_{n-1} + \delta_{n-1}x_{n-1} - \delta_{n-1}u_{n-1}\| + \|u_{n-1} - u_n\| \\
&\le \alpha_{n-1}\|\gamma f(x_{n-1}) - Bu_{n-1}\| + \delta_{n-1}\|x_{n-1} - u_{n-1}\| + \|u_{n-1} - u_n\|.
\end{aligned}$$

Thus, by (C1) and (C2), we obtain

$$\|x_n - u_n\| \to 0, n \to \infty. \tag{3.2.18}$$

For $v \in \mathfrak{F}$, using Lemma 2.6.18, we obtain

$$\begin{aligned}
\|y_n - v\|^2 &= \|F_{r_n}y_n - F_{r_n v}\|^2 \\
&\le \langle F_{r_n}x_n - F_{r_n}v, x_n - v\rangle \\
&\le \langle y_n - v, x_n - v\rangle \\
&= \frac{1}{2}(\|y_n - v\|^2 + \|x_n - v\|^2 - \|x_n - y_n\|^2)
\end{aligned}$$

and

$$\|y_n - v\|^2 \le \|x_n - v\|^2 - \|x_n - y_n\|^2. \tag{3.2.19}$$

Therefore, from (3.2.1), the convexity of $\|\cdot\|^2$, (3.2.2) and (3.2.19), we get

$$\begin{aligned}
\|x_{n+1} - v\|^2 &= \|\alpha_n\gamma f(x_n) + \delta_n x_n + [(1-\delta_n)I - \alpha_n B]T_{r_n}y_n - v\|^2 \\
&= \|(1-\delta_n)(T_{r_n}y_n - v) + \delta_n(x_n - v) + \alpha_n(\gamma f(x_n) - BT_{r_n}y_n)\|^2 \\
&\le \|(1-\delta_n)(T_{r_n}y_n - v) + \delta_n(x_n - v)\|^2 + 2\alpha_n\langle\gamma f(x_n) - BT_{r_n}y_n, x_{n+1} - v\rangle \\
&\le (1-\delta_n)\|(y_n - v)\|^2 + \delta_n\|(x_n - v)\|^2 + 2\alpha_n L^2
\end{aligned}$$

where L is constant such that $L = \sup\|\gamma f(x_n) - BT_{r_n}y_n, x_{n+1} - v\|$ and hence

$$(1-\delta_n)\|(y_n - v)\|^2 \le \delta_n\|(x_n - v)\|^2 - \|(x_{n+1} - v)\|^2 + 2\alpha_n L^2. \tag{3.2.20}$$

So, we have $\|y_n - v\| \to 0$ as $n \to \infty$. Consequently, from (3.2.17) and (3.2.19), we obtain

$$\|u_n - y_n\| \leq \|u_n - x_n\| + \|x_n - y_n\| \to 0 \text{ as } n \to \infty.$$

Step 3 We will show that

$$\limsup_{n\to\infty} \langle (\gamma f - B)z, x_n - z\rangle \leq 0. \tag{3.2.21}$$

Let $Q = P_{\mathfrak{F}}$, and since, $Q(I - B + \gamma f)$ is contraction on H into C (see also [65]:p.18) and H is complete. Thus, by Banach Contraction Principle, then there exist a unique element z of H such that $z = Q(I - B + \gamma f)z$.
We choose subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle (\gamma f - B)z, x_n - z\rangle = \lim_{n\to\infty} \langle \gamma f z - Bz, x_{n_i} - z\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a sequence $\{x_{n_{ij}}\}$ of $\{x_{n_i}\}$ and $y \in C$ such that $\{x_{n_{ij}}\} \rightharpoonup y$. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup y$. Since C is closed and convex it is weakly closed and hence $y \in C$. Since $x_n - y_n \to 0$ as $n \to \infty$ we have that $y_{n_i} \rightharpoonup y$. Now, we show that $y \in \mathfrak{F}$ Since $y_n = F_{r_n}$, Lemma 2.6.18 and using (3.2.6), we get

$$\langle y - y_n, Ay_n\rangle + \left\langle y - y_n, \frac{y_n - x_n}{r_n}\right\rangle \geq 0, \quad \forall y \in C, \tag{3.2.22}$$

and

$$\langle y - y_{n_i}, Ay_{n_i}\rangle + \left\langle y - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \geq 0, \quad \forall y \in C. \tag{3.2.23}$$

Set $v_t = tv + (1-t)y$ for all $t \in (0, 1]$ and $v \in C$. Consequently, we get $v_i \in C$. From (3.2.19), it follows that

$$\begin{aligned} \langle v_t - y_{n_i}\rangle &\geq \langle v_t - y_{n_i}, Av_t\rangle - \langle v_t - y_{n_i}, Av_t\rangle - \left\langle v_t - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_n}\right\rangle \\ &= \langle v_t - y_{n_i}, Av_t - Ay_{n_i}\rangle - \left\langle v_t - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_n}\right\rangle, \end{aligned}$$

from the fact that $y_{n_i} - x_{n_i} \to 0$ as $i \to \infty$ we obtain that $\frac{u_{n_i} - x_{n_i}}{r_n} \to 0$ as $i \to \infty$. Since A is monotone, we also have that $\langle v_t - y_{n_i}, Av_t - Ay_{n_i}\rangle \geq 0$. Thus, if follows that

$$0 \leq \lim_{i\to\infty} \langle v_t - y_{n_i}, Av_t\rangle = \langle v_t - w, Av_t\rangle,$$

and hence $\langle v-y, Av_t\rangle \geq 0, \quad \forall v \in C$.

If $t \to 0$, the continuity of A gives that

$$\langle v-y, Ay\rangle \geq 0, \quad \forall v \in C.$$

This implies that $y \in VI(C,A)$.

Furthermore, since $u_n = T_{r_n} y_n$, Lemma 2.6.18 and using (3.2.9), we get

$$\langle y-u_{n_i}, Tu_{n_i}\rangle - \frac{1}{r_n}\langle y-u_{n_i}, (r_{n_i}+1)u_{n_i} - y_{n+1}\rangle \leq 0, \quad \forall y \in C. \tag{3.2.24}$$

Put $z_t = t(v) + (1-t)y$ for all $t \in (0,1]$ and $v \in C$. Then, $z_t \in C$ and from (3.2.20) and pseudo-contractivity of T, we get

$$\begin{aligned} \|u_{n_i} - z_t, Tz_t\| &= \langle u_{n_i} - z_t, Tz_t\rangle + \langle z_t - u_{n_i}, Tu_i\rangle - \frac{1}{r_n}\langle z_t - u_{n_i}, (1+r_{n_i})u_{n_i} - y_{n_i}\rangle \\ &= -\langle z_t - u_{n_i}, Tz_t\rangle - \frac{1}{r_{n_i}}\langle z_t - u_{n_i}, u_{n_i} - y_{n_i}\rangle - \langle z_t - u_{n_i}, u_{n_i}\rangle \\ &\geq \|z_t - u_{n_i}\|^2 - \frac{1}{r_{n_i}}\langle z_t - u_{n_i}, u_{n_i} - y_{n_i}\rangle - \langle z_t - u_{n_i}, u_{n_i}\rangle \\ &= -\langle z_t - u_{n_i}, z_t\rangle - \left\langle z_t - u_{n_i}, \frac{u_{n_i} - y_{n_i}}{r_{n_i}}\right\rangle. \end{aligned}$$

Thus, since $u_n - y_n \to 0$, as $n \to \infty$ we obtain that $\frac{u_{n_i} - y_{n_i}}{r_{n_i}} \to 0$ as $i \to \infty$.

Therefore, as $i \to \infty$, it follows that

$$\langle y - z_t, Tz_t\rangle \geq \langle y - z_t, z_t\rangle$$

and hence

$$-\langle v-y, Tz_t\rangle \geq -\langle v-y, z_t\rangle, \quad \forall v \in C.$$

Taking $t \to 0$ and since T is continuous we obtain

$$-\langle v-y, Ty\rangle \geq -\langle v-y, y\rangle, \quad \forall v \in C.$$

Now, we get $v = Ty$. Then we obtain that $y = Ty$ and hence $y \in F(T)$. Therefore, $y \in F(T) \cap VI(C,A)$ and since $z = P_{\mathfrak{F}}(I - B + \gamma f)z$, Lemma 2.6.16 implies that

$$\begin{aligned} \limsup_{n\to\infty}\langle(\gamma f - B)z, x_n - z\rangle &= \lim_{i\to\infty}\langle(I - B + \gamma f)z - z, x_{n_i} - z\rangle \\ &= \langle(\gamma f - B)z, y - z\rangle \leq 0. \end{aligned} \tag{3.2.25}$$

Step 4 Finally, we will show that $x_n \to z$ as $n \to \infty$, where $z = P_{\mathfrak{F}}(I - B + rf)z$.

From (3.2.1) and (3.2.2) we observe that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \langle \alpha_n \gamma f(x_n) + \delta_n x_n + [(1-\delta_n)I - \alpha_n B]T_{r_n} y_n - z, x_{n+1} - z\rangle \\
&= \alpha_n \langle \gamma f(x_n) - Bz, x_{n+1} - z\rangle + \delta_n \langle x_n - z, x_{n+1} - z\rangle \\
&\quad + \langle [(1-\delta_n)I - \alpha_n B](T_{r_n} - z), x_{n+1} - z\rangle \\
&\leq \alpha_n \gamma \langle f(x_n) - f(z), x_{n+1} - z\rangle + \alpha_n \langle \gamma f(z) - Bz, x_{n+1} - z\rangle \\
&\quad + \delta_n \|x_n - z\|\|x_{n+1} - z\| + (1 - \delta_n - \alpha_n \overline{\beta})\|z_n - z\|\|x_{n+1} - z\| \\
&\leq \alpha_n \gamma K \|x_n - z\|\|x_{n+1} - z\| + \alpha_n \langle \gamma f(z) - Bz, x_{n+1} - z\rangle \\
&\quad + \delta_n \|x_n - z\|\|x_{n+1} - z\| + (1 - \delta_n - \alpha_n \overline{\beta})\|z_n - z\|\|x_{n+1} - z\| \\
&= \alpha_n \gamma K \|x_n - z\|\|x_{n+1} - z\| + \alpha_n \langle \gamma f(z) - Bz, x_{n+1} - z\rangle \\
&\quad + (1 - \alpha_n \overline{\beta})\|x_n - z\|\|x_{n+1} - z\| \\
&\leq \frac{\gamma k}{2}\alpha_n(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \alpha_n \langle \gamma f(z) - Bz, x_{n+1} - z\rangle \\
&\quad + (1 - \alpha_n \overline{\beta})(\|x_n - z\|\|x_{n+1} - z\|) \\
&\leq \frac{\gamma k}{2}\alpha_n(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \alpha_n \langle \gamma f(z) - Bz, x_{n+1} - z\rangle \\
&\quad + \frac{(1 - \alpha_n \overline{\beta})}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\
&\leq \frac{1 - \alpha_n(\overline{\beta} - k\gamma)}{2}\|x_n - z\|^2 + \frac{1}{2}\|x_{n+1} - z\|^2 \\
&\quad + \alpha_n \langle \gamma f(x) - Bz, x_{n+1} - z\rangle,
\end{aligned}$$

which implies that

$$\|x_{n+1} + z\|^2 \leq [1 - \alpha_n(\overline{\beta} - k\gamma)]\|x_n - z\|^2 + 2\alpha_n \langle \gamma f(z) - Bz, x_{n+1} - z\rangle.$$

By the condition (*C1*), (3.2.25) and using Lemma 2.1.23, we see that $\lim_{n\to\infty} \|x_n - z\| = 0$. This complete the proof. □

CHAPTER 4 ITERATIVE ALGORITHMS FOR SOLVING THE SYSTEM OF MIXED EQUILIBRIUM, FIXED POINT AND VARIATIONAL INCLUSIONS PROBLEMS

4.1 Strong Convergence Theorem for Inverse-Strongly Monotone operators

In this section, we prove a strong convergence theorem for solving a common solution of the set of solutions of fixed point for an infinite family of nonexpansive mappings, the set of solution of a system of mixed equilibrium problems and the set of solutions of the variational inclusion for an β-inverse-strongly monotone mapping in a real Hilbert space. we show for some application and numerical example.

Theorem 4.1.1. *Let H be a real Hilbert space, C a close convex subset of H and B be an β-inverse-strongly monotone operator. Let $\varphi : C \longrightarrow \mathbb{R}$ be a convex and lower semicontinuous function, $f : C \longrightarrow C$ be a contraction mapping with coefficient α $(0 < \alpha < 1)$, $M : H \longrightarrow 2^H$ be a maximal monotone operator. Let A be a strongly positive linear bounded operator of H into itself with coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$ and $\lambda \in (0, 2\beta)$. Let $\{T_n\}$ be a family of nonexpansive mappings of H into itself such that*

$$\theta := \cap_{n=1}^{\infty} F(T_n) \cap (\cap_{k=1}^{N} SMEP(F_k)) \cap I(B, M) \neq \emptyset.$$

Suppose that $\{x_n\}$ is a sequence generated by the following algorithm for $x_0 \in C$ arbitrarily and

$$\begin{cases} u_n = K_{r_n,n}^{F_N} \cdot K_{r_{n-1},n}^{F_{N-1}} \cdot K_{r_{n-2},n}^{F_{N-2}} \cdot \ldots \cdot K_{r_2,n}^{F_2} \cdot K_{r_1,n}^{F_1} \cdot x_n, \ \forall n \in N \\ x_{n+1} = P_C[\epsilon_n \gamma f(x_n) + (I - \epsilon_n A) W_n J_{M,\lambda}(u_n - \lambda B u_n)] \end{cases} \tag{4.1.1}$$

for all n = 1,2,3,..., where

$$K^{F_i}_{r_{i,n}}(x) = \{u_n \in C := F_i(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_{i,n}}\langle y - u_n, u_n - x_n\rangle \geq 0,$$
$$\forall y \in C\},\ i = 1, 2, 3, \ldots, N,$$

and the following conditions are satisfied

(C1): $\{\epsilon_n\} \subset (0,1),\ \lim_{n\to 0} \epsilon_n = 0,\ \sum_{n=1}^{\infty} \epsilon_n = \infty,\ \sum_{n=1}^{\infty} |\epsilon_{n+1} - \epsilon_n| < \infty;$

(C2): $\{r_{k,n}\} \subset [c, d]$ *with* $c, d \in (0, 2\sigma)$ *and* $\sum_{n=1}^{\infty} |r_{k,n+1} - r_{k,n}| < \infty.$

Then, the sequence $\{x_n\}$ *converges strongly to* $q \in \theta$*, where* $q = P_\theta(\gamma f + I - A)(q)$ *which solves the following variational inequality:*

$$\langle(\gamma f - A)q, p - q\rangle \leq 0, \quad \forall p \in \theta, \tag{4.1.2}$$

which is the optimality condition for the minimization problem

$$\min_{q\in\theta} \frac{1}{2}\langle Aq, q\rangle - h(q), \tag{4.1.3}$$

where h is a potential function for γf *(i.e.,* $h'(q) = \gamma f(q)$ *for* $q \in H$*).*

Proof. Since condition (C1), we may assume without loss of generality, then $\epsilon_n \in (0, \| A \|^{-1})$ for all n. By Lemma 2.6.9 we have $\|I - \epsilon_n A\| \leq 1 - \epsilon_n \bar{\gamma}$. Next, we will assume that $\|I - A\| \leq \|1 - \bar{\gamma}\|$.

Next, we will divide the proof into six steps.

Step 1. First, will show that $\{x_n\}$ and $\{u_n\}$ are bounded. Since B is β - inverse strongly monotone mappings, we have

$$\begin{aligned}
\|(I - \lambda B)x - (I - \lambda B)y\|^2 &= \|Ix - \lambda Bx - Iy + \lambda By\|^2 \\
&= \|x - y - \lambda Bx + \lambda By\|^2 \\
&= \|(x - y) - \lambda(Bx + By)\|^2 \\
&\leq \|x - y\|^2 - 2\lambda\langle x - y\rangle\langle Bx + By\rangle \\
&\quad +\lambda^2\|Bx - By\|^2 \\
&\leq \|x - y\|^2 - 2\lambda\beta\|Bx + By\|^2 \\
&\quad +\lambda^2\|Bx - By\|^2 \\
&\leq \|x - y\|^2 + \lambda(\lambda - 2\beta\|Bx + By\|^2 \\
&\leq \|x - y\|^2
\end{aligned} \tag{4.1.4}$$

if $0 < \lambda < 2\beta$ *and* $0 < r_n < 2\sigma$, *then* $I - \lambda B$ is all nonexpansive.

Put $y_n = J_{M,\lambda}(u_n - \lambda Bu_n),\ n \geq 0$. It follows that

$$\begin{aligned} \|y_n - q\| &= \|J_{M,\lambda}(u_n - \lambda Bu_n) - J_{M,\lambda}(q - \lambda Bq)\| \\ &\leq \|(u_n - \lambda Bu_n) - (q - \lambda Bq)\| \\ &\leq \|u_n - q\|. \end{aligned} \tag{4.1.5}$$

By Lemma 2.6.13, we have

$$\begin{aligned} u_n &= K^{F_N}_{r_n,n} \cdot K^{F_{N-1}}_{r_{n-1},n} \cdot K^{F_{N-2}}_{r_{n-2},n} \cdot \ldots \cdot K^{F_2}_{r_2,n} \cdot K^{F_1}_{r_1,n} \cdot x_n,\ \ for\ n \geq 0 \\ \tau^k_n &= K^{F_k}_{r_k,n} \cdot K^{F_{k-1}}_{r_{k-1},n} \cdot \ldots \cdot K^{F_2}_{r_2,n} \cdot K^{F_1}_{r_1,n},\quad for\ k \in \{0, 1, 2, ..., N\} \end{aligned}$$

and $\tau^0_n = I$ for all $n \in N$, $\quad q = \tau^{F_k}_{r_k,n} q, \quad u_n = \tau^N_{r_k,N} x_n$ Then, we have

$$\begin{aligned} \|u_n - q\|^2 &= \|\tau^N_{r_k,n} x_n - \tau^{F_k}_{r_k,n} q\|^2 \\ &= \|x_n - q\|^2. \end{aligned} \tag{4.1.6}$$

Hence, we get

$$\|y_n - q\| \leq \|x_n - q\|. \tag{4.1.7}$$

From (4.1.1), we deduce that

$$\begin{aligned} \|x_{n+1} - q\| &= \|P_C(\epsilon_n \gamma f(x_n) + (I - \epsilon_n A)W_n y_n) - P_C q\| \\ &\leq \|\epsilon_n(\gamma f(x_n) - Aq) + (I - \epsilon_n A)(W_n y_n - q)\| \\ &\leq \epsilon_n \|\gamma f(x_n) - Aq)\| + (1 - \epsilon_n \bar{\gamma})\|(y_n)) - q\| \\ &\leq \epsilon \gamma \epsilon_n \|x_n - q\| + \epsilon_n \|\gamma f(g) - Aq\| \\ &\quad + (1 - \epsilon_n \bar{\gamma})\|x_n - q\| \qquad (4.1.8) \\ &= (1 - (\bar{\gamma} - \gamma\epsilon)\epsilon_n)\|x_n - q\| - \epsilon_n \|\gamma f(q) - Aq\| \\ &= (1 - (\bar{\gamma} - \gamma\epsilon)\epsilon_n)\|x_n - q\| + (\bar{\gamma} - \gamma\epsilon)\epsilon_n \frac{\|\gamma f(q) - Aq\|}{\bar{\gamma} - \gamma\epsilon} \\ &\quad \vdots \\ &\leq \max\Big\{\|x_n - q\|,\ \frac{\|\gamma f(q) - Aq\|}{\bar{\gamma} - \gamma\epsilon}\Big\}. \end{aligned}$$

It follows by induction that

$$\|x_n - q\| \leq \max\{\|x_0 - q\|,\ \frac{\|\gamma f(q) - Aq\|}{\bar{\gamma} - \gamma\epsilon}\}, n \geq 0. \tag{4.1.9}$$

Therefore $\{x_n\}$ is bounded, so are $\{y_n\},\ \{Bu_n\},\ \{f(x_n)\}$ and $\{AW_ny_n\}$.

Step 2. We claim that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ and $\lim_{n\to\infty}\|y_{n+1}-y_n\|=0$. From (4.1.1), we have

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|P_C(\epsilon_n\gamma f(x_n)+(I-\epsilon_nA)W_ny_n)-P_C(\epsilon_{n-1}\gamma f(x_{n-1})\\ &\quad +(I-\epsilon_{n-1}A)W_ny_{n-1}\| \qquad (4.1.10)\\ &\leq \|(I-\epsilon_nA)(W_ny_n-W_ny_{n-1})-(\epsilon_n-\epsilon_{n-1})AW_ny_{n-1}+\\ &\quad \gamma\epsilon_n(f(x_n)-f(x_{n-1}))+\gamma(\epsilon_n-\epsilon_{n-1})f(x_{n-1})\|\\ &\leq (1-\epsilon_n\bar{\gamma})\|y_n-y_{n-1}\|+|\epsilon_n-\epsilon_{n-1}|\|AW_ny_n\|+\gamma\epsilon\epsilon_n\|x_n-x_{n-1}\|\\ &\quad +\gamma|\epsilon_n-\epsilon_{n-1}|\|f(x_{n-1})\|. \qquad (4.1.11)\end{aligned}$$

Since $I-\lambda B$ are nonexpansive, we also have

$$\begin{aligned}\|y_n-y_{n-1}\| &= \|J_{M,\lambda}(u_n-\lambda Bu_n)-J_{M,\lambda}(u_{n-1}-\lambda Bu_{n-1})\|\\ &\leq \|(u_n-\lambda Bu_n)-(u_{n-1}-\lambda Bu_{n-1})\| \qquad (4.1.12)\\ &\leq \|u_n-u_{n-1}\|.\end{aligned}$$

On the other hand, from $u_{n-1}=\tau^N_{r_k,n-1}x_{n-1}$ and $u_n=\tau^N_{r_k,n}x_n$, it follows that

$$F(u_{n-1},y)+\varphi(y)-\varphi(u_{n-1})+\frac{1}{r_{n-1}}\langle y-u_{n-1},u_{n-1}-x_{n-1}\rangle \geq 0, \quad \forall y\in C \qquad (4.1.13)$$

and

$$F(u_n,y)+\varphi(y)-\varphi(u_n)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \geq 0, \quad \forall y\in C. \qquad (4.1.14)$$

Substituting $y=u_n$ into (4.1.13) and $y=u_{n-1}$ into (4.1.14), we get

$$F(u_{n-1},u_n)+\varphi(u_n)-\varphi(u_{n-1})+\frac{1}{r_{n-1}}\langle u_n-u_{n-1},u_{n-1}-x_{n-1}\rangle \geq 0 \qquad (4.1.15)$$

and

$$F(u_n,u_{n-1})+\varphi(u_{n-1})-\varphi(u_n)+\frac{1}{r_n}\langle u_{n-1}-u_n,un-x_n\rangle \geq 0. \qquad (4.1.16)$$

From (A2), we obtain

$$\langle u_n-u_{n-1},\frac{u_{n-1}-x_{n-1}}{r_{n-1}}-\frac{u_n-x_n}{r_n}\rangle \geq 0, \qquad (4.1.17)$$

and then

$$\langle u_n - u_{n-1}, u_{n-1} - x_{n-1} - \frac{r_{n-1}}{r_n}(u_n - x_n)\rangle \geq 0, \tag{4.1.18}$$

so

$$\langle u_n - u_{n-1}, u_{n-1} - u_n + u_n - x_{n-1} - \frac{r_{n-1}}{r_n}(u_n - x_n)\rangle \geq 0. \tag{4.1.19}$$

It follows that

$$\langle u_n - u_{n-1}, u_{n-1} - u_n + u_n - x_n - \frac{r_{n-1}}{r_n}(u_n - x_n)\rangle \geq 0,$$

$$\langle u_n - u_{n-1}, u_{n-1} - u_n\rangle + \langle u_n - u_{n-1}, (1 - \frac{r_{n-1}}{r_n})(u_n - x_n)\rangle \geq 0. \tag{4.1.20}$$

Without loss of generality, let us assume that there exists a real number c such that $r_{n-1} > c > 0$, for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned} \|u_n - u_{n-1}\|^2 &\leq \Big\langle u_n - u_{n-1}, \Big(1 - \frac{r_{n-1}}{r_n}\Big)(u_n - x_n)\Big\rangle \\ &\leq \|u_n - u_{n-1}\|\Big\{|1 - \frac{r_{n-1}}{r_n}\Big|\|u_n - x_n\|\Big\} \end{aligned}$$

and hence

$$\begin{aligned} \|u_n - u_{n-1}\| &\leq \|x_n - x_{n-1}\| + \frac{1}{r_n}|r_n - r_{n-1}|\|u_n - x_n\| \\ &\leq \|x_n - x_{n-1}\| + \frac{M_1}{c}|r_n - r_{n-1}|, \end{aligned} \tag{4.1.21}$$

where $M_1 = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$. Substituting(4.1.21) into (4.1.12), we have

$$\|y_n - y_{n-1}\| \leq \|x_n - x_{n-1}\| + \frac{M_1}{c}|r_n - r_{n-1}|. \tag{4.1.22}$$

Substituting 4.1.22 into 4.1.10, we get

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq (1 - \epsilon_n\bar{\gamma})\Big(\|x_n - x_{n-1}\| + \frac{M_1}{c}|r_n - r_{n-1}|\Big) \\ &\quad + |\epsilon_n - \epsilon_{n-1}|\|AW_n y_{n-1}\| + \gamma\epsilon\epsilon_n\|x_n - x_{n-1}\| \\ &\quad + \gamma|\epsilon_n - \epsilon_{n-1}|\|f(x_{n-1})\| \\ &= (1 - \epsilon_n\bar{\gamma})\|x_n - x_{n-1}\| + (1 - \epsilon_n\bar{\gamma})\frac{M_1}{c}|r_n - r_{n-1}| \\ &\quad + |\epsilon_n - \epsilon_{n-1}|\|AW_n y_{n-1}\| \\ &\quad + \gamma\epsilon\epsilon_n\|x_n - x_{n-1}\| + \gamma|\epsilon_n - \epsilon_{n-1}|\|f(x_{n-1})\| \qquad (4.1.23) \\ &\leq (1 - (\bar{\gamma} - \gamma\epsilon)\epsilon_n)\|x_n - x_{n-1}\| + \frac{M_1}{c}|r_n - r_{n-1}| \\ &\quad + |\epsilon_n - \epsilon_{n-1}|\|AW_n y_{n-1}\| + \gamma|\epsilon_n - \epsilon_{n-1}|\|f(x_{n-1})\| \\ &\leq (1 - (\bar{\gamma} - \gamma\epsilon)\epsilon_n)\|x_n - x_{n-1}\| + \frac{M_1}{c}|r_n - r_{n-1}| + M_2|\epsilon_n - \epsilon_{n-1}|, \end{aligned}$$

where $M_2 = \sup\big\{\max\{\|AW_n y_{n-1}\|, \|f(x_{n-1})\| : n \in \mathbb{N}\}\big\}$. Since conditions (C1)-(C2) by Lemma 2.1.23, we have $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. From (4.1.22), we also have that $\|y_{n+1} - y_n\| \to 0$ as $n \to \infty$.

Step 3. Next, we show that $\lim_{n\to\infty} \|Bu_n - Bq\| = 0$.

For $q \in \theta$ and $q = J_{M,\lambda}(q - \lambda Bq)$. By (4.1.4) and (4.1.6), we get

$$\begin{aligned}
\|y_n - q\|^2 &= \|J_{M,\lambda}(u_n - \lambda Bu_n) - J_{M,\lambda}(q - \lambda Bq)\|^2 \\
&\leq \|(u_n - \lambda Bu_n) - (q - \lambda Bq)\|^2 \\
&\leq \|u_n - q\|^2 + \lambda(\lambda - 2\beta)\|Bu_n - Bq\|^2 \\
&\leq \|x_n - q\|^2 + \lambda(\lambda - 2\beta)\|Bu_n - Bq\|^2. \qquad (4.1.24)
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|P_C(\epsilon_n \gamma f(x_n) + (I - \epsilon_n A)W_n y_n) - P_C(q)\|^2 \\
&\leq \|\epsilon_n(\gamma f(x_n) - Aq) + (I - \epsilon_n A)(W_n y_n - q)\|^2 \\
&\leq \big(\epsilon_n \|\gamma f(x_n) - Aq\| + (1 - \epsilon_n \bar{\gamma})\|y_n - q\|\big)^2 \\
&\leq \epsilon_n \|\gamma f(x_n) - Aq\|^2 + (1 - \epsilon_n \bar{\gamma})\|y_n - q\|^2 \\
&\quad + 2\epsilon_n(1 - \epsilon_n \bar{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\| \qquad (4.1.25) \\
&\leq \epsilon_n \|\gamma f(x_n) - Aq\|^2 + 2\epsilon_n(1 - \epsilon_n \bar{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\| \\
&\quad + (1 - \epsilon_n \bar{\gamma})\Big(\|x_n - q\|^2 + \lambda(\lambda - 2\beta)\|Bu_n - Bq\|^2\Big) \\
&\leq \epsilon_n \|\gamma f(x_n) - Aq\|^2 + 2\epsilon_n(1 - \epsilon_n \bar{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\| \\
&\quad + \|x_n - q\|^2 + (1 - \epsilon_n \bar{\gamma})\lambda(\lambda - 2\beta)\|Bu_n - Bq\|^2.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
&(1 - \epsilon_n \bar{\gamma})\lambda(2\beta - \lambda)\|Bu_n - Bq\|^2 \\
\leq\ &\epsilon_n \|\gamma f(x_n) - Aq\|^2 + \|x_n - x_{n+1}\|\big(\|x_n - q\| + \|x_{n+1} - q\|\big) + \xi_n, \qquad (4.1.26)
\end{aligned}$$

where $\xi_n = 2\epsilon_n(1 - \epsilon_n \bar{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\|$. By conditions (C1),(C3) and $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, then we obtain that $\|Bu_n - Bq\| \to 0$ as $n \to \infty$.

Step 4. We show the followings:

(i) $\lim_{n\to\infty} \|x_n - u_n\| = 0$;

(ii) $\lim_{n\to\infty} \|u_n - y_n\| = 0$;

(iii) $\lim_{n\to\infty} \|y_n - W_n y_n\| = 0$.

Since $K_{r_n}(x)$ is firmly nonexpansive and Lemma 2.1.12(iii), we observe that

$$\begin{aligned}
\|u_n - q\|^2 &= \|\tau^N_{r_n,n}\, x_n - \tau^N_{r_n,n}\, q\|^2 \\
&\leq \langle x_n - q, u_n - q\rangle \\
&= \frac{1}{2}\Big(\|x_n - q\|^2 + \|u_n - q\|^2 - \|x_n - q - u_n - q\|^2\Big) \qquad (4.1.27) \\
&\leq \frac{1}{2}\Big(\|x_n - q\|^2 + \|u_n - q\|^2 - \|x_n - u_n\|^2\Big)
\end{aligned}$$

it follows that

$$\|u_n - q\|^2 \leq \|x_n - q\|^2 - \|x_n - u_n\|^2.$$

Since $J_{M,\lambda}$ is 1-inverse-strongly monotone and by Lemma 2.1.12(iii), we compute

$$\begin{aligned}
\|y_n - q\|^2 &= \|J_{M,\lambda}(u_n - \lambda Bu_n) - J_{M,\lambda}(q - \lambda Bq)\|^2 \\
&\leq \langle (u_n - \lambda Bu_n) - (q - \lambda Bq), y_n - q\rangle \\
&= \frac{1}{2}\Big(\|(u_n - \lambda Bu_n) - (q - \lambda Bq)\|^2 + \|y_n - q\|^2 \\
&\quad -\|(u_n - \lambda Bu_n) - (q - \lambda Bq) - (y_n - q)\|^2\Big) \qquad (4.1.28) \\
&\leq \frac{1}{2}\Big(\|u_n - q\|^2 + \|y_n - q\|^2 - \|(u_n - y_n) - \lambda(Bu_n - Bq)\|^2\Big) \\
&= \frac{1}{2}\Big(\|u_n - q\|^2 + \|y_n - q\|^2 - \|u_n - y_n\|^2 \\
&\quad +2\lambda\langle u_n - y_n, Bu_n - Bq\rangle - \lambda^2\|Bu_n - Bq\|^2\Big),
\end{aligned}$$

which implies that

$$\|y_n - q\|^2 \leq \|u_n - q\|^2 - \|u_n - y_n\|^2 + 2\lambda\|u_n - y_n\|\|Bu_n - Bq\|. \qquad (4.1.29)$$

Substitute (4.1.29) into (4.1.25), we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \epsilon_n\|\gamma f(x_n) - Aq\|^2 + \|y_n - q\|^2 + 2\epsilon_n(1 - \epsilon_n\bar{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\| \\
&\leq \epsilon_n\|\gamma f(x_n) - Aq\|^2 \\
&\quad +\Big(\|u_n - q\|^2 - \|u_n - y_n\|^2 + 2\lambda_n\|u_n - y_n\|\|Bu_n - Bq\|\Big) \\
&\quad +2\epsilon_n(1 - \epsilon_n\bar{\gamma})\|\gamma f(x_n) - Aq\|\|y_n - q\|. \qquad (4.1.30)
\end{aligned}$$

Then, we derive

$$\begin{aligned}
\|x_n-u_n\|^2+\|u_n-y_n\|^2 &\le \epsilon_n\|\gamma f(x_n)-Aq\|^2+\|x_n-q\|^2-\|x_{n+1}-q\|^2 \\
&\quad +2\lambda\|u_n-y_n\|\|Bu_n-Bq\| \\
&\quad +2\epsilon_n(1-\epsilon_n\bar{\gamma})\|\gamma f(x_n)-Aq\|\|y_n-q\| \\
&= \epsilon_n\|\gamma f(x_n)-Aq\|^2 \\
&\quad +\|x_n-x_{n+1}\|(\|x_n-q\|+\|x_{n+1}-q\|) \\
&\quad +2\lambda\|u_n-y_n\|\|Bu_n-Bq\| \\
&\quad +2\epsilon_n(1-\epsilon_n\bar{\gamma})\|\gamma f(x_n)-Aq\|\|y_n-q\|. \qquad (4.1.31)
\end{aligned}$$

By condition (C1), $\lim_{n\to\infty}\|x_n-x_{n+1}\|=0$ and $\lim_{n\to\infty}\|Bu_n-Bq\|=0$. So, we have $\|x_n-u_n\|\to 0,\ \|u_n-y_n\|\to 0$ as $n\to\infty$. It follows that

$$\|x_n-y_n\| \le \|x_n-u_n\|+\|u_n-y_n\|\to 0, \text{ as } n\to\infty. \qquad (4.1.32)$$

From (4.1.1), we have

$$\begin{aligned}
\|x_n-W_ny_n\| &\le \|x_n-W_ny_{n-1}\|+\|W_ny_{n-1}-W_ny_n\| \\
&\le \big\|P_C\big(\epsilon_{n-1}\gamma f(x_{n-1})+(I-\alpha_{n-1}A)W_ny_{n-1}\big)-P_C(W_ny_{n-1})\big\| \\
&\quad +\|y_{n-1}-y_n\| \qquad (4.1.33) \\
&\le \epsilon_{n-1}\|\gamma f x_{n-1}-AW_ny_{n-1}\|+\|y_{n-1}-y_n\|.
\end{aligned}$$

By condition (C1) and $\lim_{n\to\infty}\|y_{n-1}-y_n\|=0$, then we obtain that $\|x_n-W_ny_n\|\to 0$ as $n\to\infty$.

Hence, we have

$$\begin{aligned}
\|x_n-W_nx_n\| &\le \|x_n-W_ny_n\|+\|W_ny_n-W_nx_n\| \\
&\le \|x_n-W_ny_n\|+\|y_n-x_n\|. \qquad (4.1.34)
\end{aligned}$$

By (4.1.32) and $\lim_{n\to\infty}\|x_n-W_ny_n\|=0$ we obtain $\|x_n-W_nx_n\|\to 0$ as $n\to\infty$. Moreover, we also that

$$\|y_n-W_ny_n\|\le\|y_n-x_n\|+\|x_n-W_ny_n\|.$$

By (4.1.32) and $\lim_{n\to\infty}\|x_n-W_ny_n\|=0$ then we obtain $\|y_n-W_ny_n\|\to 0$ as $n\to\infty$.

Step 5. We show that $q \in \theta := \bigcap_{n=1}^{\infty} F(T_n) \cap (\bigcap_{k=1}^{N} SMEP(F_k))$ and $\limsup_{n\to\infty} \langle (\gamma f - A)q, W_n y_n - q\rangle \leq 0$. It is easy to see that $P_\theta(\gamma f + (I-A))$ is a contraction of H into itself. Indeed, since $0 < \gamma < \frac{\bar{\gamma}}{\epsilon}$, we have that

$$\begin{aligned} & \|P_\theta(\gamma f + (I-A))x - P_\theta(\gamma f + (I-A))y\| \\ \leq\ & \gamma\|f(x) - f(y)\| + \|I - A\|\|x - y\| \\ \leq\ & \gamma\epsilon\|x - y\| + (1-\bar{\gamma})\|x - y\| \\ \leq\ & (1 - \bar{\gamma} + \gamma\epsilon)\|x - y\|. \end{aligned} \tag{4.1.35}$$

Hence H is complete, there exists a unique fixed point $q \in H$ such that $q = P_\theta(\gamma f + (I-A))(q)$. By Lemma 2.1.12 we obtain that $\langle (\gamma f - A)q, w - q\rangle \leq 0$ for all $w \in \theta$.

Next, we show that $\limsup_{n\to\infty}\langle (\gamma f - A)q, W_n y_n - q\rangle \leq 0$, where $q = P_\theta(\gamma f + I - A)(q)$ is the unique solution of the variational inequality $\big\langle (\gamma f - A)q, w - q\big\rangle \geq 0,\ \ \forall w \in \theta$. We can choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle (\gamma f - A)q, W_n y_n - q\rangle = \lim_{i\to\infty}\langle (\gamma f - A)q, W_n y_{n_i} - q\rangle. \tag{4.1.36}$$

As $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to w. We may assume without loss of generality that $y_{n_i} \rightharpoonup w$.
We claim that $w \in \theta$. Since $\|y_n - W_n y_n\| \to 0$,$\|x_n - W_n x_n\| \to 0$ and $\|x_n - y_n\| \to 0$ and by Lemma 2.6.24, we have $w \in \bigcap_{n=1}^{\infty} F(T_n)$.

Next, we show that $w \in \bigcap_{k=1}^{\infty} SMEP(F_k)$.

Since $u_n = \tau_{r_k,n}^{N}\, x_n$, for k = 1, 2, 3, ..., N, we know that

$$F_k(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C. \tag{4.1.37}$$

It follows by (A2) that

$$\varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F_k(y, u_n), \quad \forall y \in C. \tag{4.1.38}$$

Hence, for k = 1, 2, 3, ..., N, we get

$$\varphi(y) - \varphi(u_{n_i}) + \frac{1}{r_{n_i}}\langle y - u_{n_i}, u_{n_i} - x_{n_i}\rangle \geq F_k(y, u_{n_i}), \quad \forall y \in C. \tag{4.1.39}$$

For $t \in (0,1]$ and $y \in H$, let $y_t = ty + (1-t)w$. From (4.1.39), we have

$$0 \ \geq\ \varphi(y_t) + \varphi(u_{n_i}) - \frac{1}{r_{n_i}}\langle y_t - u_{n_i}, u_{n_i} - x_{n_i}\rangle + F_k(y_t, u_{n_i}) \tag{4.1.40}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, from (A4) and the weakly lower semicontinuity of φ, $\frac{(u_{n_i} - x_{n_i})}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup w$. From (A1), (A4) and we have

$$\begin{aligned}
0 &= F_k(y_t, y_t) - \varphi(y_t) + \varphi(y_t) \\
&\leq tF_k(y_t, y) + (1-t)F_k(y_t, w) + t\varphi(y) + (1-t)\varphi(w) - \varphi(y_t) \\
&\leq t[F_k(y_t, y) + \varphi(y) - \varphi(y_t)].
\end{aligned}$$

Deviding by t, we get

$$F_k(y_t, y) + \varphi(y) - \varphi(y_t) \geq 0.$$

The weakly lower semicontinuity of φ for k = 1, 2, 3, ..., N,

$$F_k(w, y) + \varphi(y) \geq \varphi(w).$$

So, we have

$$F_k(w, y) + \varphi(y) - \varphi(w) \geq 0, \ \forall k = 1, 2, 3, \ldots, N.$$

This implies that $w \in \bigcap_{k=1}^{N} SMEP(F_k)$.

Lastly, we show that $w \in I(B, M)$. In fact, since B is a β-inverse-strongly monotone, hence B is a monotone and Lipschitz continuous mapping. It follows from Lemma 2.6.22 that $M + B$ is a maximal monotone. Let $(v, g) \in G(M + B)$, since $g - Bv \in M(v)$. Again since $y_{n_i} = J_{M,\lambda}(u_{n_i} - \lambda B u_{n_i})$, we have $u_{n_i} - \lambda B u_{n_i} \in (I + \lambda M)(y_{n_i})$, that is, $\frac{1}{\lambda}(u_{n_i} - y_{n_i} - \lambda B u_{n_i}) \in M(y_{n_i})$. By virtue of the maximal monotonicity of $M + B$, we have

$$\langle v - y_{n_i}, g - Bv - \frac{1}{\lambda}(u_{n_i} - y_{n_i} - \lambda B u_{n_i})\rangle \geq 0,$$

and hence

$$\begin{aligned}
\langle v - y_{n_i}, g\rangle &\geq \left\langle v - y_{n_i}, Bv + \frac{1}{\lambda}(u_{n_i} - y_{n_i} - \lambda B u_{n_i})\right\rangle \\
&= \langle v - y_{n_i}, Bv - By_{n_i}\rangle + \langle v - y_{n_i}, By_{n_i} - Bu_{n_i}\rangle \qquad (4.1.41) \\
&\quad + \left\langle v - y_{n_i}, \frac{1}{\lambda}(u_{n_i} - y_{n_i})\right\rangle.
\end{aligned}$$

It follows from $\lim_{n\to\infty} \|u_n - y_n\| = 0$, we have $\lim_{n\to\infty} \|Bu_n - By_n\| = 0$ and $y_{n_i} \rightharpoonup w$ that

$$\limsup_{n\to\infty} \langle v - y_n, g\rangle = \langle v - w, g\rangle \geq 0. \qquad (4.1.42)$$

It follows from the maximal monotonicity of $B+M$ that $\theta \in (M+B)(w)$, that is, $w \in I(B,M)$. Therefore, $w \in \theta$. It follows that

$$\begin{aligned}
\limsup_{n\to\infty}\langle(\gamma f - A)q, W_n y_n - q\rangle &= \lim_{i\to\infty}\langle(\gamma f - A)q, W_n y_{n_i} - q\rangle && (4.1.43)\\
&= \langle(\gamma f - A)q, w - q\rangle \leq 0. && (4.1.44)
\end{aligned}$$

Step 6. Finally, we prove $x_n \to q$. By using (4.1.1) and together with Schwarz inequality, we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|P_C\big(\epsilon_n\gamma f(x_n) + (I - \epsilon_n A)W_n y_n\big) - P_C(q)\|^2\\
&\leq \|\epsilon_n(\gamma f(x_n) - Aq) + (I - \epsilon_n A)(W_n y_n - q)\|^2\\
&\leq (I - \epsilon_n A)^2\|(W_n y_n - q)\|^2 + \epsilon_n^2\|\gamma f(x_n) - Aq\|^2\\
&\quad +2\epsilon_n\langle(I - \epsilon_n A)(W_n y_n - q), \gamma f(x_n) - Aq\rangle\\
&\leq (1 - \epsilon_n\bar{\gamma})^2\|y_n - q\|^2 + \epsilon_n^2\|\gamma f(x_n) - Aq\|^2\\
&\quad +2\epsilon_n\langle W_n y_n - q, \gamma f(x_n) - Aq\rangle - 2\epsilon_n^2\langle A(W_n y_n - q), \gamma f(x_n) - Aq\rangle\\
&\leq (1 - \epsilon_n\bar{\gamma})^2\|x_n - q\|^2 + \epsilon_n^2\|\gamma f(x_n) - Aq\|^2 && (4.1.45)\\
&\quad +2\epsilon_n\langle W_n y_n - q, \gamma f(x_n) - \gamma f(q)\rangle\\
&\quad +2\epsilon_n\langle W_n y_n - q, \gamma f(q) - Aq\rangle - 2\epsilon_n^2\langle A(W_n y_n - q), \gamma f(x_n) - Aq\rangle\\
&\leq (1 - \epsilon_n\bar{\gamma})^2\|x_n - q\|^2 + \epsilon_n^2\|\gamma f(x_n) - Aq\|^2 && (4.1.46)\\
&\quad +2\epsilon_n\|W_n y_n - q\|\|\gamma f(x_n) - \gamma f(q)\|\\
&\quad +2\epsilon_n\langle W_n y_n - q, \gamma f(q) - Aq\rangle - 2\epsilon_n^2\langle A(W_n y_n - q), \gamma f(x_n) - Aq\rangle\\
&\leq (1 - \epsilon_n\bar{\gamma})^2\|x_n - q\|^2 + \epsilon_n^2\|\gamma f(x_n) - Aq\|^2 + 2\gamma\epsilon\epsilon_n\|y_n - q\|\|x_n - q\|\\
&\quad +2\epsilon_n\langle W_n y_n - q, \gamma f(q) - Aq\rangle - 2\epsilon_n^2\langle A(W_n y_n - q), \gamma f(x_n) - Aq\rangle\\
&\leq (1 - \epsilon_n\bar{\gamma})^2\|x_n - q\|^2 + \epsilon_n^2\|\gamma f(x_n) - Aq\|^2 + 2\gamma\epsilon\epsilon_n\|x_n - q\|^2\\
&\quad +2\epsilon_n\langle W_n y_n - q, \gamma f(q) - Aq\rangle - 2\epsilon_n^2\langle A(W_n y_n - q), \gamma f(x_n) - Aq\rangle\\
&\leq \big((1 - \epsilon_n\bar{\gamma})^2 + 2\gamma\epsilon\epsilon_n\big)\|x_n - q\|^2 + \epsilon_n\Big\{\epsilon_n\|\gamma f(x_n) - Aq\|^2\\
&\quad +2\langle W_n y_n - q, \gamma f(q) - Aq\rangle - 2\epsilon_n\|A(W_n y_n - q)\|\|\gamma f(x_n) - Aq\|\Big\}\\
&= \big(1 - 2(\bar{\gamma} - \gamma\epsilon)\epsilon_n\big)\|x_n - q\|^2 + \epsilon_n\Big\{\epsilon_n\|\gamma f(x_n) - Aq\|^2\\
&\quad +2\langle W_n y_n - q, \gamma f(q) - Aq\rangle - 2\epsilon_n\|A(W_n y_n - q)\|\ |\gamma f(x_n) - Aq\|\\
&\quad +\epsilon_n\bar{\gamma}^2\|x_n - q\|^2\Big\}. && (4.1.47)
\end{aligned}$$

Since $\{x_n\}$ is bounded, where $\eta \geq \|\gamma f(x_n) - Aq\|^2 - 2\|A(W_n y_n - q)\|\|\gamma f(x_n) -$

$Aq\| + \bar{\gamma}^2\|x_n - q\|^2$ for all $n \geq 0$. It follows that

$$\|x_{n+1} - q\|^2 \leq \big(1 - 2(\bar{\gamma} - \gamma\epsilon)\epsilon_n\big)\|x_n - q\|^2 + \epsilon_n\delta_n, \tag{4.1.48}$$

where $\delta_n = 2\langle W_n y_n - q, \gamma f(q) - Aq\rangle + \eta\alpha_n$. Since $\limsup_{n\to\infty}\langle(\gamma f - A)q, W_n y_n - q\rangle \leq 0$, we get $\limsup_{n\to\infty}\delta_n \leq 0$. Applying Lemma 2.1.23, we can conclude that $x_n \to q$. This completes the proof. □

4.2 Some Applications to Minimization Problems

In this section, we apply the iterative scheme (4.1.1) for finding a common fixed point of a nonexpansive mapping and a strictly pseudocontractive mapping.

Using Theorem 4.1.1, we first prove a strongly convergence theorem for finding a common fixed point of a nonexpansive mapping and a strictly pseudo-contraction, which is the solution of minimization problem.

Theorem 4.2.1. *Let H be a real Hilbert space, C be a closed convex subset of H and B be an β-inverse-strongly monotone, $\varphi : C \to \mathbb{R}$ is convex and lower semicontinuous function, $f : C \to C$ be a contraction with coefficient α $(0 < \alpha < 1)$ and A be a strongly positive linear bounded operator of H into itself with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\{T_n\}$ be a family of nonexpansive mappings of H into itself and let S be a κ-strictly pseudo-contraction of C into itself such that*

$$\theta := \cap_{n=1}^{\infty} F(T_n) \cap (\cap_{k=1}^{N} SMEP(F_k)) \cap F(S) \neq 0.$$

Suppose $\{x_n\}$ is a sequence generated by the following algorithm for $x_0, u_n \in C$ arbitrarily:

$$\begin{cases} u_n = K_{r_n,n}^{F_N} \cdot K_{r_{n-1},n}^{F_{N-1}} \cdot K_{r_{n-2},n}^{F_{N-2}} \cdot \ldots \cdot K_{r_2,n}^{F_2} \cdot K_{r_1,n}^{F_1} \cdot x_n, \ \forall n \in N \\ x_{n+1} = P_C[\epsilon_n\gamma f(x_n) + (I - \epsilon_n A)W_n(1-\lambda)x_n + \lambda S_{x_n}] \end{cases} \tag{4.2.1}$$

for all $n = 0, 1, 2, \ldots,$ and the conditions (C1)-(C2) in Theorem 4.1.1 are satisfied.

Then, the sequence $\{x_n\}$ converges strongly to $q \in \theta$, where $q = P_\theta(\gamma f + I - A)(q)$ which solves the following variational inequality:

$$\langle(\gamma f - A)q, p - q\rangle \leq 0, \ \forall p \in \theta$$

which is the optimality condition for the minimization problem

$$\min_{q\in\theta} \frac{1}{2}\langle Aq, q\rangle - h(q), \tag{4.2.2}$$

where h is a potential function for γf (i.e., $h'(q) = \gamma f(q)$ for $q \in H$).

Proof. Put $B \equiv I - T$, then B is $\frac{1-\kappa}{2}$ inverse-strongly monotone and $F(S) = I(B, M)$ and $J_{M,\lambda}(x_n - \lambda B x_n) = (1-\lambda)x_n + \lambda T x_n$. So by Theorem 4.1.1, we obtain the desired result. □

4.3 Numerical Example

Now, we give a real numerical example in which the condition satisfy the ones of theorem 4.1.1 and some numerical experiment results to explain the main result theorem 4.1.1 as follows:

Example. Let $H = R, C = [-1, 1], T_n = I,\ \lambda_n = \beta \in (0, 1),\ n \in N,\ F_k(x, y) = 0, \forall x, y \in C, r_{n,n} = 1,\ k \in \{1, 2, 3, ..., N\},\ \varphi(x) = 0,\ \forall x \in C,\ B = A = I,\ f(x) = \frac{1}{5}x,\ \forall x \in H,\ \lambda = \frac{1}{2}$ with contraction coefficient $\alpha = \frac{1}{10},\ \epsilon_n = \frac{1}{n}$ for every n $\in N$ and $\gamma = 1$. Then $\{x_n\}$ is the sequence generated by

$$x_{n+1} = (\frac{1}{2} - \frac{3}{10n})x_n \tag{4.3.1}$$

and $x_n \longrightarrow 0$ as $n \longrightarrow \infty$, where 0 is the unique solution of the minimization problem

$$\min_{x\in C} = \frac{2}{5}x^2 + q.$$

Proof. We prove the Example 4.3.1 by step 1, step 2, step 3. By step 4, we give two numerical experiment results which can directly explain the sequence $\{x_n\}$ strongly converges to 0.

Step 1. We show

$$K^{F_N}_{r_n,n}x = P_C x,\ \forall x \in H, F_N \in \{1, 2, 3, ..., N\}, \tag{4.3.2}$$

where

$$P_C x = \begin{cases} \frac{x}{|x|}, & x \in H \backslash C \\ x, & x \in C. \end{cases} \tag{4.3.3}$$

Indeed, since $F_k(x,y) = 0,\ \forall x, y \in C,\ n \in \{1, 2, 3, ..., N\}$, due to the definition of K_r(x), $\forall x \in H$, as lemma 2.6.13, we have

$$K_r(x) = \Big\{ u \in C : \langle y - u, u - x \rangle \geq 0, \forall y \in C \Big\}.$$

Also by the equivalent property (2.4.1) of the nearest projection P_C from $H \longrightarrow C$, we obtain this conclusion, when we take $x \in C$, $K^{F_N}_{r_n,n} x = P_C x = Ix$. By (iii) in lemma 2.6.13, we have

$$\bigcap_{k=1}^{N} SMEP(F_k) = C. \tag{4.3.4}$$

Step 2. We show

$$W_n = I. \tag{4.3.5}$$

Indeed. By (2.4.10), we have

$$W_1 = U_{11} = \lambda_1 T_1 U_{12} + (1 - \lambda_1) I = \lambda_1 T_1 + (1 - \lambda_1) I, \tag{4.3.6}$$

$$\begin{aligned} W_2 = U_{21} &= \lambda_1 T_1 U_{22} + (1 - \lambda_1) I = \lambda_1 T_1 (\lambda_2 T_2 U_{23} + (1 - \lambda_2) I) + (1 - \lambda_1) I \\ &= \lambda_1 \lambda_2 T_1 T_2 + \lambda_1 (1 - \lambda_2) T_1 + (1 - \lambda_1) I, \\ W_3 = U_{31} &= \lambda_1 T_1 U_{32} + (1 - \lambda_1) I = \lambda_1 T_1 (\lambda_2 T_2 U_{33} + (1 - \lambda_2) I) + (1 - \lambda_1) I \\ &= \lambda_1 \lambda_2 T_1 T_2 U_{33} + \lambda_1 (1 - \lambda_2) T_1 + (1 - \lambda_1) I, \\ &= \lambda_1 \lambda_2 T_1 T_2 (\lambda_3 T_3 U_{34} + (1 - \lambda_3 I)) + \lambda_1 (1 - \lambda_2) T_1 + (1 - \lambda_1) I, \\ &= \lambda_1 \lambda_2 \lambda_3 T_1 T_2 T_3 + \lambda_1 \lambda_2 (1 - \lambda_3 T_1 T_2 + \lambda_1 (1 - \lambda_2) T_1 + (1 - \lambda_1) I. \end{aligned}$$

Compute in this way by (2.4.10), we obtain

$$\begin{aligned} W_n = U_{n1} &= \lambda_1 \lambda_2 \cdots \lambda_n T_1 T_2 \cdots T_n + \lambda_1 \lambda_2 \cdots \lambda_{n-1} (1 - \lambda_n) T_1 T_2 \cdots T_{n-1} \\ &\quad + \lambda_1 \lambda_2 \cdots \lambda_{n-2} (1 - \lambda_{n-1}) T_1 T_2 \cdots T_{n-2} + \cdots + \lambda_1 (1 - \lambda_2) T_1 + (1 - \lambda_1) I. \end{aligned}$$

Since $T_n = I,\ \lambda_n = \beta,\ n \in N$, thus

$$W_n = [\beta^n + \beta^{n-1}(1-\beta) + \cdots + \beta(1-\beta) + (1-\beta)]I = I.$$

Step 3. We show

$$x_{n+1} = (\frac{1}{2} - \frac{3}{10n})x_n \ and\ x_{n+1} \longrightarrow 0,\ as\ n \longrightarrow \infty, \tag{4.3.7}$$

where 0 is the unique solution of the minimization problem

$$\min_{x\in C} = \frac{2}{5}x^2 + q.$$

Indeed, we can see A = I is a strongly position bounded linear operator with coefficient $\bar{\gamma} = \frac{1}{2}$, γ is a real number such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$, so we can take $\gamma = 1$. Due to (4.3.1), (4.3.3) and (4.3.5), we can obtain an special sequence $\{x_n\}$ of (4.1.1) in theorem 4.1.1 as follows:

$$x_{n+1} = (\frac{1}{2} - \frac{3}{10n})x_n.$$

Since $T_n = I,\ n \in N$, so,

$$\cap_{n=1}^{\infty} F(T_n) = H,$$

combining with (4.3.4), we have

$$\theta := \cap_{n=1}^{\infty} F(T_n) \cap (\cap_{k=1}^{N} SMEP(F_k)) \cap I(B, M) = C = [-1, 1].$$

By Lemma 2.1.23, it is obviously that $z_n \longrightarrow 0$, 0 is the unique solution of the minimization problem

$$\min_{x\in C} = \frac{2}{5}x^2 + q,$$

where q is a constant number.

Step 4. We give the numerical experiment results using software Mathlab 7.0 and get the table 1 to table 2, which show that the iteration process of the sequence $\{x_n\}$ is a monotone decreasing sequence and converges to 0, but the more the iteration steps are, the more showily the sequence $\{x_n\}$ converges to 0.

Now we turn to realizing (4.1.1) for approximating a fixed point of T. We take the initial valued $x_1 = 1$ and $x_1 = 1/2$, respectively. All the numerical results are given in Tables 4.1 and 4.2. The corresponding graph appears in Figure 4.1 (i) and (ii).

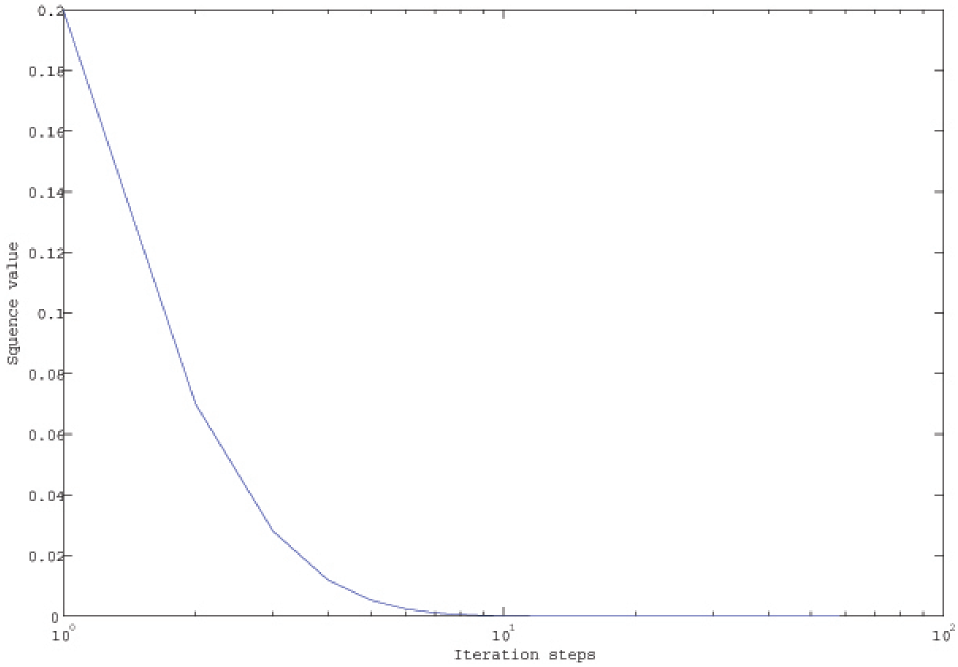
Table 4.1: This table shows the value of sequence $\{x_n\}$ on each iteration steps (initial value $x_1 = 1$)

n	x_n	n	x_n
1	1.000000000000000	31	0.000000000054337
2	0.200000000000000	32	0.000000000026643
3	0.070000000000000	33	0.000000000013072
4	0.028000000000000	34	0.000000000006417
⋮	⋮	⋮	⋮
19	0.000000301580666	39	0.000000000000184
20	0.000000146028533	40	0.000000000000091
21	0.000000070823839	41	0.000000000000045
⋮	⋮	⋮	⋮
29	0.000000000226469	47	0.000000000000001
30	0.000000000110892	48	0.000000000000000

Table 4.2: This table shows the value of sequence $\{x_n\}$ on each iteration steps (initial value $x_1 = \frac{1}{2}$)

n	x_n	n	x_n
1	0.500000000000000	31	0.000000000027168
2	0.100000000000000	32	0.000000000013321
3	0.035000000000000	33	0.000000000006536
4	0.014000000000000	34	0.000000000003208
⋮	⋮	⋮	⋮
19	0.000000150790333	39	0.000000000000092
20	0.000000073014267	40	0.000000000000045
21	0.000000035411919	41	0.000000000000022
⋮	⋮	⋮	⋮
29	0.000000000113235	46	0.000000000000001
30	0.000000000055446	47	0.000000000000000

(i)



(ii)

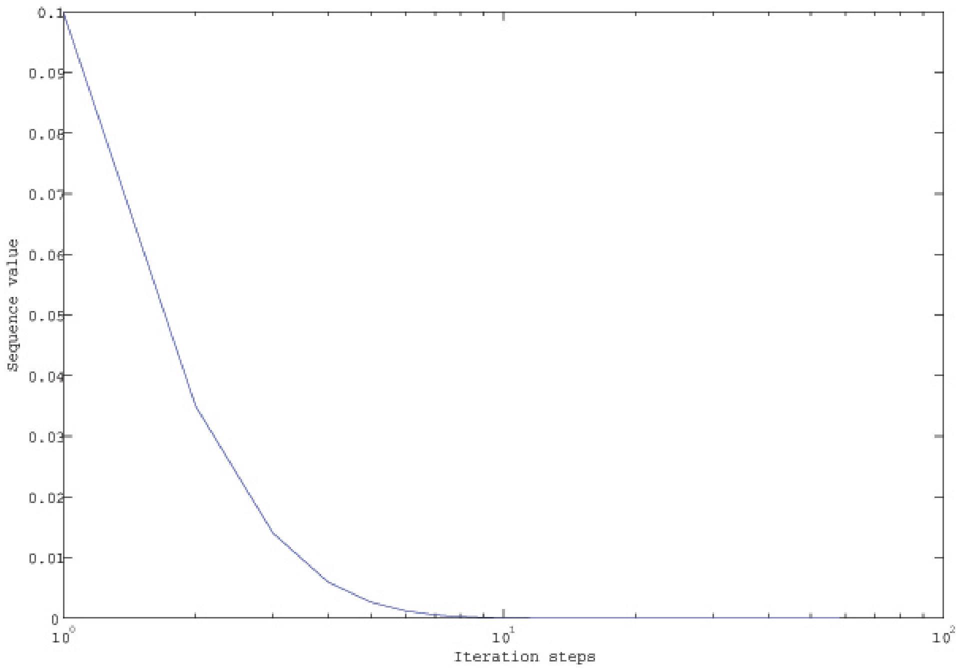


Figure 4.1 The iteration comparison chart of different initial values. (i) $x_1 = 1$ and (ii) $x_1 = \frac{1}{2}$.

CHAPTER 5 HIERARCHICAL FIXED POINTS, EQUILIBRIUM AND VARIATIONAL INEQUALITY PROBLEMS

In this section, we introduce a new iterative scheme that converges strongly to a common fixed point of a countable family of strictly pseudo-contractive mappings in a real Hilbert space which is also a solution to variational inequality problems related to quadratic minimization problems. Also a new hybrid extragradient iterative algorithm for solving a common element of the set of fixed points satisfying equilibrium problems, variational inequality problems and fixed point problems of a strict pseudocontraction mapping in Hilbert spaces are obtain.

5.1 Hierarchical Fixed Points and Variational Inequality Problems

Let H be a real Hilbert space, $T : C \to H$ be a mapping. The following problem is called a *hierarchical fixed point problem*: Find $x^* \in F(T)$ such that

$$\langle x^* - Sx^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T). \tag{5.1.1}$$

where $S : C \to H$ be a mapping.

Let us consider the net iterative scheme as follows:

$$\begin{cases} y_n = P_C[\beta_n S x_n + (1-\beta_n)x_n], \\ x_{n+1} = P_C[\alpha_n f(x_n) + \sum\limits_{i=1}^{n} (\alpha_{i-1} - \alpha_i) V_i y_n], \quad \forall n \geq 1, \end{cases} \tag{5.1.2}$$

where $V_i = k_i I + (1-k_i)T_i$, $f : C \to H$ is a ρ-contraction mapping, $S : C \to H$ is a nonexpansive mapping, $\{T_i\}_{i=1}^{\infty} : C \to C$ is a countable family of k_i-strict pseudo-contraction mappings and $\bigcap\limits_{i=1}^{\infty} F(T_i) \neq \emptyset$. Set $\alpha_0 = 1$, $\{\alpha_n\} \subset (0,1)$ is a strictly decreasing sequence and $\{\beta_n\} \subset (0,1)$. As we will see the convergence of the scheme depends on the choice of the parameters $\{\alpha_n\}$ and $\{\beta_n\}$. We list some possible hypotheses on them:

(H1) there exists $\gamma > 0$ such that $\beta_n \leq \gamma\alpha_n$;

(H2) $\lim_{n\to\infty} \beta_n/\alpha_n = \tau \in [0, \infty)$;

(H3) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(H4) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$;

(H5) $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$;

(H6) $\lim_{n\to\infty} |\alpha_n - \alpha_{n-1}|/\alpha_n = 0$;

(H7) $\lim_{n\to\infty} |\beta_n - \beta_{n-1}|/\beta_n = 0$;

(H8) $\lim_{n\to\infty} [|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|]/\alpha_n\beta_n = 0$;

(H9) there exists a constant $K > 0$ such that $\frac{1}{\alpha_n}|\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}| \leq K$.

Proposition 5.1.1. *Assume that (H1) holds. Then $\{x_n\}$ and $\{y_n\}$ are bounded.*

Proof (1) Let $z \in \bigcap_{i=1}^{\infty} F(T_i) = \bigcap_{i=1}^{\infty} F(V_i)$. Then we have

$$\begin{aligned}
\|x_{n+1} - z\| &= \left\| P_C[\alpha_n f(x_n) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) V_i y_n] - P_C[z] \right\| \\
&\leq \left\| \alpha_n f(x_n) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) V_i y_n - z \right\| \\
&= \left\| \alpha_n (f(x_n) - z) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i)(V_i y_n - z) \right\| \\
&\leq \alpha_n \|f(x_n) - f(z)\| + \alpha_n \|f(z) - z\| \\
&\quad + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \|V_i y_n - z\| \\
&\leq \alpha_n \rho \|x_n - z\| + \alpha_n \|f(z) - z\| \\
&\quad + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \|y_n - z\| \\
&\leq \alpha_n \rho \|x_n - z\| + \alpha_n \|f(z) - z\| \\
&\quad + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \|\beta_n S x_n + (1 - \beta_n) x_n - z\|
\end{aligned}$$

$$
\begin{aligned}
&\le \alpha_n\rho\|x_n - z\| + \alpha_n\|f(z) - z\| \\
&\quad + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)(\beta_n\|Sx_n - Sz\| + \beta_n\|Sz - z\| \\
&\quad + (1-\beta_n)\|x_n - z\|) \\
&\le \alpha_n\rho\|x_n - z\| + \alpha_n\|f(z) - z\| \\
&\quad + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)(\beta_n\|x_n - z\| + \beta_n\|Sz - z\| \\
&\quad + (1-\beta_n)\|x_n - z\|) \\
&= \alpha_n\rho\|x_n - z\| + \alpha_n\|f(z) - z\| \\
&\quad + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)(\|x_n - z\| + \beta_n\|Sz - z\|) \\
&= \alpha_n\rho\|x_n - z\| + \alpha_n\|f(z) - z\| \\
&\quad + (1-\alpha_n)(\|x_n - z\| + \beta_n\|Sz - z\|) \\
&= (1 - \alpha_n(1-\rho))\|x_n - z\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)\beta_n\|Sz - z\| \\
&\le (1 - \alpha_n(1-\rho))\|x_n - z\| + \alpha_n\|f(z) - z\| + \beta_n\|Sz - z\| \\
&\le (1 - \alpha_n(1-\rho))\|x_n - z\| + \alpha_n[\|f(z) - z\| + \gamma\|Sz - z\|]. \qquad (5.1.3)
\end{aligned}
$$

So, by induction, one can obtain that

$$\|x_n - z\| \le \max\left\{\|x_0 - z\|, \frac{1}{1-\rho}[\|f(z) - z\| + \gamma\|Sz - z\|\right\}. \qquad (5.1.4)$$

Hence $\{x_n\}$ is bounded. Of course $\{y_n\}$ is bounded too.

□

Proposition 5.1.2. *Suppose that (H1) and (H3) hold. Also, assume that either (H4) and (H5) hold, or (H6) and (H7) hold. Then*

(1) $\{x_n\}$ *is asymptotically regular, that is,*

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0, \qquad (5.1.5)$$

(2) the weak cluster points set $\omega_w(x_n) \subset \bigcap_{i=1}^{\infty} F(T_i)$.

proof (2) Set $u_n = \alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)V_i y_n$. From (5.1.2) and since P_C is a nonexpansive mapping, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|P_C[u_n] - P_C[u_{n-1}]\| \\
&\leq \|u_n - u_{n-1}\| && (5.1.6) \\
&= \Bigg\|\alpha_n(f(x_n) - f(x_{n-1})) + (\alpha_n - \alpha_{n-1})f(x_{n-1}) \\
&\quad + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)(V_i y_n - V_i y_{n-1}) + (\alpha_{n-1} - \alpha_n)V_n y_{n-1}\Bigg\| \\
&\leq \alpha_n\|f(x_n) - f(x_{n-1})\| + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|y_n - y_{n-1}\| \\
&\quad |\alpha_n - \alpha_{n-1}|(\|f(x_{n-1})\| + \|V_n y_{n-1}\|) \\
&\leq \alpha_n\rho\|x_n - x_{n-1}\| + (1 - \alpha_n)\|y_n - y_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|(\|f(x_{n-1})\| + \|V_n y_{n-1}\|). && (5.1.7)
\end{aligned}$$

By definition of y_n one obtain that

$$\begin{aligned}
\|y_n - y_{n-1}\| &= \|P_C[\beta_n S x_n + (1 - \beta_n)x_n] - P_C[\beta_{n-1} S x_{n-1} + (1 - \beta_{n-1})x_{n-1}]\| \\
&\leq \|(\beta_n S x_n + (1 - \beta_n)x_n) - (\beta_{n-1} S x_{n-1} + (1 - \beta_{n-1})x_{n-1})\| \\
&= \|\beta_n(S x_n - S x_{n-1}) + (\beta_n - \beta_{n-1})S x_{n-1} \\
&\quad + (1 - \beta_{n-1})(x_n - x_{n-1}) + (\beta_{n-1} - \beta_n)x_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|S x_{n-1}\| + \|x_{n-1}\|). && (5.1.8)
\end{aligned}$$

So, substituting (5.1.8) in (5.1.7), we obtain

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \alpha_n\rho\|x_n - x_{n-1}\| \\
&\quad + (1 - \alpha_n)[\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|S x_{n-1}\| + \|x_{n-1}\|)] \\
&\quad + |\alpha_n - \alpha_{n-1}|(\|f(x_{n-1})\| + \|V_n y_{n-1}\|) \\
&\leq (1 - (1 - \rho)\alpha_n)\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|S x_{n-1}\| + \|x_{n-1}\|) \\
&\quad + |\alpha_n - \alpha_{n-1}|(\|f(x_{n-1})\| + \|V_n y_{n-1}\|). && (5.1.9)
\end{aligned}$$

By Proposition 5.1.1, we say

$$M := \max\left\{\sup_{n\geq 1}\{\|S x_{n-1}\| + \|x_{n-1}\|\}, \sup_{n\geq 1}\{\|f(x_{n-1})\| + \|V_n y_{n-1}\|\}\right\}.$$

So, we have

$$\|x_{n+1} - x_n\| \leq (1 - (1 - \rho)\alpha_n)\|x_n - x_{n-1}\| + M[|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|]. \quad (5.1.10)$$

So, if (H4) and (H5) hold, we obtain the asymptotic regularity by Lemma 2.1.23, if instead, (H6) and (H7) hold, from (H1), we can write

$$\begin{aligned}\|x_{n+1}-x_n\| &\le (1-(1-\rho)\alpha_n)\|x_n-x_{n-1}\| \\ &\quad + M\alpha_n\left[\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n}+\frac{|\beta_n-\beta_{n-1}|}{\alpha_n}\right] \\ &\le (1-(1-\rho)\alpha_n)\|x_n-x_{n-1}\| \\ &\quad + M\alpha_n\left[\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n}+\gamma\frac{|\beta_n-\beta_{n-1}|}{\beta_n}\right]. \qquad (5.1.11)\end{aligned}$$

By Lemma 2.1.23, we obtain the asymptotic regularity.

In order to prove (2), since $V_i x_n \in C$ for each $i \geq 1$ and $\sum_{n=1}^{\infty}(\alpha_{n-1}-\alpha_n)+\alpha_n = 1$, we have

$$\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i x_n + \alpha_n p \in C, \quad \forall p \in C. \qquad (5.1.12)$$

Now, fixing a $p \in \bigcap_{i=1}^{\infty} F(Vi)$, from (5.1.2), we have

$$\begin{aligned}&\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(x_n - V_i x_n) \\ = \ & P_C[u_n] + (1-\alpha_n)x_n - \left(\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i x_n + \alpha_n p\right) + \alpha_n p - x_{n+1} \\ = \ & P_C[u_n] - P_C\left[\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i x_n + \alpha_n p\right] + (1-\alpha_n)(x_n - x_{n+1}) \\ & + \alpha_n(p - x_{n+1}).\end{aligned}$$

It follows that

$$\begin{aligned}&\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle x_n - V_i x_n, x_n - z\rangle \\ = \ & \left\langle P_C[u_n] - P_C\left[\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i x_n + \alpha_n p\right], x_n - z\right\rangle \\ & + (1-\alpha_n)\langle x_n - x_{n+1}, x_n - z\rangle + \alpha_n\langle p - x_{n+1}, x_n - z\rangle \\ \le \ & \left\| u_n - \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i x_n + \alpha_n p\right\| \|x_n - z\| \\ & + (1-\alpha_n)\|x_n - x_{n+1}\|\|x_n - z\| + \alpha_n\|p - x_{n+1}\|\|x_n - z\| \\ = \ & \left\|\alpha_n(f(x_n) - p) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(V_i y_n - V_i x_n)\right\| \|x_n - z\| \\ & + (1-\alpha_n)\|x_n - x_{n+1}\|\|x_n - z\| + \alpha_n\|p - x_{n+1}\|\|x_n - z\|\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n\|f(x_n)-p\|\|x_n-z\|+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|y_n-x_n\|\|x_n-z\| \\
&\quad +(1-\alpha_n)\|x_n-x_{n+1}\|\|x_n-z\|+\alpha_n\|p-x_{n+1}\|\|x_n-z\| \\
&\leq \alpha_n\|f(x_n)-p\|\|x_n-z\|+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\beta_n\|Sx_n-x_n\|\|x_n-z\| \\
&\quad +(1-\alpha_n)\|x_n-x_{n+1}\|\|x_n-z\|+\alpha_n\|p-x_{n+1}\|\|x_n-z\| \\
&= \alpha_n\|f(x_n)-p\|\|x_n-z\|+(1-\alpha_n)\beta_n\|Sx_n-x_n\|\|x_n-z\| \\
&\quad +(1-\alpha_n)\|x_n-x_{n+1}\|\|x_n-z\|+\alpha_n\|p-x_{n+1}\|\|x_n-z\|. \qquad (5.1.13)
\end{aligned}
$$

Now, from Lemma 2.6.33 and (5.1.13), we get

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|x_n-V_ix_n\|^2 \\
\leq\ & \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle x_n-V_ix_n, x_n-z\rangle \\
\leq\ & \alpha_n\|f(x_n)-p\|\|x_n-z\|+(1-\alpha_n)\beta_n\|Sx_n-x_n\|\|x_n-z\| \\
& +(1-\alpha_n)\|x_n-x_{n+1}\|\|x_n-z\|+\alpha_n\|p-x_{n+1}\|\|x_n-z\|.
\end{aligned}
$$

By (H1) and (H3), it follows that $\beta_n \to 0$, as $n\to\infty$, so that

$$\lim_{n\to\infty}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|x_n-V_ix_n\|^2=0. \qquad (5.1.14)$$

Since $(\alpha_{i-1}-\alpha_i)\|x_n-V_ix_n\|^2 \leq \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|x_n-V_ix_n\|^2$ for each $i\geq 1$ and $\{\alpha_n\}$ is strictly decreasing, one has

$$\lim_{n\to\infty}\|x_n-V_ix_n\|=0, \quad \forall i\geq 1. \qquad (5.1.15)$$

Hence, we obtain

$$\lim_{n\to\infty}\|x_n-T_ix_n\|=\lim_{n\to\infty}\frac{\|x_n-V_ix_n\|}{(1-k_i)}=0, \quad \forall i\geq 1.$$

Since $\{x_n\}$ is asymptotically regular and demiclosedness principle, we obtain the proposition.

Corollary 5.1.3. *Suppose that the hypotheses of Proposition 5.1.2 hold. Then*

(i) $\lim_{n\to\infty}\|x_n-y_n\|=0$;

(ii) $\lim_{n\to\infty}\|x_n-V_iy_n\|=0, \quad \forall i\geq 1$;

(iii) $\lim_{n\to\infty} \|y_n - V_i y_n\| = 0, \quad \forall i \geq 1.$

Proof. To prove (i), we can observe that

$$\|x_n - y_n\| \leq \beta_n \|x_n - Sx_n\|.$$

Since $\beta_n \to 0$ as $n \to \infty$, we obtain (i).

To prove (ii), we observe that

$$\|y_n - V_i x_n\| \leq \|y_n - x_n\| + \|x_n - V_i x_n\|, \ \ \forall i \geq 1$$

and

$$\|x_n - V_i y_n\| \leq \|x_n - y_n\| + \|y_n - V_i x_n\|, \ \ \forall i \geq 1.$$

Since $\|y_n - x_n\| \to 0$ and $\|x_n - V_i x_n\| \to 0$ as $n \to \infty$, $\forall i \geq 1$, then $\|y_n - V_i x_n\| \to 0$, that is, we obtain (ii). To prove (iii), we can observe that

$$\|y_n - V_i y_n\| \leq \|x_n - y_n\| + \|x_n - V_i y_n\|, \ \ \forall i \geq 1.$$

By (i) and (ii), we obtain (iii). □

Theorem 5.1.4. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $f : C \to H$ be a ρ-contraction mapping, $S : C \to H$ be a nonexpansive mapping and $\{T_i\}_{i=1}^{\infty} : C \to C$ be a countable family of k_i-strict pseudo-contraction mappings and $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\alpha_0 = 1$, and $x_1 \in C$ and define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = P_C[\beta_n S x_n + (1 - \beta_n) x_n], \\ x_{n+1} = P_C[\alpha_n f(x_n) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) V_i y_n], \quad \forall n \geq 1, \end{cases} \tag{5.1.16}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\alpha_n\}$ is a strictly decreasing sequence, $V_i = k_i I + (1 - k_i) T_i$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions (H2) with $\tau = 0$, (H3), either (H4) and (H5), or (H6) and (H7). Then the sequence $\{x_n\}$ converges strongly to a point $z \in \mathcal{F}$, which is the unique solution of the variational inequality:

$$\langle (I - f) z, x - z \rangle \geq 0, \quad \forall x \in \mathcal{F}. \tag{5.1.17}$$

Proof. First of all, since $P_{\mathcal{F}} f$ is a contraction. By Banach contraction principle, so there exists a unique $z \in \mathcal{F}$ such that $z = P_{\mathcal{F}} f(z)$, Moreover, from Lemma 2.1.12 (1), we have

$$\langle f(z) - z, y - z \rangle \leq 0, \quad \forall y \in \mathcal{F}.$$

Since (H2) implies (H1), thus $\{x_n\}$ is bounded. Moreover, since either (H4) and (H5), or (H6) and (H7), then $\{x_n\}$ is asymptotically regular. Similarly, by Proposition 5.1.2, the weak cluster points set of x_n, that is, $\omega_w(x_n)$, is a subset of $\mathcal{F}$.
Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(z)-z, x_n-z\rangle = \lim_{k\to\infty}\langle f(z)-z, x_{n_k}-z\rangle,$$

and $x_{n_k}\to x'$. By Proposition 5.1.2 it follows that $x'\in\mathcal{F}$. Then

$$\lim_{k\to\infty}\langle f(z)-z, x_{n_k}-z\rangle = \langle f(z)-z, x'-z\rangle \le 0.$$

Set $u_n = \alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i y_n$, we obtain

$$\|x_{n+1}-z\|^2 = \langle P_C[u_n]-u_n, P_C[u_n]-z\rangle + \langle u_n-z, x_{n+1}-z\rangle. \tag{5.1.18}$$

By Lemma 2.1.12 (1), we have

$$\langle P_C[u_n]-u_n, P_C[u_n]-z\rangle \le 0. \tag{5.1.19}$$

From (5.1.18) and (5.1.19), it follows that

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\le \langle u_n-z, x_{n+1}-z\rangle \\
&= \alpha_n\langle f(x_n)-f(z), x_{n+1}-z\rangle + \alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle V_i y_n - z, x_{n+1}-z\rangle \\
&\le \alpha_n\rho\|x_n-z\|\|x_{n+1}-z\| + \alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad (1-\alpha_n)\|y_n-z\|\|x_{n+1}-z\| \\
&\le \alpha_n\rho\|x_n-z\|\|x_{n+1}-z\| + \alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad (1-\alpha_n)\|\beta_n Sx_n + (1-\beta_n)x_n - z\|\|x_{n+1}-z\| \\
&\le \alpha_n\rho\|x_n-z\|\|x_{n+1}-z\| + \alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad (1-\alpha_n)\|x_n-z\|\|x_{n+1}-z\| + (1-\alpha_n)\beta_n\|Sz-z\|\|x_{n+1}-z\| \\
&= [1-\alpha_n(1-\rho)]\|x_n-z\|\|x_{n+1}-z\| + \alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad +(1-\alpha_n)\beta_n\|Sz-z\|\|x_{n+1}-z\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left[\frac{1-\alpha_n(1-\rho)}{2}\right]\left[\|x_n-z\|^2+\|x_{n+1}-z\|^2\right] \\
&\quad +\alpha_n\langle f(z)-z, x_{n+1}-z\rangle+(1-\alpha_n)\beta_n\|Sz-z\|\|x_{n+1}-z\| \\
&\leq \left[1-\frac{2(1-\rho)\alpha_n}{1+(1-\rho)\alpha_n}\right]\|x_n-z\|^2+\left[\frac{2\alpha_n}{1+(1-\rho)\alpha_n}\right] \\
&\quad \times\langle f(z)-z, x_{n+1}-z\rangle+\left[\frac{2(1-\alpha_n)\beta_n}{1+(1-\rho)\alpha_n}\right]\|Sz-z\|\|x_{n+1}-z\| \\
&= \left[1-\frac{2(1-\rho)\alpha_n}{1+(1-\rho)\alpha_n}\right]\|x_n-z\|^2+\left[\frac{2(1-\rho)\alpha_n}{1+(1-\rho)\alpha_n}\right] \\
&\quad \times\left\{\frac{1}{1-\rho}\langle f(z)-z, x_{n+1}-z\rangle+\frac{(1-\alpha_n)\beta_n}{(1-\rho)\alpha_n}\|Sz-z\|\|x_{n+1}-z\|\right\}.
\end{aligned}
$$

Let $\gamma_n = \frac{2(1-\rho)\alpha_n}{1+(1-\rho)\alpha_n}$ and $\delta_n = \frac{2(1-\rho)\alpha_n}{1+(1-\rho)\alpha_n}\left\{\frac{1}{1-\rho}\langle f(z)-z, x_{n+1}-z\rangle + \frac{(1-\alpha_n)\beta_n}{(1-\rho)\alpha_n}\|Sz - z\|\|x_{n+1}-z\|\right\}$ for all $n \geq 1$. Since

$$
\limsup_{n\to\infty}\left\{\frac{1}{1-\rho}\langle f(z)-z, x_{n+1}-z\rangle+\frac{(1-\alpha_n)\beta_n}{(1-\rho)\alpha_n}\|Sz-z\|\|x_{n+1}-z\|\right\}\leq 0,
$$

$\sum_{i=1}^{\infty}\alpha_n = \infty$ and $\frac{2(1-\rho)\alpha_n}{1+(1-\rho)\alpha_n} \geq (1-\rho)\alpha_n$, we have

$$
\sum_{n=1}^{\infty}\gamma_n=\infty \text{ and } \limsup_{n\to\infty}\frac{\delta_n}{\gamma_n}\leq 0.
$$

Hence, by Lemma 2.1.23, we conclude that $x_n \to z$ as $n \to \infty$. This completes the proof. □

Remark 5.1.5. In the iterative scheme (5.1.16), if we set $f \equiv 0$, then we get $x_n \to z = P_{\mathcal{F}}0$. In this case, from (5.1.17), it follows that

$$
\langle z, z-x\rangle \leq 0, \quad \forall x\in\mathcal{F}.
$$

That is

$$
\|z\|^2 \leq \langle z, x\rangle \leq \|z\|\|x\|, \quad \forall x\in\mathcal{F}.
$$

Therefore, the point z is the unique solution to the following quadratic minimization problem:

$$
z = arg\min_{x\in\mathcal{F}}\|x\|^2.
$$

By changing the restrictions on parameters in Theorem 5.1.4, we obtain the following results.

Theorem 5.1.6. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $f : C \to H$ be a ρ-contraction mapping, $S : C \to C$ be a nonexpansive mapping and $\{T_i\}_{i=1}^{\infty} : C \to C$ be a countable family of k_i-strict pseudo-contraction mappings and $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\alpha_0 = 1$, and $x_1 \in C$ and define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = P_C[\beta_n S x_n + (1-\beta_n)x_n] = \beta_n S x_n + (1-\beta_n)x_n \\ x_{n+1} = P_C[\alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)V_i y_n], \quad \forall n \geq 1 \end{cases} \tag{5.1.20}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\alpha_n\}$ is a strictly decreasing sequence, $V_i = k_i I + (1-k_i)T_i$, $\{\beta_n\} \subset (0,1)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions (H2) with $\tau \in (0,\infty)$, (H3), (H8) and (H9). Then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \mathcal{F}$, which is the unique solution of the variational inequality:

$$\langle \frac{1}{\tau}(I-f)x^* + (I-S)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{F}. \tag{5.1.21}$$

Proof. First, we shows that (5.1.21) has the unique solution. Let x' and x^* be two solutions. Then, since x' is solution, for $y = x^*$ one has

$$\langle (I-f)x', x' - x^* \rangle \leq \tau \langle (I-S)x', x^* - x' \rangle \tag{5.1.22}$$

and

$$\langle (I-f)x^*, x^* - x' \rangle \leq \tau \langle (I-S)x^*, x' - x^* \rangle. \tag{5.1.23}$$

Adding (5.1.22) and (5.1.23), we obtain

$$\begin{aligned} (1-\rho)\|x' - x^*\|^2 &\leq \langle (I-f)x' - (I-f)x^*, x' - x^* \rangle \\ &\leq -\rho \langle (I-S)x' - (I-S)x^*, x' - x^* \rangle \leq 0 \end{aligned}$$

so $x' = x^*$. Also now the condition (H2) with $0 < \tau < \infty$ implies (H1) so the sequence $\{x_n\}$ is bounded. Moreover, since (H8) implies (H6) and (H7), then $\{x_n\}$ is asymptotically regular.

Similarly, by Proposition 5.1.2, the weak cluster points set of x_n, i.e., $\omega_w(x_n)$, is a subset of $\mathcal{F}$.

From (5.1.6)-(5.1.10), we observe that

$$\begin{aligned}
\frac{\|x_{n+1}-x_n\|}{\beta_n} &\leq \frac{\|u_n-u_{n-1}\|}{\beta_n}\\
&\leq [1-(1-\rho))\alpha_n]\frac{\|x_n-x_{n-1}\|}{\beta_n}+M\left[\frac{|\alpha_n-\alpha_{n-1}|}{\beta_n}+\frac{|\beta_n-\beta_{n-1}|}{\beta_n}\right]\\
&= [1-(1-\rho))\alpha_n]\frac{\|x_n-x_{n-1}\|}{\beta_{n-1}}+[1-(1-\rho))\alpha_n]\|x_n-x_{n-1}\|\\
&\quad\times\left[\frac{1}{\beta_n}-\frac{1}{\beta_{n-1}}\right]+M\left[\frac{|\alpha_n-\alpha_{n-1}|}{\beta_n}+\frac{|\beta_n-\beta_{n-1}|}{\beta_n}\right]\\
&\leq [1-(1-\rho))\alpha_n]\frac{\|x_n-x_{n-1}\|}{\beta_{n-1}}+\|x_n-x_{n-1}\|\left[\frac{1}{\beta_n}-\frac{1}{\beta_{n-1}}\right]\\
&\quad+M\left[\frac{|\alpha_n-\alpha_{n-1}|}{\beta_n}+\frac{|\beta_n-\beta_{n-1}|}{\beta_n}\right]\\
&\leq [1-(1-\rho))\alpha_n]\frac{\|x_n-x_{n-1}\|}{\beta_{n-1}}+\alpha_n K\|x_n-x_{n-1}\|\\
&\quad+M\left[\frac{|\alpha_n-\alpha_{n-1}|}{\beta_n}+\frac{|\beta_n-\beta_{n-1}|}{\beta_n}\right]\\
&\leq [1-(1-\rho))\alpha_n]\frac{\|u_n-u_{n-1}\|}{\beta_{n-1}}+\alpha_n K\|x_n-x_{n-1}\|\\
&\quad+M\left[\frac{|\alpha_n-\alpha_{n-1}|}{\beta_n}+\frac{|\beta_n-\beta_{n-1}|}{\beta_n}\right].
\end{aligned}$$

Let $\gamma_n=(1-\rho)\alpha_n$ and $\delta_n=\alpha_n K\|x_n-x_{n-1}\|+M\left[\frac{|\alpha_n-\alpha_{n-1}|}{\beta_n}+\frac{|\beta_n-\beta_{n-1}|}{\beta_n}\right]$.
From condition (H3) and (H8), we have

$$\sum_{i=1}^{\infty}\gamma_n=\infty \quad and \quad \lim_{n\to\infty}\frac{\delta_n}{\gamma_n}=0.$$

By Lemma 2.1.23, we obtain

$$\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\beta_n}=0,\quad \lim_{n\to\infty}\frac{\|u_{n+1}-u_n\|}{\beta_n}=\lim_{n\to\infty}\frac{\|u_{n+1}-u_n\|}{\alpha_n}=0.$$

From (5.1.20), we have

$$\begin{aligned}
x_n-x_{n-1} &= (1-\alpha_n)x_n-\Bigg[P_C[u_n]-u_n+\alpha_n f(x_n)\\
&\quad+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(V_iy_n-y_n)+(1-\alpha_n)y_n\Bigg]\\
&= (1-\alpha_n)\beta_n(x_n-Sx_n)+(u_n-P_C[u_n])\\
&\quad+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(y_n-V_iy_n)+\alpha_n(x_n-f(x_n)).
\end{aligned}$$

It follows that

$$\begin{aligned}\frac{x_n - x_{n-1}}{(1-\alpha_n)\beta_n} &= (x_n - Sx_n) + \frac{1}{(1-\alpha_n)\beta_n}(u_n - P_C[u_n]) \\ &+\frac{1}{(1-\alpha_n)\beta_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(y_n - V_i y_n) \\ &+\frac{\alpha_n}{(1-\alpha_n)\beta_n}(x_n - f(x_n)).\end{aligned}$$

Let $v_n = \frac{x_n - x_{n-1}}{(1-\alpha_n)\beta_n}$. For all $z \in \mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) = \bigcap_{i=1}^{\infty} F(V_i)$, we get

$$\begin{aligned}\langle v_n, x_n - z\rangle &= \frac{1}{(1-\alpha_n)\beta_n}\langle u_n - P_C[u_n], P_C[u_{n-1}] - z\rangle \qquad (5.1.24)\\ &+\frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)x_n, x_n - z\rangle + \langle x_n - Sx_n, x_n - z\rangle \\ &+\frac{1}{(1-\alpha_n)\beta_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle y_n - V_i y_n, x_n - z\rangle. \qquad (5.1.25)\end{aligned}$$

By Lemma 2.6.29, we have

$$\begin{aligned}\langle x_n - Sx_n, x_n - z\rangle &= \langle (I-S)x_n - (I-S)z, x_n - z\rangle + \langle (I-S)z, x_n - z\rangle \\ &\geq \langle (I-S)z, x_n - z\rangle, \qquad (5.1.26)\end{aligned}$$

$$\begin{aligned}\langle (I-f)x_n, x_n - z\rangle &= \langle (I-f)x_n - (I-f)z, x_n - z\rangle + \langle (I-f)z, x_n - z\rangle \\ &\geq (1-\rho)\|x_n - z\|^2 + \langle (I-f)z, x_n - z\rangle \qquad (5.1.27)\end{aligned}$$

and

$$\begin{aligned}\langle y_n - V_i y_n, x_n - z\rangle &= \langle (I-V_i)y_n - (I-V_i)z, x_n - y_n\rangle \qquad (5.1.28)\\ &+\langle (I-V_i)y_n - (I-V_i)z, y_n - z\rangle \\ &\geq \langle (I-V_i)y_n - (I-V_i)z, x_n - y_n\rangle \\ &= \beta_n\langle (I-V_i)y_n, x_n - Sx_n\rangle, \quad \forall i \geq 1. \qquad (5.1.29)\end{aligned}$$

By Lemma 2.1.12(1), we obtain

$$\begin{aligned}\langle u_n - P_C[u_n], P_C[u_{n-1}] - z\rangle &= \langle u_n - P_C[u_n], P_C[u_{n-1}] - P_C[u_n]\rangle \quad (5.1.30)\\ &+\langle u_n - P_C[u_n], P_C[u_n] - z\rangle \\ &\geq \langle u_n - P_C[u_n], P_C[u_{n-1}] - P_C[u_n]\rangle. \quad (5.1.31)\end{aligned}$$

Now, from(5.1.24)-(5.1.30), it follows that

$$\begin{aligned}\langle v_n, x_n - z\rangle &\geq \frac{1}{(1-\alpha_n)\beta_n}\langle u_n - P_C[u_n], P_C[u_{n-1}] - P_C[u_n]\rangle \\ &+\frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)z, x_n - z\rangle + \langle (I-S)z, x_n - z\rangle \\ &+\frac{1}{(1-\alpha_n)}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle (I-V_i)y_n, x_n - Sx_n\rangle \\ &+\frac{(1-\rho)\alpha_n}{(1-\alpha_n)\beta_n}\|x_n - z\|^2. \end{aligned} \tag{5.1.32}$$

We observe from (5.1.32) that

$$\begin{aligned}\|x_n - z\|^2 &\leq \frac{(1-\alpha_n)\beta_n}{(1-\rho)\alpha_n}\Big[\langle v_n, x_n - z\rangle - \langle (I-S)z, x_n - z\rangle\Big] \\ &+\frac{\|u_{n-1}-u_n\|}{(1-\rho)\alpha_n}\|u_n - P_C[u_n]\| - \frac{1}{1-\rho}\langle (I-f)z, x_n - z\rangle \\ &-\frac{\beta_n}{(1-\rho)\alpha_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle (I-V_i)y_n, x_n - Sx_n\rangle, \end{aligned} \tag{5.1.33}$$

since $v_n \to 0$ and $(I-V_i)y_n \to 0$, as $n \to \infty$, then every weak cluster point of $\{x_n\}$ is also a strong cluster point. By Proposition 5.1.2, $\{x_n\}$ is bounded, thus there exists a subsequence $\{x_{n_k}\}$ converging to x^*.

For all $z \in \mathcal{F}$ by (5.1.24), we compute

$$\begin{aligned}\langle (I-f)x_{n_k}, x_{n_k} - z\rangle &= \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle v_{n_k}, x_{n_k} - z\rangle \\ &-\frac{1}{\alpha_{n_k}}\langle u_{n_k} - P_C[u_{n_k}], P_C[u_{n_k-1}] - z\rangle \\ &-\frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle x_{n_k} - Sx_{n_k}, x_{n_k} - z\rangle \\ &-\frac{1}{\alpha_{n_k}}\sum_{i=1}^{n_k}(\alpha_{i-1}-\alpha_i)\langle y_{n_k} - V_i y_{n_k}, x_{n_k} - z\rangle \\ &\leq \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle v_{n_k}, x_{n_k} - z\rangle \\ &-\frac{\beta_{n_k}}{\alpha_{n_k}}\sum_{i=1}^{n_k}(\alpha_{i-1}-\alpha_i\langle (I-V_i)y_{n_k}, x_{n_k} - Sx_{n_k}\rangle \\ &-\frac{1}{\alpha_{n_k}}\|u_{n_k-1}-u_{n_k}\|\|u_{n_k} - P_C[u_{n_k}]\| \\ &-\frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle (I-S)z, x_{n_k} - z\rangle. \end{aligned} \tag{5.1.34}$$

Since $v_n \to 0, (I-V_i)y_n \to 0$ for all $i \geq 1$, and $\|u_n - u_{n-1}\|/\alpha_n \to 0$, letting $k \to \infty$ in (3.30), we obtain

$$\langle (I-f)x^*, x^* - z\rangle \leq -\tau\langle (I-S)z, x^* - z\rangle, \quad \forall z \in \mathcal{F}.$$

Since (5.1.21) has the unique solution, it follows that $\omega_w(x_n) = \{x^*\}$. Since every weak cluster point of $\{x_n\}$ is also a strong cluster point, we conclude that $x_n \to x^*$ as $n \to \infty$. This completes the proof. $\square$

5.2 Hybrid Extragradient Method for Fixed Points and Variational Inequality Problems

Let C be a nonempty, closed and convex subset of a real Hilbert space H, let F be a bifunction $F : C \times C \to \mathbb{R}$ satisfying conditions $(A1) - (A5)$ (Lemma 2.6.34), let A be an α-inverse-strongly monotone mapping of C into H. Let S be a ξ-strict pseudocontraction mapping from C to C.

we consider the sequences $\{x_n\}, \{y_n\}, \{z_n\}, \{w_n\}$ and $\{t_n\}$ generated by $x_0 \in C$ and

$$\begin{cases} y_n = \arg\min_{y \in C}\{\lambda_n F(x_n, y) + \frac{1}{2}\|y - x_n\|^2\}, \\ z_n = \arg\min_{y \in C}\{\lambda_n F(y_n, y) + \frac{1}{2}\|y - x_n\|^2\}, \\ w_n = P_C(z_n - \lambda_n A z_n), \\ t_n = \alpha_n x_n + (1-\alpha_n)[(1-\mu)S w_n + \mu P_C(1-\beta_n) w_n], \\ C_n = \{z \in C : \|t_n - z\| \leq \|x_n - z\|\}, \\ D_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \end{cases} \tag{5.2.1}$$

for every $n \in N$, where μ be a constant in $(0,1)$, $\{\alpha_n\} \subset [0,1), \{\beta_n\} \subset (0,1), \{\lambda_n\} \subset (0,1]$.

Algorithm 1

Choose the sequences $\{\alpha_n\} \subset [0,1), \{\beta_n\} \subset (0,1), \{\lambda_n\} \subset (0,1]$ and μ be a constant in $(0,1)$.

1. Let $x_0 \in C$. Set $n = 0$.

2. Solve successively the strongly convex programs
$\arg\min_{y \in C}\{\lambda_n F(x_n, y) + \frac{1}{2}\|y - x_n\|^2\}$ and
$\arg\min_{y \in C}\{\lambda_n F(y_n, y) + \frac{1}{2}\|y - x_n\|^2\}$
to obtain the unique optimal solution y_n and z_n, respectively.

3. Compute $w_n = P_C(z_n - \lambda_n A z_n)$.

4. Compute
$t_n = \alpha_n x_n + (1-\alpha_n)[(1-\mu)S w_n + \mu P_C(1-\beta_n) w_n]$. If $y_n = x_n$ and $t_n = x_n$, then STOP:
$x_n \in EP(F) \cap Fix(S) \cap VI(C, A)$. Otherwise, go to Step 5.

5. Compute $x_{n+1} = P_{C_n \cap D_n} x_0$, where
$C_n = \{z \in C : \|t_n - z\| \leq \|x_n - z\|\}$,
$D_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}$.

6. Set $n := n + 1$, and go to Step 3.

In the sequel, we also suppose that the sequences of parameters $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}, \xi$ and μ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [\lambda_{\min}, \lambda_{\max}]$, where $0 < \lambda_{\min} \leq \lambda_{\max} < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$;

(ii) $\{\alpha_n\} \subset [0, c]$ for some $c < 1$;

(iii) $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iv) ξ and μ be constant, where $0 \leq \xi < \mu < 1$.

Now, let $\{x_n\}, \{y_n\}, \{z_n\}$, and $\{w_n\}$ be the sequences generated by combination of the hybrid extragradient method, variational inequality by projection method and the fixed point method described at the begining of this section. These sequence satisfy the following properties. Here we start our main theorem.

Theorem 5.2.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H, let F be a bifunction $F : C \times C \to \mathbb{R}$ satisfying conditions $(A1) - (A5)$, let A be an α-inverse-strongly monotone mapping of C into H. Let S be a ξ-strict pseudocontraction mapping from C to C and such that $\Theta := EP(F) \cap Fix(S) \cap VI(C, A) \neq \emptyset$. Suppose that the sequences $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}$ and μ satisfying the conditions $(i) - (iv)$. Then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to the projection of x_0 onto the set Θ.*

Proof. ***Step 1.*** We show that sequence $\{x_n\}$ is well defined. From definition of C_n and D_n, it is obvious that C_n is closed and D_n is closed and convex for every $n \in \mathbb{N}$. We prove that C_n is convex. Since $\|t_n - z\| \leq \|x_n - z\|$ is equivalent to

$$\|t_n - x_n\|^2 + 2\langle t_n - x_n, x_n - z \rangle \leq 0$$

it follows that C_n is convex. So $C_n \cap D_n$ is closed convex subset of H for any $n \in \mathbb{N}$. Let $x^* \in \Theta$.

Then $x^* = P_C(x^* - \lambda_n Ax^*)$. Since $w_n = P_C(z_n - \lambda_n Az_n)$, we consider

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|[P_C(z_n - \lambda_n Az_n)] - [P_C(x^* - \lambda_n Ax^*)]\|^2 \\
&\leq \|(z_n - \lambda_n Az_n) - (x^* - \lambda_n Ax^*)\|^2 \\
&= \|z_n - x^*\|^2 - 2\lambda_n\langle z_n - x^*, Az_n - Ax^*\rangle + \lambda_n^2\|Az_n - Ax^*\|^2 \\
&\leq \|z_n - x^*\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Az_n - Ax^*\|^2 \qquad (5.2.2) \\
&\leq \|z_n - x^*\|^2.
\end{aligned}$$

By Proposition 2.6.35 (ii), we have

$$\begin{aligned}
&\|z_n - x^*\|^2 \\
\leq\ &\|x_n - x^*\|^2 - (1 - 2\lambda_n c_1)\|y_n - x_n\|^2 - (1 - 2\lambda_n c_2)\|z_n - y_n\|^2 \qquad (5.2.3)
\end{aligned}$$

that is, we obtain $\|z_n - x^*\| \leq \|x_n - x^*\|$ and $\|w_n - x^*\| \leq \|x_n - x^*\|$.
Set $u_n := (1-\mu)Sw_n + \mu P_C(1-\beta_n)w_n,$ for all $n \geq 0$. Then, we have $t_n = \alpha_n x_n + (1-\alpha_n)u_n$, It follows that

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|(1-\mu)Sw_n + \mu P_C(1-\beta_n)w_n - x^*\|^2 \\
&\leq \|(1-\mu)(Sw_n - x^*) + \mu\big[(1-\beta_n)w_n - x^*\big]\|^2 \\
&\leq \|w_n - x^*\|^2 - (1-\mu)(\mu-\xi)\|Sw_n - w_n\|^2 \\
&\quad -\beta_n\mu^2\|w_n\|^2 \qquad (5.2.4) \\
&\leq \|w_n - x^*\|^2 \leq \|x_n - x^*\|^2
\end{aligned}$$

that is, $\|u_n - x^*\| \leq \|x_n - x^*\|$. Since $t_n = \alpha_n x_n + (1-\alpha_n)u_n$ for every $x^* \in \Theta$, we have

$$\begin{aligned}
\|t_n - x^*\|^2 &= \|\alpha_n x_n + (1-\alpha_n)u_n - x^*\|^2 \\
&= \|\alpha_n(x_n - x^*) + (1-\alpha_n)(u_n - x^*)\|^2 \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2 \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 \\
&\leq \|x_n - x^*\|^2.
\end{aligned}$$

Hence $\|t_n - x^*\| \leq \|x_n - x^*\|$ for every $n \geq 0$ and $x^* \in C_n$. So, we have

$$\Theta := EP(F) \cap Fix(S) \cap VI(A, C) \subset C_n, \qquad \forall n \in \mathbb{N}. \qquad (5.2.5)$$

Next, we will show that

$$\Theta := EP(F) \cap Fix(S) \cap VI(A, C) \subset C_n \cap D_n, \forall n \in \mathbb{N}. \tag{5.2.6}$$

We prove this by induction.

For $n = 0$, we have $x_0 = x \in C$, $\quad \Theta \subset C_0$ and $D_0 = C$. So, we get $\Theta \subset C_0 \cap D_0$. Suppose that $\Theta \subset C_k \cap D_k$ for some $k \in \mathbb{N}$. Since $C_k \cap D_k$ is closed and convex, we can define $x_{k+1} = P_{C_k \cap D_k}(x_0)$. From $x_{k+1} = P_{C_k \cap D_k}(x_0)$, we also have

$$\langle x_{k+1} - z, x_0 - x_{k+1} \rangle \geq 0, \quad \forall z \in C_k \cap D_k. \tag{5.2.7}$$

Since $\Theta \subset C_k \cap D_k(x_0)$, we also have

$$\langle x_{k+1} - x^*, x_0 - x_{k+1} \rangle \geq 0, \quad \forall x^* \in \Theta. \tag{5.2.8}$$

So, we get $\Theta \subset D_{k+1}$. Then we obtain $\Theta \subset C_{k+1} \cap D_{k+1}$. This implies that $\{x_n\}$ is well defined.

Step 2. Next, let us show that $\{x_n\}, \{w_n\}$ are bounded. Put $z_0 = P_\Theta x_0$. From $x_{n+1} = P_{C_n \cap D_n}(x_0)$, we get

$$\|x_{n+1} - x_0\| \leq \|z - x_0\|, \quad \forall z \in C_n \cap D_n. \tag{5.2.9}$$

From $z_0 \in \Theta \subset C_n \cap D_n$, we also have

$$\|x_{n+1} - x_0\| \leq \|z_0 - x_0\| \tag{5.2.10}$$

for all $n \in \mathbb{N} \cup \{0\}$ and hence $\{x_n\}$ is bounded. Since $\|w_n - x^*\| \leq \|x_n - x^*\|$ then $\{w_n\}$ also bounded.

Step 3. We will show that $\lim_{n\to\infty} \|Sw_n - w_n\| = 0$.

Since $x_{n+1} \in C_n \cap D_n \subset D_n$ and $x_n = P_{D_n}(x_0)$, we get $\|x_n - x_0\| \leq \|x_{n+1} - x_0\|$. From the boundedness of $\{x_n\}$, we get that $\lim_{n\to\infty} \|x_n - x_0\|$ exists. So, we obtain $\left(\|x_n - x_0\|^2 - \|x_{n+1} - x_0\|^2\right) \to 0$. On the other hand, from $x_{n+1} \in D_n$, we have

$$\langle x_n - x_{n+1}, x_0 - x_n \rangle \geq 0. \tag{5.2.11}$$

So, for $n \in \mathbb{N} \cup \{0\}$, we get

$$\begin{aligned} \|x_{n+1} - x_0\|^2 &= \|(x_{n+1} - x_n) + (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_n\|^2 + \|x_n - x_0\|^2 + 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \end{aligned}$$

from (5.2.11), we have

$$\|x_{n+1} - x_0\|^2 \;\; \geq \;\; \|x_{n+1} - x_n\|^2 + \|x_n - x_0\|^2$$

and

$$\|x_{n+1} - x_n\|^2 \;\; \leq \;\; \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.$$

This implies

$$\|x_{n+1} - x_n\| \to 0. \tag{5.2.12}$$

Since $x_{n+1} \in C_n$, we have

$$C_n = \{z \in C : \|t_n - z\| \leq \|x_n - z\|\};$$

$$\|t_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$$

and

$$\begin{aligned} \|x_n - t_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - t_n\| \\ &\leq 2\|x_n - x_{n+1}\|. \end{aligned}$$

By (5.2.12), we obtain

$$\lim_{n\to\infty} \|x_n - t_n\| = 0. \tag{5.2.13}$$

For $x^* \in \Theta$, from (5.2.2), (5.2.3) and (5.2.4), we can choose a constant $M > 0$ such that, $\sup_n\{\|w_n\|^2\} \leq M$. We observe that

$$\begin{aligned} \|t_n - x^*\|^2 &\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2 \\ &\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\big[\|w_n - x^*\|^2 \\ &\quad -(1-\mu)(\mu-\xi)\|Sw_n - w_n\|^2 - \beta_n\mu^2\|w_n\|^2\big] \\ &= \|x_n - x^*\|^2 - (1-\alpha_n)(1-2\lambda_n c_1)\|y_n - x_n\|^2 \\ &\quad -(1-\alpha_n)(1-2\lambda_n c_2)\|z_n - y_n\|^2 + (1-\alpha_n)\lambda_n(\lambda_n - 2\alpha)\|Az - Ax^*\|^2 \\ &\quad -(1-\alpha_n)(1-\mu)(\mu-\xi)\|Sw_n - w_n\|^2 - (1-\alpha_n)\beta_n\mu^2 M. \end{aligned}$$

Therefore, we get

$$\begin{aligned} &(1-\alpha_n)(1-2\lambda_n c_1)\|y_n - x_n\|^2 \\ &\quad \leq \|x_n - x^*\|^2 - \|t_n - x^*\|^2 - (1-\alpha_n)\beta_n\mu^2 M \\ &\quad = \big[\|x_n - x^*\| + \|t_n - x^*\|\big]\big[\|x_n - t_n\| \\ &\qquad -(1-\alpha_n)\beta_n\mu^2 M\big]. \end{aligned}$$

Since $\lim_{n\to\infty}\|x_n - t_n\| = 0, \lim_{n\to\infty}\beta_n = 0, 1-\alpha_n \geq 1-c > 0, 1-2\lambda_n c_1 > 1-2\lambda_{\max}c_1 > 0$ and the sequence $\{x_n\}, \{t_n\}$ are bounded, we get $\lim_{n\to\infty}\|y_n - x_n\| = 0$. By similar way since $\lim_{n\to\infty}\beta_n = 0, 1-\alpha_n \geq 1-c > 0$ and $1-2\lambda_n c_2 > 1-2\lambda_{\max}c_2 > 0$, we have $\lim_{n\to\infty}\|z_n - y_n\| = 0$. Since $\lim_{n\to\infty}\beta_n = 0, 1-\alpha_n \geq 1-c > 0$ and $-\lambda_n(\lambda_n - 2\alpha) > 0$, we obtain

$$\lim_{n\to\infty}\|Az - Ax^*\| = 0.$$

By $\lim_{n\to\infty}\beta_n = 0, \quad 1-\alpha_n \geq 1-c > 0$ and $(1-\mu)(\mu-\xi) > 0$, we have

$$\lim_{n\to\infty}\|Sw_n - w_n\| = 0.$$

Step 4. We will show that $\tilde{x} \in \Theta$.

(4.1). We will show that $\tilde{x} \in EP(F)$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which $x_{n_i} \rightharpoonup \tilde{x}$ and $\lim_{n\to\infty}\|x_n - y_n\| = 0$, we have that $y_{n_i} \rightharpoonup \tilde{x}$. On the other hand, by using Proposition 2.6.35, we have, for every $y \in C$ and for every $i \in \mathbb{N}$, that

$$\langle x_{n_i} - y_{n_i}, y - y_{n_i}\rangle \leq \lambda_{n_i}F(x_{n_i}, y) - \lambda_{n_i}F(x_{n_i}, y_{n_i}). \tag{5.2.14}$$

Since $\|x_{n_i} - y_{n_i}\| \to 0$ and $y - y_{n_i} \rightharpoonup y - \tilde{x}$ as $i \to \infty$ and since $\forall i \in \mathbb{N}, \quad 0 < \lambda_{\min} \leq \lambda_{n_i} \leq \lambda_{\max}$. As $i \to \infty$, we get $F(\tilde{x}, y) \geq 0, \quad \forall y \in C$. It means that $\tilde{x} \in EP(F)$.

(4.2). We will show that $\tilde{x} \in Fix(S)$. Since $\{w_n\}$ is bounded then there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ which converges weakly to $\tilde{x}$. Since S is a ξ-strict pseudocontraction mapping, we know that the mapping $I - S$ is demiclosed at zero. From $\|Sw_n - w_n\| \to 0$ as $n \to \infty$ and $\{w_{n_i}\} \rightharpoonup \tilde{x}$. Thus, we obtain that $\tilde{x} \in Fix(S)$.

(4.3). Finally, we show that $\tilde{x} \in VI(C, A)$. Define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{5.2.15}$$

Then, we have that T is maximal monotone operator,

we have

$$\begin{aligned} x \in Tv = Av + N_C v \quad &\Leftrightarrow \quad x - Av \in N_C v \\ &\Leftrightarrow \quad \langle v - u, x - Av\rangle \geq 0, (\forall u \in C). \end{aligned}$$

So, for $w_n \in C$, we have that

$$\langle v - w_n, x - Av\rangle \geq 0, \quad (n = 1, 2, 3, ...).$$

We also have

$$\begin{aligned} w_n = P_C(z_n - \lambda_n Az_n) &\Leftrightarrow \langle w_n - u, z_n - \lambda_n Az_n - w_n\rangle \geq 0, \quad \forall u \in C \\ &\Leftrightarrow \langle u - w_n, w_n - (z_n - \lambda_n Az_n)\rangle \geq 0, \quad \forall u \in C \\ &\Leftrightarrow \Big\langle u - w_n, \frac{w_n - z_n}{\lambda_n} + Az_n\Big\rangle \geq 0, \quad \forall u \in C. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \langle v - w_{n_i}, x\rangle &\geq \langle v - w_{n_i}, Av\rangle \\ &\geq \langle v - w_{n_i}, Av\rangle - \Big\langle v - w_{n_i}, \frac{w_{n_i} - z_{n_i}}{\lambda_{n_i}} + Az_{n_i}\Big\rangle \\ &= \langle v - w_{n_i}, Av - Aw_{n_i}\rangle + \langle v - w_{n_i}, Aw_{n_i} - Az_{n_i}\rangle \\ &\quad -\Big\langle v - w_{n_i}, \frac{w_{n_i} - z_{n_i}}{\lambda_{n_i}}\Big\rangle \\ &\geq \langle v - w_{n_i}, Aw_{n_i} - Az_{n_i}\rangle - \Big\langle v - w_{n_i}, \frac{w_{n_i} - z_{n_i}}{\lambda_{n_i}}\Big\rangle. \end{aligned}$$

Using $w_{n_i} \to \tilde{x}$ and $\|w_{n_i} - z_{n_i}\| \to 0$ which A is Lipshitz continuous implies that

$$\langle v - \tilde{x}, x\rangle \geq 0, \quad \text{as} \quad i \to \infty. \tag{5.2.16}$$

Since T is maximal monotone, we have $\tilde{x} \in T^{-1}(0)$, That is $\tilde{x} \in VI(A, C)$. So, we have $\tilde{x} \in \Theta$.

Finally, we show that $x_n \to \tilde{x}$, where $\tilde{x} = P_\Theta x_0$. Since $x_n = P_{D_n}x_0$ and $\tilde{x} \in \Theta \subset D_n$, we have

$$\|x_n - x_0\| \leq \|\tilde{x} - x_0\|. \tag{5.2.17}$$

It follows from $x^* = P_\Theta x_0$ and the lower semicontinuity of norm that

$$\begin{aligned} \|x^* - x_0\| &\leq \|\tilde{x} - x_0\| \leq \liminf_{i\to\infty} \|x_{n_i} - x_0\| \\ &\leq \limsup_{i\to\infty} \|x_{n_i} - x_0\| \leq \|x^* - x_0\|. \end{aligned}$$

Thus, we obtain that

$$\lim_{i\to\infty} \|x_{n_i} - x_0\| = \|\tilde{x} - x_0\| = \|x^* - x_0\|. \tag{5.2.18}$$

Using the Kadec-Klee property of H, we obtain that $\lim_{i\to\infty} x_{n_i} = \tilde{x} = x^*$. Since $\{x_{n_i}\}$ is an arbitrary subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to $P_\Theta x_0$.

□

CHAPTER 6 CONCLUSIONS

In this dissertation, we establish the following results. First of all, we introduce an explicit method for finding the least norm of fixed points for strict pseudo mappings by using the projection technique. We provide algorithm which strong convergence theorems are obtained in Hilbert spaces.

Secondly, we introduce a new iterative method for finding a common element of the set of common solution of fixed point of a pseudo-contractive mapping and the set of common solutions of variational inequalities in Hilbert spaces and we prove a strong convergence theorem for finding a common element of the set of fixed points for a continuous pseudo-contractive mapping and the solution set of a variational inequality problem governed by a continuous monotone mapping.

Thirdly, we introduce a new iterative algorithm for solving a common solution of the set of solutions of fixed point for an infinite family of nonexpansive mappings, the set of solution of a system of mixed equilibrium problems and the set of solutions of the variational inclusion for an β-inverse-strongly monotone mapping in a real Hilbert space and we show a strong convergence theorem which solves the problem of finding a common element of the common fixed points, the common solution of a system of mixed equilibrium problem and variational inclusion of inverse-strongly monotone mappings in a Hilbert space.

Fourthly, we introduce a new iterative scheme that converges strongly to a common fixed point of a countable family of strictly pseudo-contractive mappings in a real Hilbert space which is also a solution to variational inequality problem related to quadratic minimization problems.

Finally, we introduce a new hybrid extragradient iterative algorithm we prove strong convergence of an iterative algorithm for finding a common element of the set of fixed points satisfying the equilibrium problem, variational inequalities and fixed point problems of a strict pseudocontraction mapping by the hybrid extragradient projection method in Hilbert spaces.

The following results are all main theorems of this dissertation:

(1). Let C be a closed convex of a real *Hilbert space* H. Let $S : C \to C$ be a nonexpansive mapping and $\Omega := F(S) \cap VI(C, A) \neq \emptyset$. Assume that a sequence $\{\alpha_n\} \subset (0, 1)$ satisfies the conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\Sigma_{n=0}^{\infty} \alpha_n = +\infty$.

Then the sequence $\{x_n\}$ generated by the following algorithm

$$x_{n+1} = (1 - \alpha_n)[\lambda S x_n + (1 - \lambda)x_n] \tag{6.0.1}$$

converges strongly to a fixed point of S which is of minimal norm which the unique solution of the variational inequality:

$$x^* \in \Omega, \langle x^*, x - x^* \rangle \geq 0, \forall x \in \Omega.$$

(2). Assume that the solution set of (3.1.1) is nonempty. Let the objective function f be convex, fréchet differentiable and it gradient ∇f is Lipschitz continuous with Lipschitz constant L. In addition, if $0 \in C$ or C is closed convex cone. Let $\mu \in (0, \frac{2}{L})$ and define a sequence $\{x_n\}$ by following

$$x_{n+1} = (1 - \alpha_n)((I - \mu\nabla f)(x_n) + (1 - \lambda)x_n), \quad n \geq 0$$

where $\lambda \in (0, 1)$ and the sequence $\{\alpha_n\} \subset (0, 1)$ satisfies conditions in Theorem 3.1.1. Then the sequence $\{x_n\}$ converges strongly to the minimum-norm solution of the minimization (3.1.1)

(3). Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a continuous pseudo-contractive mapping and $A : C \to H$ be a continuous monotone mapping such that $\mathfrak{F} := F(T) \cap VI(C, A) \neq \emptyset$. For $x \in H$, define T_{r_n} and F_{r_n} as follows:

$$T_r x := \{z \in C : \langle y - z, Tz \rangle + \frac{1}{r}\langle y - z, (1 + r)z - x \rangle \leq 0, \quad \forall y \in C\}$$

and

$$F_r x := \{z \in C : \langle y - z, Az \rangle + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C\}.$$

Let $B : H \to H$ be a strongly positive linear bounded self-adjoint operator with coefficients $\overline{\beta} > 0$ and let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$\begin{cases} y_n = F_{r_n} x_n \\ x_{n+1} = \alpha_n u + \delta_n x_n + [(1-\delta_n)I - \alpha_n B]T_{r_n} y_n, \end{cases} \tag{6.0.2}$$

where $\{\alpha_n\}, \{\delta_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ such that

(C1) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty;$

(C2) $\lim_{n\to\infty} \delta_n = 0, \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty;$

(C3) $\liminf_{n\to\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$

Then, the sequence $\{x_n\}_{n\geq 1}$ converges strongly to $z \in \mathfrak{F}$, which is the unique solution of the variational inequality:

$$\langle (B-f)z, x-z \rangle \geq 0, \quad \forall x \in \mathfrak{F}. \tag{6.0.3}$$

Equivalently, $z = P_{\mathfrak{F}}(I - B + f)z$.

(4). Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $A : C \to H$ be a continuous monotone mapping such that $\mathfrak{F} := VI(C, A) \neq \emptyset$. For $x \in H$, define F_{r_n} as follows:

$$F_r x := \{z \in C : \langle y - z, Az \rangle + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C\}.$$

Let f be a contraction of H into itself with a contraction constant β and let $B : H \to H$ be a strongly positive linear bounded self-adjoint operator with coefficients $\overline{\beta} > 0$ and let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + [(1-\delta_n)I - \alpha_n B]F_{r_n} x_n, \tag{6.0.4}$$

where $\{\alpha_n\}, \{\delta_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ such that

(C1) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty;$

(C2) $\lim_{n\to\infty} \delta_n = 0, \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty;$

(C3) $\liminf_{n\to\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$

Then, the sequence $\{x_n\}_{n\geq 1}$ converges strongly to $z \in \mathfrak{F}$, which is the unique solution of the variational inequality:

$$\langle (B - \gamma f)z, x - z\rangle \geq 0, \quad \forall x \in \mathfrak{F} \tag{6.0.5}$$

Equivalently, $z = P_{\mathfrak{F}}(I - B + \gamma f)z$.

(5). Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a continuous pseudo-contractive mapping and $\mathfrak{F} := F(T) \neq \emptyset$. For $x \in H$, define T_{r_n} as follows:

$$T_r x := \{z \in C : \langle y - z, Tz\rangle + \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0, \quad \forall y \in C\}.$$

Let f be a contraction of H into itself with a contraction constant β and let $B : H \to H$ be a strongly positive linear bounded self-adjoint operator with coefficients $\overline{\beta} > 0$ and let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + [(1 - \delta_n)I - \alpha_n B]T_{r_n} x_n, \tag{6.0.6}$$

where $\{\alpha_n\}, \{\delta_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ such that

(C1) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty;$

(C2) $\lim_{n\to\infty} \delta_n = 0, \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty;$

(C3) $\liminf_{n\to\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$

Then, the sequence $\{x_n\}_{n\geq 1}$ converges strongly to $z \in F$, which is the unique solution of the variational inequality:

$$\langle (B - \gamma f)z, x - z\rangle \geq 0, \quad \forall x \in \mathfrak{F}. \tag{6.0.7}$$

Equivalently, $z = P_{\mathfrak{F}}(I - B + \gamma f)z$.

(6). Let H be a real Hilbert space. Let $T_n : H \to H$ be a continuous pseudo-contractive mapping and $A : H \to H$ be a continuous monotone mapping such that $\mathfrak{F} := F(T) \cap A^{-1}(0) \neq \emptyset$. For $x \in H$, define T_{r_n} and F_{r_n} as follows:

$$T_r x := \{z \in C : \langle y - z, Tz\rangle + \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0, \quad \forall y \in C\}$$

and

$$F_r x := \{z \in C : \langle y - z, Az\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\}.$$

Let f be a contraction of C into itself with a contraction constant β and let $B : H \to H$ be a strongly positive linear bounded self-adjoint operator with coefficients $\overline{\beta} > 0$ and let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$\begin{cases} y_n = F_{r_n} x_n \\ x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + [(1-\delta_n)I - \alpha_n B]T_{r_n} y_n \end{cases} \tag{6.0.8}$$

where $\{\alpha_n\}, \{\delta_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ such that

(C1) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\lim_{n\to\infty} \delta_n = 0, \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$;

(C3) $\liminf_{n\to\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then, the sequence $\{x_n\}_{n\geq 1}$ converges strongly to $z \in \mathfrak{F}$, which is the unique solution of the variational inequality:

$$\langle (B - \gamma f)z, x - z\rangle \geq 0, \quad \forall x \in \mathfrak{F} \tag{6.0.9}$$

Equivalently, $z = P_{\mathfrak{F}}(I - B + \gamma f)z$.

(7). Let H be a real Hilbert space, C a close convex subset of H and B be an β-inverse-strongly monotone operator. Let $\varphi : C \to \mathbb{R}$ be a convex and lower semicontinuous function, $f : C \to C$ be a contraction mapping with coefficient α $(0 < \alpha < 1)$, $M : H \to 2^H$ be a maximal monotone operator. Let A be a strongly positive linear bounded operator of H into itself with coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$ and $\lambda \in (0, 2\beta)$. Let $\{T_n\}$ be a family of nonexpansive mappings of H into itself such that

$$\theta := \cap_{n=1}^{\infty} F(T_n) \cap (\cap_{k=1}^{N} SMEP(F_k)) \cap I(B, M) \neq \emptyset.$$

Suppose $\{x_n\}$ is a sequence generated by the following algorithm for $x_0, \ u_n \in C$ arbitrarily:

$$\begin{cases} u_n = K_{r_n,n}^{F_N} \cdot K_{r_{n-1},n}^{F_{N-1}} \cdot K_{r_{n-2},n}^{F_{N-2}} \cdot \ldots \cdot K_{r_2,n}^{F_2} \cdot K_{r_1,n}^{F_1} \cdot x_n, \ \forall n \in N \\ x_{n+1} = P_C[\epsilon_n f(x_n) + (I - \epsilon_n) W_n J_{M,\lambda}(u_n - \lambda B u_n)] \end{cases} \tag{6.0.10}$$

for all $n = 0, 1, 2, ...,$ and the conditions (C1)-(C2) in Theorem 4.1.1 are satisfied.

Then, the sequence $\{x_n\}$ converges strongly to $q \in \theta$, where $q = P_\theta(f + I)(q)$ which solves the following variational inequality:

$$\langle (f - I)q, p - q\rangle \leq 0, \ \forall p \in \theta.$$

(8). Let H be a real Hilbert space, C a close convex subset of H and B be an β-inverse-strongly monotone operator. Let $\varphi : C \to \mathbb{R}$ be a convex and lower semicontinuous function, $f : C \to C$ be a contraction mapping with coefficient α $(0 < \alpha < 1)$, $M : H \to 2^H$ be a maximal monotone operator. Let A be a strongly positive linear bounded operator of H into itself with coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$ and $\lambda \in (0, 2\beta)$. Let $\{T_n\}$ be a family of nonexpansive mappings of H into itself such that

$$\theta := \cap_{n=1}^{\infty} F(T_n) \cap (\cap_{k=1}^{N} SMEP(F_k)) \cap I(B, M) \neq \emptyset.$$

Suppose $\{x_n\}$ is a sequence generated by the following algorithm for $x_0, u \in C$ and $u_n \in C$:

$$\begin{cases} u_n = K_{r_n,n}^{F_N} \cdot K_{r_{n-1},n}^{F_{N-1}} \cdot K_{r_{n-2},n}^{F_{N-2}} \cdot ... \cdot K_{r_2,n}^{F_2} \cdot K_{r_1,n}^{F_1} \cdot x_n, \ \forall n \in N \\ x_{n+1} = P_C[\epsilon_n u + (I - \epsilon_n) W_n J_{M,\lambda}(u_n - \lambda B u_n)] \end{cases} \tag{6.0.11}$$

for all $n = 0, 1, 2, ...,$ and the conditions (C1)-(C2) in Theorem 4.1.1 are satisfied.

Then, the sequence $\{x_n\}$ converges strongly to $q \in \theta$, where $q = P_\theta(q)$ which solves the following variational inequality:

$$\langle u - q, p - q\rangle \leq 0, \ \forall p \in \theta.$$

(9). Let H be a real Hilbert space, C a close convex subset of H and B be an β-inverse-strongly monotone operator. Let $f : C \to C$ be a contraction mapping with coefficient α $(0 < \alpha < 1)$, $M : H \to 2^H$ be a maximal monotone operator. Let A be a strongly positive linear bounded operator of H into itself with coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\{T_n\}$ be a family of nonexpansive mappings of H into itself such that

$$\theta := \cap_{n=1}^{\infty} F(T_n) \cap VI(C, B) \neq \emptyset.$$

Suppose $\{x_n\}$ is a sequence generated by the following algorithm for $x_0 \in C$ arbitrarily:

$$x_{n+1} = P_C\Big[\epsilon_n \gamma f(x_n) + (I - \epsilon_n A)W_n P_C(x_n - \lambda B x_n)\Big] \tag{6.0.12}$$

for all $n = 0, 1, 2, ...$, and the conditions (C1)-(C2) in Theorem 4.1.1 are satisfied.

Then, the sequence $\{x_n\}$ converges strongly to $q \in \theta$, where $q = P_\theta(\gamma f + I - A)(q)$ which solves the following variational inequality:

$$\langle(\gamma f - A)q, p - q\rangle \leq 0, \ \forall p \in \theta.$$

(10). Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $f : C \to H$ be a ρ-contraction mapping, $S : C \to H$ be a nonexpansive mapping and $T : C \to C$ be a k-strict pseudo-contraction mapping such that $F(T) \neq \emptyset$. Let $x_1 \in C$ and define the sequence $\{x_n\}$ by

$$\begin{cases} y_n = P_C[\beta_n S x_n + (1 - \beta_n)x_n], \\ x_{n+1} = P_C[\alpha_n f(x_n) + (1 - \alpha_n)V y_n], \quad \forall n \geq 1, \end{cases} \tag{6.0.13}$$

where $V = kI + (1-k)T$, $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\} \subset (0, 1)$ are sequences satisfying the conditions (H2) with $\tau = 0$, (H3), either (H4) and (H5) , or (H6) and (H7). Then the sequence $\{x_n\}$ converges strongly to a point $z \in F(T)$, which is the unique solution of the variational inequality:

$$\langle(I - f)z, x - z\rangle \geq 0, \quad \forall x \in F(T).$$

(11). Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $f : C \to H$ be a ρ-contraction mapping, $S : C \to H$ be a nonexpansive mapping and $\{T_i\}_{i=1}^{\infty} : C \to C$ be a countable family of nonexpansive mappings and $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\alpha_0 = 1$, $x_1 \in C$ and define the sequence $\{x_n\}$ by

$$\begin{cases} y_n = P_C[\beta_n S x_n + (1 - \beta_n)x_n], \\ x_{n+1} = P_C[\alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)T_i y_n], \quad \forall n \geq 1, \end{cases} \tag{6.0.14}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\alpha_n\}$ is a strictly decreasing sequence, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions (H2) with $\tau = 0$, (H3), either (H4) and (H5) , or (H6) and (H7). Then the sequence $\{x_n\}$ converges strongly to a point $z \in \mathcal{F}$, which is the unique solution of the variational inequality:

$$\langle(I - f)z, x - z\rangle \geq 0, \quad \forall x \in \mathcal{F}.$$

(12). Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $f : C \to H$ be a ρ-contraction mapping, $S : C \to H$ be a nonexpansive mapping and $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $x_1 \in C$ and define the sequence $\{x_n\}$ by

$$\begin{cases} y_n = P_C[\beta_n S x_n + (1-\beta_n)x_n], \\ x_{n+1} = P_C[\alpha_n f(x_n) + (1-\alpha_n)Ty_n], \quad \forall n \geq 1, \end{cases} \tag{6.0.15}$$

where $\{\alpha_n\} \subset (0,1), \{\beta_n\} \subset (0,1)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions (H2) with $\tau = 0$, (H3), either (H4) and (H5) , or (H6) and (H7). Then the sequence $\{x_n\}$ converges strongly to a point $z \in F(T)$, which is the unique solution of the variational inequality:

$$\langle (I-f)z, x-z \rangle \geq 0, \quad \forall x \in F(T).$$

(13). Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $f : C \to H$ be a ρ-contraction mapping, $S : C \to C$ be a nonexpansive mapping and $T : C \to C$ be a k-strict pseudo-contraction mapping and $\mathcal{F} = F(T) \neq \emptyset$. Let $x_1 \in C$ and define the sequence $\{x_n\}$ by

$$\begin{cases} y_n = \beta_n S x_n + (1-\beta_n)x_n, \\ x_{n+1} = P_C[\alpha_n f(x_n) + (1-\alpha_n)Vy_n], \quad \forall n \geq 1, \end{cases} \tag{6.0.16}$$

where $V = kI + (1-k)T$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions (H2) with $\tau \in (0,\infty)$, (H3), (H8) and (H9). Then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \mathcal{F}$, which is the unique solution of the variational inequality:

$$\langle \frac{1}{\tau}(I-f)x^* + (I-S)x^*, x-x^* \rangle \geq 0, \quad \forall x \in \mathcal{F}. \tag{6.0.17}$$

(14). Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $f : C \to H$ be a ρ-contraction mapping, $S : C \to C$ be a nonexpansive mapping and $\{T_i\}_{i=1}^{\infty} : C \to C$ be a countable family of nonexpansive mappings and $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\alpha_0 = 1$, $x_1 \in C$ and define the sequence $\{x_n\}$ by

$$\begin{cases} y_n = \beta_n S x_n + (1-\beta_n)x_n, \\ x_{n+1} = P_C[\alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)T_i y_n], \quad \forall n \geq 1, \end{cases} \tag{6.0.18}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\alpha_n\}$ is a strictly decreasing sequence, $\{\beta_n\} \subset (0,1)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions (H2) with $\tau \in (0,\infty)$, (H3), (H8) and (H9). Then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \mathcal{F}$, which is the unique solution of the variational inequality:

$$\langle \frac{1}{\tau}(I-f)x^* + (I-S)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{F}. \tag{6.0.19}$$

(15). Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $f: C \to H$ be a ρ-contraction mapping, $S, T : C \to C$ be nonexpansive mappings and $\mathcal{F} = F(T) \neq \emptyset$. Let $x_1 \in C$ and define the sequence $\{x_n\}$ by

$$\begin{cases} y_n = \beta_n S x_n + (1-\beta_n)x_n, \\ x_{n+1} = P_C[\alpha_n f(x_n) + (1-\alpha_n)Ty_n], \quad \forall n \geq 1, \end{cases} \tag{6.0.20}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions (H2) with $\tau \in (0,\infty)$, (H3), (H8) and (H9). Then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \mathcal{F}$, which is the unique solution of the variational inequality:

$$\langle \frac{1}{\tau}(I-f)x^* + (I-S)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{F}. \tag{6.0.21}$$

(16). Let C be a nonempty, closed and convex subset of a real Hilbert space H, let F be a bifunction $F : C \times C \to \mathbb{R}$ satisfying conditions $(A1) - (A5)$, let A be an α-inverse-strongly monotone mapping of C into H. Let S be a ξ-strict pseudocontraction mapping from C to C and such that $\Theta := EP(F) \cap Fix(S) \cap VI(C,A) \neq \emptyset$. Suppose that the sequences $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}$ and μ satisfying the condition $(i) - (iv)$. Then the sequences $\{x_n\}$ generated by Algorithm 1 converges strongly to the projection of x_0 on to the set Θ.

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