CHAPTER 5 HIERARCHICAL FIXED POINTS, EQUILIBRIUM AND VARIATIONAL INEQUALITY PROBLEMS

In this section, we introduce a new iterative scheme that converges strongly to a common fixed point of a countable family of strictly pseudo-contractive mappings in a real Hilbert space which is also a solution to variational inequality problems related to quadratic minimization problems. Also a new hybrid extragradient iterative algorithm for solving a common element of the set of fixed points satisfying equilibrium problems, variational inequality problems and fixed point problems of a strict pseudocontraction mapping in Hilbert spaces are obtain.

5.1 Hierarchical Fixed Points and Variational Inequality Problems

Let H be a real Hilbert space, $T : C \to H$ be a mapping. The following problem is called a *hierarchical fixed point problem*: Find $x^* \in F(T)$ such that

$$\langle x^* - Sx^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T). \tag{5.1.1}$$

where $S : C \to H$ be a mapping.

Let us consider the net iterative scheme as follows:

$$\begin{cases} y_n = P_C[\beta_n S x_n + (1-\beta_n)x_n], \\ x_{n+1} = P_C[\alpha_n f(x_n) + \sum\limits_{i=1}^{n}(\alpha_{i-1} - \alpha_i)V_i y_n], \quad \forall n \geq 1, \end{cases} \tag{5.1.2}$$

where $V_i = k_i I + (1-k_i)T_i$, $f : C \to H$ is a ρ-contraction mapping, $S : C \to H$ is a nonexpansive mapping, $\{T_i\}_{i=1}^{\infty} : C \to C$ is a countable family of k_i-strict pseudo-contraction mappings and $\bigcap\limits_{i=1}^{\infty} F(T_i) \neq \emptyset$. Set $\alpha_0 = 1$, $\{\alpha_n\} \subset (0,1)$ is a strictly decreasing sequence and $\{\beta_n\} \subset (0,1)$. As we will see the convergence of the scheme depends on the choice of the parameters $\{\alpha_n\}$ and $\{\beta_n\}$. We list some possible hypotheses on them:

(H1) there exists $\gamma > 0$ such that $\beta_n \leq \gamma\alpha_n$;

(H2) $\lim\limits_{n\to\infty} \beta_n/\alpha_n = \tau \in [0, \infty)$;

(H3) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(H4) $\sum\limits_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$;

(H5) $\sum\limits_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$;

(H6) $\lim\limits_{n\to\infty} |\alpha_n - \alpha_{n-1}|/\alpha_n = 0$;

(H7) $\lim\limits_{n\to\infty} |\beta_n - \beta_{n-1}|/\beta_n = 0$;

(H8) $\lim\limits_{n\to\infty} [|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|]/\alpha_n\beta_n = 0$;

(H9) there exists a constant $K > 0$ such that $\frac{1}{\alpha_n}|\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}| \leq K$.

Proposition 5.1.1. *Assume that (H1) holds. Then $\{x_n\}$ and $\{y_n\}$ are bounded.*

Proof (1) Let $z \in \bigcap\limits_{i=1}^{\infty} F(T_i) = \bigcap\limits_{i=1}^{\infty} F(V_i)$. Then we have

$$\begin{aligned}
\|x_{n+1} - z\| &= \left\| P_C[\alpha_n f(x_n) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) V_i y_n] - P_C[z] \right\| \\
&\leq \left\| \alpha_n f(x_n) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) V_i y_n - z \right\| \\
&= \left\| \alpha_n (f(x_n) - z) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i)(V_i y_n - z) \right\| \\
&\leq \alpha_n \|f(x_n) - f(z)\| + \alpha_n \|f(z) - z\| \\
&\quad + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \|V_i y_n - z\| \\
&\leq \alpha_n \rho \|x_n - z\| + \alpha_n \|f(z) - z\| \\
&\quad + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \|y_n - z\| \\
&\leq \alpha_n \rho \|x_n - z\| + \alpha_n \|f(z) - z\| \\
&\quad + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \|\beta_n S x_n + (1 - \beta_n) x_n - z\|
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n\rho\|x_n - z\| + \alpha_n\|f(z) - z\| \\
&\quad + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)(\beta_n\|Sx_n - Sz\| + \beta_n\|Sz - z\| \\
&\quad + (1 - \beta_n)\|x_n - z\|) \\
&\leq \alpha_n\rho\|x_n - z\| + \alpha_n\|f(z) - z\| \\
&\quad + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)(\beta_n\|x_n - z\| + \beta_n\|Sz - z\| \\
&\quad + (1 - \beta_n)\|x_n - z\|) \\
&= \alpha_n\rho\|x_n - z\| + \alpha_n\|f(z) - z\| \\
&\quad + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)(\|x_n - z\| + \beta_n\|Sz - z\|) \\
&= \alpha_n\rho\|x_n - z\| + \alpha_n\|f(z) - z\| \\
&\quad + (1 - \alpha_n)(\|x_n - z\| + \beta_n\|Sz - z\|) \\
&= (1 - \alpha_n(1 - \rho))\|x_n - z\| + \alpha_n\|f(z) - z\| + (1 - \alpha_n)\beta_n\|Sz - z\| \\
&\leq (1 - \alpha_n(1 - \rho))\|x_n - z\| + \alpha_n\|f(z) - z\| + \beta_n\|Sz - z\| \\
&\leq (1 - \alpha_n(1 - \rho))\|x_n - z\| + \alpha_n[\|f(z) - z\| + \gamma\|Sz - z\|]. \qquad (5.1.3)
\end{aligned}$$

So, by induction, one can obtain that

$$\|x_n - z\| \leq \max\left\{\|x_0 - z\|, \frac{1}{1-\rho}[\|f(z) - z\| + \gamma\|Sz - z\|]\right\}. \qquad (5.1.4)$$

Hence $\{x_n\}$ is bounded. Of course $\{y_n\}$ is bounded too.

□

Proposition 5.1.2. *Suppose that (H1) and (H3) hold. Also, assume that either (H4) and (H5) hold, or (H6) and (H7) hold. Then*

(1) $\{x_n\}$ is asymptotically regular, that is,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0, \qquad (5.1.5)$$

(2) the weak cluster points set $\omega_w(x_n) \subset \bigcap_{i=1}^{\infty} F(T_i)$.

proof (2) Set $u_n = \alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)V_i y_n$. From (5.1.2) and since P_C is a nonexpansive mapping, we have

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|P_C[u_n]-P_C[u_{n-1}]\| \\
&\le \|u_n-u_{n-1}\| \qquad (5.1.6)\\
&= \Bigg\|\alpha_n(f(x_n)-f(x_{n-1}))+(\alpha_n-\alpha_{n-1})f(x_{n-1}) \\
&\quad +\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(V_iy_n-V_iy_{n-1})+(\alpha_{n-1}-\alpha_n)V_ny_{n-1}\Bigg\| \\
&\le \alpha_n\|f(x_n)-f(x_{n-1})\|+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|y_n-y_{n-1}\| \\
&\quad |\alpha_n-\alpha_{n-1}|(\|f(x_{n-1})\|+\|V_ny_{n-1}\|) \\
&\le \alpha_n\rho\|x_n-x_{n-1}\|+(1-\alpha_n)\|y_n-y_{n-1}\| \\
&\quad +|\alpha_n-\alpha_{n-1}|(\|f(x_{n-1})\|+\|V_ny_{n-1}\|). \qquad (5.1.7)
\end{aligned}$$

By definition of y_n one obtain that

$$\begin{aligned}
\|y_n-y_{n-1}\| &= \|P_C[\beta_nSx_n+(1-\beta_n)x_n]-P_C[\beta_{n-1}Sx_{n-1}+(1-\beta_{n-1})x_{n-1}]\| \\
&\le \|(\beta_nSx_n+(1-\beta_n)x_n)-(\beta_{n-1}Sx_{n-1}+(1-\beta_{n-1})x_{n-1})\| \\
&= \|\beta_n(Sx_n-Sx_{n-1})+(\beta_n-\beta_{n-1})Sx_{n-1} \\
&\quad +(1-\beta_{n-1})(x_n-x_{n-1})+(\beta_{n-1}-\beta_n)x_{n-1}\| \\
&\le \|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|(\|Sx_{n-1}\|+\|x_{n-1}\|). \qquad (5.1.8)
\end{aligned}$$

So, substituting (5.1.8) in (5.1.7), we obtain

$$\begin{aligned}
\|x_{n+1}-x_n\| &\le \alpha_n\rho\|x_n-x_{n-1}\| \\
&\quad +(1-\alpha_n)[\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|(\|Sx_{n-1}\|+\|x_{n-1}\|)] \\
&\quad +|\alpha_n-\alpha_{n-1}|(\|f(x_{n-1})\|+\|V_ny_{n-1}\|) \\
&\le (1-(1-\rho)\alpha_n)\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|(\|Sx_{n-1}\|+\|x_{n-1}\|) \\
&\quad +|\alpha_n-\alpha_{n-1}|(\|f(x_{n-1})\|+\|V_ny_{n-1}\|). \qquad (5.1.9)
\end{aligned}$$

By Proposition 5.1.1, we say

$$M:=\max\left\{\sup_{n\ge1}\{\|Sx_{n-1}\|+\|x_{n-1}\|\},\sup_{n\ge1}\{\|f(x_{n-1})\|+\|V_ny_{n-1}\|\}\right\}.$$

So, we have

$$\|x_{n+1}-x_n\|\le(1-(1-\rho)\alpha_n)\|x_n-x_{n-1}\|+M[|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|]. \qquad (5.1.10)$$

So, if (H4) and (H5) hold, we obtain the asymptotic regularity by Lemma 2.1.23, if instead, (H6) and (H7) hold, from (H1), we can write

$$\begin{aligned}\|x_{n+1}-x_n\| &\leq (1-(1-\rho)\alpha_n)\|x_n-x_{n-1}\|\\ &\quad +M\alpha_n\left[\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n}+\frac{|\beta_n-\beta_{n-1}|}{\alpha_n}\right]\\ &\leq (1-(1-\rho)\alpha_n)\|x_n-x_{n-1}\|\\ &\quad +M\alpha_n\left[\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n}+\gamma\frac{|\beta_n-\beta_{n-1}|}{\beta_n}\right]. \qquad (5.1.11)\end{aligned}$$

By Lemma 2.1.23, we obtain the asymptotic regularity.

In order to prove (2), since $V_i x_n \in C$ for each $i \geq 1$ and $\sum\limits_{n=1}^{\infty}(\alpha_{n-1}-\alpha_n)+\alpha_n = 1$, we have

$$\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i x_n+\alpha_n p \in C, \quad \forall p \in C. \qquad (5.1.12)$$

Now, fixing a $p \in \bigcap\limits_{i=1}^{\infty} F(Vi)$, from (5.1.2), we have

$$\begin{aligned}&\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(x_n-V_i x_n)\\ &= P_C[u_n]+(1-\alpha_n)x_n-\left(\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i x_n+\alpha_n p\right)+\alpha_n p-x_{n+1}\\ &= P_C[u_n]-P_C\left[\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i x_n+\alpha_n p\right]+(1-\alpha_n)(x_n-x_{n+1})\\ &\quad +\alpha_n(p-x_{n+1}).\end{aligned}$$

It follows that

$$\begin{aligned}&\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle x_n-V_i x_n, x_n-z\rangle\\ &= \left\langle P_C[u_n]-P_C\left[\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i x_n+\alpha_n p\right], x_n-z\right\rangle\\ &\quad +(1-\alpha_n)\langle x_n-x_{n+1}, x_n-z\rangle+\alpha_n\langle p-x_{n+1}, x_n-z\rangle\\ &\leq \left\|u_n-\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i x_n+\alpha_n p\right\|\|x_n-z\|\\ &\quad +(1-\alpha_n)\|x_n-x_{n+1}\|\|x_n-z\|+\alpha_n\|p-x_{n+1}\|\|x_n-z\|\\ &= \left\|\alpha_n(f(x_n)-p)+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(V_i y_n-V_i x_n)\right\|\|x_n-z\|\\ &\quad +(1-\alpha_n)\|x_n-x_{n+1}\|\|x_n-z\|+\alpha_n\|p-x_{n+1}\|\|x_n-z\|\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n\|f(x_n)-p\|\|x_n-z\|+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|y_n-x_n\|\|x_n-z\| \\
&\quad +(1-\alpha_n)\|x_n-x_{n+1}\|\|x_n-z\|+\alpha_n\|p-x_{n+1}\|\|x_n-z\| \\
&\leq \alpha_n\|f(x_n)-p\|\|x_n-z\|+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\beta_n\|Sx_n-x_n\|\|x_n-z\| \\
&\quad +(1-\alpha_n)\|x_n-x_{n+1}\|\|x_n-z\|+\alpha_n\|p-x_{n+1}\|\|x_n-z\| \\
&= \alpha_n\|f(x_n)-p\|\|x_n-z\|+(1-\alpha_n)\beta_n\|Sx_n-x_n\|\|x_n-z\| \\
&\quad +(1-\alpha_n)\|x_n-x_{n+1}\|\|x_n-z\|+\alpha_n\|p-x_{n+1}\|\|x_n-z\|. \qquad (5.1.13)
\end{aligned}
$$

Now, from Lemma 2.6.33 and (5.1.13), we get

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|x_n-V_ix_n\|^2 \\
\leq\ & \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle x_n-V_ix_n, x_n-z\rangle \\
\leq\ & \alpha_n\|f(x_n)-p\|\|x_n-z\|+(1-\alpha_n)\beta_n\|Sx_n-x_n\|\|x_n-z\| \\
& +(1-\alpha_n)\|x_n-x_{n+1}\|\|x_n-z\|+\alpha_n\|p-x_{n+1}\|\|x_n-z\|.
\end{aligned}
$$

By (H1) and (H3), it follows that $\beta_n \to 0$, as $n \to \infty$, so that

$$\lim_{n\to\infty}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|x_n-V_ix_n\|^2=0. \qquad (5.1.14)$$

Since $(\alpha_{i-1}-\alpha_i)\|x_n-V_ix_n\|^2 \leq \sum\limits_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|x_n-V_ix_n\|^2$ for each $i \geq 1$ and $\{\alpha_n\}$ is strictly decreasing, one has

$$\lim_{n\to\infty}\|x_n-V_ix_n\|=0, \quad \forall i \geq 1. \qquad (5.1.15)$$

Hence, we obtain

$$\lim_{n\to\infty}\|x_n-T_ix_n\|=\lim_{n\to\infty}\frac{\|x_n-V_ix_n\|}{(1-k_i)}=0, \quad \forall i \geq 1.$$

Since $\{x_n\}$ is asymptotically regular and demiclosedness principle, we obtain the proposition.

Corollary 5.1.3. *Suppose that the hypotheses of Proposition 5.1.2 hold. Then*

(i) $\lim\limits_{n\to\infty}\|x_n-y_n\|=0;$

(ii) $\lim\limits_{n\to\infty}\|x_n-V_iy_n\|=0, \quad \forall i \geq 1;$

(iii) $\lim\limits_{n\to\infty}\|y_n - V_iy_n\| = 0, \quad \forall i \geq 1.$

Proof. To prove (i), we can observe that

$$\|x_n - y_n\| \leq \beta_n\|x_n - Sx_n\|.$$

Since $\beta_n \to 0$ as $n \to \infty$, we obtain (i).
To prove (ii), we observe that

$$\|y_n - V_ix_n\| \leq \|y_n - x_n\| + \|x_n - V_ix_n\|, \;\; \forall i \geq 1$$

and

$$\|x_n - V_iy_n\| \leq \|x_n - y_n\| + \|y_n - V_ix_n\|, \;\; \forall i \geq 1.$$

Since $\|y_n - x_n\| \to 0$ and $\|x_n - V_ix_n\| \to 0$ as $n \to \infty$, $\forall i \geq 1$, then $\|y_n - V_ix_n\| \to 0$, that is, we obtain (ii). To prove (iii), we can observe that

$$\|y_n - V_iy_n\| \leq \|x_n - y_n\| + \|x_n - V_iy_n\|, \;\; \forall i \geq 1.$$

By (i) and (ii), we obtain (iii). □

Theorem 5.1.4. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $f : C \to H$ be a ρ-contraction mapping, $S : C \to H$ be a nonexpansive mapping and $\{T_i\}_{i=1}^{\infty} : C \to C$ be a countable family of k_i-strict pseudo-contraction mappings and $\mathcal{F} = \bigcap\limits_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\alpha_0 = 1$, and $x_1 \in C$ and define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = P_C[\beta_nSx_n + (1-\beta_n)x_n], \\ x_{n+1} = P_C[\alpha_nf(x_n) + \sum\limits_{i=1}^{n}(\alpha_{i-1} - \alpha_i)V_iy_n], \;\; \forall n \geq 1, \end{cases} \tag{5.1.16}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\alpha_n\}$ is a strictly decreasing sequence, $V_i = k_iI + (1-k_i)T_i$, $\{\beta_n\} \subset (0,1)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions (H2) with $\tau = 0$, (H3), either (H4) and (H5), or (H6) and (H7). Then the sequence $\{x_n\}$ converges strongly to a point $z \in \mathcal{F}$, which is the unique solution of the variational inequality:

$$\langle (I-f)z, x-z\rangle \geq 0, \;\; \forall x \in \mathcal{F}. \tag{5.1.17}$$

Proof. First of all, since $P_{\mathcal{F}}f$ is a contraction. By Banach contraction principle, so there exists a unique $z \in \mathcal{F}$ such that $z = P_{\mathcal{F}}f(z)$, Moreover, from Lemma 2.1.12 (1), we have

$$\langle f(z) - z, y - z\rangle \leq 0, \;\; \forall y \in \mathcal{F}.$$

Since (H2) implies (H1), thus $\{x_n\}$ is bounded. Moreover, since either (H4) and (H5), or (H6) and (H7), then $\{x_n\}$ is asymptotically regular. Similarly, by Proposition 5.1.2, the weak cluster points set of x_n, that is, $\omega_w(x_n)$, is a subset of $\mathcal{F}$.

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(z)-z, x_n-z\rangle = \lim_{k\to\infty}\langle f(z)-z, x_{n_k}-z\rangle,$$

and $x_{n_k}\to x'$. By Proposition 5.1.2 it follows that $x'\in\mathcal{F}$. Then

$$\lim_{k\to\infty}\langle f(z)-z, x_{n_k}-z\rangle = \langle f(z)-z, x'-z\rangle \le 0.$$

Set $u_n = \alpha_n f(x_n) + \sum\limits_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i y_n$, we obtain

$$\|x_{n+1}-z\|^2 = \langle P_C[u_n]-u_n, P_C[u_n]-z\rangle + \langle u_n - z, x_{n+1}-z\rangle. \tag{5.1.18}$$

By Lemma 2.1.12 (1), we have

$$\langle P_C[u_n]-u_n, P_C[u_n]-z\rangle \le 0. \tag{5.1.19}$$

From (5.1.18) and (5.1.19), it follows that

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\le \langle u_n - z, x_{n+1}-z\rangle \\
&= \alpha_n\langle f(x_n)-f(z), x_{n+1}-z\rangle + \alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle V_i y_n - z, x_{n+1}-z\rangle \\
&\le \alpha_n\rho\|x_n-z\|\|x_{n+1}-z\| + \alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad (1-\alpha_n)\|y_n-z\|\|x_{n+1}-z\| \\
&\le \alpha_n\rho\|x_n-z\|\|x_{n+1}-z\| + \alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad (1-\alpha_n)\|\beta_n Sx_n + (1-\beta_n)x_n - z\|\|x_{n+1}-z\| \\
&\le \alpha_n\rho\|x_n-z\|\|x_{n+1}-z\| + \alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad (1-\alpha_n)\|x_n-z\|\|x_{n+1}-z\| + (1-\alpha_n)\beta_n\|Sz-z\|\|x_{n+1}-z\| \\
&= [1-\alpha_n(1-\rho)]\|x_n-z\|\|x_{n+1}-z\| + \alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad +(1-\alpha_n)\beta_n\|Sz-z\|\|x_{n+1}-z\|
\end{aligned}$$

$$\begin{aligned}
&\leq \left[\frac{1-\alpha_n(1-\rho)}{2}\right]\left[\|x_n - z\|^2 + \|x_{n+1} - z\|^2\right] \\
&\quad +\alpha_n\langle f(z) - z, x_{n+1} - z\rangle + (1-\alpha_n)\beta_n\|Sz - z\|\|x_{n+1} - z\| \\
&\leq \left[1 - \frac{2(1-\rho)\alpha_n}{1+(1-\rho)\alpha_n}\right]\|x_n - z\|^2 + \left[\frac{2\alpha_n}{1+(1-\rho)\alpha_n}\right] \\
&\quad \times\langle f(z) - z, x_{n+1} - z\rangle + \left[\frac{2(1-\alpha_n)\beta_n}{1+(1-\rho)\alpha_n}\right]\|Sz - z\|\|x_{n+1} - z\| \\
&= \left[1 - \frac{2(1-\rho)\alpha_n}{1+(1-\rho)\alpha_n}\right]\|x_n - z\|^2 + \left[\frac{2(1-\rho)\alpha_n}{1+(1-\rho)\alpha_n}\right] \\
&\quad \times\left\{\frac{1}{1-\rho}\langle f(z) - z, x_{n+1} - z\rangle + \frac{(1-\alpha_n)\beta_n}{(1-\rho)\alpha_n}\|Sz - z\|\|x_{n+1} - z\|\right\}.
\end{aligned}$$

Let $\gamma_n = \frac{2(1-\rho)\alpha_n}{1+(1-\rho)\alpha_n}$ and $\delta_n = \frac{2(1-\rho)\alpha_n}{1+(1-\rho)\alpha_n}\left\{\frac{1}{1-\rho}\langle f(z) - z, x_{n+1} - z\rangle + \frac{(1-\alpha_n)\beta_n}{(1-\rho)\alpha_n}\|Sz - z\|\|x_{n+1} - z\|\right\}$ for all $n \geq 1$. Since

$$\limsup_{n\to\infty}\left\{\frac{1}{1-\rho}\langle f(z) - z, x_{n+1} - z\rangle + \frac{(1-\alpha_n)\beta_n}{(1-\rho)\alpha_n}\|Sz - z\|\|x_{n+1} - z\|\right\} \leq 0,$$

$\sum\limits_{i=1}^{\infty}\alpha_n = \infty$ and $\frac{2(1-\rho)\alpha_n}{1+(1-\rho)\alpha_n} \geq (1-\rho)\alpha_n$, we have

$$\sum_{n=1}^{\infty}\gamma_n = \infty \text{ and } \limsup_{n\to\infty}\frac{\delta_n}{\gamma_n} \leq 0.$$

Hence, by Lemma 2.1.23, we conclude that $x_n \to z$ as $n \to \infty$. This completes the proof. □

Remark 5.1.5. In the iterative scheme (5.1.16), if we set $f \equiv 0$, then we get $x_n \to z = P_{\mathcal{F}}0$. In this case, from (5.1.17), it follows that

$$\langle z, z - x\rangle \leq 0, \quad \forall x \in \mathcal{F}.$$

That is

$$\|z\|^2 \leq \langle z, x\rangle \leq \|z\|\|x\|, \quad \forall x \in \mathcal{F}.$$

Therefore, the point z is the unique solution to the following quadratic minimization problem:

$$z = arg\min_{x\in\mathcal{F}}\|x\|^2.$$

By changing the restrictions on parameters in Theorem 5.1.4, we obtain the following results.

Theorem 5.1.6. *Let* C *be a nonempty closed and convex subset of a real Hilbert space* H. *Let* $f : C \to H$ *be a* ρ*-contraction mapping,* $S : C \to C$ *be a nonexpansive mapping and* $\{T_i\}_{i=1}^{\infty} : C \to C$ *be a countable family of* k_i*-strict pseudo-contraction mappings and* $\mathcal{F} = \bigcap\limits_{i=1}^{\infty} F(T_i) \neq \emptyset$. *Let* $\alpha_0 = 1$, *and* $x_1 \in C$ *and define the sequence* $\{x_n\}$ *by*

$$\begin{cases} y_n = P_C[\beta_n S x_n + (1-\beta_n)x_n] = \beta_n S x_n + (1-\beta_n)x_n \\ x_{n+1} = P_C[\alpha_n f(x_n) + \sum\limits_{i=1}^{n}(\alpha_{i-1}-\alpha_i)V_i y_n], \quad \forall n \geq 1 \end{cases} \tag{5.1.20}$$

where $\{\alpha_n\} \subset (0,1)$ *and* $\{\alpha_n\}$ *is a strictly decreasing sequence,* $V_i = k_i I + (1-k_i)T_i$, $\{\beta_n\} \subset (0,1)$ *and* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences satisfying the conditions (H2) with* $\tau \in (0,\infty)$, *(H3), (H8) and (H9). Then the sequence* $\{x_n\}$ *converges strongly to a point* $x^* \in \mathcal{F}$, *which is the unique solution of the variational inequality:*

$$\langle \frac{1}{\tau}(I-f)x^* + (I-S)x^*, x - x^*\rangle \geq 0, \quad \forall x \in \mathcal{F}. \tag{5.1.21}$$

Proof. First, we shows that (5.1.21) has the unique solution. Let x' and x^* be two solutions. Then, since x' is solution, for $y = x^*$ one has

$$\langle (I-f)x', x' - x^*\rangle \leq \tau\langle (I-S)x', x^* - x'\rangle \tag{5.1.22}$$

and

$$\langle (I-f)x^*, x^* - x'\rangle \leq \tau\langle (I-S)x^*, x' - x^*\rangle. \tag{5.1.23}$$

Adding (5.1.22) and (5.1.23), we obtain

$$\begin{aligned} (1-\rho)\|x' - x^*\|^2 &\leq \langle (I-f)x' - (I-f)x^*, x' - x^*\rangle \\ &\leq -\rho\langle (I-S)x' - (I-S)x^*, x' - x^*\rangle \leq 0 \end{aligned}$$

so $x' = x^*$. Also now the condition (H2) with $0 < \tau < \infty$ implies (H1) so the sequence $\{x_n\}$ is bounded. Moreover, since (H8) implies (H6) and (H7), then $\{x_n\}$ is asymptotically regular.

Similarly, by Proposition 5.1.2, the weak cluster points set of x_n, i.e., $\omega_w(x_n)$, is a subset of $\mathcal{F}$.

From (5.1.6)-(5.1.10), we observe that

$$\begin{aligned}
\frac{\|x_{n+1}-x_n\|}{\beta_n} &\leq \frac{\|u_n-u_{n-1}\|}{\beta_n} \\
&\leq [1-(1-\rho))\alpha_n]\frac{\|x_n-x_{n-1}\|}{\beta_n}+M\left[\frac{|\alpha_n-\alpha_{n-1}|}{\beta_n}+\frac{|\beta_n-\beta_{n-1}|}{\beta_n}\right] \\
&= [1-(1-\rho))\alpha_n]\frac{\|x_n-x_{n-1}\|}{\beta_{n-1}}+[1-(1-\rho))\alpha_n]\|x_n-x_{n-1}\| \\
&\quad \times\left[\frac{1}{\beta_n}-\frac{1}{\beta_{n-1}}\right]+M\left[\frac{|\alpha_n-\alpha_{n-1}|}{\beta_n}+\frac{|\beta_n-\beta_{n-1}|}{\beta_n}\right] \\
&\leq [1-(1-\rho))\alpha_n]\frac{\|x_n-x_{n-1}\|}{\beta_{n-1}}+\|x_n-x_{n-1}\|\left[\frac{1}{\beta_n}-\frac{1}{\beta_{n-1}}\right] \\
&\quad +M\left[\frac{|\alpha_n-\alpha_{n-1}|}{\beta_n}+\frac{|\beta_n-\beta_{n-1}|}{\beta_n}\right] \\
&\leq [1-(1-\rho))\alpha_n]\frac{\|x_n-x_{n-1}\|}{\beta_{n-1}}+\alpha_n K\|x_n-x_{n-1}\| \\
&\quad +M\left[\frac{|\alpha_n-\alpha_{n-1}|}{\beta_n}+\frac{|\beta_n-\beta_{n-1}|}{\beta_n}\right] \\
&\leq [1-(1-\rho))\alpha_n]\frac{\|u_n-u_{n-1}\|}{\beta_{n-1}}+\alpha_n K\|x_n-x_{n-1}\| \\
&\quad +M\left[\frac{|\alpha_n-\alpha_{n-1}|}{\beta_n}+\frac{|\beta_n-\beta_{n-1}|}{\beta_n}\right].
\end{aligned}$$

Let $\gamma_n=(1-\rho)\alpha_n$ and $\delta_n=\alpha_n K\|x_n-x_{n-1}\|+M\left[\frac{|\alpha_n-\alpha_{n-1}|}{\beta_n}+\frac{|\beta_n-\beta_{n-1}|}{\beta_n}\right]$. From condition (H3) and (H8), we have

$$\sum_{i=1}^{\infty}\gamma_n=\infty \quad \textit{and} \quad \lim_{n\to\infty}\frac{\delta_n}{\gamma_n}=0.$$

By Lemma 2.1.23, we obtain

$$\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\beta_n}=0, \quad \lim_{n\to\infty}\frac{\|u_{n+1}-u_n\|}{\beta_n}=\lim_{n\to\infty}\frac{\|u_{n+1}-u_n\|}{\alpha_n}=0.$$

From (5.1.20), we have

$$\begin{aligned}
x_n-x_{n-1} &= (1-\alpha_n)x_n-\Bigg[P_C[u_n]-u_n+\alpha_n f(x_n) \\
&\quad +\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(V_i y_n-y_n)+(1-\alpha_n)y_n\Bigg] \\
&= (1-\alpha_n)\beta_n(x_n-Sx_n)+(u_n-P_C[u_n]) \\
&\quad +\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(y_n-V_i y_n)+\alpha_n(x_n-f(x_n)).
\end{aligned}$$

It follows that

$$\begin{aligned}\frac{x_n - x_{n-1}}{(1-\alpha_n)\beta_n} &= (x_n - Sx_n) + \frac{1}{(1-\alpha_n)\beta_n}(u_n - P_C[u_n]) \\ &+\frac{1}{(1-\alpha_n)\beta_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(y_n - V_i y_n) \\ &+\frac{\alpha_n}{(1-\alpha_n)\beta_n}(x_n - f(x_n)).\end{aligned}$$

Let $v_n = \frac{x_n - x_{n-1}}{(1-\alpha_n)\beta_n}$. For all $z \in \mathcal{F} = \bigcap\limits_{i=1}^{\infty} F(T_i) = \bigcap\limits_{i=1}^{\infty} F(V_i)$, we get

$$\begin{aligned}\langle v_n, x_n - z\rangle &= \frac{1}{(1-\alpha_n)\beta_n}\langle u_n - P_C[u_n], P_C[u_{n-1}] - z\rangle \qquad (5.1.24)\\ &+\frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)x_n, x_n - z\rangle + \langle x_n - Sx_n, x_n - z\rangle \\ &+\frac{1}{(1-\alpha_n)\beta_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle y_n - V_i y_n, x_n - z\rangle. \qquad (5.1.25)\end{aligned}$$

By Lemma 2.6.29, we have

$$\begin{aligned}\langle x_n - Sx_n, x_n - z\rangle &= \langle (I-S)x_n - (I-S)z, x_n - z\rangle + \langle (I-S)z, x_n - z\rangle \\ &\geq \langle (I-S)z, x_n - z\rangle, \qquad (5.1.26)\end{aligned}$$

$$\begin{aligned}\langle (I-f)x_n, x_n - z\rangle &= \langle (I-f)x_n - (I-f)z, x_n - z\rangle + \langle (I-f)z, x_n - z\rangle \\ &\geq (1-\rho)\|x_n - z\|^2 + \langle (I-f)z, x_n - z\rangle \qquad (5.1.27)\end{aligned}$$

and

$$\begin{aligned}\langle y_n - V_i y_n, x_n - z\rangle &= \langle (I-V_i)y_n - (I-V_i)z, x_n - y_n\rangle \qquad (5.1.28)\\ &+\langle (I-V_i)y_n - (I-V_i)z, y_n - z\rangle \\ &\geq \langle (I-V_i)y_n - (I-V_i)z, x_n - y_n\rangle \\ &= \beta_n\langle (I-V_i)y_n, x_n - Sx_n\rangle, \quad \forall i \geq 1. \qquad (5.1.29)\end{aligned}$$

By Lemma 2.1.12(1), we obtain

$$\begin{aligned}\langle u_n - P_C[u_n], P_C[u_{n-1}] - z\rangle &= \langle u_n - P_C[u_n], P_C[u_{n-1}] - P_C[u_n]\rangle \quad (5.1.30)\\ &+\langle u_n - P_C[u_n], P_C[u_n] - z\rangle \\ &\geq \langle u_n - P_C[u_n], P_C[u_{n-1}] - P_C[u_n]\rangle. \quad (5.1.31)\end{aligned}$$

Now, from(5.1.24)-(5.1.30), it follows that

$$\begin{aligned}\langle v_n, x_n - z\rangle &\geq \frac{1}{(1-\alpha_n)\beta_n}\langle u_n - P_C[u_n], P_C[u_{n-1}] - P_C[u_n]\rangle \\ &+\frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)z, x_n - z\rangle + \langle (I-S)z, x_n - z\rangle \\ &+\frac{1}{(1-\alpha_n)}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle (I-V_i)y_n, x_n - Sx_n\rangle \\ &+\frac{(1-\rho)\alpha_n}{(1-\alpha_n)\beta_n}\|x_n - z\|^2.\end{aligned} \tag{5.1.32}$$

We observe from (5.1.32) that

$$\begin{aligned}\|x_n - z\|^2 &\leq \frac{(1-\alpha_n)\beta_n}{(1-\rho)\alpha_n}\Big[\langle v_n, x_n - z\rangle - \langle (I-S)z, x_n - z\rangle\Big] \\ &+\frac{\|u_{n-1}-u_n\|}{(1-\rho)\alpha_n}\|u_n - P_C[u_n]\| - \frac{1}{1-\rho}\langle (I-f)z, x_n - z\rangle \\ &-\frac{\beta_n}{(1-\rho)\alpha_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle (I-V_i)y_n, x_n - Sx_n\rangle,\end{aligned} \tag{5.1.33}$$

since $v_n \to 0$ and $(I-V_i)y_n \to 0$, as $n \to \infty$, then every weak cluster point of $\{x_n\}$ is also a strong cluster point. By Proposition 5.1.2, $\{x_n\}$ is bounded, thus there exists a subsequence $\{x_{n_k}\}$ converging to x^*.

For all $z \in \mathcal{F}$ by (5.1.24), we compute

$$\begin{aligned}\langle (I-f)x_{n_k}, x_{n_k} - z\rangle &= \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle v_{n_k}, x_{n_k} - z\rangle \\ &-\frac{1}{\alpha_{n_k}}\langle u_{n_k} - P_C[u_{n_k}], P_C[u_{n_k-1}] - z\rangle \\ &-\frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle x_{n_k} - Sx_{n_k}, x_{n_k} - z\rangle \\ &-\frac{1}{\alpha_{n_k}}\sum_{i=1}^{n_k}(\alpha_{i-1}-\alpha_i)\langle y_{n_k} - V_i y_{n_k}, x_{n_k} - z\rangle \\ &\leq \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle v_{n_k}, x_{n_k} - z\rangle \\ &-\frac{\beta_{n_k}}{\alpha_{n_k}}\sum_{i=1}^{n_k}(\alpha_{i-1}-\alpha_i\langle (I-V_i)y_{n_k}, x_{n_k} - Sx_{n_k}\rangle \\ &-\frac{1}{\alpha_{n_k}}\|u_{n_k-1} - u_{n_k}\|\|u_{n_k} - P_C[u_{n_k}]\| \\ &-\frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle (I-S)z, x_{n_k} - z\rangle.\end{aligned} \tag{5.1.34}$$

Since $v_n \to 0, (I-V_i)y_n \to 0$ for all $i \geq 1$, and $\|u_n - u_{n-1}\|/\alpha_n \to 0$, letting $k \to \infty$ in (3.30), we obtain

$$\langle (I-f)x^*, x^* - z\rangle \leq -\tau\langle (I-S)z, x^* - z\rangle, \quad \forall z \in \mathcal{F}.$$

Since (5.1.21) has the unique solution, it follows that $\omega_w(x_n) = \{x^*\}$. Since every weak cluster point of $\{x_n\}$ is also a strong cluster point, we conclude that $x_n \to x^*$ as $n \to \infty$. This completes the proof. $\square$

5.2 Hybrid Extragradient Method for Fixed Points and Variational Inequality Problems

Let C be a nonempty, closed and convex subset of a real Hilbert space H, let F be a bifunction $F : C \times C \to \mathbb{R}$ satisfying conditions $(A1) - (A5)$ (Lemma 2.6.34), let A be an α-inverse-strongly monotone mapping of C into H. Let S be a ξ-strict pseudocontraction mapping from C to C.

we consider the sequences $\{x_n\}, \{y_n\}, \{z_n\}, \{w_n\}$ and $\{t_n\}$ generated by $x_0 \in C$ and

$$\begin{cases} y_n = \arg\min_{y\in C}\{\lambda_n F(x_n, y) + \frac{1}{2}\|y - x_n\|^2\}, \\ z_n = \arg\min_{y\in C}\{\lambda_n F(y_n, y) + \frac{1}{2}\|y - x_n\|^2\}, \\ w_n = P_C(z_n - \lambda_n A z_n), \\ t_n = \alpha_n x_n + (1-\alpha_n)[(1-\mu)Sw_n + \mu P_C(1-\beta_n)w_n], \\ C_n = \{z \in C : \|t_n - z\| \le \|x_n - z\|\}, \\ D_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n\cap D_n} x_0, \end{cases} \tag{5.2.1}$$

for every $n \in N$, where μ be a constant in $(0, 1)$, $\{\alpha_n\} \subset [0, 1), \{\beta_n\} \subset (0, 1), \{\lambda_n\} \subset (0, 1]$.

Algorithm 1

Choose the sequences $\{\alpha_n\} \subset [0, 1), \{\beta_n\} \subset (0, 1), \{\lambda_n\} \subset (0, 1]$ and μ be a constant in $(0, 1)$.

1. Let $x_0 \in C$. Set $n = 0$.

2. Solve successively the strongly convex programs
$\arg\min_{y\in C}\{\lambda_n F(x_n, y) + \frac{1}{2}\|y - x_n\|^2\}$ and
$\arg\min_{y\in C}\{\lambda_n F(y_n, y) + \frac{1}{2}\|y - x_n\|^2\}$
to obtain the unique optimal solution y_n and z_n, respectively.

3. Compute $w_n = P_C(z_n - \lambda_n A z_n)$.

4. Compute
$t_n = \alpha_n x_n + (1-\alpha_n)[(1-\mu)Sw_n + \mu P_C(1-\beta_n)w_n]$. If $y_n = x_n$ and $t_n = x_n$, then STOP:
$x_n \in EP(F) \cap Fix(S) \cap VI(C, A)$. Otherwise, go to Step 5.

5. Compute $x_{n+1} = P_{C_n \cap D_n} x_0$, where
$C_n = \{z \in C : \|t_n - z\| \leq \|x_n - z\|\}$,
$D_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}$.

6. Set $n := n + 1$, and go to Step 3.

In the sequel, we also suppose that the sequences of parameters $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}, \xi$ and μ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [\lambda_{\min}, \lambda_{\max}]$, where $0 < \lambda_{\min} \leq \lambda_{\max} < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$;

(ii) $\{\alpha_n\} \subset [0, c]$ for some $c < 1$;

(iii) $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iv) ξ and μ be constant, where $0 \leq \xi < \mu < 1$.

Now, let $\{x_n\}, \{y_n\}, \{z_n\}$, and $\{w_n\}$ be the sequences generated by combination of the hybrid extragradient method, variational inequality by projection method and the fixed point method described at the begining of this section. These sequence satisfy the following properties. Here we start our main theorem.

Theorem 5.2.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H, let F be a bifunction $F : C \times C \to \mathbb{R}$ satisfying conditions $(A1) - (A5)$, let A be an α-inverse-strongly monotone mapping of C into H. Let S be a ξ-strict pseudocontraction mapping from C to C and such that $\Theta := EP(F) \cap Fix(S) \cap VI(C, A) \neq \emptyset$. Suppose that the sequences $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}$ and μ satisfying the conditions $(i) - (iv)$. Then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to the projection of x_0 onto the set Θ.*

Proof. ***Step 1.*** We show that sequence $\{x_n\}$ is well defined. From definition of C_n and D_n, it is obvious that C_n is closed and D_n is closed and convex for every $n \in \mathbb{N}$. We prove that C_n is convex. Since $\|t_n - z\| \leq \|x_n - z\|$ is equivalent to

$$\|t_n - x_n\|^2 + 2\langle t_n - x_n, x_n - z \rangle \leq 0$$

it follows that C_n is convex. So $C_n \cap D_n$ is closed convex subset of H for any $n \in \mathbb{N}$.
Let $x^* \in \Theta$.
Then $x^* = P_C(x^* - \lambda_n Ax^*)$. Since $w_n = P_C(z_n - \lambda_n Az_n)$, we consider

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|[P_C(z_n - \lambda_n Az_n)] - [P_C(x^* - \lambda_n Ax^*)]\|^2 \\
&\leq \|(z_n - \lambda_n Az_n) - (x^* - \lambda_n Ax^*)\|^2 \\
&= \|z_n - x^*\|^2 - 2\lambda_n\langle z_n - x^*, Az_n - Ax^*\rangle + \lambda_n^2\|Az_n - Ax^*\|^2 \\
&\leq \|z_n - x^*\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Az_n - Ax^*\|^2 \qquad (5.2.2) \\
&\leq \|z_n - x^*\|^2.
\end{aligned}$$

By Proposition 2.6.35 (ii), we have

$$\begin{aligned}
&\|z_n - x^*\|^2 \\
\leq\ &\|x_n - x^*\|^2 - (1 - 2\lambda_n c_1)\|y_n - x_n\|^2 - (1 - 2\lambda_n c_2)\|z_n - y_n\|^2 \qquad (5.2.3)
\end{aligned}$$

that is, we obtain $\|z_n - x^*\| \leq \|x_n - x^*\|$ and $\|w_n - x^*\| \leq \|x_n - x^*\|$.
Set $u_n := (1-\mu)Sw_n + \mu P_C(1-\beta_n)w_n, \quad \text{for all} \quad n \geq 0$. Then, we have
$t_n = \alpha_n x_n + (1-\alpha_n)u_n$, It follows that

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|(1-\mu)Sw_n + \mu P_C(1-\beta_n)w_n - x^*\|^2 \\
&\leq \|(1-\mu)(Sw_n - x^*) + \mu\big[(1-\beta_n)w_n - x^*\big]\|^2 \\
&\leq \|w_n - x^*\|^2 - (1-\mu)(\mu - \xi)\|Sw_n - w_n\|^2 \\
&\quad -\beta_n\mu^2\|w_n\|^2 \qquad (5.2.4) \\
&\leq \|w_n - x^*\|^2 \leq \|x_n - x^*\|^2
\end{aligned}$$

that is, $\|u_n - x^*\| \leq \|x_n - x^*\|$. Since $t_n = \alpha_n x_n + (1-\alpha_n)u_n$ for every $x^* \in \Theta$, we have

$$\begin{aligned}
\|t_n - x^*\|^2 &= \|\alpha_n x_n + (1-\alpha_n)u_n - x^*\|^2 \\
&= \|\alpha_n(x_n - x^*) + (1-\alpha_n)(u_n - x^*)\|^2 \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2 \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 \\
&\leq \|x_n - x^*\|^2.
\end{aligned}$$

Hence $\|t_n - x^*\| \leq \|x_n - x^*\|$ for every $n \geq 0$ and $x^* \in C_n$. So, we have

$$\Theta := EP(F) \cap Fix(S) \cap VI(A, C) \subset C_n, \qquad \forall n \in \mathbb{N}. \qquad (5.2.5)$$

Next, we will show that

$$\Theta := EP(F) \cap Fix(S) \cap VI(A, C) \subset C_n \cap D_n, \forall n \in \mathbb{N}. \tag{5.2.6}$$

We prove this by induction.

For $n = 0$, we have $x_0 = x \in C, \quad \Theta \subset C_0$ and $D_0 = C$. So, we get $\Theta \subset C_0 \cap D_0$. Suppose that $\Theta \subset C_k \cap D_k$ for some $k \in \mathbb{N}$. Since $C_k \cap D_k$ is closed and convex, we can define $x_{k+1} = P_{C_k \cap D_k}(x_0)$. From $x_{k+1} = P_{C_k \cap D_k}(x_0)$, we also have

$$\langle x_{k+1} - z, x_0 - x_{k+1} \rangle \geq 0, \quad \forall z \in C_k \cap D_k. \tag{5.2.7}$$

Since $\Theta \subset C_k \cap D_k(x_0)$, we also have

$$\langle x_{k+1} - x^*, x_0 - x_{k+1} \rangle \geq 0, \quad \forall x^* \in \Theta. \tag{5.2.8}$$

So, we get $\Theta \subset D_{k+1}$. Then we obtain $\Theta \subset C_{k+1} \cap D_{k+1}$. This implies that $\{x_n\}$ is well defined.

Step 2. Next, let us show that $\{x_n\}, \{w_n\}$ are bounded. Put $z_0 = P_\Theta x_0$. From $x_{n+1} = P_{C_n \cap D_n}(x_0)$, we get

$$\|x_{n+1} - x_0\| \leq \|z - x_0\|, \quad \forall z \in C_n \cap D_n. \tag{5.2.9}$$

From $z_0 \in \Theta \subset C_n \cap D_n$, we also have

$$\|x_{n+1} - x_0\| \leq \|z_0 - x_0\| \tag{5.2.10}$$

for all $n \in \mathbb{N} \cup \{0\}$ and hence $\{x_n\}$ is bounded. Since $\|w_n - x^*\| \leq \|x_n - x^*\|$ then $\{w_n\}$ also bounded.

Step 3. We will show that $\lim_{n\to\infty} \|Sw_n - w_n\| = 0$.

Since $x_{n+1} \in C_n \cap D_n \subset D_n$ and $x_n = P_{D_n}(x_0)$, we get $\|x_n - x_0\| \leq \|x_{n+1} - x_0\|$. From the boundedness of $\{x_n\}$, we get that $\lim_{n\to\infty} \|x_n - x_0\|$ exists. So, we obtain $\left(\|x_n - x_0\|^2 - \|x_{n+1} - x_0\|^2\right) \to 0$. On the other hand, from $x_{n+1} \in D_n$, we have

$$\langle x_n - x_{n+1}, x_0 - x_n \rangle \geq 0. \tag{5.2.11}$$

So, for $n \in \mathbb{N} \cup \{0\}$, we get

$$\begin{aligned}
\|x_{n+1} - x_0\|^2 &= \|(x_{n+1} - x_n) + (x_n - x_0)\|^2 \\
&= \|x_{n+1} - x_n\|^2 + \|x_n - x_0\|^2 + 2\langle x_{n+1} - x_n, x_n - x_0 \rangle
\end{aligned}$$

from (5.2.11), we have

$$\|x_{n+1}-x_0\|^2 \;\geq\; \|x_{n+1}-x_n\|^2+\|x_n-x_0\|^2$$

and

$$\|x_{n+1}-x_n\|^2 \;\leq\; \|x_{n+1}-x_0\|^2-\|x_n-x_0\|^2.$$

This implies

$$\|x_{n+1}-x_n\| \to 0. \tag{5.2.12}$$

Since $x_{n+1}\in C_n$, we have

$$C_n=\{z\in C:\|t_n-z\|\leq\|x_n-z\|\};$$

$$\|t_n-x_{n+1}\|\leq\|x_n-x_{n+1}\|$$

and

$$\begin{aligned}\|x_n-t_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-t_n\|\\ &\leq 2\|x_n-x_{n+1}\|.\end{aligned}$$

By (5.2.12), we obtain

$$\lim_{n\to\infty}\|x_n-t_n\|=0. \tag{5.2.13}$$

For $x^*\in\Theta$, from (5.2.2), (5.2.3) and (5.2.4), we can choose a constant $M>0$ such that, $\sup_n\{\|w_n\|^2\}\leq M$. We observe that

$$\begin{aligned}\|t_n-x^*\|^2 &\leq \alpha_n\|x_n-x^*\|^2+(1-\alpha_n)\|u_n-x^*\|^2\\ &\leq \alpha_n\|x_n-x^*\|^2+(1-\alpha_n)\big[\|w_n-x^*\|^2\\ &\quad -(1-\mu)(\mu-\xi)\|Sw_n-w_n\|^2-\beta_n\mu^2\|w_n\|^2\big]\\ &= \|x_n-x^*\|^2-(1-\alpha_n)(1-2\lambda_nc_1)\|y_n-x_n\|^2\\ &\quad -(1-\alpha_n)(1-2\lambda_nc_2)\|z_n-y_n\|^2+(1-\alpha_n)\lambda_n(\lambda_n-2\alpha)\|Az-Ax^*\|^2\\ &\quad -(1-\alpha_n)(1-\mu)(\mu-\xi)\|Sw_n-w_n\|^2-(1-\alpha_n)\beta_n\mu^2M.\end{aligned}$$

Therefore, we get

$$\begin{aligned}&(1-\alpha_n)(1-2\lambda_nc_1)\|y_n-x_n\|^2\\ &\quad\leq \|x_n-x^*\|^2-\|t_n-x^*\|^2-(1-\alpha_n)\beta_n\mu^2M\\ &\quad= \big[\|x_n-x^*\|+\|t_n-x^*\|\big]\big[\|x_n-t_n\|\\ &\qquad -(1-\alpha_n)\beta_n\mu^2M\big].\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n - t_n\| = 0, \lim_{n\to\infty}\beta_n = 0, 1-\alpha_n \geq 1-c > 0, 1-2\lambda_n c_1 > 1-2\lambda_{\max}c_1 > 0$ and the sequence $\{x_n\}, \{t_n\}$ are bounded, we get $\lim_{n\to\infty}\|y_n - x_n\| = 0$. By similar way since $\lim_{n\to\infty}\beta_n = 0, 1-\alpha_n \geq 1-c > 0$ and $1-2\lambda_n c_2 > 1-2\lambda_{\max}c_2 > 0$, we have $\lim_{n\to\infty}\|z_n - y_n\| = 0$. Since $\lim_{n\to\infty}\beta_n = 0, 1-\alpha_n \geq 1-c > 0$ and $-\lambda_n(\lambda_n - 2\alpha) > 0$, we obtain

$$\lim_{n\to\infty}\|Az - Ax^*\| = 0.$$

By $\lim_{n\to\infty}\beta_n = 0, \quad 1-\alpha_n \geq 1-c > 0$ and $(1-\mu)(\mu-\xi) > 0$, we have

$$\lim_{n\to\infty}\|Sw_n - w_n\| = 0.$$

Step 4. We will show that $\tilde{x} \in \Theta$.

(4.1). We will show that $\tilde{x} \in EP(F)$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which $x_{n_i} \rightharpoonup \tilde{x}$ and $\lim_{n\to\infty}\|x_n - y_n\| = 0$, we have that $y_{n_i} \rightharpoonup \tilde{x}$. On the other hand, by using Proposition 2.6.35, we have, for every $y \in C$ and for every $i \in \mathbb{N}$, that

$$\langle x_{n_i} - y_{n_i}, y - y_{n_i}\rangle \leq \lambda_{n_i}F(x_{n_i}, y) - \lambda_{n_i}F(x_{n_i}, y_{n_i}). \tag{5.2.14}$$

Since $\|x_{n_i} - y_{n_i}\| \to 0$ and $y - y_{n_i} \rightharpoonup y - \tilde{x}$ as $i \to \infty$ and since $\forall i \in \mathbb{N}, \quad 0 < \lambda_{\min} \leq \lambda_{n_i} \leq \lambda_{\max}$. As $i \to \infty$, we get $F(\tilde{x}, y) \geq 0, \quad \forall y \in C$. It means that $\tilde{x} \in EP(F)$.

(4.2). We will show that $\tilde{x} \in Fix(S)$. Since $\{w_n\}$ is bounded then there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ which converges weakly to $\tilde{x}$. Since S is a ξ-strict pseudocontraction mapping, we know that the mapping $I - S$ is demiclosed at zero. From $\|Sw_n - w_n\| \to 0$ as $n \to \infty$ and $\{w_{n_i}\} \rightharpoonup \tilde{x}$. Thus, we obtain that $\tilde{x} \in Fix(S)$.

(4.3). Finally, we show that $\tilde{x} \in VI(C, A)$. Define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{5.2.15}$$

Then, we have that T is maximal monotone operator,
we have

$$\begin{aligned} x \in Tv = Av + N_C v \quad &\Leftrightarrow \quad x - Av \in N_C v \\ &\Leftrightarrow \quad \langle v - u, x - Av\rangle \geq 0, (\forall u \in C). \end{aligned}$$

So, for $w_n \in C$, we have that

$$\langle v - w_n, x - Av\rangle \geq 0, \quad (n = 1, 2, 3, ...).$$

We also have

$$\begin{aligned} w_n = P_C(z_n - \lambda_n A z_n) \;&\Leftrightarrow\; \langle w_n - u, z_n - \lambda_n A z_n - w_n\rangle \geq 0, \quad \forall u \in C \\ &\Leftrightarrow\; \langle u - w_n, w_n - (z_n - \lambda_n A z_n)\rangle \geq 0, \quad \forall u \in C \\ &\Leftrightarrow\; \Big\langle u - w_n, \frac{w_n - z_n}{\lambda_n} + A z_n\Big\rangle \geq 0, \quad \forall u \in C. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \langle v - w_{n_i}, x\rangle &\geq \langle v - w_{n_i}, Av\rangle \\ &\geq \langle v - w_{n_i}, Av\rangle - \Big\langle v - w_{n_i}, \frac{w_{n_i} - z_{n_i}}{\lambda_{n_i}} + A z_{n_i}\Big\rangle \\ &= \langle v - w_{n_i}, Av - A w_{n_i}\rangle + \langle v - w_{n_i}, A w_{n_i} - A z_{n_i}\rangle \\ &\quad - \Big\langle v - w_{n_i}, \frac{w_{n_i} - z_{n_i}}{\lambda_{n_i}}\Big\rangle \\ &\geq \langle v - w_{n_i}, A w_{n_i} - A z_{n_i}\rangle - \Big\langle v - w_{n_i}, \frac{w_{n_i} - z_{n_i}}{\lambda_{n_i}}\Big\rangle. \end{aligned}$$

Using $w_{n_i} \to \tilde{x}$ and $\|w_{n_i} - z_{n_i}\| \to 0$ which A is Lipshitz continuous implies that

$$\langle v - \tilde{x}, x\rangle \geq 0, \quad \text{as} \;\; i \to \infty. \tag{5.2.16}$$

Since T is maximal monotone, we have $\tilde{x} \in T^{-1}(0)$, That is $\tilde{x} \in VI(A, C)$. So, we have $\tilde{x} \in \Theta$.

Finally, we show that $x_n \to \tilde{x}$, where $\tilde{x} = P_\Theta x_0$. Since $x_n = P_{D_n} x_0$ and $\tilde{x} \in \Theta \subset D_n$, we have

$$\|x_n - x_0\| \leq \|\tilde{x} - x_0\|. \tag{5.2.17}$$

It follows from $x^* = P_\Theta x_0$ and the lower semicontinuity of norm that

$$\begin{aligned} \|x^* - x_0\| &\leq \|\tilde{x} - x_0\| \leq \lim_{i\to\infty} \inf \|x_{n_i} - x_0\| \\ &\leq \lim_{i\to\infty} \sup \|x_{n_i} - x_0\| \leq \|x^* - x_0\|. \end{aligned}$$

Thus, we obtain that

$$\lim_{i\to\infty} \|x_{n_i} - x_0\| = \|\tilde{x} - x_0\| = \|x^* - x_0\|. \tag{5.2.18}$$

Using the Kadec-Klee property of H, we obtain that $\lim_{i\to\infty} x_{n_i} = \tilde{x} = x^*$. Since $\{x_{n_i}\}$ is an arbitrary subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to $P_{\Theta}x_0$.

□